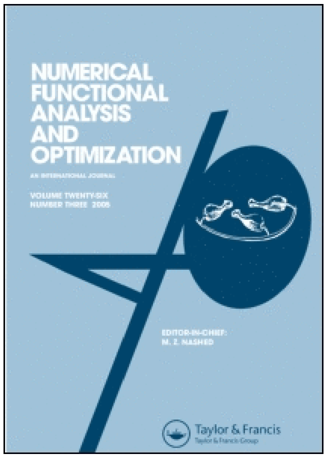

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A Class of Sparse Invertible Matrices and Their Use for Nonlinear Prediction of Nearly Periodic Time Series with Fixed Period

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A CLASS OF SPARSE INVERTIBLE MATRICES AND THEIR USE FOR NONLINEAR PREDICTION OF NEARLY PERIODIC TIME SERIES WITH FIXED PERIOD

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□ *We introduce a class of sparse matrices $U_m(A_{p_1})$ of order m by m, where m is a composite natural number, p_1 is a divisor of m, and A_{p_1} is a set of nonzero real numbers of length p_1. The construction of $U_m(A_{p_1})$ is achieved by iteration, involving repetitive dilation operations and block-matrix operations. We prove that the matrices $U_m(A_{p_1})$ are invertible and we compute the inverse matrix $(U_m(A_{p_1}))^{-1}$ explicitly. We prove that each row of the inverse matrix $(U_m(A_{p_1}))^{-1}$ has only two nonzero entries with alternative signs, located at specific positions, related to the divisors of m. We use the structural properties of the matrix $(U_m(A_{p_1}))^{-1}$ in order to build a nonlinear estimator for prediction of nearly periodic time series of length m with fixed period.*

Keywords Prediction; Sparse matrices; Time series.

AMS Subject Classification 65F50; 65F30; 15A09; 60G25.

1. INTRODUCTION

A *time series* is a sequence of observations taken sequentially in time. Many sets of data appear as time series: hourly observations made on the yield of a chemical process, a weekly series of the number of road accidents, and so on. Examples of time series abound in such fields as economics, engineering, geophysics, meteorology, social sciences, etc. An intrinsic feature of a time series is that, typically, adjacent observations are dependent. The nature of this dependence among observations is of considerable practical interest. As an example of this nature, one can consider the periodicity with which data appear. We suppose that observations are available at discrete, equi-spaced intervals of time (more about time series can be seen in [4, 12]).

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Some of the main goals of time series analysis are predicting, modeling, and characterization. In this direction, matrix analysis and linear algebra techniques (see [2]) have contributed a lot, as data are usually stored via a matrix. Sparse matrices have a "small" number of nonzero elements (see [6, 11]), so they provide fast computations and computational saving methods. They are mainly used for graph algorithms, neural networks, numerical solution of partial differential equations, and they could also be very useful in the process of extracting local information.

Basically, the aim of *predicting* is to predict the sort-term evolution of a system, that is to "predict" future values of a process, given a record of its past values. Obviously, for the process of predicting the future values, we wish to make use of the given information. This problem is clearly of interest in the context of most branches of sciences, like economics (for example, to predict future values of the stock market prices), weather analysis (for example, to forecast the weather), geophysics (for example, to predict future values of the ozone of the atmosphere on different layers), and so forth. For surveys and perspectives for time series prediction, see [3, 4, 7, 8, 10, 12–15].

The aim of this work is

(a) To build a linear invertible transform on data of length m, with the ability to extract local information at different scales. The particular transform is based on the construction of a class of sparse invertible matrices $U_m(A_{p_1})$ of order m by m (generalizing our work in [1]), such that:

- $U_m(A_{p_1})$ is built via an iteration process on matrices, starting from an initial set $A_{p_1} = \{a_1, \dots, a_{p_1}\}$ of nonzero numbers (p_1 is a divisor of m) and using repetitive, properly selected dilation operations and block matrix operations.
- $U_m(A_{p_1})$ is invertible and the inverse matrix $(U_m(A_{p_1}))^{-1}$ is also a sparse matrix with entries $1/a_i, 0, -1/a_i$, $(i = 1, \dots, p_1)$ and it is constructed via a recursion equation on matrices. It presents interesting properties, listed in Section 3.

(b) To use the transform corresponding with the matrix $U_m(A_{p_1})$ for prediction of nearly periodic time series with fixed period. We say that a sequence $\{t_k,\ k = 1, \dots, m\}$ is nearly periodic with fixed period N, if:

(i) t_k has the same repeating pattern of length N, but with different scaling over different periods, or

(ii) the sequence t_k has nearly repeating patterns with different scaling factors over different periods (see [5]).

Our basic idea for prediction is based on the fact that the extension $\widetilde{T}$ of a data T as defined in Definition 4.4, reflects on equality of most of their corresponding transform elements (see Proposition 4.5).

In Section 2, Definitions 2.1, 2.3, 2.5, and 2.6, we present some new dilation operations and block matrix operations on matrices. In Definition 2.11, we introduce the iteration process to construct the matrix $U_m(A_{p_1})$. In Proposition 2.13, we prove that these matrices are invertible. In Theorem 2.15, we prove a recursion equation for computing the inverse matrix $(U_m(A_{p_1}))^{-1}$.

In Section 3, Proposition 3.1, we demonstrate the structure of $(U_m(A_{p_1}))^{-1}$ and we list its properties.

Finally, in Section 4, we build an algorithm, giving rise to a nonlinear estimator for prediction of nearly periodic time series.

2. CONSTRUCTION AND PROPERTIES OF $U_M(A_{P_1})$

Notation (see also [9]). Let $\mathsf{M}_{n,m}$ be the set of all matrices of order m by m over the field of complex numbers. If $n = m$, then $\mathsf{M}_{n,m}$ is abbreviated to M_n. We shall use the symbolism $A = [A_{ij}]$ to denote a matrix A with elements A_{ij}. The notation

$$A_i = \{A_{ij} : j = 1, \ldots, m\}$$

shall be used to denote the i-row of a matrix $A \in \mathsf{M}_{n,m}$. We use the notation A^T to denote the transpose of a matrix A. A square matrix $A \in \mathsf{M}_n$ is invertible, if there is a unique square matrix $A^{-1} \in \mathsf{M}_n$ called the inverse matrix of A, such that $AA^{-1} = \mathbf{I}_n$, where $\mathbf{I}_n$ is the identity matrix. A matrix having a small number of nonzero elements is called *sparse*. $P \in \mathsf{M}_n$ is a *permutation* matrix, if it is formed from the identity matrix $\mathbf{I}_n$ by reordering its columns (or rows). The determinant of a permutation matrix P is given by:

$$Det(P) = sgn\, \sigma,$$

where $\sigma = \{\sigma(i) : i = 1, \ldots, n\}$ is the permutation of its columns and the signature $sgn\, \sigma$ equals $(-1)^r$, where r is the number of transpositions of pairs of columns that must be composed to build up the permutation. In practice, in order to estimate r, we compute the number of elements $\sigma(i) : \sigma(1) > \sigma(i),\ i = 2, \ldots, n$, then we compute the number of elements $\sigma(i) : \sigma(2) > \sigma(i),\ i = 3, \ldots, n$, and so forth, and finally we sum all previously computed numbers.

The *ceiling* of a real number x shall be denoted by $\lceil x \rceil = inf\{n \in \mathbf{Z} : x \le n\}$ ($\mathbf{Z}$ is the set of integers). If p, q are natural numbers, we denote by $Mod(p, q)$ the remainder of the division of p by q, and we shall use the symbolism $[q]_p = \{q + tp : t \in \mathbf{Z}\}$ to denote the residue class of q *modulo* p.

We define the following matrix dilation operations D_p and H_p on the set $\mathsf{M}_{n,m}$, where $p = 2, 3, \ldots$.

Definition 2.1. Let $D_p : \mathsf{M}_{n,m} \to \mathsf{M}_{n,pm}$, such that:

$$D_p(M) = \left\{ M_{i,\lceil \frac{j}{p} \rceil}, i = 1, \ldots, n, j = 1, \ldots, pm \right\}.$$

Notice that D_p can be written as a block matrix:

$$D_p(M) = \begin{pmatrix} D_p(M_{11}) & \ldots & D_p(M_{1m}) \\ \vdots & \ddots & \vdots \\ D_p(M_{n1}) & \ldots & D_p(M_{nm}) \end{pmatrix}, \tag{2.1}$$

where $D_p(M_{ij}) \in \mathsf{M}_{1,p}$: $D_p(M_{ij}) = \{M_{ij}, M_{ij}, \ldots, M_{ij}\}$.

Example 2.2.

$$D_2\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right) = \begin{pmatrix} a_{11} & a_{11} & a_{12} & a_{12} \\ a_{21} & a_{21} & a_{22} & a_{22} \end{pmatrix},$$
$$D_3\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right) = \begin{pmatrix} a_{11} & a_{11} & a_{11} & a_{12} & a_{12} & a_{12} \\ a_{21} & a_{21} & a_{21} & a_{22} & a_{22} & a_{22} \end{pmatrix}.$$

Definition 2.3. Let $H_p : \mathsf{M}_{n,m} \to \mathsf{M}_{pn,m}$:

$$H_p(M) = \left\{ \begin{cases} M_{\lceil \frac{i}{p} \rceil, j}, & \text{whenever } i \in [0]_p \\ 0, & \text{whenever } i \notin [0]_p \end{cases}, \quad i = 1, \ldots, pn, j = 1, \ldots, m \right\}.$$

Example 2.4.

$$H_2\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ a_{11} & a_{12} \\ 0 & 0 \\ a_{21} & a_{22} \end{pmatrix}, H_3\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ a_{11} & a_{12} \\ 0 & 0 \\ 0 & 0 \\ a_{21} & a_{22} \end{pmatrix}.$$

Definition 2.5. Let $S(., \ldots, .) : \mathsf{M}_{n_1,m} \times \cdots \times \mathsf{M}_{n_k,m} \to \mathsf{M}_{n_1+\cdots+n_k,m}$ be the following block matrix operation:

$$S(M_1, \ldots, M_k) = \begin{pmatrix} M_1 \\ \vdots \\ M_k \end{pmatrix}.$$

Definition 2.6. Let p, w be positive integers such that w is a divisor of p and let $A_w = \{a_1, \ldots, a_w\}$ be a set of nonzero real numbers. We define the following block matrix operator: $Q_{p,A_w} : \mathsf{M}_{n,m} \to \mathsf{M}_{pn,pm}$:

$$Q_{p,A_w}(M) = \underbrace{a_1 M \oplus \cdots \oplus a_1 M}_{p/w\text{-times}} \oplus \cdots \oplus \underbrace{a_w M \oplus \cdots \oplus a_w M}_{p/w\text{-times}}$$
$$= \begin{pmatrix} a_1 M & & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & & a_w M \end{pmatrix}, \tag{2.2}$$

where $\oplus$ is the *direct sum* of matrices and $\mathbf{O}$ is the zero matrix of order $n \times m$.

Example 2.7.

(1) $$Q_{4,\{a_1,a_2\}}((1\ 0)) = \begin{pmatrix} a_1 \cdot (1\ 0) & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & a_1 \cdot (1\ 0) & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & a_2 \cdot (1\ 0) & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & a_2 \cdot (1\ 0) \end{pmatrix}$$
$$= \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 \end{pmatrix},$$

where $\mathbf{O}$ is the zero matrix of order 1×2.

(2) $$Q_{3,\{a_1,a_2,a_3\}}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} a_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & a_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & a_3\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 \\ -a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & a_3 \\ 0 & 0 & -a_3 \end{pmatrix},$$

where $\mathbf{O}$ is the zero matrix of order 2×1.

Definition 2.8. Let p, q, w be positive integers such that $q > 1$ and w is a divisor of p. Let $A_w = \{a_1, \ldots, a_w\}$ be a set of nonzero real numbers. We define the following matrix: $R(p, q, A_w) \in \mathsf{M}_{p(q-1),pq}$:

$$R(p, q, A_w) = \begin{cases} S\big(a_1 \cdot \mathbf{e}_1^q, \ldots, a_{q-1} \cdot \mathbf{e}_{q-1}^q\big), & p = 1 \\ S\big(Q_{p,A_w}(\mathbf{e}_1^q), \ldots, Q_{p,A_w}(\mathbf{e}_{q-1}^q)\big), & p > 1 \end{cases},$$

where $\mathbf{e}_i^q$ is the i row of the identity matrix $\mathbf{I}_q$ and Q_{p,A_w} is given in Definition 2.6.

Example 2.9. Let $w = 2$, (i.e., $A = \{a_1, a_2\}$), then:

$$R(2,2,A_2) = Q_{2,\{a_1,a_2\}}(\mathbf{e}_1^2) = \begin{pmatrix} a_1\mathbf{e}_1^2 & \mathbf{O} \\ \mathbf{O} & a_2\mathbf{e}_1^2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 \end{pmatrix},$$

where $\mathbf{e}_1^2 = \{1,0\}$, $\mathbf{O} = \{0,0\}$, and $R(2,3,A_2) =$

$$S\big(Q_{2,\{a_1,a_2\}}(\mathbf{e}_1^3), Q_{2,\{a_1,a_2\}}(\mathbf{e}_2^3)\big) = \begin{pmatrix} a_1\mathbf{e}_1^3 & \mathbf{O} \\ \mathbf{O} & a_2\mathbf{e}_1^3 \\ a_1\mathbf{e}_2^3 & \mathbf{O} \\ \mathbf{O} & a_2\mathbf{e}_2^3 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 \end{pmatrix},$$

where $\mathbf{e}_1^3 = \{1,0,0\}$, $\mathbf{e}_2^3 = \{0,1,0\}$ and $\mathbf{O} = \{0,0,0\}$.

Remark 2.10.

(i) Let r, s be positive integers. It is easy to see that $D_r D_s(M) = D_{rs}(M)$. The same is also true for the operator H_p.
(ii) Because the matrix $Q_{p,A_w}(\mathbf{e}_q^q)$ has not been used in the construction of the matrix $R(p,q,A_w)$, we have $(R(p,q,A_w))_{\cdot,lq} = 0$, for any $l = 1,\ldots,p$.

From now on, we consider a composite positive integer $m = lp_1$, where $l \geq 1$. Moreover, we assume that m can be written as:

$$m = p_1 p_2 \ldots p_N, \tag{2.3}$$

where $p_2 \geq p_3 \geq \cdots \geq p_N$ are prime factors of m/p_1. Notice that p_1 is not necessarily prime. We denote:

$$J(0) = 1, \quad J(n) = \prod_{r=1}^{n} p_r, \quad n = 1,\ldots,N \tag{2.4}$$

$$A(i) = \prod_{r=i}^{N} p_r, \quad i = 1,\ldots,N, \quad A(N+1) = 1. \tag{2.5}$$

Definition 2.11. We consider the factorization (2.3) of a positive integer m and we define a sequence of block matrices $U_m(n, A_{p_1}) \in \mathsf{M}_{J(n)}$ ($J(n)$ is defined in (2.4)), where $n = 0, \dots, N$, by using the following iteration:

$$U_m(n, A_{p_1}) = \begin{cases} (a_1 \dots a_{p_1}), & n = 0 \\ S(U_m(0, A_{p_1}), R(1, p_1, A_{p_1})), & n = 1 \\ S(D_{p_n}(U_m(n-1, A_{p_1})), R(J(n-1), p_n, A_{p_1})), & n = 2, \dots, N \end{cases},$$

where $A_{p_1} = \{a_1, \dots, a_{p_1}\}$ is a set of nonzero real numbers. In case where $n = N$, we shall write $U_m(N, A_{p_1}) = U_m(A_{p_1})$.

Example 2.12.

$$U_p(1, A_p) = \begin{pmatrix} a_1 & a_2 & \dots & a_{p-1} & a_p \\ a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & a_{p-1} & 0 \end{pmatrix}. \tag{2.6}$$

Let $m = 12$, take $p_1 = 3$, then $m = p_1 p_2 p_3$, where $p_2 = 2$, $p_3 = 2$, so:

$$U_{12}(0, A_3) = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}, \quad U_{12}(1, A_3) = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & 0 & 0 \\ 0 & a_2 & 0 \end{pmatrix},$$

$$U_{12}(2, A_3) = \begin{pmatrix} a_1 & a_1 & a_2 & a_2 & a_3 & a_3 \\ a_1 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & a_2 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_3 & 0 \end{pmatrix} \text{ and}$$

$$U_{12}(3, A_3) = \begin{pmatrix} a_1 & a_1 & a_1 & a_1 & a_2 & a_2 & a_2 & a_2 & a_3 & a_3 & a_3 & a_3 \\ a_1 & a_1 & a_1 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & a_2 & a_2 & a_2 & 0 & 0 & 0 & 0 \\ a_1 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & a_3 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 \end{pmatrix}.$$

Now, let $j = 1, \dots, p_n - 1$, $(n = 1, \dots, N)$, we define the following column matrices $V_j^{p_n} = \{v_{kj}^{p_n} : k = 1, \dots, p_n\}$:

$$v_{kj}^{p_n} = \begin{cases} 1, & \text{whenever } k = j \\ -1, & \text{whenever } k = p_n \\ 0, & \text{elsewhere} \end{cases}. \tag{2.7}$$

Proposition 2.13. *Let $\{p_n : n = 1, \dots, N\}$ be a sequence of factors of a composite positive integer m as in (2.3) with the corresponding sequence $J(n)$ defined in (2.4), then:*

$$Det(U_m(n, A_{p_1})) = \begin{cases} (-1)^{p_1+1} a_1 \cdots a_{p_1}, & n = 1 \\ (-1)^{q(n)} a_1^{\frac{(p_n-1)J(n-1)}{p_1}} \cdots a_{p_1}^{\frac{(p_n-1)J(n-1)}{p_1}} Det(U_m(n-1, A_{p_1})), & n > 1 \end{cases},$$

where $q_n = \frac{p_n-1}{4} J(n-1)(J(n) - p_n + 4)$.

Proof. Let $n = 1$, we use (2.6) to get: $Det(U_m(1, A_{p_1})) = (-1)^{1+p_1} \cdot a_{p_1} \cdot Det(M^{1,p_1})$, where M^{1,p_1} is a minor of the matrix $U_m(1, A_{p_1})$. Because M^{1,p_1} is a diagonal matrix (see (2.6)), we get that $Det(U_m(1, A_{p_1})) = (-1)^{1+p_1} a_1 \dots a_{p_1}$.

Let $n > 1$ and let $\mathbf{e}_i^{p_n}$ be the i row of the identity matrix $\mathbf{I}_{p_n}$. We consider the set $\widetilde{A}_{p_1} = \{1/a_1, \dots, 1/a_{p_1}\}$ and we define the following block matrix

$$C(n, \widetilde{A}_{p_1}) \in \mathsf{M}_{J(n)} : C\big(n, \widetilde{A}_{p_1}\big) = \left(Q_{J(n-1), \{\underbrace{1,\dots,1}_{p_n\text{-times}}\}}\big((\mathbf{e}_{p_n}^{p_n})^T\big) \quad Q_{J(n-1), \widetilde{A}_{p_1}}\big(V_1^{p_n}\big) \quad \cdots \quad Q_{J(n-1), \widetilde{A}_{p_1}}\big(V_{p_n-1}^{p_n}\big) \right),$$

where the block submatrices $Q_{J(n-1), \{\underbrace{1,\dots,1}_{p_n\text{-times}}\}}((\mathbf{e}_{p_n}^{p_n})^T)$ and $Q_{J(n-1), \widetilde{A}_{p_1}}(V_j^{p_n})$, $j = 1, \dots, p_n - 1$ are in $\mathsf{M}_{J(n), J(n-1)}$ (the column matrices $V_j^{p_n}$ are given in (2.7)). The block matrix multiplication $U_m(n, A_{p_1}) C(n, \widetilde{A}_{p_1})$ derives the following block diagonal matrix (for a proof of (2.8), see Appendix I):

$$U_m(n, A_{p_1}) C(n, \widetilde{A}_{p_1}) = \begin{pmatrix} D_{p_n}(U_m(n-1, A_{p_1})) \\ Q_{J(n-1), A_{p_1}}(\mathbf{e}_1^{p_n}) \\ \vdots \\ Q_{J(n-1), A_{p_1}}(\mathbf{e}_{p_n-1}^{p_n}) \end{pmatrix}$$

$$\cdot\left(Q_{J(n-1),\left\{\underbrace{1,\ldots,1}_{p_n\text{-times}}\right\}}\left(\left(\mathbf{e}^{p_n}_{p_n}\right)^T\right)\quad Q_{J(n-1),\widetilde{A}_{p_1}}(V_1^{p_n})\quad\cdots\quad Q_{J(n-1),\widetilde{A}_{p_1}}\left(V^{p_n}_{p_n-1}\right)\right)$$
$$=\begin{pmatrix} U_m(n-1,A_{p_1}) & & & \mathbf{O}\\ & \mathbf{I}_{J(n-1)} & & \\ & & \ddots & \\ \mathbf{O} & & & \mathbf{I}_{J(n-1)}\end{pmatrix}, \tag{2.8}$$

where the zero matrix $\mathbf{O}$ in the right-hand side of (2.8) belongs in $\mathsf{M}_{J(n-1)}$. As a result we get

$$Det(U_m(n,A_{p_1}))Det(C(n,\widetilde{A}_{p_1}))=Det(U_m(n-1,A_{p_1})).$$

The computation of $Det\big(C\big(n,\widetilde{A}_{p_1}\big)\big)$ is equivalent to computing $Det(K(n,\widetilde{A}_{p_1}))$, where $K(n,\widetilde{A}_{p_1})$

$$=\left(Q_{J(n-1),\left\{\underbrace{1,\ldots,1}_{p_n\text{-times}}\right\}}\left(\left(\mathbf{e}^{p_n}_{p_n}\right)^T\right)\quad Q_{J(n-1),\widetilde{A}_{p_1}}\left(\left(\mathbf{e}^{p_n}_1\right)^T\right)\quad\cdots\quad Q_{J(n-1),\widetilde{A}_{p_1}}\left(\left(\mathbf{e}^{p_n}_{p_n-1}\right)^T\right)\right)$$

is a block matrix in $\mathsf{M}_{J(n)}$ resulting from $C(n,\widetilde{A}_{p_1})$, by replacing each block submatrix $Q_{J(n-1),\widetilde{A}_{p_1}}(V_j^{p_n})$ with the linear combination:

$$Q_{J(n-1),\widetilde{A}_{p_1}}(V_j^{p_n})+Q_{J(n-1),\widetilde{A}_{p_1}}\left(\left(\mathbf{e}^{p_n}_{p_n}\right)^T\right)=Q_{J(n-1),\widetilde{A}_{p_1}}\left(\left(\mathbf{e}^{p_n}_{j}\right)^T\right),\ \ j=1,\ldots,p_n-1.$$

$K(n,\widetilde{A}_{p_1})$ is a generalized permutation matrix with the only nonzero elements in each row either 1, or an element of the set $\widetilde{A}_{p_1}$, so:

$$Det(K(n,\widetilde{A}_{p_1}))=a_1^{-\frac{(p_n-1)J(n-1)}{p_1}}\cdots a_{p_1}^{-\frac{(p_n-1)J(n-1)}{p_1}}\cdot sgn\,\sigma_n,$$

where $\frac{(p_n-1)J(n-1)}{p_1}$ is the number of a_i's $(i=1,\ldots,p_1)$ appearing in $K(n,\widetilde{A}_{p_1})$ (see (2.2)) and σ_n is the permutation of its columns (in order to obtain the identity matrix), thus:

$$Det(U_m(n,A_{p_1}))=a_1^{\frac{(p_n-1)J(n-1)}{p_1}}\cdots a_{p_1}^{\frac{(p_n-1)J(n-1)}{p_1}}sgn\,\sigma_n\cdot Det(U_m(n-1,A_{p_1})).$$

The permutation $\sigma_n=\{\sigma_n(1),\ldots,\sigma_n(J(n))\}$ of the columns of the matrix $K(n,\widetilde{A}_{p_1})$ can be written as:

$$\sigma_n=\Upsilon_{0,n}\bigcup_{i=1}^{p_n-1}\Upsilon_{i,n},$$

where $\Upsilon_{0,n}=\{tp_n:1\le t\le J(n-1)\}$ and $\Upsilon_{i,n}=\{i+tp_n:0\le t\le J(n-1)-1\}$.

In Appendix II we prove that $sgn\,\sigma_n = (-1)^{q_n}$, where $q_n = \frac{p_n-1}{4} J(n-1)(J(n)-p_n+4)$ and we complete the proof. □

Lemma 2.14. *The inverse matrix of $U_m(1, A_{p_1})$ satisfies:*

$$(U_m(1, A_{p_1}))^{-1} = \begin{pmatrix} 0 & \frac{1}{a_1} & 0 & \dots & \dots & 0 \\ 0 & 0 & \frac{1}{a_2} & 0 & \dots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & 0 & \frac{1}{a_{p_1-1}} \\ \frac{1}{a_{p_1}} & -\frac{1}{a_{p_1}} & \dots & -\frac{1}{a_{p_1}} & -\frac{1}{a_{p_1}} & -\frac{1}{a_{p_1}} \end{pmatrix}_{p_1 \times p_1}.$$

Proof. Elementary calculation. □

Theorem 2.15. *The inverse matrix of $U_m(n, A_{p_1})$ is given by the following recursion equation:*

$$(U_m(n, A_{p_1}))^{-1} = \Big(H_{p_n}((U_m(n-1, A_{p_1}))^{-1}) \quad Q_{J(n-1),\widetilde{A}_{p_1}}(V_1^{p_n}) \quad \dots \quad Q_{J(n-1),\widetilde{A}_{p_1}}\big(V_{p_n-1}^{p_n}\big)\Big), \quad n > 1.$$

Proof. We multiply both sides of (2.8) with the block diagonal matrix:

$$\begin{pmatrix} (U_m(n-1, A_{p_1}))^{-1} & & & \mathbf{O} \\ & \mathbf{I}_{J(n-1)} & & \\ & & \ddots & \\ \mathbf{O} & & & \mathbf{I}_{J(n-1)} \end{pmatrix}$$

whose block submatrices are in $\mathsf{M}_{J(n-1)}$ and we deduce that the inverse matrix $(U_m(n, A_{p_1}))^{-1}$ results from the following block matrix multiplication:

$$(U_m(n, A_{p_1}))^{-1} = \Bigg(Q_{J(n-1),\{\underbrace{1,\dots,1}_{p_n\text{-times}}\}}((\mathbf{e}_{p_n}^{p_n})^T) \quad Q_{J(n-1),\widetilde{A}_{p_1}}(V_1^{p_n}) \quad \dots \quad Q_{J(n-1),\widetilde{A}_{p_1}}\big(V_{p_n-1}^{p_n}\big)\Bigg)$$
$$\cdot \begin{pmatrix} (U_m(n-1, A_{p_1}))^{-1} & & & \mathbf{O} \\ & \mathbf{I}_{J(n-1)} & & \\ & & \ddots & \\ \mathbf{O} & & & \mathbf{I}_{J(n-1)} \end{pmatrix}$$
$$= \Bigg(Q_{J(n-1),\{\underbrace{1,\dots,1}_{p_n\text{-times}}\}}((\mathbf{e}_{p_n}^{p_n})^T) \cdot (U_m(n-1, A_{p_1}))^{-1} \quad Q_{J(n-1),\widetilde{A}_{p_1}}(V_1^{p_n}) \quad \dots \quad Q_{J(n-1),\widetilde{A}_{p_1}}\big(V_{p_n-1}^{p_n}\big)\Bigg)$$

$$= \Big(H_{p_n}(\mathbf{I}_{J(n-1)}) \cdot (U_m(n-1, A_{p_1}))^{-1} \quad Q_{J(n-1),\widetilde{A}_{p_1}}(V_1^{p_n}) \quad \ldots \quad Q_{J(n-1),\widetilde{A}_{p_1}}(V_{p_n-1}^{p_n})\Big)$$
$$= \Big(H_{p_n}((U_m(n-1, A_{p_1}))^{-1}) \quad Q_{J(n-1),\widetilde{A}_{p_1}}(V_1^{p_n}) \quad \ldots \quad Q_{J(n-1),\widetilde{A}_{p_1}}(V_{p_n-1}^{p_n})\Big). \qquad \square$$

3. SOME PROPERTIES OF THE MATRIX $((U_m(A_{p_1}))^{-1})^T$

Let V_m be the Euclidean space consisting of all real-valued sequences of length m, where m satisfies (2.3).

Proposition 3.1. *Let e_l, $l = 1, \ldots, m$, be a row of $((U_m(A_{p_1}))^{-1})^T$, such that $J(i) + 1 \le l \le J(i+1)$, where $i = 0, \ldots, N-1$. Take $l = kJ(i) + r$, where $k = 1, \ldots, p_{i+1} - 1$, $r = 1, \ldots, J(i)$, then we have:*

$$e_{l,n} = e_{kJ(i)+r,n} = \begin{cases} 1/a_{\lceil \frac{np_1}{m} \rceil}, & \text{whenever } n = (r-1)A(i+1) + kA(i+2) \\ -1/a_{\lceil \frac{np_1}{m} \rceil}, & \text{whenever } n = rA(i+1) \\ 0, & \text{for all others } n\text{'s} \end{cases},$$

where the sequences $J(n)$ and $A(i)$ have been defined in (2.4) *and* (2.5), *respectively.*

Proof. Take $l = kJ(i) + r$, where $k = 1, \ldots, p_{i+1} - 1$ and $r = 1, \ldots, J(i)$. For $i = 0$, we have $k = 1, \ldots, p_1 - 1$ and $r = 1$, so by Theorem 2.15, we get that:

$$e_{l,n} = \big((U_m(A_{p_1}))^{-1}\big)_{n,l} = H_{A(2)}\big((U_m(1, A_{p_1}))^{-1}\big)_{n,k+1}$$
$$= \begin{cases} \big((U_m(1, A_{p_1}))^{-1}\big)_{z,k+1}, & \text{whenever } n = zA(2) \\ 0, & \text{whenever } n \ne zA(2) \end{cases}, \quad z = 1, \ldots, p_1$$
$$= \begin{cases} \begin{cases} 1/a_k, & \text{whenever } z = k \\ -1/a_{p_1}, & \text{whenever } z = p_1 \\ 0, & \text{for all other } z\text{'s} \end{cases}, & \text{whenever } n = zA(2) \\ 0, & \text{whenever } n \ne zA(2) \end{cases},$$
$$z = 1, \ldots, p_1 \quad \text{(see Lemma 2.14)}$$
$$= \begin{cases} 1/a_k, & \text{whenever } n = kA(2) \\ -1/a_{p_1}, & \text{whenever } n = p_1A(2) \\ 0, & \text{for all other } n\text{'s} \end{cases}.$$

$$= \begin{cases} 1/a_{\lceil \frac{n}{A(2)} \rceil} = 1/a_{\lceil \frac{np_1}{m} \rceil}, & \text{whenever } n = kA(2) \\ -1/a_{\lceil \frac{n}{A(2)} \rceil} = -1/a_{\lceil \frac{np_1}{m} \rceil}, & \text{whenever } n = A(1) \\ 0, & \text{for all other } n\text{'s} \end{cases} .$$

For any $i = 1, \ldots, N-1$, we use the recursive equation of Theorem 2.15 to deduce that:

$$e_{l,n} = \big((U_m(A_{p_1}))^{-1}\big)_{n,l} = H_{A(i+2)}\big(Q_{J(i),\widetilde{A}_{p_1}}(V_k^{p_{i+1}})\big)_{n,l-J(i)}$$
$$= \begin{cases} (Q_{J(i),\widetilde{A}_{p_1}}(V_k^{p_{i+1}}))_{z,l-J(i)}, & \text{whenever } n = zA(i+2) \\ 0, & \text{whenever } n \neq zA(i+2) \end{cases},$$
$$z = 1, \ldots, J(i+1).$$

Obviously, $l - J(i) = (k-1)J(i) + r$, thus: $e_{kJ(i)+r,n}$

$$= \begin{cases} \begin{cases} 1/a_{\lceil \frac{rp_1}{J(i)} \rceil} (V_k^{p_{i+1}})_{Mod(z-1,p_{i+1})+1, \lceil \frac{p_1((k-1)J(i)+r)}{J(i)} \rceil}, & \text{whenever } r = \left\lceil \frac{z}{p_{i+1}} \right\rceil \\ 0, & \text{whenever } r \neq \left\lceil \frac{z}{p_{i+1}} \right\rceil \end{cases}, \\ \quad n = zA(i+2) \\ 0, \quad n \neq zA(i+2) \end{cases},$$

for $z = 1, \ldots, J(i+1)$

$$= \begin{cases} \begin{cases} 1/a_{\lceil \frac{rp_1}{J(i)} \rceil} v^{p_{i+1}}_{\sigma, p_1(k-1) + \lceil \frac{rp_1}{J(i)} \rceil}, & \text{whenever } z = (r-1)p_{i+1} + \sigma, \sigma = 1, \ldots, p_{i+1} \\ 0, & \text{whenever } z \neq (r-1)p_{i+1} + \sigma \end{cases} \\ \quad n = zA(i+2) \\ 0, \quad n \neq zA(i+2) \end{cases},$$

for $z = 1, \ldots, J(i+1)$

$$= \begin{cases} \begin{cases} \begin{cases} 1/a_{\lceil \frac{rp_1}{J(i)} \rceil}, & \text{for } \sigma \neq p_{i+1} \text{ and } \sigma = k \\ -1/a_{\lceil \frac{rp_1}{J(i)} \rceil}, & \text{for } \sigma = p_{i+1} \\ 0, & \text{otherwise} \end{cases}, \\ \quad \text{for } z = (r-1)p_{i+1} + \sigma, \sigma = 1, \ldots, p_{i+1} \\ 0, \quad \text{for } z \neq (r-1)p_{i+1} + \sigma \end{cases}, \\ \quad n = zA(i+2) \\ 0 \quad n \neq zA(i+2) \end{cases},$$

for $z = 1, \ldots, J(i+1)$

$$= \begin{cases} \begin{cases} 1/a_{\left\lceil \frac{rp_1}{J(i)} \right\rceil}, & \text{whenever } z = (r-1)p_{i+1} + k \\ -1/a_{\left\lceil \frac{rp_1}{J(i)} \right\rceil}, & \text{whenever } z = rp_{i+1} \\ 0, & \text{for other } z\text{'s} \end{cases}, & z = 1, \ldots, J(i+1). \\ \quad n = zA(i+2) & \\ 0, \quad n \neq zA(i+2) & \end{cases} \tag{3.1}$$

If $z = (r-1)p_{i+1} + k$, $z \neq lp_{i+1}$, then we have:

$$\left\lceil \frac{rp_1}{J(i)} \right\rceil = \left\lceil \frac{\left(\frac{z-k}{p_{i+1}} + 1\right)p_1}{J(i)} \right\rceil = \left\lceil \frac{zp_1}{J(i+1)} + \frac{p_1}{J(i)}\left(1 - \frac{k}{p_{i+1}}\right) \right\rceil = \left\lceil \frac{zp_1}{J(i+1)} \right\rceil,$$

thus (3.1) becomes:

$$e_{kJ(i)+r,n} = \begin{cases} \begin{cases} 1/a_{\left\lceil \frac{zp_1}{J(i+1)} \right\rceil}, & \text{for } z = (r-1)p_{i+1} + k \\ -1/a_{\left\lceil \frac{zp_1}{J(i+1)} \right\rceil}, & \text{for } z = rp_{i+1} \\ 0, & \text{for other } z\text{'s} \end{cases}, & n = zA(i+2) \\ 0, & n \neq zA(i+2) \end{cases},$$
$$z = 1, \ldots, J(i+1).$$

Because $n = zA(i+2)$ for all nonzero terms, we substitute z inside the brackets to get the result. □

Example 3.2. Let $m = 12$ and $p_1 = 3$, then $p_2 = 2$, $p_3 = 2$, so:

$$\left((U_{12}(A_3))^{-1}\right)^T$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/a_3 \\ 0 & 0 & 0 & 1/a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/a_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/a_2 & 0 & 0 & 0 & -1/a_3 \\ 0 & 1/a_1 & 0 & -1/a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/a_2 & 0 & -1/a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/a_3 & 0 & -1/a_3 \\ 1/a_1 & -1/a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/a_1 & -1/a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/a_2 & -1/a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/a_2 & -1/a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/a_3 & -1/a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/a_3 & -1/a_3 \end{pmatrix}.$$

Remark 3.3. Proposition 3.1 clarifies the structure of the matrix $((U_m(A_{p_1}))^{-1})^T$ in the following sense:

(a) Each row e_i of the matrix $((U_m(A_{p_1}))^{-1})^T$ (except for the first row) has only two nonzero entries with alternative signs. The first row is always of the form $(0\ \ldots\ 0\ 1/a_{p_1})$.
(b) $((U_m(A_{p_1}))^{-1})^T = V_0 \bigcup_{i=1}^{N-1} \bigcup_{k=1}^{p_{i+1}-1} V_{i,k}$, where $V_{i,k} = \{e_{kJ(i)+r} : r = 1, \ldots, J(i)\}$, $V_0 = \{e_1\}$.
(c) For any $i \geq 1$, there holds $V_{i,k} = \bigcup_{\mu=0}^{p_1-1} Q_{j,k,\mu}$, where:

$$Q_{j,k,\mu} = \{e_{kJ(i)+\mu(p_2\ldots p_i)+v} : v = 1, \ldots, (p_2 \ldots p_i)\}.$$

Moreover, every row of $Q_{j,k,\mu}$ has always its two nonzero entries in the form $\pm 1/a_\mu$.

Corollary 3.4. *Let t be a real valued sequence of length m. Take $l = kJ(i) + r$, where $k = 1, \ldots, p_{i+1} - 1$, $r = 1, \ldots, J(i)$ $(i = 0, \ldots, N-1)$ and let e_l be the l row of the matrix $((U_m(A_{p_1}))^{-1})^T$, then:*

$$\langle t, e_l \rangle = \langle t, e_{kJ(i)+r} \rangle = \frac{t_\mu}{a_{\lceil \frac{\mu p_1}{m} \rceil}} - \frac{t_v}{a_{\lceil \frac{v p_1}{m} \rceil}}.$$

where $\mu = (r-1)A(i+1) + kA(i+2)$, $v = rA(i+1)$.

Proof. Straightforward application of Proposition 3.1. □

4. A PREDICTION METHOD FOR NEARLY PERIODIC TIME SERIES

In this section, we consider either periodic data, or nearly repeating patterns that may be scaled differently, which we call nearly periodic data. We present a new method for prediction of such data, by using the structural properties of the matrix $((U_m(A_{p_1}))^{-1})^T$. Clearly, the invertibility of the matrix $U_m(A_{p_1})$ gives rise to a discrete transform, introduced below:

Lemma 4.1. *Let u_i, e_i, $(i = 1, \ldots, m)$, be rows of the matrices $U_m(A_{p_1})$, $((U_m(A_{p_1}))^{-1})^T$ respectively, then any element $t \in V_m$ can be written as:*

$$t_n = \sum_{i=1}^{m} y_i u_{i,n},$$

where $y_i = \langle t, e_i \rangle$, ($\langle .,. \rangle$ is the usual inner product of V_m).

Proof. Obvious. □

Definition 4.2. Let $y_i = \langle t, e_i \rangle$ be defined in Lemma 4.1, we define the following sets:

$$W_{i,k} = \{y_{kJ(i)+r} : r = 1, \dots, J(i)\},$$

where $i = 0, \dots, N-1$, $k = 1, \dots, p_{i+1} - 1$, $r = 1, \dots, J(i)$ and $J(i)$ has been defined in (2.4).

Example 4.3. Let $m = 72$, take $p_1 = 4$, then $p_2 = 3$, $p_3 = 3$, $p_4 = 2$, so we have $N = 4$. The corresponding sets $W_{i,k}$ of Definition 4.2 are the following:

For $i = 0$ we have $k = 1, 2, 3$ and $J(0) = 1$, so

$$W_{0,1} = \{y_2\}, \quad W_{0,2} = \{y_3\}, \quad W_{0,3} = \{y_4\}.$$

For $i = 1$ we have $k = 1, 2$ and $J(1) = 4$, so:

$$W_{1,1} = \{y_n : n = 5, \dots, 8\}, \quad W_{1,2} = \{y_n : n = 9, \dots, 12\}.$$

For $i = 2$ we have $k = 1, 2$ and $J(2) = 12$, so:

$$W_{2,1} = \{y_n : n = 13, \dots, 24\}, \quad W_{2,2} = \{y_n : n = 25, \dots, 36\}.$$

For $i = 3$ we have $k = 1$ and $J(3) = 36$, so:

$$W_{3,1} = \{y_n : n = 37, \dots, 72\}.$$

Now, we consider a positive integer m as defined in (2.3). Obviously, m can be written as:

$$m = p_1 c, \quad c = p_2 \dots p_N. \tag{4.1}$$

We consider another integer m_1 such that:

$$m_1 = m + c = (p_1 + 1)c. \tag{4.2}$$

Definition 4.4. Let m, m_1 be defined in (4.1) and (4.2) and let $U_m(A_{p_1}), U_{m_1}(A_{p_1+1})$ be the corresponding matrices with initial sets $\{a_1, \dots, a_{p_1}\}$, $\{a_1, \dots, a_{p_1}, a_{p_1+1}\}$ respectively. Let $T = \{t_1, \dots, t_m\}$, we call *c-extension of* T, the data $\widetilde{T} = \{\tilde{t}_1, \dots, \tilde{t}_{m_1}\}$ satisfying:

$$\tilde{t}_i = \begin{cases} t_i, & \text{whenever } i = 1, \dots, m \\ \tilde{t}_i, & \text{whenever } i = m+1, \dots, m+c-1 \\ \dfrac{a_{p_1+1}}{a_{p_1}} t_m, & \text{whenever } i = m + c = m_1 \end{cases}.$$

Proposition 4.5. *Let $\widetilde{T} = \{\tilde{t}_1, \ldots, \tilde{t}_{m_1}\}$ be the c-extension of a data $T = \{t_1, \ldots, t_m\}$. We define $y_i = \langle t, e_i \rangle$ and $\tilde{y}_j = \langle \tilde{t}, \tilde{e}_j \rangle$, where $e_i \in M_{1,m}$, $\tilde{e}_j \in M_{1,\underset{\sim}{m_1}}$ are rows of the matrices $((U_m(A_{p_1}))^{-1})^T$, $((U_{m_1}(A_{p_1+1}))^{-1})^T$ respectively. Let $\widetilde{W}_{i,k} = \{\tilde{y}_{k\widetilde{J}(i)+\tilde{r}} : \tilde{r} = 1, \ldots, \widetilde{J}(i)\}$, where $\widetilde{J}(i) = (p_1 + 1)p_2 \cdots p_i$, then:*

(i) *For $i = 0$ we have: $\tilde{y}_1 = y_1$, $\tilde{y}_2 = y_2, \ldots, \tilde{y}_{p_1} = y_{p_1}$, $\tilde{y}_{p_1+1} = 0$.*
(ii) *For $i = 1, \ldots, N-1$ we have:*

$$\widetilde{W}_{i,k} = \begin{cases} y_{kJ(i)+\tilde{r}}, & \text{whenever } \tilde{r} = 1, \ldots, J(i) \\ \tilde{y}_{k\widetilde{J}(i)+\tilde{r}}, & \text{whenever } \tilde{r} = J(i)+1, \ldots, \widetilde{J}(i) \end{cases}.$$

Proof. (i) Because $\tilde{e}_1 = (0, \ldots, 0, \frac{1}{a_{p_1+1}})$ we have $\tilde{y}_1 = \langle \tilde{t}, \tilde{e}_1 \rangle = \frac{\tilde{t}_{m_1}}{a_{p_1+1}} = \frac{t_m}{a_{p_1}} = y_1$ (see Definition 4.4). Now, for $\lambda = 2, \ldots, p_1 + 1$, we use Definition 4.4 and Corollary 3.4 for $i = 0$, $k = 1, \ldots, p_1$, $r = 1$ to get:

$$\begin{aligned} \tilde{y}_\lambda = \langle \tilde{t}, \tilde{e}_\lambda \rangle = \langle \tilde{t}, \tilde{e}_{k+1} \rangle &= \frac{\tilde{t}_{kA(2)}}{a_k} - \frac{\tilde{t}_{m_1}}{a_{p_1+1}} = \frac{t_{kA(2)}}{a_k} - \frac{t_m}{a_{p_1}} \\ &= \begin{cases} \langle t, e_{k+1} \rangle, & \text{whenever } k = 1, \ldots, p_1 - 1 \\ 0, & \text{whenever } k = p_1 \end{cases}. \\ &= \begin{cases} \langle t, e_\lambda \rangle, & \text{whenever } \lambda = 2, \ldots, p_1 \\ 0, & \text{whenever } \lambda = p_1 + 1 \end{cases}. \\ &= \begin{cases} y_\lambda, & \text{whenever } \lambda = 2, \ldots, p_1 \\ 0, & \text{whenever } \lambda = p_1 + 1 \end{cases}. \end{aligned}$$

(ii) Now, let $i = 1, \ldots, N-1$, as $m_1 = (p_1 + 1)p_2 \ldots p_N$, by Corollary 3.4 we get:

$$\begin{aligned} \tilde{y}_{k\widetilde{J}(i)+\tilde{r}} = \langle \tilde{t}, \tilde{e}_{k\widetilde{J}(i)+\tilde{r}} \rangle &= \frac{\tilde{t}_\mu}{a_{\lceil \frac{\mu(p_1+1)}{m_1} \rceil}} - \frac{\tilde{t}_\nu}{a_{\lceil \frac{\nu(p_1+1)}{m_1} \rceil}} \\ &= \frac{\tilde{t}_\mu}{a_{\lceil \frac{\mu}{p_2 \cdots p_N} \rceil}} - \frac{\tilde{t}_\nu}{a_{\lceil \frac{\nu}{p_2 \cdots p_N} \rceil}} = \frac{\tilde{t}_\mu}{a_{\lceil \frac{\mu p_1}{m} \rceil}} - \frac{\tilde{t}_\nu}{a_{\lceil \frac{\nu p_1}{m} \rceil}}, \end{aligned}$$

where $\mu = (\tilde{r} - 1)A(i+1) + kA(i+2)$, $\nu = \tilde{r}A(i+1)$, $k = 1, \ldots, p_{i+1} - 1$, $\tilde{r} = 1, \ldots, \widetilde{J}(i)$. Because $\tilde{t}_n = t_n$ for $n = 1, \ldots, m$ it is clear that the equality above becomes:

$$\tilde{y}_{k\widetilde{J}(i)+\tilde{r}} = \frac{\tilde{t}_\mu}{a_{\lceil \frac{\mu p_1}{m} \rceil}} - \frac{\tilde{t}_\nu}{a_{\lceil \frac{\nu p_1}{m} \rceil}} = \frac{t_\mu}{a_{\lceil \frac{\mu p_1}{m} \rceil}} - \frac{t_\nu}{a_{\lceil \frac{\nu p_1}{m} \rceil}} = \langle t, e_{kJ(i)+\tilde{r}} \rangle = y_{kJ(i)+\tilde{r}},$$

whenever $\tilde{r} = 1, \ldots, J(i)$. □

Now we are able to present our prediction method:

1. We consider a nearly periodic data $T = \{t_n : n = 1, \ldots, m\}$ of positive real numbers with fixed period P and frequency ω, such that:

$$m = \omega P.$$

We write $P = p_2 \ldots p_N$, where $p_2 \geq \cdots \geq p_N$ are primes some of them being possibly equal, so m satisfies (4.1) with $p_1 = \omega$. Notice that we require that ω must be greater than or equal to 5 for computational reasons.

2. We compute the matrix $U_m(A_{p_1})$ by using an initial set $A_{p_1} = \{a_1, \ldots, a_{p_1}\}$ whose elements are defined in the following equality:

$$a_i = \sum_{k=(i-1)P+1}^{iP} t_k.$$

3. We compute the $U_m(A_{p_1})$-transform elements:

$$y_i = \langle t, e_i \rangle, \quad i = 1, \ldots, m$$

as defined in Lemma 4.1.

4. Let $m_1 = m + P$, where P has been defined in step 1. It is clear that m_1 satisfies (4.2). We use Proposition 4.5 to define the $U_{m_1}(A_{p_1+1})$-transform $\widetilde{Y} = \{\tilde{y}_1, \ldots, \tilde{y}_{m_1}\}$ of a P-periodic extension data $\widetilde{T} = \{t_1, \ldots, t_m, \tilde{t}_{m+1}, \ldots, \tilde{t}_{m_1-1}, t_m a_{p_1+1}/a_{p_1}\}$ of T. Because:

$$\widetilde{Y} = y_1 \bigcup_{i=1}^{N-1} \bigcup_{k=1}^{p_{i+1}-1} \widetilde{W}_{i,k},$$

where the sets $\widetilde{W}_{i,k}$ have been defined in Definition 4.2, Proposition 4.5 states that:

(i) For $i = 0$: $\tilde{y}_1 = y_1, \tilde{y}_2 = y_2, \ldots, \tilde{y}_{p_1} = y_{p_1}, \tilde{y}_{p_1+1} = 0$.
(ii) For any $i = 1, \ldots, N-1$:

$$\widetilde{W}_{i,k} = \begin{cases} y_{kJ(i)+\tilde{r}}, & \text{whenever } \tilde{r} = 1, \ldots, J(i) \\ \tilde{y}_{k\widetilde{J}(i)+\tilde{r}}, & \text{whenever } \tilde{r} = J(i)+1, \ldots, \widetilde{J}(i) \end{cases},$$

where $\widetilde{J}(i) = (p_1 + 1)p_2 \ldots p_i$. Obviously, we need to fulfill the unknown elements $\tilde{y}_{k\widetilde{J}(i)+\tilde{r}}$, $\tilde{r} = J(i)+1, \ldots, \widetilde{J}(i)$. Because T is nearly periodic, we can assume that the sets:

$$Y_{i,k,m} = \{y_{kJ(i)+l(\widetilde{J}(i)-J(i))+m} : l = 0, \ldots, p_1 - 1\},$$

where $k = 1, \ldots, p_{i+1} - 1, m = 1, \ldots, (\widetilde{J}(i) - J(i))$ (see Remark 3.3(c)) correspond with a stationary processes, because:

$$y_{kJ(i)+l(\widetilde{J}(i)-J(i))+m} = \frac{t_{Pq_{k,l,m}}}{a_l} - \frac{t_{Pq_{p_1,l,m}}}{a_l},$$

where $q_{k,l,m}, q_{p_1,l,m}$ are integers that can be explicitly calculated in Corollary 3.4, so the unknown elements $\tilde{y}_{k\widetilde{J}(i)+\tilde{r}}$ can be efficiently approximated by the mean:

$$\tilde{y}_{k\widetilde{J}(i)+\tilde{r}} = \frac{1}{p_1} \sum_{l=0}^{p_1-1} y_{kJ(i)+l(\widetilde{J}(i)-J(i))+(\tilde{r}-J(i))}, \quad \tilde{r} = J(i)+1, \ldots, \widetilde{J}(i).$$

5. We assume that the set $A_{p_1} = \{a_1, \ldots, a_{p_1}\}$, as defined in step 2, can be considered either as a stationary process or as a nonstationary process exhibiting some sort of homogeneity (i.e., there exists a positive integer $k_0 \leq p_1/4$ such that $a_i - a_{i+k_0}$ becomes stationary for any $i = 1, \ldots, p_1$). In any case, we use a properly selected autoregressive model to predict a new element a_{p_1+1}.

Example: In many cases, such a model could be of the form

$$a_i - \mu = \phi_1(a_{i-1} - \mu) + \cdots + \phi_{p_1-2}(a_{i-p_1+2} - \mu) + \varepsilon_i,$$

where μ is the mean of a_i, ε_i is a white noise process, and the coefficients $\phi_1, \ldots, \phi_{p_1-2}$ are calculated via an equation $\phi_k = \Theta^{-1} \cdot \theta_k$, where Θ is the autocorrelation matrix and θ_k are the autocorrelations (see [4]). An estimator for a_{p_1+1} could be:

$$a_{p_1+1} = \mu + \phi_1(a_{p_1} - \mu) + \cdots + \phi_{p_1-2}(a_2 - \mu).$$

6. We compute the matrix $U_{m_1}(A_{p_1+1})$, where the first p_1 elements of the set $A_{p_1+1} = \{a_1, \ldots, a_{p_1}, a_{p_1+1}\}$ have been calculated in step 2 and the element a_{p_1+1} is computed in step 5.

7. We compute the P-extension data of T:

$$\begin{aligned} \tilde{t}_n &= \sum_{i=1}^{m_1} \tilde{y}_i \tilde{u}_{i,n} \\ &= \begin{cases} a_{p_1+1}\tilde{y}_1, & \text{whenever } Mod(n-m, p_N) = 0 \\ a_{p_1+1}\left(\tilde{y}_1 + \tilde{y}_{\widetilde{J}(N-1)Mod(n-m,p_N)+\lceil \frac{n-m}{p_N} \rceil}\right), & \text{whenever } Mod(n-m, p_N) \neq 0 \end{cases}, \end{aligned}$$

where $n = m+1, \ldots, m+P$ and this is a nonlinear estimator for predicting T one period ahead.

Example 4.6. We consider the function $f(x) = e^x(cos(2\pi 10x) + 0.5cos(2\pi 15x))$, $x \in [0,1]$ and we take $T = \{f(k/500) : k = 0, \ldots, 499\}$. We observe that T is nearly periodic with fixed period equal to 100. We apply the above prediction method for $m = 500$, $p_1 = 5$ and we get Figure 1.

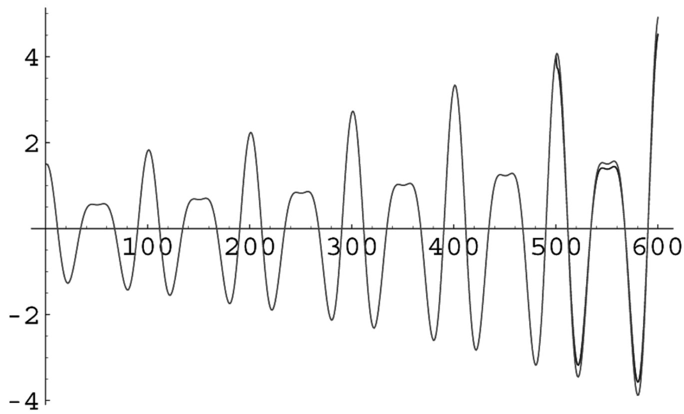

FIGURE 1 Plot of the function $f(x)$ (see clear-sighted line) together with the prediction of $f(x)$ period ahead (indistinguishable line).

5. APPENDICES

Appendix I

Let $\mathbf{e}_i^{p_n}$ be rows of the identity matrix $\mathbf{I}_{p_n}$ and $V_j^{p_n}$, $j = 1, \ldots, p_n - 1$ are column matrices defined in (2.7), then:

$$\begin{pmatrix} D_{p_n}(U_m(n-1, A_{p_1})) \\ Q_{J(n-1),A_{p_1}}\left(\mathbf{e}_1^{p_n}\right) \\ \vdots \\ Q_{J(n-1),A_{p_1}}\left(\mathbf{e}_{p_n-1}^{p_n}\right) \end{pmatrix}$$

$$\cdot \left(Q_{J(n-1),\left\{\underbrace{1,\ldots,1}_{p_n\text{-times}}\right\}}\left(\left(\mathbf{e}_{p_n}^{p_n}\right)^T\right) \quad Q_{J(n-1),\widetilde{A}_{p_1}}\left(V_1^{p_n}\right) \quad \cdots \quad Q_{J(n-1),\widetilde{A}_{p_1}}\left(V_{p_n-1}^{p_n}\right) \right)$$

$$= \begin{pmatrix} U_m(n-1, A_{p_1}) & & & \mathbf{O} \\ & \mathbf{I}_{J(n-1)} & & \\ & & \ddots & \\ \mathbf{O} & & & \mathbf{I}_{J(n-1)} \end{pmatrix}.$$

Proof. It suffices to prove that:

(i) $D_{p_n}(U_m(n-1, A_{p_1}))Q_{J(n-1),\{\underbrace{1,\ldots,1}_{p_n\text{-times}}\}}\left(\left(\mathbf{e}_{p_n}^{p_n}\right)^T\right) = U^m(n-1, A_{p_1})$.

(ii) $D_{p_n}(U_m(n-1, A_{p_1}))Q_{J(n-1),\widetilde{A}_{p_1}}(V_j^{p_n}) = \mathbf{O}$, $j = 1, \ldots, p_n - 1$, where $\mathbf{O}$ is the zero matrix in $\mathsf{M}_{J(n-1)}$.

(iii) $Q_{J(n-1),A_{p_1}}(\mathbf{e}_j^{p_n})Q_{J(n-1),\{\underbrace{1,\ldots,1}_{p_n\text{-times}}\}}((\mathbf{e}_{p_n}^{p_n})^T) = \mathbf{O}$, $j = 1, \ldots, p_n - 1$, where $\mathbf{O}$ is the zero matrix in $\mathsf{M}_{J(n-1)}$.

(iv) $Q_{J(n-1),A_{p_1}}(\mathbf{e}_j^{p_n})Q_{J(n-1),\widetilde{A}_{p_1}}(V_l^{p_n}) = \delta_{j,l}\mathbf{I}_{J(n-1)}$, $j, l = 1, \ldots, p_n - 1$ and $\delta_{j,l}$ is the Kroneckers's delta.

Indeed we have:

(i) We use relations (2.1) and (2.2) to perform the following block matrix multiplication:

$$D_{p_n}(U_m(n-1, A_{p_1}))Q_{J(n-1),\{\underbrace{1,\ldots,1}_{p_n\text{-times}}\}}\left(\left(\mathbf{e}_{p_n}^{p_n}\right)^T\right)$$
$$= \left[D_{p_n}\big((U_m(n-1, A_{p_1}))_{i,j}\big)\left(\mathbf{e}_{p_n}^{p_n}\right)^T\right]_{i,j=1}^{J(n-1)} = U_m(n-1, A_{p_1}).$$

(ii) We observe that all column matrices $V_j^{p_n}$ have zero mean, so the block matrix multiplication leads to:

$$D_{p_n}(U_m(n-1, A_{p_1}))Q_{J(n-1,\widetilde{A}_{p_1})}(V_j^{p_n})$$
$$= \left[D_{p_n}\big((U_m(n-1, A_{p_1}))_{k,l}\big) \cdot \frac{1}{a_j} \cdot \sum_{r=1}^{p_n} v_{rj}^{p_n}\right]_{k,l=1}^{J(n-1)} = 0.$$

(iii) Obvious consequence of the fact that $\mathbf{e}_j^{p_n}(\mathbf{e}_{p_n}^{p_n})^T = 0$, $j = 1, \ldots, p_n - 1$.

(iv) $Q_{J(n-1),A_{p_1}}(\mathbf{e}_j^{p_n})Q_{J(n-1),\widetilde{A}_{p_1}}(V_l^{p_n}) = \begin{pmatrix} a_1 \cdot \mathbf{e}_j^{p_n} \cdot \frac{1}{a_1} \cdot V_l^{p_n} & & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & & a_{p_1}\mathbf{e}_j^{p_n} \cdot \frac{1}{a_{p_1}} \cdot V_l^{p_n} \end{pmatrix}$, $j, l = 1, \ldots, p_n - 1$. Because $\mathbf{e}_j^{p_n} V_l^{p_n} = \sum_{k=1}^{p_n} \delta_{k,j} v_{k,l}^{p_n} = v_{j,l}^{p_n} = \delta_{j,l}$ we get the result. □

Appendix II

Let σ_n be the permutation defined in Proposition 2.13, then:

$$sgn\,\sigma_n = \frac{p_n - 1}{2}J(n-1)\left(1 + J(n-1) + \frac{p_n - 1}{2}(J(n-1) - 1)\right).$$

TABLE 1 Inversion vector elements corresponding to the permutation $\sigma_n(i)$ of the i-row of the matrix $U_m(n, A_{p_1})$ of Proposition 2.13

i	$\sigma_n(i)$	Inversion vector elements $IV_{\sigma_n}(i)$
$1, \ldots, J(n-1)$	ip_n	$i(p_n - 1)$
$J(n-1)+1, \ldots, 2J(n-1)$	$1 + Mod(i-1, J(n-1))p_n$	$Mod(i-1, J(n-1))(p_n - 2)$
$\ldots$	$\ldots$	$\ldots$
$(p_n - 2)J(n-1)$ $+1, \ldots, (p_n-1)J(n-1)$	$p_n - 2 + Mod(i-1, J(n-1))p_n$	$Mod(i-1, J(n-1))$
$(p_n - 1)J(n-1) + 1, \ldots, J(n)$	$p_n - 1 + Mod(i-1, J(n-1))p_n$	0 for all i's

Proof. $sgn\, \sigma_n = (-1)^{q_n}$, where q_n equals the number of all inversions in the permutation σ_n. A pair of elements $(\sigma_n(i), \sigma_n(j))$ is called an inversion, if $i < j$ and $\sigma_n(i) > \sigma_n(j)$. The number of elements less than i to the right of i in σ_n gives the ith element of the *inversion vector* IV_{σ_n} corresponding with σ_n and q_n equals the sum of all inversion vector elements.

The last column of Table 1 gives the elements of the inversion vector: Now we have: $sgn\, \sigma_n = (-1)^{q_n}$, where

$$
\begin{aligned}
q_n &= \sum_{i=1}^{J(n)} IV_{\sigma_n}(i) \\
&= \sum_{i=1}^{J(n-1)} i(p_n - 1) + \sum_{i=J(n-1)+1}^{2J(n-1)} Mod(i-1, J(n-1))(p_n - 2) + \ldots \\
&= (p_n - 1) \sum_{i=1}^{J(n-1)} i + (p_n - 2) \sum_{i=1}^{J(n-1)-1} i + \cdots + \sum_{i=1}^{J(n-1)-1} i \\
&= (p_n - 1) \frac{J(n-1)(1 + J(n-1))}{2} \\
&\quad + \frac{J(n-1)(J(n-1)-1)}{2} \frac{(p_n - 2)(p_n - 1)}{2}
\end{aligned}
$$

and elementary calculations yield the result. □

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