Chapter 17

A WALSHP-TYPE MULTIRESOLUTION ANALYSIS

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Abstract
We introduce a class of orthonormal matrices $U^{(n)}$ of order $p^n \times p^n$, $p = 2, \ldots, n = 1, \ldots$. The construction of those matrices is achieved in different scales by an iteration process, determined by a repetitive block matrix operation, involving the cross product of properly selected sub-matrices. For the case $p = 2$ we get the well known Walsh system. This particular construction also induces a multiscale transform on $L_2(T)$, reminiscent (although different) of a multiresolution analysis of $L_2(T)$.

Keywords: Discrete transforms, Walsh system

1. Introduction

In order to provide efficient multiscale analysis on finite data, we seek for linear transforms whose corresponding matrices have the ability to detect specific characteristics from those data. In [4], we introduced a class of weighted sparse matrices for the purpose of prediction of almost periodic time series, while in [5] we built sparse matrices capable of revealing local information at different scales. In [2], we introduced a new class of sparse invertible matrices $H(m)$ of order $m \times m$, suitable for grammar detection of symbolic sequences. In fact, the matrices $H(m)$ may be considered as a generalization of the usual Haar matrices, since their construction was based on dilation and translation operations on unbalanced Haar matrices. Thus, we obtained a generalized Haar transform:

$$\{t_n : n = 1, \ldots, m\} \leftrightarrow \{< t, h_n > : n = 1, \ldots, m\},$$

where $\langle,\rangle$ is the usual inner product of the Euclidean space $R^m$ and where $h_n$ are the rows of $H(m)$. 
In this work we dealt with the problem: what happens if we use dilation and replication operations, instead of using dilation and translation operations on matrices?

In Section 2, we build a discrete transform on finite data by using an iteration in scales. The cross product of matrices plays a central role in our construct, because it can be used either as a dilation or replication operator. So, we start from an initial matrix $U$ of order $p \times p$. In every step of the iteration process we create a new matrix $U^{(n)}$ of order $p^n \times p^n$. $U^{(n)}$ is a block matrix, whose block sub-matrices are defined from the cross product $U^{(n-1)} \otimes U_i$ (see below). In Theorem 17.1, we prove that the matrices $U^{(n)}$ are orthonormal, whenever the initial matrix $U$ is orthonormal. Thus, we obtain a discrete transform:

$$\{t_i : i = 1, \ldots, p^n\} \leftrightarrow \{< t, U^{(n)}_i > : i = 1, \ldots, p^n\},$$

where $U^{(n)}_i$ are the rows of $U^{(n)}$. For a suitable selection of the matrix $U$ we see that the resulting orthonormal system is the Walsh system.

Since to any row of the matrix $U^{(n)}$ there corresponds a step function on $T$, an orthonormal set $	ilde{M}_n = \{\tilde{m}_k(\gamma) : k = 1, \ldots, p^n\}$ of functions of $L^2(T)$ emerges naturally from the matrix $U^{(n)}$. In Section 3 we see that the set $\tilde{M}_n$ is produced by successive dilations and replicas of a generator set of functions $M = \{m_i(\gamma) : i = 0, \ldots, p - 1\}$:

$$m_i(\gamma) = \sum_{j=1}^{p} U_{i+1,j} 1_{\left[\frac{j}{p}, \frac{j+1}{p}\right)}, \quad i = 0, \ldots, p - 1.$$  

Indeed:

$$\tilde{M}_n = \left\{ \tilde{m}_k(\gamma) = \prod_{j=0}^{n-1} m_{\varepsilon_j}(p^j \gamma) : k = 1 + \sum_{j=0}^{n-1} \varepsilon_j p^j, \quad \varepsilon_j \in \{0, \ldots, p - 1\} \right\}.$$  

Finally, we see that our multiscale construction naturally extends to an invertible transform on $L^2(T)$.

2. **A class of Walsh-type discrete transforms**

**Notation:** Let $M_{n,m}$ be the set of all matrices of order $n \times m$ over the field of complex numbers. If $n = m$, then $M_{n,m}$ is abbreviated to $M_n$. We shall use the symbolism $A = [A_{ij}]$ to denote a matrix $A$ with elements $A_{ij}$. The notation

$$A_i = \{A_{i,j} : j = 1, \ldots, m\}$$

shall be used to denote the $i$-row of a matrix $A$. We define the following operators:
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Definition 17.1 For $p = 2, \ldots$, the tensor product of two matrices $A \in M_{n,m}$ and $B \in M_{k,l}$ is a block matrix $A \otimes B \in M_{nk,ml}$:

$$A \otimes B = \begin{pmatrix}
a_{11}B & \cdots & a_{1n}B \\
\vdots & \ddots & \vdots \\
a_{m1}B & \cdots & a_{mn}B
\end{pmatrix}.$$ 

Definition 17.2 Let $S : M_{n_1,m_1} \times \ldots \times M_{n_k,m_k} \to M_{n_1+\ldots+n_k,m_1+\ldots+m_k}$ be the following block matrix operator:

$$S(M_1, \ldots, M_k) = \begin{pmatrix} M_1 \\ \vdots \\ M_k \end{pmatrix}.$$ 

Definition 17.3 Let $U$ be an orthonormal matrix of order $p \times p$, we define a sequence of block matrices $U^{(n)}$, where $n = 1, \ldots, N$, by using the following iteration:

$$U^{(n)} = \begin{cases} U, & n = 1 \\ S \left( U^{(n-1)} \otimes U_1, \ldots, U^{(n-1)} \otimes U_p \right), & n = 2, \ldots, N, \end{cases} \quad (17.1)$$

where $U_i$ is the $i$ row of $U$.

Theorem 17.1 The matrix $U^{(n)}$, $(n = 1, \ldots, N)$ is orthonormal.

Proof. We work inductively. Clearly, the theorem is true for $n = 1$. We suppose that the matrix $U^{(n-1)}$ is orthonormal, so it suffices to prove that $\langle U^{(n)}_j, U^{(n)}_l \rangle = \delta_{j,l}$, where $\langle, \rangle$ is the usual inner product of the Euclidean space $\mathbb{R}^{p^n}$. Let $j = mp^n - 1 + \zeta$, $l = qp^n - 1 + \sigma$, where $m, q = 0, \ldots, p - 1$, $\zeta, \sigma = 1, \ldots, p^{n-1}$, then:

$$\langle U^{(n)}_j, U^{(n)}_l \rangle = \sum_{r=1}^{p^n} U^{(n)}_{j,ir} U^{(n)}_{rl} = \sum_{\nu=0}^{p-1} \sum_{\mu=1}^{p^{n-1}} U^{(n)}_{j,\nu p^{n-1}+\mu} U^{(n)}_{\nu p^{n-1}+\mu,l}$$

$$= \sum_{\nu=0}^{p-1} \sum_{\mu=1}^{p^{n-1}} U_{m+1,\nu+1}^{(n-1)} U^{(n-1)}_{\nu+1,q+1} U^{(n-1)}_{\mu,\sigma}$$

$$= \left( \sum_{\nu=1}^{p} U_{m+1,\nu} U_{\nu,q+1} \right) \left( \sum_{\mu=1}^{p^{n-1}} U^{(n-1)}_{\zeta,\mu} U^{(n-1)}_{\mu,\sigma} \right)$$

$$= \delta_{m,q} \delta_{\zeta,\sigma} \delta_{j,l} = \delta_{j,l}.$$
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It is clear that the inverse matrix of $U^{(n)}$ coincides to its transpose $\left(U^{(n)}\right)^T$.

The following multiresolution structure arises from the matrices $U^{(n)}$:

Let $V_{p^n}$ be the space of all real-valued sequences of length $p^n$ and let $U_i^{(n)}$ be the $i$-row of the matrix $U^{(n)}$, then any element $t \in V_{p^n}$ can be written as:

$$t_i = \sum_{i=1}^{p^n} \langle t, U_i^{(n)} \rangle U_i^{(n)}.$$ 

Let $j = 1, \ldots, n - 1$, $k = 1, \ldots, p - 1$, we define

$$W_{j,k} = \text{span}\{U_{kp^j+s}^{(n)} : s = 1, \ldots, p^j\},$$

then, we have the decomposition:

$$V_{p^n} = V_0 \oplus_{j=1}^{n-1} \oplus_{k=1}^{p-1} W_{j,k},$$

where $V_0 = \text{span}\{U_s^{(n)} : s = 1, \ldots, p\}$.

**Example 17.1** Let $p = 3$, $n = 3$, then $V_{3^3} = V_0 \oplus_{j=1}^{2} \oplus_{k=1}^{2} W_{j,k}$, where:

$$W_{1,1} = \text{span}\{U_4^{(3)}, \ldots, U_6^{(3)}\}, \quad W_{1,2} = \text{span}\{U_7^{(3)}, \ldots, U_9^{(3)}\}, \quad W_{2,1} = \text{span}\{U_10^{(3)}, \ldots, U_{18}^{(3)}\}, \quad W_{2,2} = \text{span}\{U_19^{(3)}, \ldots, U_{27}^{(3)}\}.$$

**Definition 17.4** Let $p \geq 2$, we define the following matrix $\Psi^{(p)}$ of order $p \times p$:

$$\psi_{ij}^{(p)} = \begin{cases} 
\frac{1}{\sqrt{p}}, & \text{whenever } i = 1 \\
\frac{1}{\sqrt{p+i+1}} \frac{1}{\sqrt{p+i+2}}, & \text{whenever } 1 \leq j \leq p - i + 1, \quad i, j = 1, \ldots, p. \\
0, & \text{whenever } p - i + 2 < j \leq p.
\end{cases}$$

**Example 17.2** $\Psi^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\Psi^3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$.

**Proposition 17.1** (see [1])

The matrix $\Psi^{(p)}$ satisfies the following properties:

(i) $\sum_{j=1}^{p} \psi_{ij}^{(p)} = 0$, $i = 2, \ldots, p$. 

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(ii) $\psi_i^{(p)} \psi_j^{(p)} = \psi_{i+1}^{(p)} \psi_j^{(p)}$, whenever $i < j, i, j = 1, \ldots, p$.

(iii) The matrix $\Psi^{(p)}$ is orthonormal.

Observation 17.1 If we consider the iteration (17.1) with initial matrix $U = \Psi^{(2)}$, then we obtain the Walsh system (see [5]). Indeed:

$$U^{(1)} = \Psi^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, U^{(2)} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \ldots.$$ Whenever $p > 2$, we get a Walsh-type construction.

Observation 17.2 If we consider the iteration (17.1) with initial matrix $U = \Psi^{(p)} = \left( e^{2\pi ikl/p} \right)^{p-1}_{k,l=0}$, then we obtain the Generalized Walsh system as defined in [5].

3. A multiscale transform on $L_2(T)$

We denote by $\mathbf{R}$ the additive group of real numbers and by $\mathbf{Z}$ the subgroup consisting of the integers. The group $T$ is defined as the quotient $\mathbf{R}/\mathbf{Z}$. Since there is an obvious identification between functions on $T$ and 1-periodic functions on $\mathbf{R}$, from now on we identify the elements of the space $L_2(T)$ of all complex valued Lebesgue square integrable functions on $T$, as 1-periodic functions on $\mathbf{R}$.

Since any row $U_{k}^{(n)}$, $k = 1, \ldots, p^n$ of the matrix $U^{(n)}$ defined in Theorem 17.1 can be assigned to a step function $\tilde{m}_k(\gamma)$ on $T$ such that

$$\tilde{m}_k(\gamma) = m_{kj}, \quad \gamma \in \Omega_{j,n} = \left[ \frac{j-1}{p^n}, \frac{j}{p^n} \right], \quad j = 1, \ldots, p^n,$$ an orthonormal set of functions of $L_2(T)$ emerges naturally from the construction presented in section 2:

$$\tilde{M}_n = \left\{ \tilde{m}_k(\gamma) : \tilde{m}_k(\gamma) = \sum_{j=1}^{p^n} U_{k,j}^{(n)} \mathbf{1}_{\Omega_{j,n}}(\gamma), \quad k = 1, \ldots, p^n. \right\}$$

Moreover, if $U$ is the initial orthonormal matrix of the iteration process (17.1), by defining:

$$m_i(\gamma) = \sum_{j=1}^{p} U_{i+1,j} \mathbf{1}_{\Omega_{j,n}}(\gamma), \quad i = 0, \ldots, p - 1,$$
we can see that the set $\tilde{M}_n$ can be produced by successive dilations of the functions $m_i(\gamma)$ in the following:

$$\tilde{M}_n = \left\{ \tilde{m}_k(\gamma) = \prod_{j=0}^{n-1} m_{\varepsilon_j}(p^j\gamma), \ k = 1 + \sum_{j=0}^{n-1} \varepsilon_j p^j, \ \varepsilon_j \in \{0, \ldots, p-1\}\right\}. \tag{17.2}$$

Moreover, we can prove:

**Theorem 17.2** Let $\{V_n : V_n \subset V_{n+1}, \ n \geq 1\}$ be a nested sequence of $p^n$-dimensional subspaces of $L_2(T)$, whose orthonormal basis is the set $\tilde{M}_n$ defined in (17.2), then:

$$\bigcup_{n \geq 1} V_n = L_2(T).$$

**Proof.** See [1].

**Acknowledgments**

Research supported by the Joint Research Project within the Bilateral S&T Cooperation between the Hellenic Republic and the Republic of Bulgaria (2004-2006).

**References**


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