A1] $f(x,y)$ με $x(u,w)$ και $y(u,w)$. Για να υπολογίσουμε την $2^{η}$ παράγωγο της σύνθετης συνάρτησης $f(x(u,w),y(u,w))$ ως προς u, $\frac{\partial^2 f}{\partial u^2} \equiv \partial_u^2 f \equiv f_{uu}$, υπολογίζουμε πρώτα την μερική παράγωγο, $\frac{\partial f}{\partial u} \equiv \partial_u f \equiv f_u$ και μετά εφαρμόζουμε διαδοχικά πάλι $\frac{\partial}{\partial u}$ στην f_u:

$$\partial_u f = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial w}\right) \quad (*)$$

$$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right) = \frac{\partial}{\partial u}\left[\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right)\right] + \frac{\partial}{\partial u}\left[\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial w}\right)\right] =$$

$$= \left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial u} f\right)\right] \cdot \left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial f}{\partial x}\right)\left[\frac{\partial^2}{\partial u^2} x\right]$$

$$+ \left[\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial u}\right]\right]\left(\frac{\partial y}{\partial w}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial^2 y}{\partial u^2}\right),$$

όπου $\frac{\partial f}{\partial u}$ από σχέση (*). $\Longrightarrow$

$$\frac{\partial^2 f}{\partial u^2} = \left[\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial^2 f}{\partial x \partial y}\right)\left(\frac{\partial y}{\partial w}\right)\right] \cdot \left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial^2 x}{\partial u^2}\right)$$

$$+ \left[\left(\frac{\partial^2 f}{\partial x \partial y}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial^2 f}{\partial y^2}\right)\left(\frac{\partial y}{\partial w}\right)\right] \cdot \left(\frac{\partial y}{\partial u}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial^2 y}{\partial u^2}\right)$$

$\Rightarrow$ (2)

$$f_{uu} = \underline{f_{xx} \cdot (x_u)^2} + f_{xy} \cdot y_u \cdot x_u + f_x \cdot x_{uu}$$

$$+ f_{xy} \cdot x_u \cdot y_u + \underline{f_{yy} \cdot (y_u)^2} + f_y \cdot y_{uu}$$

ή

$$f_{uu} = f_{xx} \cdot (x_u)^2 + f_{yy} \cdot (y_u)^2 + 2 \cdot f_{xy} \cdot x_u \cdot y_u + f_x \cdot x_{uu} + f_y \cdot y_{uu}$$

✓

Α2 Δίνεται $f(x,y)$ με $x = \varphi_1(u,w) = u+w$
$y = \varphi_2(u,w) = u-w$

(α) Υπολογίζουμε

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \quad (1)$$

$$\boxed{\begin{array}{l} x_u = 1, \ y_u = 1 \\ x_w = 1, \ y_w = -1 \end{array}} \quad \frac{\partial f}{\partial w} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial w} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial w} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \quad (2)$$

$$\Rightarrow (1) \times (2): \left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial f}{\partial w}\right) = (f_x + f_y)(f_x - f_y) \underset{(*)}{=} (f_x)^2 - (f_y)^2$$

✓

(*) Εδώ ισχύει το Θεώρημα Schwarz, $f_{xy} = f_{yx}$, επειδή υπάρχουν συνεχείς παράγωγοι της f.

(β) → βλέπε τελευταία σελίδα.

A3 Για $V(r)$ με $r \equiv |\vec{r}| = \sqrt{x^2+y^2+z^2}$ έχουμε

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} = V_r \cdot \left(-\frac{1}{2}\right)\left(x^2+y^2+z^2\right)^{\frac{1}{2}-1} \cdot (2x)$$

$$\Rightarrow \frac{\partial V}{\partial x} = -V_r \cdot \frac{x}{r}$$

Ανάλογα είναι $\frac{\partial V}{\partial y} = -V_r \frac{y}{r}$ και $\frac{\partial V}{\partial z} = -V_r \cdot \frac{z}{r}$.

$$\Rightarrow (V_x)^2 + (V_y)^2 + (V_z)^2 = (V_r)^2 \left(\frac{x^2+y^2+z^2}{r^2}\right) = \left(\frac{\partial V}{\partial r}\right)^2 \cdot \frac{r^2}{r^2} = \left(\frac{\partial V}{\partial r}\right)^2$$

✓ ■

A4 (a)

$$f_1 = \frac{5+x}{y} . \quad f_1(\lambda x, \lambda y) = \frac{5+\lambda x}{\lambda y} \neq \lambda^m f(x,y)$$

→ δεν είναι ομογενής.

$$f_2(\lambda x, \lambda y, \lambda z) = \lambda y + \lambda x e^{\frac{\lambda^2 x^2}{\lambda^2 yz}} + \sqrt{\lambda^2 x^2 + \lambda^2 xy + \lambda^2 z^2}$$

$$= \lambda\left[y + x e^{\frac{x^2}{yz}} + \sqrt{x^2+xy+z^2}\right] = \lambda \cdot f(x,y,z)$$

→ ομογενής 1ου βαθμού.

$f_3(\lambda x, \lambda y, \lambda z) = \lambda z + e^{\frac{\lambda^2 x^2}{\lambda y}} = \lambda z + e^{\frac{\lambda x^2}{y}} \neq \lambda f(x,y,z)$ → δεν είναι ομογ.

$f_4(\lambda x, \lambda y) = (\lambda x)^{3/4} + (\lambda y)^{3/4} = \lambda^{3/4}\left(x^{3/4} + y^{3/4}\right)$ → ομογενής βαθμού $\frac{3}{4}$.

(β) $f(x,y,z)$ ομογενής βαθμού m $\Rightarrow$

$$f(\lambda x, \lambda y, \lambda z) = \lambda^m \cdot f(x,y,z).$$

Θέτω $\lambda = \frac{1}{x}$ και έχω

$$f\left(1, \frac{y}{x}, \frac{z}{x}\right) = \frac{1}{x^m} f(x,y,z) \Rightarrow f(x,y,z) = x^m \cdot f\left(1, \frac{y}{x}, \frac{z}{x}\right).$$

$\boxed{A5}$ Taylor της $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ μέχρι και $1^{ης}$ τάξης:

$$f(x,y) \simeq f(x_0,y_0) + f_x\Big|_{(x_0,y_0)} \cdot (x-x_0) + f_y\Big|_{(x_0,y_0)} \cdot (y-y_0)$$

Εδώ:

$$f_x = \frac{\partial f}{\partial x} = -\frac{1}{\cancel{2}} \frac{\cancel{2}x}{\sqrt{x^2+y^2}^3}, \quad f_y = -\frac{1}{\cancel{2}} \frac{\cancel{2}y}{\sqrt{x^2+y^2}^3}$$

$$\Rightarrow f(x,y) \simeq \underbrace{f(1,1)}_{= \frac{1}{\sqrt{2}}} + \frac{1}{2^{3/2}} \cdot (x-1) + \frac{1}{2^{3/2}} (y-1)$$

Εν συνεχεία

$$\begin{aligned} x' &= x + \Delta x \\ y' &= y + \Delta y \end{aligned} \quad \text{με } (x,y) = (1,1) \text{ και } (\Delta x, \Delta y) = (0.02, -0.05)$$

$$\Rightarrow f(1{,}02, 0.95) = \frac{1}{\sqrt{2}} - \frac{1}{2^{3/2}} \cdot (0.02) - \frac{1}{2^{3/2}} (-0.05) \simeq \underline{\underline{0.717}}$$

(με Η/Υ: $f(1.02, 0.95) = 0.7175$)

(β) Mac-Laurin για $g(x,y)=\sin(x^2+y)$ μέχρι 2ης τάξης!

$$g(x,y)=g(0,0)+g_x\big|_{(0,0)}\cdot x+g_y\big|_{(0,0)}\cdot y$$

$$+\frac{1}{2!}\left[g_{xx}\big|_{(0,0)}x^2+g_{yy}\big|_{(0,0)}\cdot y^2+2g_{xy}\big|_{(0,0)}xy\right]$$

$$g_x=\frac{\partial g}{\partial x}=\cos(x^2+y)\cdot(2x)\Rightarrow g_x\big|_{(0,0)}=0.$$

$$g_y=\frac{\partial g}{\partial y}=\cos(x^2+y)\cdot 1\Rightarrow g_y\big|_{(0,0)}=1.$$

$$g_{xx}=\frac{\partial}{\partial x}\left[2x\cos(x^2+y)\right]=2\cos(x^2+y)+2x(-\sin(x^2+y))\cdot(2x)$$

$$\rightarrow g_{xx}\big|_{(0,0)}=2$$

$$g_{yy}=\frac{\partial}{\partial y}\left[\cos(x^2+y)\right]=-\sin(x^2+y)\cdot 1\rightarrow g_{yy}\big|_{(0,0)}=0.$$

$$g_{xy}=\frac{\partial}{\partial x}g_y=\frac{\partial}{\partial x}\left(\cos(x^2+y)\right)=-\sin(x^2+y)\cdot 2x\rightarrow g_{xy}=0.$$

Εξάγεται και από $g_{yx}=\frac{\partial}{\partial y}g_x=\frac{\partial}{\partial y}\left[2x\cdot\cos(x^2+y)\right]=-2x\sin(x^2+y)$

$$\Rightarrow g(x,y)\simeq g(0,0)+0\cdot x+1\cdot y+\frac{1}{2!}\left[2\cdot x^2+0\cdot y^2+2\cdot 0\cdot xy\right]$$

$\Rightarrow \sin(x^2+y) \approx x^2+y.$ ■

Α2(β) Επαλήθευση της σχέσης

$$\left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial f}{\partial w}\right)$$

για $f(x,y) = x^2 + 2y^3$ με $x = u+w$ και $y = u-w$:

$$\frac{\partial f}{\partial x} = 2x \ , \ \frac{\partial f}{\partial y} = 6y^2 \Rightarrow \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 = 4x^2 - 36y^4$$

$$\frac{\partial f}{\partial u} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial u}\right)$$

$$= 2x \cdot 1 + 6y^2(1) = 2x + 6y^2$$

$$\frac{\partial f}{\partial w} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial w}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial w}\right)$$

$$= 2x \cdot 1 + 6y^2 \cdot (-1) = 2x - 6y^2$$

$$\Rightarrow \left(\frac{\partial f}{\partial u}\right)\left(\frac{\partial f}{\partial w}\right) = (2x+6y^2)(2x-6y^2) = 4x^2 - 36y^4.$$

✓ ■