J Mech Behav Mater 20 (2011): 103–105 © 2012 by Walter de Gruyter • Berlin • Boston. DOI 10.1515/jmbm-2012-0002

A note on gradient elasticity and nonsingular crack fields

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Keywords: gradient elasticity; nonsingular crack fields.

A robust form of gradient elasticity theory introduced by the author at the beginning of the 1990s [1] and further established and extended upon at the beginning of the 2000s [2] has been used to eliminate strain and stress in dislocation and crack problems where classical linear or nonlinear elasticity and other forms of generalized continuum theories (gradient, nonlocal, or Cosserat type) fail to do so. This theory has also been used with success to interpret size effects observed in twisted microwires and bent microcantilever beams [3]. As the higher-order stress and strain terms introduced in the constitutive equation of the aforementioned gradient elasticity theory also require the introduction of additional mathematically consistent and physically reasonable boundary conditions, there was an initial hesitation to fully use the theory for revisiting classical elasticity problems where physically unreasonable singularities persisted and observed size effects could not be captured. Nevertheless, the explosive progress in nanosciences and nanotechnology and the need for effective approaches to address nanomechanical problems in small volumes [e.g., microelectromechanical system (MEMS)/nanoelectromechanical system (NEMS) devices] where standard mechanics tools used for macroengineering and microengineering applications do not suffice has induced an enormous publishing activity on the use of gradient elasticity theory in modeling the mechanical behavior of nanoobjects (for a review, see [4] and references quoted therein) and in determining stress and strain fields in nanovolumes. In particular, the so-called Ru-Aifantis theorem [5] has been used to derive easy-to-use nonsingular expressions for the strain field of the mode III crack problem, and these expressions have been checked against corresponding finite element calculations. Similar analytical expressions for mode III have been provided in [6, 7], and more recently, special solutions for mode I, which are valid under certain conditions, were also listed in [8] (see also [9]).

These solutions for crack problems and the special conditions for which they are valid, either exactly or as approximations, were not discussed so far, and corresponding results were kept unpublished in anticipation of additional findings by the author's students and collaborators, as well as by other researchers working on the topic. The purpose of the present note is to supplement the preliminary information released in [6, 8] (see also [9]) with some additional results and details that might be helpful to the material mechanics and fracture communities in evaluating the current status of gradient elasticity theory in relation to crack singularities. This becomes particularly important in connection with experimental findings pertaining to nonsingular distributions of strain near crack tips and the development of new techniques to analyze displacements at the nanoscale [10–13] (see also [14, 15]).

Before we proceed with the stress/strain analysis of the aforementioned crack problems, we first refer briefly to several recent articles that have dealt with this problem using generalized elasticity theories. These include earlier publications by the author and his coworkers [16, 17, see also 18, 19; 20, see also 21–24], as well as other contributors [25, see also 26, 27; 28–31], many of which [29–31] have used the author's simplified version of gradient elasticity model [1] given by the following Laplacian of strain modification of Hooke's law:

$$\sigma_{ij}=[\lambda(\varepsilon_{kk})\delta_{ij}+2\mu\varepsilon_{ij}]-c\nabla^2[\lambda(\varepsilon_{kk})\delta_{ij}+2\mu\varepsilon_{ij}]. \tag{1}$$

In several of these articles, the elastic singularities at the crack tip persist. For example, the analyses in [29–31] show that certain components of the stress and strain fields still remain singular, exhibiting a stronger singularity ($\sim r^{-3/2}$) than the classical one ($\sim r^{-1/2}$). In particular, the normal stress component σ_{22} attains a positive (tensile) maximum value at some distance away from the crack tip and then becomes negative (compressive), going to negative infinity as it approaches the crack tip. The physical meaning of these solutions will be discussed elsewhere. The modest purpose of this note is to show that it is possible to obtain explicit solutions for the stress and strain fields, which remain nonsingular as the crack tip is approached. Such nonsingular behavior for the stress field has also been discussed earlier by Elliot [32] and Eringen [33] (see also [34–38]) based on atomistic calculations and nonlocal elasticity, respectively. The main advantage of the expressions given below is that they are easy to use for revisiting a large number of engineering problems in linear fracture mechanics and for checking their validity against related experimental measurements.

Let us start with the strain field analysis for a mode III crack. The well-known classical elasticity asymptotic expressions for the nonvanishing strain components $\varepsilon^0_{\alpha 3}$ (α=1,2) read

$$\varepsilon^0_{13}=-\frac{K_{\mathrm{III}}}{2G\sqrt{2\pi r}}\sin\frac{\theta}{2}, \qquad \varepsilon^0_{23}=\frac{K_{\mathrm{III}}}{2G\sqrt{2\pi r}}\cos\frac{\theta}{2}, \tag{2}$$

where (r,θ) denotes the usual polar coordinates from the crack tip, $K_{\mathrm{III}}=\tau^\infty\sqrt{\pi a}$ is the stress intensity factor, τ^∞ denotes the antiplane shear stress applied at infinity and a is the half crack length. Then the gradient elasticity solution for the

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component ε_{23}, for example, is determined from the inhomogeneous Helmholtz equation [2]

$$\varepsilon_{23}\text{-}c\nabla^2\varepsilon_{23}=\frac{K_{\text{III}}}{2G\sqrt{2\pi r}}\cos\frac{\theta}{2}. \tag{3}$$

By setting $\varepsilon_{23}=\left(K_{\text{III}}/2G\sqrt{2\pi}\right)E(r)\cos(\theta/2)$ in Eq. (3), it follows that the unknown function $E(r)$ satisfies the differential equation

$$\left(1+\frac{c}{4r^2}\right)E(r)\text{-}c\left[E''(r)+\frac{1}{r}E'(r)\right]=\frac{1}{\sqrt{r}}, \tag{4}$$

whose solution is

$$E(r)=\frac{1}{\sqrt{r}}\left(1+Ae^{-r/\sqrt{c}}+Be^{r/\sqrt{c}}\right). \tag{5}$$

By requiring that $\varepsilon_{23}=\varepsilon_{23}^0$ for $r\to\infty$ and $\varepsilon_{23}=0$ for $r\to 0$, we conclude that B=0 and A=-1; thus,

$$\varepsilon_{23}=\frac{K_{\text{III}}}{2G\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-e^{-r/\sqrt{c}}\right), \tag{6a}$$

and a similar expression is obtained for ε_{13}, i.e.,

$$\varepsilon_{13}=\text{-}\frac{K_{\text{III}}}{2G\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(1\text{-}e^{-r/\sqrt{c}}\right). \tag{6b}$$

Motivated by these findings and adopting a more general gradient elasticity model by replacing σ_{ij} in Eq. (1) with $\bar{\sigma}_{ij}=\sigma_{ij}\text{-}c\nabla^2\sigma_{ij}$ as discussed in [2], it turns out that the corresponding stress components (σ_{23},σ_{13}) for mode III read

$$\sigma_{23}=\frac{K_{\text{III}}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1-e^{-r/\sqrt{c}}\right), \tag{7a}$$

$$\sigma_{13}=\text{-}\frac{K_{\text{III}}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left(1\text{-}e^{-r/\sqrt{c}}\right). \tag{7b}$$

Analogous results can be derived for mode I. In particular, the governing equation for the stress field turns out to be

$$\sigma_{ij}\text{-}c\nabla^2\sigma_{ij}=\sigma_{ij}^0, \tag{8}$$

where σ_{ij}^0 is the classical elastic stress field. For example, the σ_{22} component is determined from the inhomogeneous Helmholtz equation

$$\sigma_{22}\text{-}c\nabla^2\sigma_{22}=\frac{K_{\text{I}}}{\sqrt{2\pi r}}\left[\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\right], \tag{9}$$

where $K_{\text{I}}=\sigma\sqrt{\pi a}$ is the usual stress intensity factor for mode I (σ is the applied tensile stress and a is the half crack length) and (r, θ) are the usual polar coordinates with origin at the crack tip. By writing the angular component of the right-hand side of Eq. (15) as [(5/4)cos(θ/2)-(1/4)cos(5θ/2)], inserting this into Eq. (9), splitting this equation in two parts, separately solving the two resulting inhomogeneous Helmholtz equations using superposition, and finally taking into account the boundary conditions $\sigma_{ij}\to\sigma_{ij}^0$ as $r\to\infty$ and $r\to 0$, it turns out that, under certain conditions, the relevant nonsingular solution can be cast in the form

$$\sigma_{22}=\frac{K_{\text{I}}}{\sqrt{2\pi r}}\left[\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\right]\left(1\text{-}e^{-r/\sqrt{c}}\right). \tag{10}$$

Alternative forms of the mode I solution can be obtained if one may not insist on the vanishing of σ_{22} at the crack tip. One form of solution with such property turns out to be composed from the usual elastic solution and an additional term that varies as $r^{-3/2}$, thus recovering the form obtained by other authors (e.g., [31] and references quoted therein). A detailed discussion on the various forms of solutions at the crack tip (singular or not) and their relevance to experimental observations is postponed for the future.

It is pointed out that the above solution for mode I, which was also listed in [8] (see also [9]) is only valid as an approximation of the exact asymptotic expression for the stress field, which is more complex than the one given by Eq. (10). This is not the case for the mode III solution given by Eqs. (6) and (7) for the strain and the stress fields, respectively. The degree of approximation used for obtaining the simplified expression given by Eq. (10) for σ_{22} and other analogous expressions for the rest of stress and strain components will be discussed in a future publication. A detailed discussion of these results and the ones listed below can be found in [39]. It turns out that, by adopting the same procedure as the one that led to Eq. (10) and using Mathematica (Wolfram Research, Champaign, IL, USA), the exact solution for σ_{22} reads

$$\sigma_{22}=\frac{K_{\text{I}}}{4\sqrt{2\pi}r^{5/2}}\left[5\cos\frac{\theta}{2}r^2\left(1\text{-}e^{-\frac{r}{\sqrt{c}}}\right)\right.$$
$$\left.\text{-}\cos\frac{5\theta}{2}\left(\text{-}6c+r^2+2e^{-\frac{r}{\sqrt{c}}}\left(3c+3\sqrt{c}r+r^2\right)\right)\right] \tag{11}$$

The corresponding expressions for σ_{11} and σ_{33} read

$$\sigma_{11}=\frac{K_{\text{I}}}{4\sqrt{2\pi}r^{5/2}}\left[3\cos\frac{\theta}{2}r^2\left(1\text{-}e^{-\frac{r}{\sqrt{c}}}\right)\right.$$
$$\left.+\cos\frac{5\theta}{2}\left(\text{-}6c+r^2+2e^{-\frac{r}{\sqrt{c}}}\left(3c+3\sqrt{c}r+r^2\right)\right)\right], \tag{12}$$

and

$$\sigma_{33}=\frac{K_{\text{I}}v\sqrt{2}}{\sqrt{\pi r}}\cos\frac{\theta}{2}\left(1\text{-}e^{-\frac{r}{\sqrt{c}}}\right). \tag{13}$$

Finally, a series expansion in the above formulas for $r\to 0$ gives the following asymptotic forms for the finite stress components near the crack tip

$$\sigma_{11}=\frac{K_{\text{I}}}{4\sqrt{2\pi c}}\left[3\cos\left(\frac{\theta}{2}\right)r^{1/2}\right.$$
$$\left.+\frac{1}{4\sqrt{c}}\left\{\text{-}6\cos\left(\frac{\theta}{2}\right)+\cos\left(\frac{5\theta}{2}\right)\right\}r^{3/2}+O\left(r^{5/2}\right)\right]$$

$$\sigma_{22}=\frac{K_I}{4\sqrt{2\pi c}}\left[5\cos\left(\frac{\theta}{2}\right)r^{1/2} - \frac{1}{4\sqrt{c}}\left\{10\cos\left(\frac{\theta}{2}\right)+\cos\left(\frac{5\theta}{2}\right)\right\}r^{3/2}+O\left(r^{5/2}\right)\right] \quad (14)$$

$$\sigma_{33}=\frac{\sqrt{2}\nu K_I}{\sqrt{\pi c}}\left[\cos\left(\frac{\theta}{2}\right)r^{1/2}-\frac{1}{2\sqrt{c}}\cos\left(\frac{\theta}{2}\right)r^{3/2}+O\left(r^{5/2}\right)\right].$$

It thus turns out that, strictly speaking, only component σ_{33} and hydrostatic stress $\sigma_h=1/3(\sigma_{11}+\sigma_{22}+\sigma_{33})$ for mode I have the exponential dependence shown in Eq. (10), which is exact for mode III as shown in Eq. (7), but it holds only as an approximation of convenience for mode I. More details on all these issues and corresponding plots for all stress and strain components for modes I, II, and III will be provided in a forthcoming article [40].

Acknowledgments

Discussions with I. Konstantopoulos and K.E. Aifantis are gratefully acknowledged. K.E. Aifantis has kindly made available useful correspondence with Jim Rice, Harvard University, pertaining to an early form of the mode III solution given by Eq. (7) in December 2005. I. Konstantopoulos worked on the exact form of the mode I solution given by Eq. (11), and these results were presented in keynote lectures delivered by the author at the 16th European Conference of Fracture in July 2006 (organized by E.E. Gdoutos in Alexandroupolis) and at the Wieghardt-Irwin Centenary Conference in March 2007 (organized by H.P. Rossmanith in Vienna).

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