Wavelet Decomposition and Sampling for p-adic Multiresolution Analysis on spaces of $p^M$-periodic sequences

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Abstract

We define p-adic Multiresolution Analysis (MRA) on the space of $p^M$-periodic sequences, where $p, M \in \mathbb{Z}^+$. We present the Sampling Theorem on MRA subspaces and we discuss the existence of p-adic wavelets which provide a variety of new Discrete Transforms.

1. Introduction

It is well known that the Discrete Fourier Transform (DFT) is one of the most widely used tools in communication, engineering and computational mathematics. Recall the following definition of the DFT of an $N$-periodic sequence $s(n)$:

$$\hat{s}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} s(n)e^{-2\pi ink/N}, \quad k = 0, \ldots, N - 1.$$ 

The Inverse Discrete Fourier Transform is given by the formula:

$$s(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{s}(k)e^{2\pi ink/N}, \quad n = 0, \ldots, N - 1.$$ 

The DFT Analysis will be proven very useful for defining a Multiresolution Analysis over $L^2(\mathbb{Z}/(N\mathbb{Z}))$. Recall that a Multiresolution Analysis of $L^2(\mathbb{R})$ (MRA) is a nested sequence $\{V_m \subset V_{m+1}, m \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ and a scaling function $\varphi$ such that:

(i) $\bigcup_m V_m = L^2(\mathbb{R})$ and $\bigcap_m V_m = \{0\},$
(ii) $f \in V_0 \iff f(2^m) \in V_m$, $m \in \mathbb{Z}$,

(iii) the set $\{\varphi(-n), n \in \mathbb{Z}\}$ is an orthonormal basis of $V_0$.

Since in most practical applications only sampled data are available, it is natural to look for a similar construction for the space of $N$-periodic sequences. Our aim is to define a Multiresolution Analysis for the space of $N$-periodic sequences in a way such that we can be able to use two important advantages of MRA: the existence of sampling expansions for the MRA subspaces and a wavelet decomposition algorithm.

2. The $p$-adic multiresolution analysis

**Definition 1** Let $p, M > 1$ be positive integers, we define $V_M = \{s(n), n = 0, ..., p^M - 1\}$ to be the $p^M$-dimensional vector space of all complex $p^M$-periodic sequences with the usual orthonormal basis $e_k(n) = \delta_{n,k}$, where $\delta_{n,k}$ is the Kronecker’s delta.

The main difficulty for defining a Multiresolution Analysis for spaces of $p^M$-periodic sequences is the notion of the Dilation operator for sequences. We give the following:

**Definition 2** Let $p, M > 1$ be positive integers and let $\varphi_0$ be a $p^M$-periodic sequence, the dilation operator with respect to the sequence $\varphi_0$ is:

$$D_{\varphi_0} : V_M \rightarrow V_M, s \rightarrow (D_{\varphi_0}s)(n) = \frac{1}{\sqrt{p^M}} \sum_{k=0}^{p^{M-1}-1} \varphi_0(n - kp) \sum_{m=0}^{p-1} s(k + mp^{M-1}), n = 0, ..., N - 1,$$

or equivalently: $(\widehat{D_{\varphi_0}s})(n) = \hat{s}(pn)\hat{\varphi_0}(n), n = 0, ..., p^M - 1$, where $\hat{s}(n)$ is the Discrete Fourier Transform of the $p^M$-periodic sequence $s(n)$.

**Definition 3** We shall say that a $p$-adic Multiresolution Analysis ($p$-adic MRA) of $V_M$ is a nested sequence $\{V_j, j = 0, ..., M - 1\}$ of subspaces of $V_M$ and a scaling sequence $\{\varphi_1(n)\}$ such that:

(i) $\{c(n) : c(n) = (\alpha, ..., \alpha), \ \alpha \in \mathbb{C}\} = V_0 \subset V_1 \subset ... \subset V_{M-1} \subset V_M$,

(ii) $V_j$ is the linear span of an orthonormal set $\{\varphi_{M-j}(-kp^{M-j}), k = 0, ..., p^j - 1\}$, where for $j = M - 1$ the scaling sequence $\{\varphi_1(n)\}$ is the dilation of the sequence $e_0(n) = (1, 0, ..., 0)$ and for $j = 0, ..., M - 2$ the sequence $\{\varphi_{M-j}(n)\}$ is the dilation of the sequence $\{\varphi_{M-j-1}(n)\}$.
Definition 4 Given the dilation operator \((D_{\varphi_0})\) its associate sequence is given by:

\[
\varphi_{M-j}(n) = \begin{cases} 
\frac{\varphi_0(n)}{\sqrt{p^M}}, & j = M - 1 \\
(D_{\varphi_0}\varphi_{M-j-1})(n), & j = 0, \ldots, M - 2 
\end{cases}
\]

The associate sequence is given by its DFT form in the following:

\[
\varphi_{M-j}(k) = \begin{cases} 
\frac{\hat{\varphi}_0(k)}{\sqrt{p^M}}, & j = M - 1 \\
\varphi_{M-j-1}(pk)\hat{\varphi}_0(k), & j = 0, \ldots, M - 2 
\end{cases}
\]

(1)

Proposition 1 The Dilation operator \(D_{\varphi_0}\) satisfies the following:

(i) \((D_{\varphi_0}T_1 s)(n) = (T_p D_{\varphi_0} s)(n)\), where \(T_k : V_M \to V_M\), \((T_k s)(n) = s(n-k)\) is the Translation operator for sequences,

(ii) \(\|D_{\varphi_0}\|_2 = p^{1/2}\),

(iii) Let \(\hat{\varphi}_0(k_1) = \ldots = \varphi_0(k_m) = 0, 0 \leq m < p^N - 1\), then: \(\text{Ker}D_{\varphi_0} = \{s(n) : s(pm) = 0, n \neq k_i\}\),

(iv) Let \(s(n) \in V_j\), then \((D_{\varphi_0} s)(n) \in V_{j-1}\).

Proof (i) It is an immediate consequence of Definition 2.

\[
\|D_{\varphi_0}\|_2^2 = \sum_{n=0}^{p^M-1} |\hat{s}(pn)|^2 |\varphi_0(n)|^2 = \sum_{k=0}^{p-1} \sum_{m=0}^{pM-1} |\hat{s}(pm)|^2 |\varphi_0(m+kp^M)|^2
\]

\[
= p \sum_{m=0}^{pM-1} |\hat{s}(pm)|^2 \leq p\|s(.)\|_2^2.
\]

Obviously \(\|D_{\varphi_0}\|_2 = \sup_{s \in V_M} \frac{\|D_{\varphi_0} s(.)\|_2}{\|s(.)\|_2} \leq \sqrt{p}\), but we can find a sequence \(s_1(n)\) to obtain equality. In fact define \(s_1(n)\) as in the following:

\[
s_1(n) = \begin{cases} 
1, & n = pk \\
0, & n \neq pk 
\end{cases}
\]

(iii) It is an immediate consequence of the fact that \((D_{\varphi_0} s)(n) = \hat{s}(pn)\hat{\varphi}_0(n)\).

(iv) Let \(s(n) \in V_j\) then:

\[
(D_{\varphi_0} s)(n) = \sum_{k=0}^{p^j-1} c_k \sum_{k=0}^{p^j-1} c_k (T_{kp^M-j} \varphi_{M-j})(n) = \sum_{k=0}^{p^j-1} c_k (T_{kp^M-j+1} D_{\varphi_0} \varphi_{M-j})(n)
\]

\[
= \sum_{k=0}^{p^j-1} c_k \varphi_{M-j+1}(n-kp^M-j+1) = \sum_{s=0}^{p^j-1} \sum_{r=0}^{p-1} c_s c_{rM-j+1} \varphi_{M-j+1}(n-sp^M-j+1).
\]

\(\square\)
Proposition 2 Let $\varphi_0$ be a $p^M$-periodic sequence and let the collection \( \{ \varphi_{M-j}(n) \}_{j=0}^{M-1} \) be as in (1). If
\[
\sum_{s=0}^{p-1} |\hat{\varphi}_0(r + sp^{M-1})|^2 = p, \ r = 0, ..., p^{M-1} - 1,
\]
then:

(i) the collection \( \{ \varphi_{M-j}(n-kp^M-j) \}_{k=0}^{j}, (j = 0, ..., M-1) \) is an orthonormal basis of \( V_j \),

(ii) the subspaces \( V_j \) form a p-adic MRA of \( V_M \) with scaling sequence $\varphi_1(n) = \frac{\hat{\varphi}_0(n)}{\sqrt{p^M}}$.

Proof The proof was presented in [?]. We shall briefly sketch it. First we find a necessary and sufficient condition for the orthonormality of the set \( \{ \varphi_{M-j}(n-kp^M-j) \}_{m=0}^{p^M-j-1}, (m = 0, ..., p^j - 1) \), which is the following:
\[
\sum_{s=0}^{p^M-j-1} |\hat{\varphi}(r + sp^j)|^2 = \frac{1}{p^j}, \ r = 0, ..., p^j - 1.
\]

Let \( j = M - 1 \). Using (3), definition 4 and (2), we see that the p-translations of \( \varphi_1 \) form an orthonormal basis of \( V_{M-1} \). In fact:
\[
\sum_{s=0}^{p-1} |\hat{\varphi}_1(r + sp^{M-1})|^2 = \frac{\sum_{s=0}^{p-1} |\hat{\varphi}_0(r + sp^{M-1})|^2}{p^{M-1}} = \frac{p}{p^{M-1}}, \ r = 0, ..., p^{M-1} - 1,
\]
thus Proposition 2 is valid for \( j = M - 1 \). Now let \( j = 0, ..., M - 2 \). We suppose that the set \( \{ \varphi_{M-j}(n-kp^M-j) \}_{k=0}^{p^j-1} \) is an orthonormal basis of \( V_j \) and by using (3) in combination with (1) and (2) we show that the $p^{M-j+1}$ translations of the sequence $\varphi_{M-j+1}(n)$ form an orthonormal basis of $V_{j-1}$. The proof follows by induction. \( \square \)

Proposition 3 Let $\varphi(n)$ be a $p^M$-periodic sequence which satisfies the following:
\[
\sum_{m=0}^{p^{M-1}-1} \varphi(k + mp)\overline{\varphi(k + (m-r)p)} = \begin{cases} 
1/p, & r = 0 \\
0, & r = 1, ..., p^{M-1} - 1 
\end{cases}
\]

then $\varphi(n)$ is the scaling sequence of a p-adic MRA.

Proof The proof was presented in [?]. \( \square \)

3. The p-adic sampling sequence

Definition 5 We say that an M-dimensional subspace $W$ of $V$ has a sampling basis $\{ s_0, ..., s_{M-1} \}$, if there exist M positive integers $0 \leq n_1 < ... < n_M < N$ such that for any sequence $a(n) \in W$ we have:
\[
a(n) = \sum_{j=0}^{M-1} a(n_j)s_j(n), \ 0 \leq n \leq N - 1.
\]
In particular we say that \( W \) has a \textbf{sampling sequence} \( s(n) \) (which is \( N \)-periodic), if there exist \( M \) positive integers \( 0 \leq n_1 < ... < n_M < N \) such that for any sequence \( a(n) \in W \) we have:

\[
a(n) = \sum_{j=0}^{M-1} a(n_j)s(n - n_j), \ 0 \leq n \leq N - 1.
\]

**Theorem 1** Let \( \varphi_0(n) \) be a \( p^M \)-periodic sequence which produces a \( p \)-adic MRA \( \{V_j\}_{j=0}^{M-1} \). If

\[
\sum_{s=0}^{p-1} \widehat{\varphi_0}(r + sp^M) \neq 0, \ 0 \leq r \leq p^{M-1} - 1,
\]

then any sequence \( f(n) \in V_j \) has the following \textbf{sampling expansion}:

\[
f(n) = \sum_{m=0}^{p^j-1} f(mp^M - j)s_{M-j}(n - mp^M - j), \ n = 0, ..., p^M - 1,
\]

where

\[
s_{M-j}(\cdot) \leftrightarrow \frac{\sqrt{p^M}}{p^j} \sum_{r=0}^{p^{M-j-1}} \widehat{\varphi_{M-j}}(\cdot + r).
\]

Obviously the sequence \( s_{M-j}(n) \) is the \textbf{sampling sequence} of the MRA subspace \( V_j \).

**Proof** The proof was presented in [?]. We give here a slightly alternate proof. The hypothesis (4) implies that

\[
\sum_{s=0}^{p^M-1} \varphi_{M-j}(r + sp^j) \neq 0, \ 0 \leq r \leq p^j - 1 \ (j = 0, ..., M - 1).
\]

(6)

By [?] page 255, it suffices to prove that the set \( \{K_j(\cdot, mp^M - j), m = 0, ..., p^j - 1\} \) is a basis for \( V_j \), where

\[
K_j(n, m) = \sum_{r=0}^{p^j-1} \varphi_{M-j}(n - rp^M - j)\varphi_{M-j}(m - rp^M - j), \ 0 \leq n, m \leq N - 1
\]

is the reproducing kernel of \( V_j \), thus it suffices to find a positive constant \( A > 0 \) such that for any sequence of scalars \( \{a_m\}_{m=0}^{p^j-1} \), one has:

\[
A \left\| \sum_{m=0}^{p^j-1} a_m K_j(\cdot, mp^M - j) \right\|_2^2 \leq \left\| \sum_{m=0}^{p^j-1} a_m K_j(\cdot, mp^M - j) \right\|_2^2.
\]

After some calculations concerning the DFT of the kernel we find

\[
\left\| \sum_{m=0}^{p^j-1} a_m K_j(\cdot, mp^M - j) \right\|_2^2 = \frac{1}{p^M-j} \sum_{k=0}^{p^M-j-1} \sum_{l=0}^{p^j-1} |\widehat{\alpha_k}|^2 |\varphi_{M-j}(k + lp^j)|^2 \sum_{s=0}^{p^{M-j-1}} |\varphi_{M-j}(k + sp^j)|^2
\]

thus \( \{K_j(\cdot, mp^M - j), m = 0, ..., p^j - 1\} \) is a basis of \( V_j \), (see (6)) and it possesses a unique biorthonormal sequence \( \{s_{M-j, m} : m = 0, ..., p^j - 1\} \) which is a sampling basis for \( V_j \) (see [?]). Finally:

\[
\varphi_{M-j}(n) = \sum_{k=0}^{p^j-1} \varphi_{M-j}(kp^M - j)s_{M-j}(n - kp^M - j)
\]

which implies (5). □
4. Spectral wavelet decomposition

It is natural this work to provide wavelet Analysis. We believe that the wavelet sequence associated with our MRA is the following:

**Definition 6** Given the dilation operator \((D_{\varphi_0})\), then for \(l = 1, ..., p - 1\) its associate sequence is given by:

\[
\hat{\psi}_{M-j}(k) = \begin{cases} 
\hat{\varphi}_1(k + lp^{M-1}), & j = M - 1 \\
\hat{\psi}_{M-j-1}(pk)\hat{\varphi}_0(k), & j = 0, ..., M - 2
\end{cases}
\]

Suppose \(\varphi_1(n)\) be a scaling sequence of a p-adic MRA which satisfies Proposition 2. If \(W^l_j\) is the linear span of \(\{\psi_{M-j}(-kp^{M-j}), k = 0, ..., p^j - 1\}\) and if

\[
\sum_{n=0}^{p^j-1} \hat{\varphi}_1(np^{M-j} - lp^{M-j})\hat{\varphi}_1(np^{M-j} - kp^{M-j}) = 0, k \neq l, k, l = 1, ..., p - 1
\]

(the above condition implies that the spaces \(W^l_j\) are mutually orthogonal for \(l = 1, ..., p - 1\)), then we can write:

\[
V_M = V_0 \bigoplus_{j=0}^{M-1} \bigoplus_{l=1}^{p-1} W^l_j
\]

and we have the following decomposition algorithm:

\[
s(n) = \sum_{j=0}^{M-1} \sum_{k=0}^{p^j-1} \sum_{l=0}^{p-1} (s(\cdot), \psi_{M-j}(-kp^{M-j}))\psi_{M-j}(n - kp^{M-j}) + \frac{1}{p^M} \sum_{k=0}^{p^{M-1}-1} s(k).
\]

References
