Geometric Versions of Schwarz’s Lemma and Semigroups of Holomorphic Functions

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Introduction

My Ph.D. Thesis lies in the research area of Complex Analysis and has tight connections with Geometric Function Theory and Potential Theory.

The results presented here can be found in the following publications

- *Conformal Mapping, Convexity and Total Absolute Curvature*, Conformal Geometry and Dynamics, AMS,
- *Length and Area Estimates for (Hyperbolically) Convex Mappings*, Computational Methods and Function Theory
Chapter 1

Overview

The purpose of this thesis is to examine geometric versions of the classical Schwarz’s lemma. Monotonicity theorems are presented for geometric quantities with the use of both euclidean and hyperbolic geometry of the unit disk.

Moreover, in the second part of the thesis, semigroups of holomorphic self-maps of the unit disk are considered. We observe the asymptotic behavior of the trajectory of a compact set. In order to achieve this, we use several geometric and potential theoretic quantities, such as hyperbolic area, harmonic measure, Green potential etc.

The proofs of the results presented in this chapter can be found in the articles [22], [23] and [24].

1.1 Euclidean Geometry in the Unit Disk

Let $f$ be a holomorphic function in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We denote by $T = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle. Moreover, we denote by $r \mathbb{D} = \{z \in \mathbb{D} : |z| < r\}$ and $r \cdot T := \partial(r \mathbb{D}) = \{z \in \mathbb{D} : |z| = r\}$ the open disk and circle of radius $r \in (0, 1)$, respectively.

According to G. Pólya and G. Szegő [30, p.165, Problem 309], the function

$$L(r) := \frac{L(f(r \cdot T))}{L(r \cdot T)} = \frac{1}{2\pi} \int_{0}^{2\pi} |f'(re^{it})| \, dt,$$  \hspace{1cm} (1.1)

where $L$ denotes the euclidean length of a curve, is increasing for $r \in (0, 1)$. In [3] and [12], the same was proved for the function

$$A(r) := \frac{A(f(r \mathbb{D}))}{A(r \mathbb{D})} = \frac{1}{\pi r^2} A(f(r \mathbb{D})), \quad 0 < r < 1,$$  \hspace{1cm} (1.2)

where $A$ denotes the euclidean area of a domain.

The above monotonicity results make a comparison between the size of $r \mathbb{D}$ or $r \cdot T$ and their images, respectively, measuring them with the use of area and length. Several other geometric quantities have been used in order to compare $r \mathbb{D}$ or $r \cdot T$ with their images under a holomorphic function. Such geometric quantities are logarithmic capacity, diameter, condenser capacity, inner radius, etc., as we can see in [6], [7], [11], [12] and [13]. Monotonicity
1.1. Euclidean Geometry in the Unit Disk

results of this kind can be seen as geometric versions of the classical Schwarz’s Lemma. In this way, information on the growth of the image is extracted that leads to a variety of distortion theorems.

If \( f \) is a conformal mapping, the curves \( f(r\mathbb{T}) \) possess stronger geometric properties. They are simple, smooth and closed curves for every \( r \in (0, 1) \). At this point, let’s recall that a univalent function \( f \) is called convex if \( f(\mathbb{D}) \) is convex. A holomorphic and locally univalent function \( f \) is convex if and only if

\[
v_f(z) := \Re \left\{ 1 + z \frac{f'''(z)}{f'(z)} \right\} > 0, \tag{1.3}
\]

for every \( z \in \mathbb{D} \); more information on convex functions can be found in [15] and [31].

Using F.R. Keogh’s upper bound for the length of the image of \( r\mathbb{T} \) under a convex conformal mapping, in [21], it is easily obtained that

\[
L(r) \leq \frac{|f'(0)|}{1 - r^2}, \tag{1.4}
\]

with equality holding if and only if \( f \) maps \( \mathbb{D} \) conformally onto a half-plane. We call half-plane mapping a conformal function that maps \( \mathbb{D} \) onto a half-plane of \( \mathbb{C} \).

With the use of the isoperimetric inequality, we can find an upper bound for the function \( A(r) \), as well,

\[
A(r) \leq \frac{|f'(0)|^2}{(1 - r^2)^2}, \tag{1.5}
\]

for every \( r \in (0, 1) \).

We set the functions

\[
\mathcal{L}(r) := (1 - r^2)L(r) = \frac{1 - r^2}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|d\theta \tag{1.6}
\]

and

\[
\mathcal{A}(r) := (1 - r^2)^2 A(r) = \frac{(1 - r^2)^2}{\pi r^2} \int \int_{r\mathbb{D}} |f'(z)|^2 \, dA(z), \tag{1.7}
\]

where \( dA \) is the Lebesgue measure on \( \mathbb{D} \), and we obtain the following monotonicity result.

**Theorem 1.1.** Let \( f \) be a convex mapping in \( \mathbb{D} \). The functions \( \mathcal{L}(r) \) and \( \mathcal{A}(r) \) are decreasing for \( r \in (0, 1) \). Moreover, they are strictly decreasing if and only if \( f \) is not a half-plane mapping of \( \mathbb{D} \). In this case, they are constant and equal to \( |f'(0)| \) and \( |f'(0)|^2 \), respectively.

The above result provides us with an estimate on how sharp the bounds of Keogh [21] are. The following isoperimetric-type inequality concerning the image of \( r\mathbb{D} \) under a convex mapping is a consequence of Theorem 1.1.

**Corollary 1.1.** Let \( f \) be a convex function in \( \mathbb{D} \). Then

\[
L^2 f(r\mathbb{T}) < 4\pi \frac{1 + r^2}{1 - r^2} A f(r\mathbb{D}),
\]

for \( r \in (0, 1) \).
In addition, we are able to state a similar monotonicity theorem concerning the total absolute curvature. For any conformal mapping \( f \), the curve \( f(rT) \) is convex, when \( r \leq 2 - \sqrt{3} \); see [15, Theorem 2.13]. The number \( 2 - \sqrt{3} \) is called radius of convexity and it is a sharp bound regarding the convexity of the domain \( f(D) \). The question that arises is what happens when \( r \) is greater than the radius of convexity. And what if the function \( f \) is not univalent but only locally univalent?

We need some kind of measurement to show us whether \( f(rT) \) is convex or not and how much it diverges from being convex. The most suitable geometric quantity with this property is the total absolute curvature of \( f(rT) \). The total absolute curvature of \( f(rT) \) provides some kind of distance between a function and its convexity.

Let \( \gamma \) be a smooth curve in \( D \). We denote by \( \kappa(z, \gamma) \) the signed euclidean curvature of \( \gamma \) at the point \( z \in \gamma \). The total absolute curvature of \( \gamma \) is the quantity

\[
\int_{\gamma} |\kappa(z, \gamma)||dz|.
\]

It is known that the total absolute curvature of a smooth and closed curve is always greater than \( 2\pi \), with equality holding if and only if the curve \( \gamma \) is convex; see [33, Corollary 6.18].

Let \( f \) be a holomorphic and locally univalent function on \( D \). By \( \kappa(w, f(\gamma)) \) we denote the euclidean curvature of \( f(\gamma) \) at the point \( w \in f(\gamma) \). Therefore, the quantity

\[
\int_{f(\gamma)} |\kappa(w, f(\gamma))||dw|
\]

is the total absolute curvature of \( f(\gamma) \). The greater this quantity becomes, the less convex the function \( f \) is.

We should notice that the total absolute curvature of \( rT \) is constant. Since \( rT \) is a circle, and therefore a convex curve, its total absolute curvature is equal to \( 2\pi \). Set

\[
\Phi(r) := \frac{\int_{f(rT)} |\kappa(w, f(rT))||dw|}{\int_{rT} |\kappa(z, rT)||dz|} = \frac{1}{2\pi} \int_{f(rT)} |\kappa(w, f(rT))||dw|, \tag{1.8}
\]

which is the ratio of the total absolute curvature of \( f(C_r) \) to the total absolute curvature of \( rT \).

**Theorem 1.2.** Let \( f \) be a holomorphic and locally univalent function on \( D \). Then \( \Phi(r) \) is a strictly increasing function of \( r \in (0, 1) \), except when \( f \) is convex. In this case, it is constant and equal to \( 2\pi \).

### 1.2 Hyperbolic Geometric Aspects of Schwarz’s Lemma

As we stated above, monotonicity results concerning geometric quantities have been extensively examined when \( f(D) \subset \mathbb{C} \) and \( \mathbb{C} \) is equipped with the euclidean metric. But what happens when \( f(D) \) is seen from a hyperbolic perspective? Can the above geometric versions of Schwarz’s Lemma be extended in the hyperbolic geometry of the unit disk?
Let \( f \) be a holomorphic function in the unit disk \( \mathbb{D} \) with \( f(\mathbb{D}) \subset \mathbb{D} \). We suppose that the unit disk is endowed with the hyperbolic metric

\[
\lambda_{\mathbb{D}}(z)|dz| = \frac{|dz|}{1-|z|^2},
\]

where \( \lambda_{\mathbb{D}}(z) \) is the density of the hyperbolic metric.

Let \( \Omega \) be a simply connected subdomain of \( \mathbb{D} \) and \( f \) be a conformal mapping with \( f(\Omega) = \mathbb{D} \). The hyperbolic metric \( \lambda_{\Omega}(z)|dz| \) on \( \Omega \) is defined to be

\[
\lambda_{\Omega}(z) = \lambda_{\mathbb{D}}(f(z)) |f'(z)|.
\]

The hyperbolic distance between two points \( a, b \in \Omega \) is defined by

\[
d_{\Omega}(a,b) = \inf_{\gamma \subset \Omega} \int_{\gamma} \lambda_{\Omega}(z)|dz|,
\]

where \( \gamma \) is any rectifiable curve that lies in \( \Omega \) and joins \( a, b \). If the infimum is attained for a curve \( \gamma_0 \subset \Omega \), then \( \gamma_0 \) is called hyperbolic geodesic. In the unit disk, every pair of points is joined by a unique hyperbolic geodesic. The hyperbolic geodesic curves of \( \mathbb{D} \) are the arcs of euclidean circles in \( \mathbb{D} \) that are perpendicular to the boundary. The hyperbolic distance in the unit disk, for \( a, b \in \mathbb{D} \) is equal to

\[
d_{\mathbb{D}}(a,b) = \text{arctanh} \left| \frac{a-b}{1-\overline{a}b} \right|
\]

and it is invariant under any conformal automorphism of \( \mathbb{D} \). We denote the set of all conformal automorphisms of the unit disk \( \mathbb{D} \) by \( \text{Aut}(\mathbb{D}) \); this set consists of all the mappings

\[
g(z) = e^{i\theta} \frac{z-\alpha}{1-\overline{\alpha}z},
\]

where \( \alpha \in \mathbb{D} \) and \( \theta \in \mathbb{R} \). The hyperbolic metric and, in general, hyperbolic geometry of the unit disk are thoroughly examined in [4] and [26].

A subregion \( \Omega \) of the unit disk is hyperbolically convex if for every pair of points in \( \Omega \) the hyperbolic geodesic arc that joins them, lies in \( \Omega \); see [26]. Also from [26], a conformal map \( f : \mathbb{D} \to \mathbb{D} \) is called hyperbolically convex if \( f(\mathbb{D}) \) is a hyperbolically convex subregion of \( \mathbb{D} \). From the hyperbolic analogue of Study’s Theorem, which is proved in [26], arises a basic property of hyperbolically convex functions. Suppose \( f \) is a conformal map with \( f(\mathbb{D}) \subset \mathbb{D} \). If \( f \) is hyperbolically convex, then \( f \) maps every subdisk of \( \mathbb{D} \) onto a hyperbolically convex region. More specifically, for \( 0 < r < 1 \), the function \( f(rz) \) is hyperbolically convex. Also in [26], we can see that a holomorphic and locally univalent function \( f \) in \( \mathbb{D} \), with \( f(\mathbb{D}) \subset \mathbb{D} \), is hyperbolically convex if and only if

\[
u_f(z) := \text{Re} \left \{ 1 + \frac{zf''(z)}{f'(z)} + \frac{2zf'(z)f(\overline{z})}{1-|f(z)|^2} \right \} > 0,
\]
for every $z \in \mathbb{D}$. For further information on hyperbolic convexity, the reader may refer to [25], [26], [27], [28] and [29].

The hyperbolic disk centered at the origin of radius $\rho$,

$$D_h(0, \rho) = \{ z \in \mathbb{D} : \text{arctanh} |z| < \rho \} = \{ z \in \mathbb{D} : |z| < \tanh \rho \}$$

(1.10)

is a euclidean disk centered at the origin of radius $r := \tanh \rho$. Its boundary is

$$\partial D_h(0, \rho) = \{ z \in \mathbb{D} : |z| = \tanh \rho \} = \partial D(0, r) = r \mathbb{T},$$

(1.11)

where $r = \tanh \rho$.

For holomorphic functions $f : \mathbb{D} \to \mathbb{D}$, monotonicity results of the same kind as in the euclidean case have been proved, concerning some geometric quantities viewed in the hyperbolic geometry of the disk. In [8], it was proved that

$$r \mapsto \frac{R_h(f(r \mathbb{T}))}{r} \quad \text{and} \quad r \mapsto \frac{\text{caph} f(r \mathbb{T})}{r}$$

are increasing functions of $r \in (0, 1)$, where $R_h$ is the hyperbolic-area-radius of $f(r \mathbb{T})$ and caph denotes the hyperbolic capacity.

The monotonic behavior of the functions (1.1) and (1.2) plays a pivotal role in the euclidean geometry of the complex plane. However, there are not any similar results regarding length and area with respect to the hyperbolic geometry of the unit disk.

With the help of hyperbolic convexity, we present the hyperbolic analogues of the functions (1.1) and (1.2). Let’s define the function

$$L_h^h(r) := \frac{L_h f(r T)}{L_h(r T)}, \quad r \in (0, 1),$$

(1.12)

where $L_h$ is the hyperbolic length of a curve in the unit disk. We have the following outcome concerning the monotonicity of the function $L_h^h(r)$.

**Theorem 1.3.** Let $f : \mathbb{D} \to \mathbb{D}$ be a hyperbolically convex mapping. Then $L_h^h$ is a decreasing function in $(0, 1)$. Moreover, $L_h^h$ is strictly decreasing if and only if $f$ is not a conformal automorphism of the unit disk $\mathbb{D}$. In the case where $f \in \text{Aut}(\mathbb{D})$, $L_h^h$ is constant and equal to 1.

Furthermore, we have a similar theorem for the hyperbolic area. Define the function

$$A_h^h(r) := \frac{A_h f(r \mathbb{D})}{A_h(r \mathbb{D})}, \quad r \in (0, 1),$$

(1.13)

where $A_h$ is the hyperbolic area of a domain in $\mathbb{D}$.

**Theorem 1.4.** Let $f : \mathbb{D} \to \mathbb{D}$ be a hyperbolically convex mapping. Then $A_h^h$ is a decreasing function in $(0, 1)$. Moreover, $A_h^h$ is strictly decreasing if and only if $f$ is not a conformal automorphism of the unit disk $\mathbb{D}$. In the case where $f \in \text{Aut}(\mathbb{D})$, $A_h^h$ is constant and equal to 1.
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An immediate consequence of Theorem 1.4 is the following isoperimetric-type inequality for the image of \( r \mathbb{D} \).

**Corollary 1.2.** For a hyperbolically convex mapping \( f \) in \( \mathbb{D} \), it holds

\[
L_h^2 f(r T) \leq \frac{4\pi}{1 - r^2} A_h f(r \mathbb{D}),
\]

for \( r \in (0, 1) \). Equality occurs if and only if \( f \) is a conformal automorphism of \( \mathbb{D} \).

Corollary 1.2 provides an upper bound for the hyperbolic isoperimetric ratio of \( f(\mathbb{D}) \). Furthermore, Theorems 1.3 and 1.4 lead to Schwarz-type inequalities involving hyperbolic length and hyperbolic area, as well as, information on their limiting behavior.

**Corollary 1.3.** Let \( f : \mathbb{D} \to \mathbb{D} \) be a hyperbolically convex mapping. Then

\[
L_h f(r T) \leq \frac{|f'(0)|}{1 - |f(0)|^2} \frac{2\pi r}{1 - r^2} \quad \text{and} \quad L_h f(r T) = O \left( \frac{1}{1 - r^2} \right),
\]

(1.14) as \( r \to 1^- \). If \( A_h f(\mathbb{D}) < +\infty \), then

\[
L_h f(r T) = O \left( \frac{1}{1 - r^2} \right),
\]

as \( r \to 1^- \). Equality occurs in (1.14) if and only if \( f \in \text{Aut}(\mathbb{D}) \).

A similar result holds for the hyperbolic area of \( f(\mathbb{D}) \).

**Corollary 1.4.** Let \( f : \mathbb{D} \to \mathbb{D} \) be a hyperbolically convex mapping. Then

\[
A_h f(r \mathbb{D}) \leq \frac{|f'(0)|^2}{(1 - |f(0)|^2)^2} \frac{\pi r^2}{1 - r^2} \quad \text{and} \quad A_h f(r \mathbb{D}) = O \left( \frac{1}{1 - r^2} \right),
\]

(1.15) as \( r \to 1^- \). If \( A_h f(\mathbb{D}) < +\infty \), then

\[
A_h f(r \mathbb{D}) = O \left( \frac{1}{1 - r^2} \right),
\]

as \( r \to 1^- \). Equality holds in (1.15) if and only if \( f \in \text{Aut}(\mathbb{D}) \).

Moreover, the above outcomes lead to an integrated version of the classical Schwarz-Pick lemma for the class of hyperbolically convex functions.

**Corollary 1.5.** Let \( f : \mathbb{D} \to \mathbb{D} \) be a hyperbolically convex mapping. Then, for every \( r \in (0, 1) \),

\[
\frac{1}{2\pi} \int_{r T} (1 - |z|^2) \frac{|f'(z)|}{1 - |f(z)|^2} |dz| \leq \frac{|f'(0)|}{1 - |f(0)|^2} r \leq r,
\]

where equality occurs if and only if \( f \in \text{Aut}(\mathbb{D}) \).
To continue with, we will examine whether the hyperbolic analogue of Theorem 1.2 exists. Let $\gamma$ be a smooth curve in the unit disk $D$, with non-vanishing derivative, and $f$ a holomorphic and locally univalent map with $f(D) \subset D$. The hyperbolic curvature of $\gamma$ at the point $z \in \gamma$ is denoted by $\kappa_h(z, \gamma)$, whereas, the hyperbolic curvature of $f \circ \gamma$ at a point $f(z)$, $z \in \gamma$, is denoted by $\kappa_h(f(z), f \circ \gamma)$.

We should note that the hyperbolic curvature on the unit disk is invariant under conformal self-maps of $D$. For the total hyperbolic curvature of the curve $f(r T)$ in the unit disk, the following monotonicity result holds.

**Theorem 1.5.** Let $f$ be a holomorphic and locally univalent function on $D$ with $f(D) \subset D$. Then

$$r \mapsto \int_{f(r T)} \kappa_h(w, f(r T)) \lambda_D(w) |dw|, \quad 0 < r < 1,$$

(1.16)

is a strictly increasing function.

Let’s define the function

$$\Phi_h(r) := \frac{\int_{r T} |\kappa_h(w, f(r T))| ds}{\int_{r T} |\kappa_h(z, r T)| ds}, \quad 0 < r < 1,$$

(1.17)

which is the ratio of the hyperbolic total absolute curvature of $f(r T)$ to the hyperbolic total absolute curvature of $r T$. The function $\Phi_h(r)$ is the hyperbolic analogue of the function $\Phi(r)$.

**Theorem 1.6.** Let $f$ be a hyperbolically convex function in $D$, with $f(D) \subset D$. Then $\Phi_h(r)$ is a strictly decreasing function of $r \in (0, 1)$, except when $f$ is a conformal self-map of the unit disk. In that case, $\Phi_h$ is constant and equal to 1.

### 1.3 Importance of Hyperbolic Convexity

A natural question is whether Theorems 1.3 and 1.4 can be generalized for all holomorphic functions of the unit disk into itself, or at least for conformal mappings. The answer is no and hyperbolic convexity is a property which cannot be omitted in the above results.

Consider the function

$$g(z) = k^{-1}\left(\frac{1}{2}k(z)\right),$$

where $k$ is the Koebe function. The function $g$ maps $D$ conformally onto $D \setminus (-1, -p]$, where $p = 3 - 2\sqrt{2}$. It is clear that $D \setminus (-1, -p]$ is not a hyperbolically convex domain.
With the use of MATHEMATICA®, we can see below the graphs of $L^h(r)$ and $A^h(r)$.

Since for the mapping $g$, the functions $L^h(r)$ and $A^h(r)$ are not decreasing, we conclude that the assumption that $f$ is hyperbolically convex in Theorems 1.3 and 1.4 cannot be omitted.

### 1.4 Semigroups of Holomorphic Functions

One-parameter semigroups of holomorphic self-maps of the unit disk $D$ have been extensively examined in recent years due to their strong connection with dynamical systems, composition operators, Markov stochastic processes, etc. The theory of one-parameter semigroups has been introduced by Berkson and Porta in [5]. For further developments see [1,9,14,16,17,19,20,32].

A one-parameter semigroup is a family $(\phi_t)_{t \geq 0}$ of holomorphic functions in $D$, where

1. $\phi_0$ is the identity map;
2. $\phi_{t+s}(z) = \phi_t(\phi_s(z))$, for every $t, s \geq 0$ and $z \in D$;
3. $\phi_t(z) \xrightarrow{t \to 0} z$, uniformly on every compact subset of $D$.

We denote by $\gamma_z$ the trajectory of a point $z \in D$, which is the curve

$$
\gamma_z : [0, +\infty) \to D \quad \text{with} \quad \gamma_z(t) = \phi_t(z).
$$

According to the continuous Denjoy-Wolff theorem, there exists a unique point $\tau \in \overline{D}$ such that

$$
\lim_{t \to +\infty} \gamma_z(t) = \lim_{t \to +\infty} \phi_t(z) = \tau, \quad (1.18)
$$

for every $z \in D$. So, for every point in $D$, its trajectory approaches a fixed point in $D$. This point $\tau$ is called the Denjoy-Wolff point of the semigroup; see [1, Theorem 1.4.17].

Depending on the position of $\tau$, we have the following classification of one-parameter semigroups. Supposing that $\phi_t$ is not an elliptic automorphism of $D$ for any $t \geq 0$, if
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\[ \tau \in \mathbb{D}, \text{ the semigroup } (\phi_t) \text{ is called } \text{elliptic}. \]  
At the case where \( \tau \) lies on the unit circle \( \mathbb{T} \), without loss of generality, we can assume that \( \tau = 1 \). Then, the angular derivative \( \phi'_t \) at 1 is examined; see [1]. If \( \phi'_t(1) < 1 \), the semigroup \( (\phi_t) \) is called \textit{hyperbolic}, whereas, if \( \phi'_t(1) = 1 \), the semigroup \( (\phi_t) \) is called \textit{parabolic}.

The class of parabolic semigroups is divided into two subcategories. If for all \( s > 0 \) and \( z \in \mathbb{D} \), the hyperbolic distance \( d_\mathbb{D}(\phi_t(z), \phi_{t+s}(z)) \xrightarrow{t \to +\infty} 0 \), the semigroup is called \textit{parabolic of zero hyperbolic step}. If the limit is not equal to zero for some \( s > 0 \) or \( z \in \mathbb{D} \), then the parabolic semigroup is of \textit{positive hyperbolic step}.

An important property that allows us to linearize the trajectories is that for every semigroup \( (\phi_t) \) with Denjoy-Wolff point 1, there exists a conformal mapping \( h \) of \( \mathbb{D} \) onto a simply connected domain such that

\[ h(\phi_t(z)) = h(z) + t, \quad (1.19) \]

for all \( z \in \mathbb{D} \) and \( t \geq 0 \). This mapping is called the \textit{Koenigs function}.

The simply connected domain \( \Omega = h(\mathbb{D}) \) is convex in the horizontal direction, as \( \{w + s : s > 0\} \subset \Omega \), for every \( w \in \Omega \). This occurs since the trajectory of each point \( z \in \mathbb{D} \) is mapped onto a half-line; see (1.19).

From [14], the semigroup is hyperbolic if and only if \( \Omega \) is contained in a horizontal strip. Furthermore, according to [9, Theorem 1], if \( (\phi_t) \) is parabolic of positive hyperbolic step, then \( \Omega \) is contained in a horizontal half-plane. In both cases, we can find the smallest such horizontal domain. Set \( S \) to be the smallest horizontal strip that contains \( \Omega \), in the case of a hyperbolic semigroup and \( H \) to be the smallest horizontal half-plane containing \( \Omega \), when \( (\phi_t) \) is a parabolic semigroup of positive hyperbolic step. The triple \( (S, h, \phi_t) \) or \( (H, h, \phi_t) \), respectively, is called \textit{holomorphic model} of \( (\phi_t) \) and \( S \) (or \( H \), respectively) is called the \textit{base space} of the semigroup; more details in [2].

As we will observe, the behavior of these two types of semigroups is similar. We will denote by \( \Omega^* \) the associated base space of the semigroup, regardless of the type, which would be the smallest horizontal domain containing \( \Omega \).

From now on, we assume that \( (\phi_t)_{t \geq 0} \) is either a hyperbolic or a parabolic semigroup of holomorphic self-maps of the unit disk \( \mathbb{D} \) with associated Koenigs function \( h \) and base space \( \Omega^* \).

Suppose \( K \) is a compact subset of \( \mathbb{D} \) of positive logarithmic capacity. The set

\[ \gamma_K(t) := \bigcup_{z \in K} \gamma_z(t) = \bigcup_{z \in K} \phi_t(z) = \phi_t(K) \]

is called \textit{trajectory} of \( K \) and it consists of all trajectories of \( z \in K \). As \( t \) increases, \( \phi_t(K) \) approaches the Denjoy-Wolff point of the semigroup and particularly the unit circle. Our goal is to determine how the geometric and potential theoretic characteristics of \( \phi_t(K) \) behave during this approach.

We will start with the harmonic measure \( \omega(\phi_t(z), \phi_t(K), \mathbb{D}) \) of the boundary of \( \phi_t(K) \). For every \( t > 0 \), the function \( \zeta \mapsto \omega(\zeta, \phi_t(K), \mathbb{D}) \) is the Perron solution to the Dirichlet problem on \( \mathbb{D} \setminus \phi_t(K) \) with boundary values 1 on the boundary of \( \phi_t(K) \) and 0 on \( \mathbb{T} \).
Considering the harmonic measure $\omega(\phi_t(z), \phi_t(K), \mathbb{D})$ as a function of $t$, we obtain the following monotonicity result.

**Theorem 1.7.** The harmonic measure $\omega(\phi_t(z), \phi_t(K), \mathbb{D})$ is an increasing function of $t \geq 0$, for every $z \in \mathbb{D} \setminus K$.

Therefore, the limit of the harmonic measure $\omega(\phi_t(z), \phi_t(K), \mathbb{D})$, as $t$ approaches infinity, exists. This way, we can get information on the asymptotic behavior of $\phi_t(K)$.

**Theorem 1.8.** Let $(\phi_t)$ be either a hyperbolic semigroup or a parabolic semigroup of positive hyperbolic step. Then

$$
\lim_{t \to +\infty} \omega(\phi_t(z), \phi_t(K), \mathbb{D}) = \omega(h(z), h(K), \Omega^*), \quad z \in \mathbb{D} \setminus K.
$$

Let $g_\mathbb{D}$ be the Green function on the unit disk $\mathbb{D}$. Since $\phi_t(K)$ is a compact set, we can calculate its Green energy with respect to $\mathbb{D}$

$$
V(\phi_t(K), \mathbb{D}) = \inf_{\mu} \iint_{\mathbb{D} \times \mathbb{D}} g_\mathbb{D}(x, y)d\mu(x)d\mu(y),
$$

where the infimum is taken over all Borel measures with compact support on $\phi_t(K)$ and $\mu(\phi_t(K)) = 1$. The *equilibrium measure* of $\phi_t(K)$ minimizes the Green energy of $\phi_t(K)$ with respect to $\mathbb{D}$.

The reciprocal of the Green energy of $\phi_t(K)$ is called the *Green capacity* of $\phi_t(K)$. In [10], it is proved that the Green capacity $\text{cap}_\mathbb{D} \phi_t(K)$ is a decreasing function of $t$. On the other hand, according to Theorem 1.7, the harmonic measure of $\phi_t(K)$ is an increasing function of $t$. A question of particular interest is what happens to their product, which has also a potential theoretic meaning, as we will see shortly. Let $\mu_t$ be the equilibrium measure of $\phi_t(K)$. Its Green potential

$$
G_{\mu_t}^\mathbb{D}(\phi_t(z)) = \int g_\mathbb{D}(\phi_t(z), w)d\mu_t(w)
$$

is a superharmonic function on $\mathbb{D}$ and harmonic on $K$. A significant property of the Green potential is that it is in fact the product of the harmonic measure and Green capacity of $\phi_t(K)$, as for every $z \in \mathbb{D} \setminus K$ we have

$$
G_{\mu_t}^\mathbb{D}(\phi_t(z)) = \omega(\phi_t(z), \phi_t(K), \mathbb{D}) \cdot \text{cap}_\mathbb{D} \phi_t(K); \quad (1.20)
$$

see e.g. [18, p.111].

The trajectory of $K$ is mapped under the Koenigs function $h$ onto the compact set

$$
h(\phi_t(K)) = h(K) + t,
$$
due to (1.19). Suppose $\nu^*$ is the equilibrium measure of the compact set $h(K)$ with respect to $\Omega^*$. Concerning the asymptotic behavior of the Green equilibrium potential of $\phi_t(K)$, we obtain the following result.
Chapter 1. Overview

Theorem 1.9. Let \((\phi_t)\) be a hyperbolic or a parabolic semigroup of positive hyperbolic step. Then
\[
\lim_{t \to +\infty} G^D_{\mu_t}(\phi_t(z)) = G^\Omega_\nu(h(z)),
\]
for every \(z \in \mathbb{D}\), except maybe for a subset of \(\partial K\) of zero logarithmic capacity.

Let’s denote by \(\nu_t\) the push-forward measure of \(\mu_t\) under the Koenigs function \(h\) of the semigroup. Then \(\nu_t\) is the equilibrium measure of \(h(K) + t\) with respect to \(\Omega\). Hence, for every \(t > 0\), the measures \(\nu_t\) compose a family of Borel measures. Our question is whether this family converges to a measure.

Theorem 1.10. Suppose that \((\phi_t)\) is either a hyperbolic or a parabolic semigroup of positive hyperbolic step and \(\nu_t\) is the equilibrium measure of \(h(K) + t\) with respect to \(\Omega\). Then \((\nu_t)_{t \geq 0}\) converges strongly to \(\nu^*\), as \(t \to +\infty\).

One of the most natural ways to measure the size of \(\phi_t(K)\) is its hyperbolic area \(A^D_h(\phi_t(K))\) with respect to the geometry of \(\mathbb{D}\). Since \(\phi_t(K)\) is getting ‘smaller’ and approaches the Denjoy-Wolff point, its hyperbolic area decreases. In fact, the same is true for another significant geometric quantity, the hyperbolic \(n\)-th diameter. These monotonicity properties arise from the fact that the hyperbolic density is a decreasing function of \(t\). But what happens when \(t\) tends to infinity?

Theorem 1.11. Suppose \((\phi_t)\) is either hyperbolic or parabolic of positive hyperbolic step. Then
\[
\lim_{t \to \infty} A^D_h(\phi_t(K)) = A^\Omega_\nu(h(K)),
\]
and
\[
\lim_{t \to +\infty} d^D_{n,h}(\phi_t(K)) = d^\Omega_{n,h}(h(K)).
\]

The boundary behavior of parabolic semigroups of zero hyperbolic step differs from the other types. That is due to the fact that \(\Omega\) cannot be contained in a horizontal strip or half-plane. To continue with, we elaborate what happens to the asymptotic behavior of the harmonic measure, equilibrium measure and its Green potential. Moreover, there are results concerning the geometric characteristics of the compact set \(\phi_t(K)\), such as logarithmic capacity and euclidean area.

Theorem 1.12. Suppose \((\phi_t)\) is a parabolic semigroup of zero hyperbolic step. Then
1. \(\omega(\phi_t(z), \phi_t(K), \mathbb{D}) \xrightarrow{t \to +\infty} 1\), for every \(z \in \mathbb{D} \setminus K\).
2. \(G^D_{\mu_t}(\phi_t(z)) \xrightarrow{t \to +\infty} 0\), for every \(z \in \mathbb{D} \setminus K\) and \(G^D_{\mu_t}(\phi_t(z)) \xrightarrow{t \to +\infty} +\infty\), for \(z \in K\), except for a set of zero logarithmic capacity.
3. \(A^D_h(\phi_t(K)), d^D_{n,h}(\phi_t(K)), \text{cap}(\phi_t(K)), A(\phi_t(K)) \xrightarrow{t \to +\infty} 0\).
Bibliography


