Chapter 18

DISCRETE TYPE RIESZ PRODUCTS

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**Abstract**  
We factorize finite data of length \( m \), or step functions determined on the intervals \( [k/m, (k + 1)/m), k = 0, \ldots, m - 1 \) of \([0,1)\), by writing them as a discrete Riesz-type Product \( t_n = \prod_{k=1}^{m} (1 + a_k h_{k,n}) \) with respect to the rows \( h_k \) of a matrix \( H(m) \) of order \( m \times m \) and associated to a sequence of coefficients \( \{a_k : k = 1, \ldots, m\} \). We give sufficient conditions on \( H(m) \) and \( \{a_k\} \), providing invertibility of the underlying non-linear Riesz-type transform and we present examples of classes of acceptable matrices.

**Keywords:** Riesz Products, non-linear transforms.

**1. Introduction**

The original Riesz’s construction associated to a sequence of coefficients \( \{a_n\} \), was to show that there exists a continuous function \( F \) of bounded variation in \([0,2\pi)\), whose Fourier-Stieltjes coefficients do not vanish at infinity, \( F \) being the pointwise limit of the sequence of functions:

\[
F_N(x) = \int_0^x \prod_{n=1}^{N} (1 + a_n \cos(2\pi 4^n t))dt.
\]

Over the years, Riesz’s construction was generalized, by replacing the generating function \( \cos(2\pi t) \) with other generating functions such as the Rademacher, or Walsh functions, or trigonometric polynomials (see [4], [5], [6]). Recently
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in [3], multiscale Riesz Products have been constructed, based on a real valued function \( H \) on \([0, 1)\), called generating function and a dilation operator \( T : (0, 1) \to (0, 1) \), such that:

\[
\mu_m(\gamma) = \prod_{n=1}^{m} (1 + a_n H(T^{n-1}\gamma))
\]

converges weak-* to a bounded measure as \( m \to \infty \). Obviously, we can generalize the definition of \( \mu_m \), by considering partial Riesz Products of the form:

\[
\mu_m(\gamma) = \prod_{n=1}^{m} (1 + a_n H_n(\gamma)), \quad (18.1)
\]

where \( H_n(\gamma), (n = 1, \ldots, m) \) are bounded functions on \([0, 1)\). Clearly, if we denote by \( V_m \) the space of sequences of length \( m \) and by \( B[0, 1) \) the space of bounded functions on \([0, 1)\), the partial Riesz Products (18.1) induce a non-linear transform \( \mu_m : V_m \to B[0, 1) \), such that for every \( a = \{a_1, \ldots, a_m\} \in V_m \) we have:

\[
\mu_m(a)(\gamma) = \prod_{n=1}^{m} (1 + a_n H_n(\gamma)).
\]

In order to achieve invertibility for \( \mu_m \), in [1] and [2] we considered step functions \( H_n \) on the intervals \( \Omega_{n,m} = \left[\frac{n-1}{m}, \frac{n}{m}\right), n = 1, \ldots, m \):

\[
H_n(\gamma) = \sum_{i=0}^{m} h_{n,i} \mathbf{1}_{\Omega_{i,m}}(\gamma).
\]

As a consequence, we dealt with discrete Riesz-type products of the form:

\[
t_n = \prod_{k=1}^{m} (1 + a_k h_{k,n}). \quad (18.2)
\]

We proved the following:

**Theorem 18.1 (see [2])**

Let \( H(m) = \{h_{k,n} : k, n = 1, \ldots, m\} \) be a real orthonormal matrix whose first row is the constant row \( (\frac{1}{\sqrt{m}}, \ldots, \frac{1}{\sqrt{m}}) \) and all rows satisfy

\[
h_n h_l = h_n, l h_l \quad \text{whenever } n < l \quad (18.3)
\]

where \( h_n, h_l \) are rows of \( H(m) \) and \( h_n, l \) is the first non-zero entry of the \( l \)-row of the matrix \( H(m) \). If \( t = \{t_1, \ldots, t_m\} \) is a sequence of real numbers such that

\[
\langle t, h_i \rangle \neq 0, \quad i = 1, \ldots, m,
\]
where \(\langle \cdot, \cdot \rangle\) is the usual inner product of \(\mathbb{R}^m\), then there is a unique sequence of coefficients \(\{a_k : k = 1, \ldots, m\}\) such that:

\[
t_n = \prod_{k=1}^{m} (1 + a_k h_{k,n}).
\]  \hspace{1cm} (18.4)

Moreover, the coefficients \(\{a_n : n = 1, \ldots, m\}\) are computed via the following:

\[
a_n = \begin{cases} 
\langle t, h_1 \rangle - \sqrt{m} & n = 1 \\
\frac{\langle t, h_n \rangle}{\prod_{k=1}^{n-1} (1 + a_k h_{k,n_0})}, & n = 2, \ldots, m
\end{cases}
\]

where \(h_{n,n_0}\) is the first non-zero entry of the row \(h_n\).

Also, we constructed a class of unbalanced Haar matrices \(H(m)\) satisfying (18.3) of Theorem 18.1. An example is shown below:

\[
H(3) = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\end{pmatrix}
\]

\[
H(6) = \begin{pmatrix}
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

In this paper, we relax the conditions imposed on the matrix \(H(m)\) in Theorem 18.1. In Section 2, we see that Theorem 18.1 is true, if orthonormality is replaced by invertibility. Also, we show that for a particular class of exponential matrices we can drop (3) and Theorem 18.1 is valid, as long as the values of the coefficients \(\{a_k\}\) are restricted to the discrete set \(A = \{0, 1\}\).

2. Discrete Riesz Products

In this section we obtain classes of matrices, whose corresponding Riesz Products give rise to an invertible non-linear transform.

**Proposition 18.1** Let \(H(m) = \{h_{k,n} : k, n = 1, \ldots, m\}\) be a real invertible matrix satisfying (18.3) of Theorem 18.1. If \(t = \{t_1, \ldots, t_m\}\) is a sequence of real numbers such that

\[
\langle t, h_i \rangle \neq 0, \quad i = 1, \ldots, m,
\]

then there is a unique sequence of coefficients \(\{a_k : k = 1, \ldots, m\}\) such that:

\[
t_n = \prod_{k=1}^{m} (1 + a_k h_{k,n}).
\]

Moreover, the coefficients \(\{a_n : n = 1, \ldots, m\}\) are computed via the following:

\[
a_n = \begin{cases} 
\langle t, h_i^{-1} \rangle - \langle 1, h_i^{-1} \rangle & n = 1 \\
\frac{\langle t, h_i^{-1} \rangle}{\prod_{k=1}^{n-1} (1 + a_k h_{k,n_0})}, & n = 2, \ldots, m
\end{cases}
\]
where $H^{-1}(m) = [h_{j,k}^{-1}]$ is the inverse matrix of $H(m)$.

**Proof.** We expand the discrete Riesz Product and we use (18.3) to get:

\[
\begin{align*}
t_n &= 1 + \sum_{k=1}^{m} a_k h_{k,n} + \sum_{k_1=1}^{m-1} \sum_{k_2=k_1+1}^{m} a_{k_1} a_{k_2} h_{k_1,k_2} h_{k_2,n} + \ldots \\
&\quad + (a_1 \ldots a_m) \left( \prod_{j=1}^{m-1} h_{k_j,k_0} \right) h_{m,n},
\end{align*}
\]

where $h_{k_j,k_0}$ is the first non-zero entry of the row $h_{k_j}$.

The invertibility of $H(m)$ and (18.4) imply that $\langle t, h_{1,-1} \rangle = \langle 1, h_{1,-1} \rangle + a_1$.

For any $s > 1$ we have:

\[
\begin{align*}
\langle t, h_{s,-1} \rangle &= a_s \left( 1 + \sum_{k_1=1}^{s-1} a_{k_1} h_{k_1,s} + \sum_{k_1=1}^{m-2} \sum_{k_2=k_1+1}^{m-1} a_{k_1} a_{k_2} \left( \prod_{j=1}^{2} h_{k_j,s_0} \right) \right) \\
&\quad + \ldots + (a_1 \ldots a_{s-1}) \left( \prod_{j=1}^{s-1} h_{k_j,s_0} \right) \\
&= a_s \prod_{k=1}^{s-1} (1 + a_k h_{k,s_0}).
\end{align*}
\]

**Example 18.1** A class of matrices $H(m)$ satisfying Proposition 18.1 is produced by the following rules:

(a) The first row of $H(m)$ is the constant row $\{1, \ldots, 1\}$.

(b) Every other row has only two non-zero entries $0$ or $1$.

(c) If we denote by supp$\{h_k\} = \{j \in \{1, \ldots, m\} : h_{kj} \neq 0\}$, then:

\[
\text{supp}\{h_k\} \cap \text{supp}\{h_l\} = \varnothing \quad \text{or} \quad \text{supp}\{h_k\} \cap \text{supp}\{h_l\} = \text{supp}\{h_l\}
\]

whenever $k < l$.

Below, we present examples of matrices satisfying rules (a)-(c):

\[
H(3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad H(6) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.
\]
Proposition 18.2 Let $\Theta(m) = \{\theta_{n,j} : |\theta_{n,j}| < \pi, n, j = 1, \ldots, m\}$ be an invertible matrix whose columns satisfy the following:

$$-\pi \leq \sum_{n=1}^{m} \theta_{n,j} \leq \pi, j = 1, \ldots, m.$$ 

If $t = \{t_j = |t_j|e^{i\arg(t_j)}, j = 1, \ldots, m\}$ $-\pi \leq \arg(z) \leq \pi$ is a sequence of complex numbers, then there is a unique sequence of boolean coefficients $\{a_n : n = 1, \ldots, m\}$, such that:

$$t_j = \prod_{n=1}^{m} (1 + a_ne^{i\theta_{n,j}}).$$

Moreover, the coefficients $\{a_n : n = 1, \ldots, m\}$ are computed via the following matrix equation:

$$a = 2\Theta^{-1}C(t),$$

where $a = [a_n]$ and $C(t) = [\arg(t_n)\] \text{ are column matrices of order } m \times 1.$

**Proof.** Let $t_j = \prod_{n=1}^{m} (1 + a_ne^{i\theta_{n,j}})$, where $a_n \in \{0, 1\}$, then we have:

$$t_j = \prod_{n=1}^{m} (1 + a_ne^{i\theta_{n,j}}) = \prod_{n=1}^{m} (1 + e^{i\theta_{n,j}})^{a_n}.$$ 

Since

$$t_j = \prod_{n=1}^{m} (1 + e^{-i\theta_{n,j}})^{a_n} = \prod_{n=1}^{m} \left( e^{-i\theta_{n,j}} \left( e^{i\theta_{n,j}} + 1 \right) \right)^{a_n} = \prod_{n=1}^{m} \left( e^{-i\theta_{n,j}a_n} \left( 1 + e^{i\theta_{n,j}} \right)^{a_n} \right) = t_je^{-i\sum_{n=1}^{m} a_n\theta_{n,j}},$$

we get:

$$e^{-i\arg(t_j)} = e^{i\arg(t_j)} e^{-i\sum_{n=1}^{m} a_n\theta_{n,j}},$$

thus:

$$\sum_{n=1}^{m} a_n\theta_{n,j} = 2\arg(t_j) + 2\lambda_j\pi, \lambda \in \mathbb{Z}.$$ 

The hypothesis $-\pi \leq \sum_{n=1}^{m} \theta_{n,j} \leq \pi$ indicates that $\lambda_j = 0$ for every $j$ and the result follows as a consequence of the invertibility of the matrix $\Theta$.

**Example 18.2 (Haar-type unbalanced matrices)**

Since Haar type unbalanced matrices $H(m)$ as defined in [2] have rows with zero mean, except for the first row which is the constant row $\left(\frac{1}{\sqrt{p^m}}, \ldots, \frac{1}{\sqrt{p^m}}\right)$, orthogonal matrices of the form

$$\Theta(m) = \frac{\pi}{\sqrt{p^m}}H(m)$$
satisfy Proposition 18.2. We present below two examples:

$$\Theta(3) = \frac{\pi}{\sqrt{3}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{2} & 0 \end{pmatrix}, \quad H(6) = \frac{\pi}{\sqrt{6}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix}.$$ 

**Example 18.3 (Generalized Walsh-type Riesz Products)**

Since Walsh orthogonal matrices $W(2^k), k = 1, \ldots$, produced from the Walsh system $\{w_0, \ldots, w_{2^k}\}$ defined in [5] have rows with zero mean, except for the first row which is the constant row $(1, \ldots, 1)$, orthogonal matrices of the form

$$\Theta(2^k) = \frac{\pi}{2^k} W(2^k)$$

satisfy Proposition 18.2. Below, we present below two examples:

$$\Theta(2) = \frac{\pi}{2} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \quad \Theta(4) = \frac{\pi}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \ldots.$$ 

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**References**


