III. 2. Unit Root Tests

1. Testing Framework – DF and ADF tests
2. Type I and II Errors
3. Testing Sequence
4. Other Unit Root Tests

Testing for a unit root

With constant only:

II. $\Delta Y_t = \mu + \gamma Y_{t-1} + \varepsilon_t$

$H_0 : \gamma = 0; \mu = 0$ ➡ I(1) without drift

$H_{A1} : \gamma < 0; \mu = 0$ ➡ Stationary with no mean

$H_{A2} : \gamma < 0; \mu \neq 0$ ➡ Stationary with nonzero mean

$H_{A3} : \gamma = 0; \mu \neq 0$ ➡ I(1) with drift

Which alternative is appropriate? Look at your data

1. Testing Framework

- Test strategies:
  1. Joint test: H_0: $\gamma = 0; \mu = 0$
     - Test stat: $\Phi_1 \sim F$ Table 6.5, pp. 231
     - If non-rejection, move down to the most restrictive maintained regression
     - If joint null is rejected and series' mean reverting frequently enough without any trend →

       H_{A2}: I(0) with nonzero mean

1. Testing Framework

- Test strategies:

  2. H_0: $\gamma < 0$

  - t-test: $\hat{\tau}_\mu$

Table 6.6, μ=0, pp. 232

1. But if μ is not zero in the DGP, Hylleberg and Mizon(1989a): for smaller sample sizes, $\hat{\tau}_\mu$ is not normal and dependent on μ ; as μ increases $\hat{\tau}_\mu$ approaching to normal. (see Table 6.6, when T=500, μ=10 → normal)
2. For many economic time series, taking log, the range of values for μ (0.05; 0.1), there is relatively little variation in the critical value.

1. Testing Framework

- The models under the alternative hypotheses:

$$\tau: \quad Y_t = \phi Y_{t-1} + \varepsilon_t; |\phi| < 1$$

$$\tau_\mu: \quad Y_t = \mu + \phi Y_{t-1} + \varepsilon_t; |\phi| < 1$$

They may not be realistic alternatives since they cannot generate the trended behavior typical of economic time series.

- Good starting point: the maintained regression

$$Y_t = \mu + \beta t + \phi Y_{t-1} + \varepsilon_t$$

1. Testing Framework

If there is a trend in the data, the maintained regression is more appropriately as:

III. $Y_t = \mu + \phi_1 Y_{t-1} + \beta t + \varepsilon_t$

$H_0: \ (\mu, \phi_1, \beta) = (\mu, 1, 0)$

* $\boxed{H_{A1}: (\mu, \phi_1, \beta) = (\mu, \phi_1, \beta)}$

$H_{A2}: (\mu, \phi_1, \beta) = (\mu, 1, \beta)$ $\longrightarrow$ **I(1): quadratic trend**

$H_{A3}: (\mu, \phi_1, \beta) = (\mu, \phi_1, 0)$ $\longrightarrow$ **I(0): no trend**

Test statistics: $\Phi_3 \sim F$ (DF test); $\hat{\tau}_\tau$

Table 6.7 *Table 6.8*

Unit root test: the ADF test

AR(p)

$$Y_t = \phi(L)Y_{t\text{-}1} + \varepsilon_t$$

- If the true DGP is an AR(p), fitting AR(1) will cause serial correlation in error terms.
- Select a max p such that residuals are WN; for monthly data, we set $p \geqq 12$,
- LM test for serial correlation in residuals;
- Information criteria: AIC, SIC

ADF test when p=2

$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t$

- $\Delta Y_t = \mu + (\phi_1 + \phi_2 - 1) Y_{t-1} - \phi_2 \Delta Y_{t-1} + \varepsilon_t$
- We should consider ΔY_{t-1} in the regression if p=2
- Unit root test is the same as for AR(1)

$H_0 : \gamma = \phi_1 + \phi_2 - 1 = 0$

Test statistic is the t-statistic on the γ

ADF(1); test statistic: τ, τ_μ, τ_τ

(i) When there is a trend;

- Choose a right maintained regression for data; (by checking plots of the time series and ACFs of levels and differences); to be sure that this model nests null and alternative as special cases
- Starting with a regression with an intercept and time trend

$$\Delta Y_t = \mu + \gamma Y_{t-1} + \beta t + \varepsilon_t$$

$$H_0 : \beta = 0; \gamma = 0$$

$$\Phi_3 \sim F; \ \hat{\tau}_\tau$$

- What is a reasonable H_A?
- Test stat: Φ_3 and/or τ_τ
  - H_0: RW with drift (stochastic trend)
  - H_A: stationary but trended (deterministic trend)

Patterson's strategy

Cp: Perron's (1988) strategy (Patterson, p243)

H_0: $\gamma = \beta = 0$;

Φ_3

rejected →

$\gamma \neq 0$ and/or $\beta \neq 0$;

τ_β~6.8 & $t(\beta)$~6.10

$\beta(\mu)$ - null model estimation

↓ not rejected

RW with unrestricted drift

- If use one test, most researchers use the τ_β for the sake of size&power
- If data is trended and the joint null $\gamma = \beta = 0$ is rejected using Φ_3, we favor the alternative of stationarity around a deterministic trend

(ii) Data is not trended but with non-zero mean

- Maintained regression:

$$\Delta Y_t = \mu + \gamma Y_{t-1} + \varepsilon_t$$
$$H_0 : \mu = 0; \gamma = 0$$
$$\Phi_1 \sim F$$

- If data has no trend and the joint null is rejected, we favor the alternative of stationarity since μ~0 as no trend in the data; then use the test statistic τ_μ
- Perron's approach:
  - 1. τ_μ (μ=0)~Table 6.6; if not rejected, then
  - 2. H_0: $\gamma = \mu = 0$; Φ_1, if rejected suggesting $\mu \neq 0$
  - 3. Revisit τ_μ , critical values from the standard normal when the sample size and μ are large enough.

(iii) No trend and non-zero mean

- H_0: pure RW
- H_a: stationarity without an intercept and a time trend

$$\Delta Y_t = \gamma Y_{t-1} + \varepsilon_t$$

$$H_0 : \gamma = 0$$

$$\tau$$

- This is not the case in most economic time series; the maintained regression is not an appropriate starting point.

Review: **Unit Root Test**

Two ways to check if a time series is stationary.

i. Correlogram (a plot of ρ_k vs k)

- Theoretically, if a series is stationary, its population ρ_k will converge to zero.,
- In practice, with finite sample it is difficult to tell the difference between a non-stationary time series data and a slowly-converging stationary series.

- <u>Rule of Thumb</u>
  If autocorrelation drops to zero or close to zero in a few periods, we can say that the underlying series Y is stationary.

Review: **Unit Root Test**

(ii) Unit Root Test

There are various sources of Non-stationarity. Not all can be tested or detected.

- Unit root test is used to detect a unit root.
- It can be applied to the following cases of non-stationarity.

  Case 1: Random walk

  $$Y_t = Y_{t-1} + \varepsilon_t$$

  Case 2: Random walk with shift

  $$Y_t = \alpha + Y_{t-1} + \varepsilon_t$$

  Case 3: Random walk with shift and time trend

  $$Y_t = \alpha + \beta \cdot t + Y_{t-1} + \varepsilon_t$$

DF Unit Root Test

H_0: γ=0 or Y is non-stationary

- If H_0 is true, the OLS is not correct and OLS of γ is not normally distributed.
- OLS of γ, $\hat{\gamma}$,follows a non-standard Distribution. Dickey-fuller created a table to illustrate the distribution of it.
- EViews kindly provides the critical values from the table.
  Reject H_0 if $\hat{\gamma}$ is less than the critical value. Otherwise, accept H_0

DF Unit Root Test

If Y is non-stationary, try to test other forms of Y.

- Trend-stationary ⇒ detrend Y
- Unit root ⇒ use higher-order difference of Y

  If ΔY_t is still non-stationary, use $\Delta^2 Y_t$

  where $\Delta^2 Y_t = \Delta Y_t - \Delta Y_{t-1}$

  $\Delta Y_t = Y_t - Y_{t-1}$
- The unit root test in Eviews

ADF Unit Root Test

- Case 4: General Case

$$Y_t = a + b \cdot t + c_1 Y_{t-1} + c_2 Y_{t-2} + \ldots + c_m Y_{t-m} + \varepsilon_t \Rightarrow$$

$$\Delta Y_t = a + b \cdot t + \tilde{c}_1 Y_{t-1} + \tilde{c}_2 \Delta Y_{t-1} + \ldots + \tilde{c}_m \Delta Y_{t-m} + \varepsilon_t$$

$$\text{where } \tilde{c}_1 = \left(\sum_{i=1}^{m} c_i \right) - 1$$

- Tested Model: augmented DF (ADF) test

$$\Delta Y_t = a + b \cdot t + \gamma Y_{t-1} + \tilde{c}_2 \Delta Y_{t-1} + \ldots + \tilde{c}_m \Delta Y_{t-m} + \varepsilon_t$$

2. Type I Error

- We are never 100% confident on our conclusion. We must allow some probability of committing a mistake or the probability of rejecting the correct hypothesis or punishing the innocent people.
- Define:
  - α = probability of Type I Error or significance level
  - 1-α = confidence level
- Based upon the sampling distribution of the estimator for the parameter, the acceptance region and the rejection region for the statistics can be established (using the probability tables provided).

2. Type II Error

- If the real value of the parameter is different from the hypothesized value or H_0 is wrong, the evidence may lead to the wrong conclusion that H_0 is correct.
- This mistake is called **Type II Error**. However, the assumption about the real value of the parameter must be made. If the real value is not much different from the hypothesized value, then it is more likely that H_0 will be accepted while it is not correct or Type II Error will be made.
- Prob{Type II Error|real value}

  = a function of the real value.
- It is an indicator for Power of Test. If the test is vulnerable to Type II error it has low Power of Test. If the test can distinguish a false value from the real value it has Power of Test.

2. Power of Test: Type II error

- H_A:

$$Y_t = \phi_1 Y_{t-1} + \varepsilon_t$$
$$\phi_1 = 0.8, 0.9, 0.95$$
$$T = 100$$
$$\Phi_1;\ \Phi_3;\ \hat{\tau}_\mu;\ \hat{\tau}_\beta;$$

- Φ_1 ; significance level (the size): 5%
  - 80% of the simulations with $\phi_1 = 0.8,$ reject null
  - 24% of the simulations with $\phi_1 = 0.9,$ reject null
  - 9% of the simulations with $\phi_1 = 0.95,$ reject null

2. Power problem of unit root test

- When the alternative close to 1, power is low; i.e. false non-rejection of the null (type II error) is frequent $\phi_1 \to 1$
- Empirical power approaches the size (significant level) of the test as
- When significant level is low so is the power; can we reverse the role of hypothesis $H_A \rightarrow H_0$?

we still have a "Near observational equivalence"

- Raise the significance level from 5% to 10 or 20%; trade off type I and type II errors

2. Type I error: the size

- Type I error = prob (falsely rejecting the null that a time series contains a unit root)
- Testing strategies involve more than one test; τ_β, τ_μ; higher overall type I error: 5~10 %
- Reduce the significant level at each stage; but the power is also low for near unit root time series

3. Ayat and Burridge's sequential procedure (2000, J. of Econometrics, 95, pp. 71-96)

- 1. Estimate (*) $\Delta Y_t = \mu + \gamma Y_{t-1} + \beta t + \varepsilon_t$
  - H_0: $\gamma=0$ → *t-test* (DF, ADF)
- 2 (a) if not rejecting unit root, maintain this hypothesis, test for the presence of time trend
  - $\Delta y_t = \mu + \beta t + \varepsilon_t$
  - H_0: $\beta=0$ → standard *t-test*
- 2(b) if unit root is rejected, test $\beta=0$ in (*)

3. Sequential procedure

- 3(a)
  - if $\beta = 0$ rejected in 2(a) → data have a unit root and a linear trend (a RW with drift)
  - If $\beta = 0$ rejected in 2(b) → the process is stationary around a linear trend.
  - We can stop here.
- 3(b)
  - If $\beta = 0$ not rejected in 2(a) → a unit root test w/o trend is more powerful, so estimate: $\Delta Y_t = \mu + \gamma Y_{t-1} + \varepsilon_t$

  Perform a second unit root test using t-test
  - If $\beta = 0$ not rejected in 2(b) → the process is I(0) w/o trend, stop here

(*) $\Delta Y_t = \mu + \gamma Y_{t-1} + \beta t + \varepsilon_t$

H_0: $\gamma=0$ → *t-test* (DF, ADF)

Step 1

not rejecting

$\Delta y_t = \mu + \beta t + \varepsilon_t$
H_0: $\beta=0$
standard *t-test*

rejecting

test $\beta=0$ in (*)

Step 2

rejecting

a unit root and a linear trend

not rejecting

a unit root test w/o trend is more powerful, so estimate

$\Delta Y_t = \mu + \gamma Y_{t-1} + \varepsilon_t$ → Second unit root test

rejecting

I(0), no trend

RW with drift

Step 3

rejecting

I(0) around a linear trend

not rejecting

I(0), no trend

3. Sequential procedure

- How good this procedure is at identifying unit root?
  - If there is no trend, the overall significance level is twice the nominal size (size distortion problem).
  - If there is a trend, significance level is closer to nominal level. The prob. that a I(0) is correctly identified is quite low if $\phi > 0.7$. (no size problem, but low power when near unit root)

Uncontroversial practical advice

Look at the data;

If you think there might be a trend,

Include a trend in the test equation

Always include a constant.

Other tips of unit root tests

- It's useful to reverse the null (unit root) and alternative (stationary) hypotheses in some cases (KPSS test)
- Other forms of nonstationarity:
  1. I(2)
  2. I(d); d is not an integer; fractionally integrated

4. Other unit root tests

i. DF: $T\left(\phi-1\right)$

- **AR(1):** $\left(1-\phi L\right)Y_t=\varepsilon_t$
- **H_0:** $\phi=1$

Under null, T($\phi=1$)~ asym ρ (non-normal)

- **Critical values:** ρ , ρ_{μ}, ρ_{τ}

  Do not depend on the DGP (pure RW)

  Do not vary much according to T

4. Other unit root tests

ii. Phillips and Perron test

$$Y_t = \mu + \phi Y_{t-1} + \varepsilon_t$$

DF: $\varepsilon_t \sim$ iid

PP: $\varepsilon_t \sim$ serially correlated

- Add a correction factor to the DF test stat.

(ADF is to add lagged ΔY_t to 'whiten' the serially correlated residuals)

$$Z_{\hat{\rho}_\mu} = T(\hat{\phi} - 1) - CF$$

$$Z_{\hat{\tau}_\mu} = CF_1 \hat{\tau}_\mu - CF_2$$

Problem of PP test

- On the one hand, the PP tests tend to be more powerful but, on the other hand, also subject to more severe size distortions
  - Size problem: actual size is larger than the nominal one when autocorrelations of ε_t are negative
  - more sensitive to model misspecification (the order of autoregressive and moving average components).
- Plotting ACFs help us to detect the potential size problem
  - Economic time series sometimes have negative autocorrelations especially at lag one, we can use a Monte Carlo analysis to simulate the appropriate critical values, which may not be attractive to do.

4. Other unit root tests
iii. Stationarity as the null

- Structural time-series models
- Local level model

$$Y_t = \alpha_t + \xi_t$$

$$\alpha_t = \beta + \alpha_{t-1} + \eta_t$$

$$\begin{bmatrix} \xi_t \\ \eta_t \end{bmatrix} \sim \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_\xi^2 & 0 \\ 0 & \sigma_\eta^2 \end{bmatrix}$$

$$\longrightarrow \quad \Delta Y_t = \beta + \eta_t + \Delta\xi_t$$

KPSS test: equations 7.48 and 7.49, pp. 269.

- Kwiatkowski, D., P. C. B. Phillips, P. Schmidt and Y. Shin, (1992), “Testing the Null Hypothesis of Stationary Against the Alternative of a Unit Root,” *Journal of Econometrics*, 54, 159–178.

iii. Stationarity as the null

Structural model:

$$\Delta Y_t = \beta + \eta_t + \Delta\xi_t$$

$$Var(\Delta Y_t) = \sigma_\eta^2 + 2\sigma_\xi^2$$

$$\gamma(1) = -\sigma_\xi^2$$

$$\rho(1) = -\frac{\sigma_\xi^2}{(\sigma_\eta^2 + 2\sigma_\xi^2)}$$

$$\gamma(k) = 0, k > 1$$

$$\Delta Y_t \sim MA(1)$$

Reduced form:

$$\Delta Y_t = \beta + \varepsilon_t + \theta\varepsilon_{t-1}$$

$$Var(\Delta Y_t) = \sigma_\varepsilon^2(1 + \theta^2)$$

$$\gamma(1) = \theta\sigma_\varepsilon^2$$

$$\rho(1) = \frac{\theta}{(1 + \theta^2)}$$

$$\gamma(k) = 0, k > 1$$

Solve θ in terms of the signal-to-noise ratio: $q = \sigma_\eta^2 / \sigma_\xi^2$

$$\frac{\theta}{(1+\theta^2)} = \frac{-1}{2+q} \quad \Rightarrow \quad \begin{aligned} &q = 0 \Rightarrow \theta = -1 \\ &q \to \infty \Rightarrow \theta \to 0 \end{aligned}$$

$if \;\; \sigma_\eta^2 = 0$

$\Rightarrow$ Y_t is stationary with deterministic time trend

$$Y_t = \alpha_t + \xi_t$$

$$\alpha_t = \alpha_{t-1} + \beta$$

structural : $\Delta Y_t = \beta + \xi_t - \xi_{t-1}$

reduced form : $\Delta Y_t = \beta + \varepsilon_t + \theta\varepsilon_{t-1}$

$$Y_t = (1-L)^{-1}\beta + \varepsilon_t = \alpha_0 + \beta t + \varepsilon_t$$

$$H_0 : \sigma_\eta^2 = 0 \;\; \Rightarrow \;\; Y_t \sim TS$$

$$H_a : \sigma_\eta^2 \neq 0 \;\; \Rightarrow \;\; Y_t \sim DS(RW / drift)$$

iii. KPSS test

- Treat the observed series as the sum of a I(0) and a I(1) component:

  y_t = constant/trend + $x_t + v_t$

  $x_t = x_{t-1} + \varepsilon_t$; nonstationary, $\varepsilon_t \sim IID(0, \sigma_\varepsilon^2)$

  v_t = stationary$\sim IID(0, \sigma_v^2)$
- Testing the null that the variance of the I(1) component is zero:

$$H_0: \sigma_\varepsilon^2 = 0$$

iii. KPSS test

- 1. Regress y_t on a constant and trend; construct the OLS residuals, $e = [e_1, \ldots, e_T]'$
- 2. $S_t = \sum_{i=1}^{t} e_i$; the partial sum of the residuals
- 3. Test statistic: $KPSS = T^{-2} \sum_t S_t^2 \Big/ \sigma_T(l)$

  $\sigma_T(l)$ represents an estimate of the long run variance of the residuals.
- We reject the stationary null when KPSS is large, since that is evidence that the series wanders from its mean.
- As with unit root test, KPSS must be modified if v_t is serially correlated.

4. Other unit root tests
iv. Structural breaks

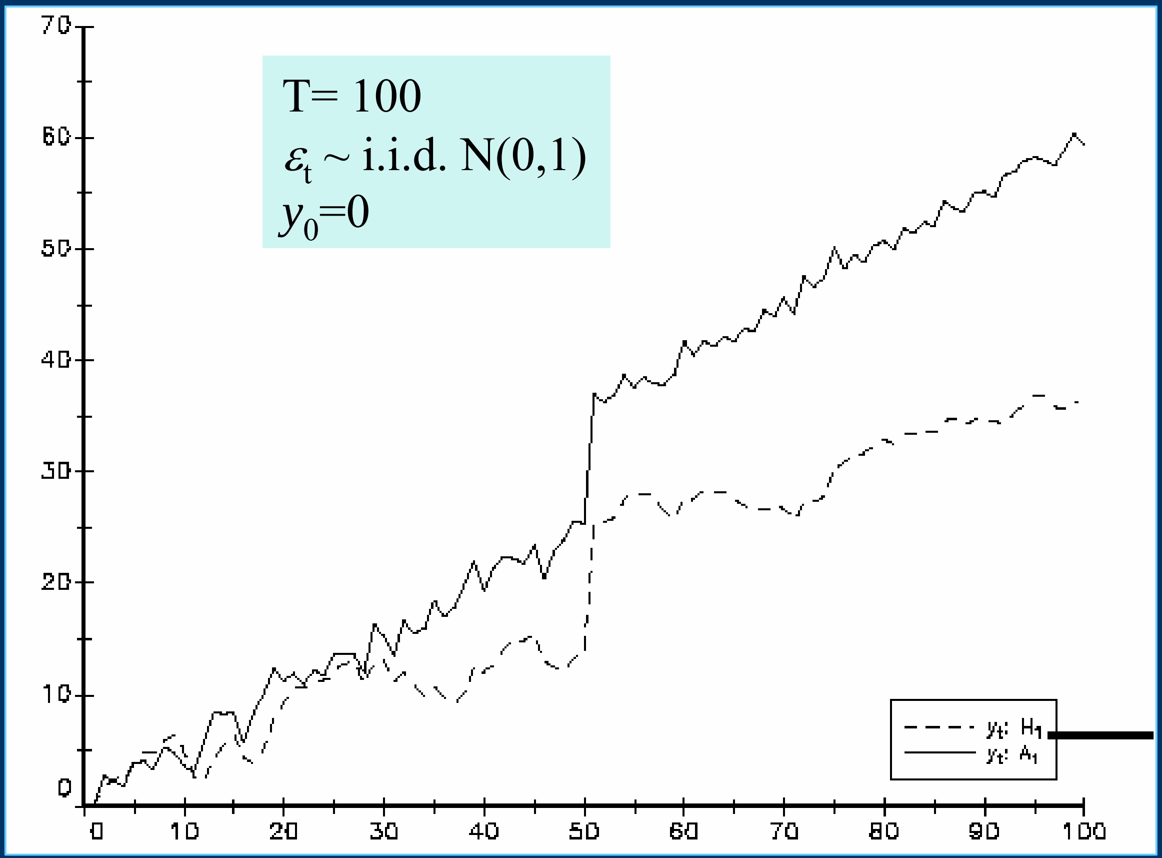
- A stationary time-series may look like nonstationary when there are structural breaks in the intercept or trend
- The unit root tests lead to false nonrejection of the null when we don't consider the structural breaks → low power
- A single breakpoint is introduced in Perron (1989) into the regression model; he (1997) extended it to a case of unknown breakpoint
- **Perron, P., (1989), "The Great Crash, the Oil Price Shock and the Unit Root Hypothesis,"** ***Econometrica*****, 57, 1361–1401.**

iv. Structural breaks

1. Consider the null and alternative hypotheses

- H_0: $y_t = a_0 + y_{t-1} + \mu_1 D_P + \varepsilon_t$
- H_A: $y_t = a_0 + a_2 t + \mu_2 D_L + \varepsilon_t$
  - Pulse break: $D_P = 1$ if $t = T_B + 1$ and zero otherwise,
  - Level break: $D_L = 0$ for $t = 1, \ldots, T_B$ and one otherwise.
- Null: y_t contains a unit root with a one–time jump in the level of the series at time $t = T_B + 1$.
- Alternative: y_t is trend stationary with a one–time jump in the intercept at time $t = T_B + 1$.

Simulated unit root and trend stationary processes with structural break.



H_0: ----

- a_0 = 0.5,
- D_P = 1 for t = 51 zero otherwise,
- μ_1 = 10.

H_A:

- a_2 = 0.5,
- D_L = 1 for t > 50.
- μ_2 = 10

Power of ADF tests: Rejection frequencies of ADF–tests

	Model: $a_0 = a_2 = 0.5$ and $\mu_2 = 10$		
	1% level	5% level	10% level
ADF–tests	0.004	0.344	0.714
	Model: $a_0 = a_2 = 0.5$ and $\mu_2 = 12$		
ADF–tests	0.000	0.028	0.264

- ADF tests are biased toward nonrejection of the null
- Rejection frequency is inversely related to the magnitude of the shift.
- Perron:

estimated values of the autoregressive parameter in the Dickey–Fuller regression was biased toward unity and that this bias increased as the magnitude of the break increased

Testing for unit roots when there are structural changes

- Perron suggests running the following OLS regression:

$$y_t = a_0 + a_1 y_{t-1} + a_2 t + \mu_2 D_L + \sum_{i=1}^{p} \beta_i \Delta y_{t-i} + \varepsilon_t$$

- H_0: $a_1 = 1$; t–ratio, DF unit root test.
- Perron shows that the asymptotic distribution of the t-statistic depends on the location of the structural break, $\lambda = T_B/T$
  - critical values are supplied in Perron (1989) for different assumptions about λ, see Table IV.B.