N-body systems

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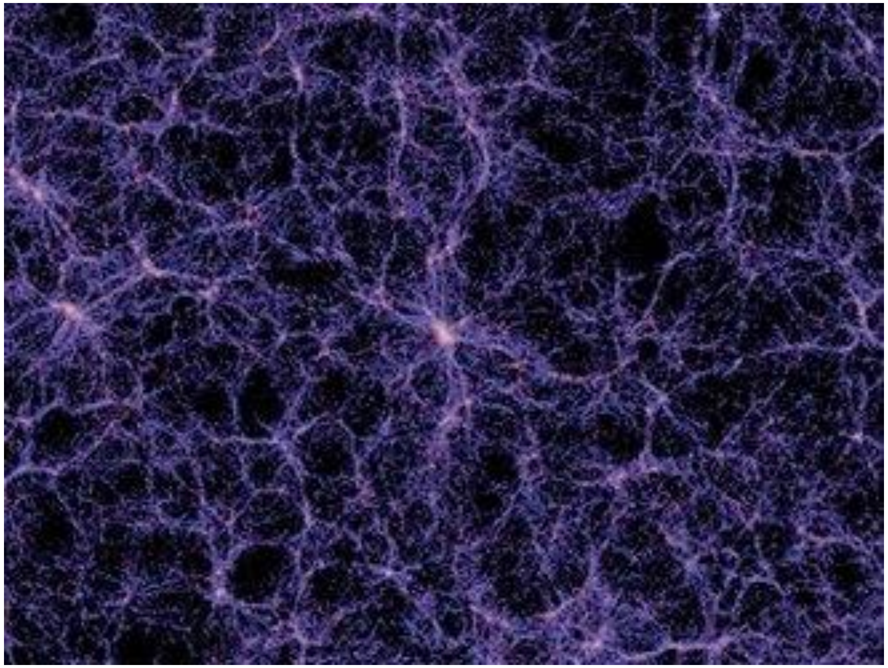
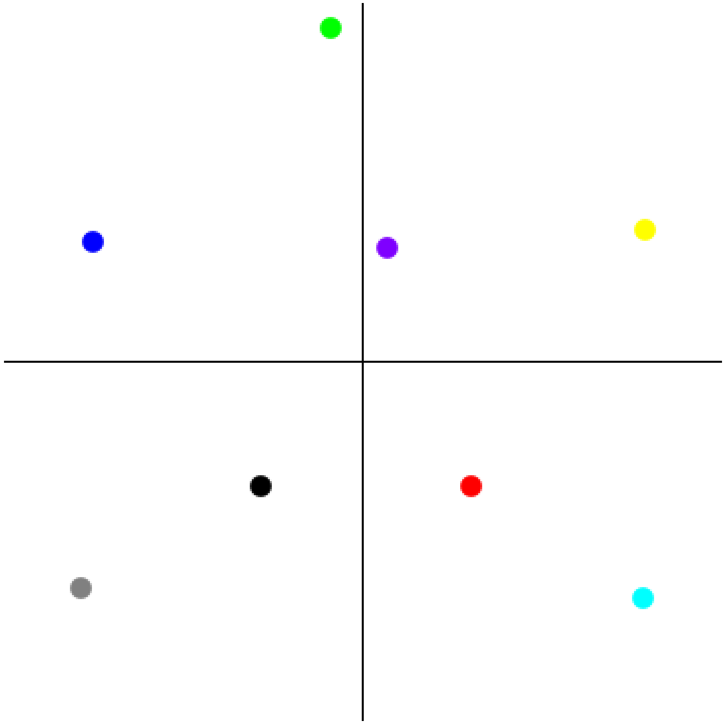
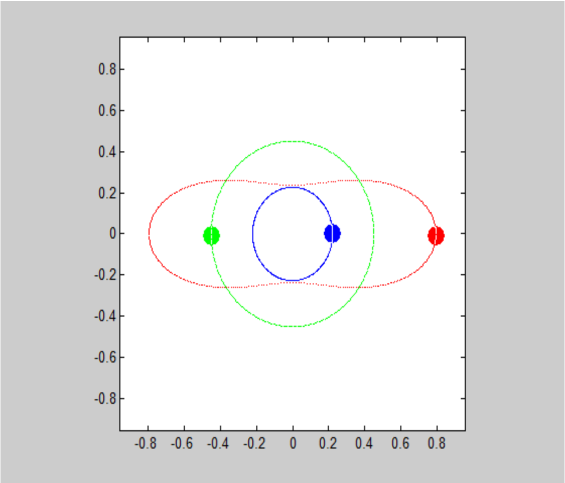
N-body simulations



The Millennium Simulation Project, N=10^9

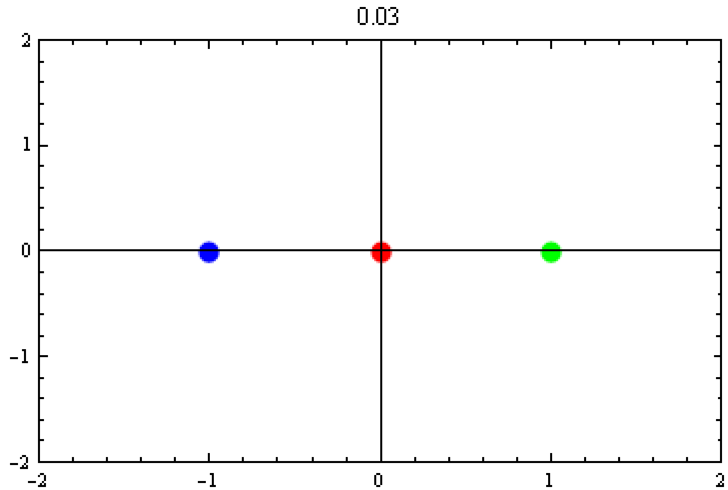
Central configurations





N-body choreographies

Euler solution



Lagrange solution

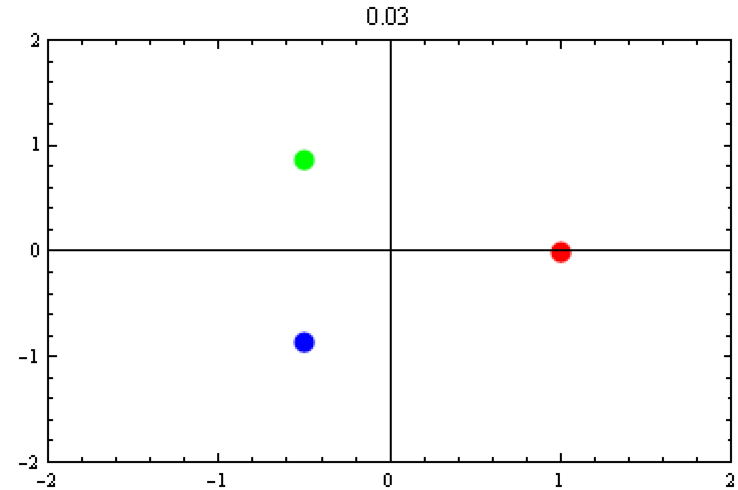
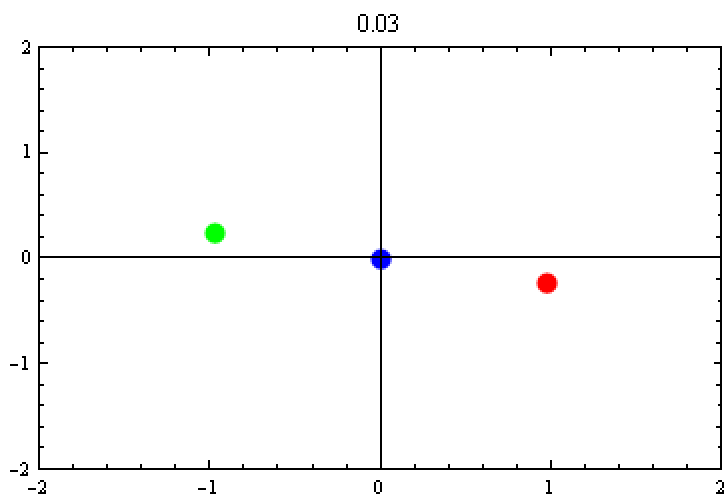
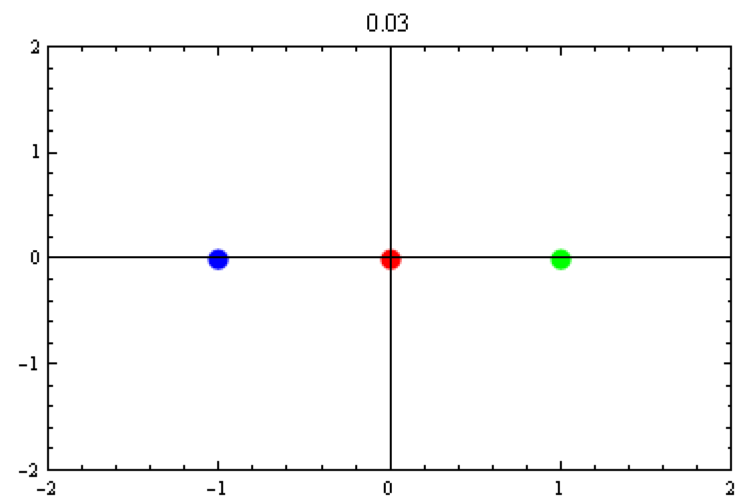


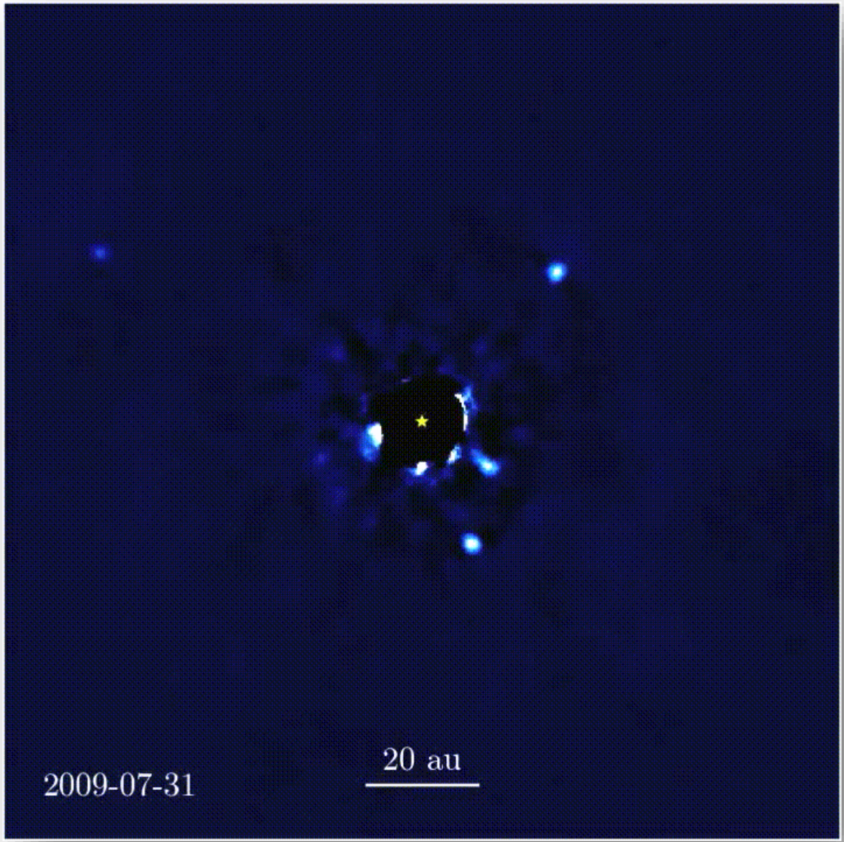
figure 8



perturbed Euler



Planetary systems

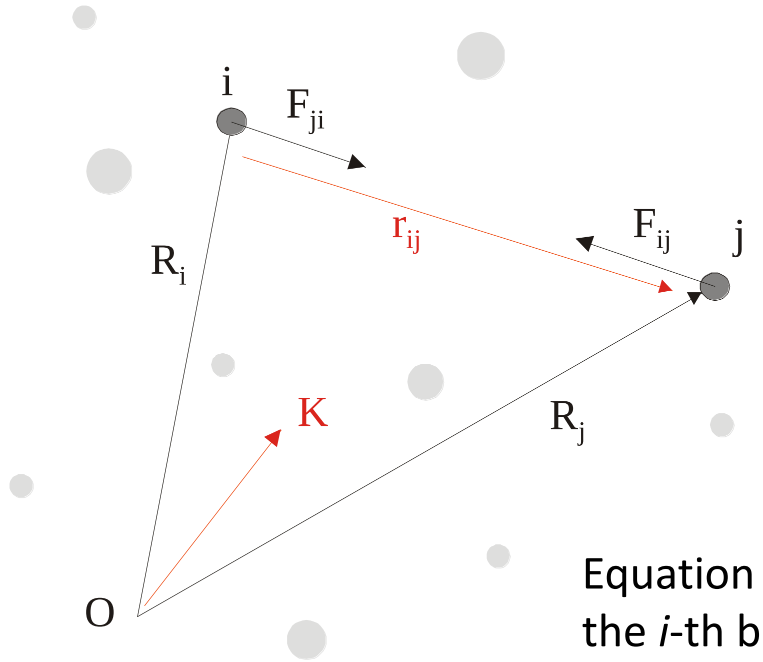


HR 8799, απόσταση 130 ly, M= 1.5 M_H

Planet	Mass (M_{Jup})	Radius (R_{Jup})	Period (day)	(y)	*a* (AU)
HR 8799 d	8.3	1.2	41054	112,5	27
HR 8799 c	8.3	1.3	82145	225	42.9
HR 8799 e	9.2	1.17	18000	50	16.4
HR 8799 b	7	1.2	164250	450	68

http://exoplanet.eu/catalog/

The N-body problem



$$\mathbf{F}_{ij} = -\mathbf{F}_{ji} = -\frac{Gm_1m_2}{r_{ij}^3}\mathbf{r}_{ij}$$

$$\mathbf{r}_{ij} = \mathbf{R}_j - \mathbf{R}_i$$

$$\mathbf{R}_K = \frac{1}{m}\sum_{i=1}^{N} m_i\mathbf{R}_i, \quad m = \sum_{i=1}^{N} m_i$$

Equation of Motion for the *i*-th body

$$m_i\ddot{\mathbf{R}}_i = \sum_{j=1,\, j\neq i}^{N} \mathbf{F}_{ji} = \sum_{j=1,\, j\neq i}^{N} \frac{Gm_im_j}{r_{ij}^3}\mathbf{r}_{ij}$$

(3N ODEs of 2nd order)

E.M. $\Rightarrow \sum_{i=1}^{N} m_i\ddot{\mathbf{R}}_i = \sum_{i}^{N}\sum_{j}^{N}\mathbf{F}_{ji} = 0 \Rightarrow m\ddot{\mathbf{R}}_K = 0 \Rightarrow \begin{cases}\dot{\mathbf{R}}_K = \mathbf{c}_1 \\ \mathbf{R}_K = \mathbf{c}_1 t + \mathbf{c}_2\end{cases}$ integrals of CM $\quad c_1 = c_2 = 0 \;\; (O \equiv K)$

E.M. $\Rightarrow \sum_{i=1}^{N} m_i\mathbf{R}_i \times \ddot{\mathbf{R}}_i = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{Gm_im_j}{r_{ij}^3}\mathbf{R}_i \times \mathbf{r}_{ij} = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{Gm_im_j}{r_{ij}^3}\mathbf{R}_i \times \mathbf{R}_j = 0 \Rightarrow$

$$\sum_{i=1}^{N} m_i\mathbf{R}_i \times \ddot{\mathbf{R}}_i = \sum_{i=1}^{N} m_i \frac{d}{dt}\left(\mathbf{R}_i \times \dot{\mathbf{R}}_i\right) = 0 \Rightarrow \sum_{i=1}^{N} m_i\mathbf{R}_i \times \dot{\mathbf{R}}_i = C$$

Angular momentum integral

The N-body problem

Potential function

$$U = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{Gm_i m_j}{r_{ij}}, \qquad r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$$

$$\mathbf{F}_i = \sum_{j=1}^{N}\mathbf{F}_{ji} = -\nabla_i U, \qquad m_i\ddot{\mathbf{R}}_i = -\nabla_i U, \qquad \nabla_i = \frac{\partial}{\partial x_i}\mathbf{i} + \frac{\partial}{\partial y_i}\mathbf{j} + \frac{\partial}{\partial z_i}\mathbf{k}$$

$$W = \sum_{i}^{N}\int_A^B \mathbf{F}_i d\mathbf{R}_i \Rightarrow \sum_{i}^{N}\int_A^B m_i\ddot{\mathbf{R}}_i d\mathbf{R}_i = -\sum_{i}^{N}\int_A^B \nabla_i U d\mathbf{R}_i \Rightarrow \sum_{i=1}^{N}\frac{1}{2}m_i\dot{R}_{i,B}^2 - \sum_{i=1}^{N}\frac{1}{2}m_i\dot{R}_{i,A}^2 = -(U_B - U_A) \Rightarrow$$

$$E = K + U = \sum_{i=1}^{N}\frac{1}{2}m_i\dot{R}_i^2 - \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{Gm_i m_j}{r_{ij}}$$

Energy integral

The N-body problem

Scaling of units

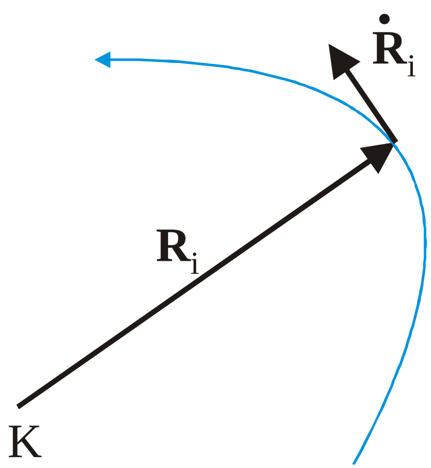
$$r' \to \alpha r \qquad m' \to \beta m \,, \qquad t' \to \gamma t \qquad \text{invariant ODEs} \Leftrightarrow \frac{\alpha^3}{\beta\gamma^2} = 1$$

Time inversion $t \to -t$: invariant ODEs

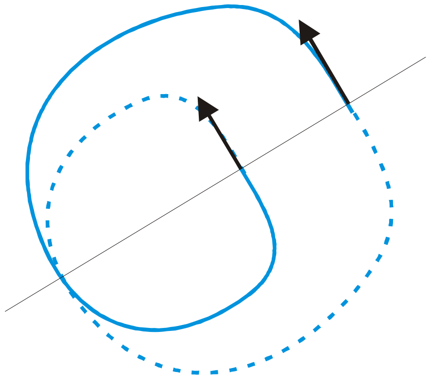
The mirror theorem

$$\dot{\mathbf{R}}_i(t_0)\cdot\mathbf{R}_i(t_0) = 0, \quad \forall i = 1,\dots,N \quad \Rightarrow \quad Orbit[t_0, t_0 + t] \equiv \begin{matrix}\text{mirror}\\ \text{Image}\end{matrix} \ \textit{of} \ Orbit[t_0 - t, t_0]$$

(mirror configuration)



Appearance of two mirror configurations at t=0 and at t=t_0 means **periodic orbit** with period T=2t_0.



The N-body problem

The Virial Theorem

Moment of inertia $$I = \sum_{i=1}^{N} m_i R_i^2$$

$$\frac{dI}{dt} = 2\sum_{i=1}^{N} m_i \mathbf{R}_i \dot{\mathbf{R}}_i \quad \Rightarrow \quad \frac{d^2 I}{dt^2} = 2\sum_{i=1}^{N} m_i \dot{\mathbf{R}}_i^2 + 2\sum_{i=1}^{N} m_i \mathbf{R}_i \ddot{\mathbf{R}}_i = 4K - 2\sum_{i=1}^{N} \mathbf{R}_i \cdot \nabla_i U$$

$$\sum_{i=1}^{N} \mathbf{R}_i \cdot \nabla_i U = \sum_{i=1}^{N} \left(x_i \frac{\partial U}{\partial x_i} + y_i \frac{\partial U}{\partial y_i} + z_i \frac{\partial U}{\partial z_i} \right) = -U$$

(homogeneous of degree -1)

$$\frac{d^2 I}{dt^2} = 4K + 2U \quad \Rightarrow \quad \frac{d^2 I}{dt^2} = 2(K + E)$$

If $E>0$ then $d^2I/dt^2>0$ and I increases indefinitely, thus at least one of the particles will escape from the system

The three-body problem(s)

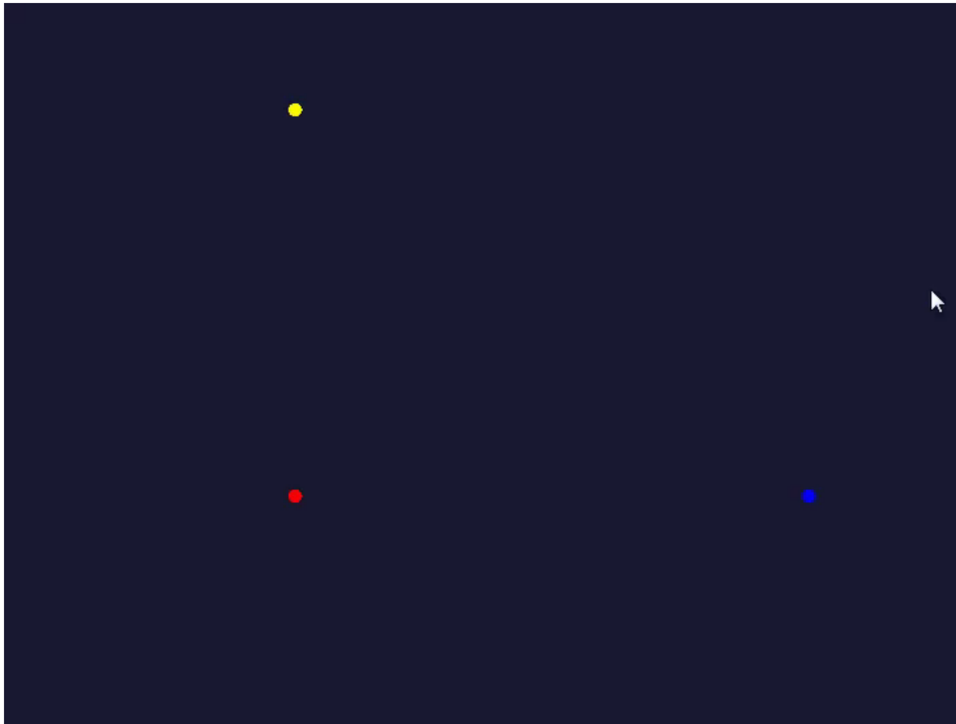
- Basic classification : **Planar** or **Spatial**

- The **General TBP** (triple systems)
  masses of all bodies are of the same order

- The **Planetary TBP**
  1 star (heavy body) + 2 planets (small masses)

- The **Restricted TBP**
  2 massive bodies (in Keplerian orbit) + massless body (or bodies)

- The **Circular Restricted TBP**
  2 massive bodies (in a planar Circular orbit) + massless body



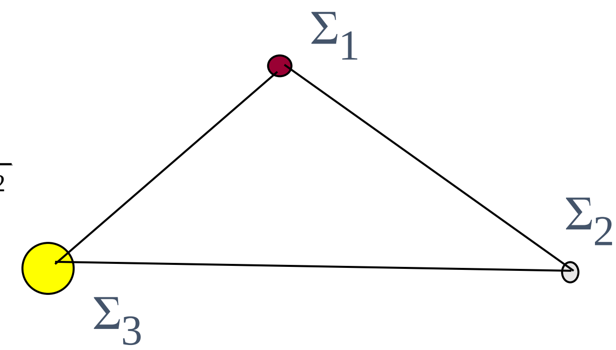
Example: The general planar three body problem

$$\dot{x}_1 = v_{x1}$$
$$\dot{y}_1 = v_{y1}$$
$$\dot{x}_2 = v_{x2}$$
$$\dot{y}_2 = v_{y2}$$
$$\dot{x}_3 = v_{x3}$$
$$\dot{y}_3 = v_{y3}$$

$$\dot{v}_{x1} = -\frac{Gm_2}{r_{12}^2}(x_1 - x_2) - \frac{Gm_3}{r_{13}^2}(x_1 - x_3)$$
$$\dot{v}_{y1} = -\frac{Gm_2}{r_{12}^2}(y_1 - y_2) - \frac{Gm_3}{r_{13}^2}(y_1 - y_3)$$
$$\dot{v}_{x2} = -\frac{Gm_2}{r_{21}^2}(x_2 - x_1) - \frac{Gm_3}{r_{23}^2}(x_2 - x_3)$$
$$\dot{v}_{y2} = -\frac{Gm_2}{r_{21}^2}(y_2 - y_1) - \frac{Gm_3}{r_{23}^2}(y_2 - y_3)$$
$$\dot{v}_{x3} = -\frac{Gm_3}{r_{31}^2}(x_3 - x_1) - \frac{Gm_1}{r_{23}^2}(x_3 - x_2)$$
$$\dot{v}_{y3} = -\frac{Gm_3}{r_{31}^2}(y_3 - y_1) - \frac{Gm_1}{r_{23}^2}(y_3 - y_2)$$

Σ_i: $r_i=(x_i,y_i)$, $v_i=(v_{xi},v_{yi})$

$$r_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$



Reduction of number of equations

$$\Sigma_3 = f_{KM}(\Sigma_1, \Sigma_2)$$

$$x_3 = -\frac{1}{m_3}(m_1 x_1 + m_2 x_2), \quad v_{x3} = -\frac{1}{m_3}(m_1 v_{x1} + m_2 v_{x2})$$
$$y_3 = -\frac{1}{m_3}(m_1 y_1 + m_2 y_2), \quad v_{y3} = -\frac{1}{m_3}(m_1 v_{y1} + m_2 v_{y2})$$

Normalization of units

$$e.g. \quad r_{13}(0) = 1, \quad m_1 + m_2 + m_3 = 1, \quad G = 1$$

see gtbp2.nb

Bibliography

A. Roy , “Orbital Motion”, Chapter 5, paragraphs 5.1 to 5.7

Reading Course : paragraph 5.9 (Lagrange Solutions)

Exercise : Upgrade code *gtbp2.nb* to run in 3-dimensions