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The four-dimensional hyperbolic twist map

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Abstract

We introduce a four-dimensional symplectic map which consists of two coupled two-dimensional hyperbolic twist maps. The central point $x = 0$ of the map is a fixed point and its stability depends on the parameters of the map. We study the topology of the space around the fixed point and the destruction of tori when the fixed point becomes unstable.

1 Introduction

The 4-dimensional hyperbolic twist map is defined by the formulae

$$X_1 = \rho_1 cos(x_1^2 + x_3^2 + \phi_1)^b x_1 + \rho_1 sin(x_1^2 + x_3^2 + \phi_1)^b x_3 \quad (1a)$$

$$X_2 = \rho_2 cos(x_2^2 + x_4^2 + \phi_2)^b x_2 + \rho_2 sin(x_2^2 + x_4^2 + \phi_2)^b x_4 \quad (1b)$$

$$X_3 = -\frac{1}{\rho_1} sin(x_1^2 + x_3^2 + \phi_1)^b x_1 + \frac{1}{\rho_1} cos(x_1^2 + x_3^2 + \phi_1)^b x_3 \quad (1c)$$

$$X_4 = -\frac{1}{\rho_2} sin(x_2^2 + x_4^2 + \phi_2)^b x_2 + \frac{1}{\rho_2} cos(x_2^2 + x_4^2 + \phi_2)^b x_4 \quad (1d)$$

where $x_i = x_n^{(i)}, X_i = x_{n+1}^{(i)}$, b is a constant (= 1.5 for our numerical study) and

$$\phi_1 = x_2^2 + x_4^2 + \theta_1 \ \ \& \ \ \phi_2 = x_1^2 + x_3^2 + \theta_2 \quad (2)$$

with θ_1, θ_2 constants.

By assuming ϕ_1 and ϕ_2 to be constants, equations (1a),(1c) and (1b),(1d) define two 2-dimensional hyperbolic twist maps, $M^{(1)}$ and $M^{(2)}$ respectively. These maps have been introduced by Easton (1979,1982) in order to study homoclinic connections and the transition to chaos. Also, they were used to describe the topology of Poincaré's sections near a periodic orbit of a Hamiltonian system of 2 d.o.f. (e.g. Hadjidemetriou 1986). The map (1) is symplectic, i.e. $N\Omega N^T = \Omega$, where $N = (\partial X_j/\partial x_i)$ is the Jacobian of the map and Ω the standard matrix of symplectic structure. So the mapping (1), in some sense, is in topological equivalence with the 4-dimensional Poincaré's section of a Hamiltonian system of 3 d.o.f. in the neighborhood of periodic orbits.

We denote as $\tilde{M}^{(i)}$ the maps $M^{(i)}$ including the coupling. The condition for the stability of the point (0,0) (for each map $\tilde{M}^{(i)}$ separately) reads as follows

$$\rho_i < \frac{1 + sin\phi_i^b}{cos\phi_i^b} \quad (3)$$

so on the plane (ϕ_i, ρ_i) we obtain two distinct regions, one for stability $(D_i^{(s)})$ and one for instability $(D_i^{(u)})$. For $\rho_i = 1$, stability results for all $\phi_i \in [0, \pi/2)$ and also $M^{(i)}$ become *integrable.*

2 Qualitative description of trajectories around fixed point

We examine, next, the evolution of trajectories with respect to the initial conditions in an open domain around the fixed point $x_o = 0$ of (1). The fixed point can be stable, simple unstable and double unstable.

When x_o is stable, $\tilde{M}^{(i)}$ can be transformed to linear maps in a domain around x_o, according to Siegel's theorem (see Moser 1973). The existence of a dense set of invariant tori follows directly. Our numerical studies, showed that a domain D around x_o is almost entirely filled by invariant tori if (ρ_i, ϕ_i) lie in the regions $D_i^{(s)}$ respectively, during the evolution of a trajectory with initial conditions $x(0) \in D$, and such a domain always exist if x_o is stable.

2.1 The case of simple instability

The fixed point x_o of (1) is simple unstable if $(\rho_1, \theta_1) \in D_1^{(u)}$ and $(\rho_2, \theta_2) \in D_2^{(s)}$ (or the inverse). We consider $\rho_2 = 1$ and the corresponding action-angle variables (J, ζ) for $M^{(1)}$ where $J = (x_2^2 + x_4^2)/2$ .For $\rho_2 \neq 1$, action-angle variables can also be defined in a dense set around x_o. The stability of the point $(0,0)$ of $\tilde{M}^{(1)}$ depends on J and it is hyperbolic (we denote it $\gamma(J)$) for $J \in U \subset R$. We form the set

$$\mu = \{(\bar{x}_{1,3}, J, \zeta) \in R^2 \times R \times S^1 \mid \bar{x}_{1,3} = \gamma(J), \forall J \in U\}$$

where (for $\rho_2 = 1$)

$$U = \{J \in R \mid (2J + \theta_1)^b < \theta^*, \;\; \theta^* = \arccos(2\frac{\rho_1}{1+\rho_1^2}), \;\; 0 \leq \theta^* < \frac{\pi}{2}, \;\; \rho_1 > 1\}$$

The set μ has the structure of a 1-parameter family of tori $\tau(J)$ but every $\tau(J)$ has 1-dimensional stable and unstable manifolds $W^s(\tau(J))$ and $W^u(\tau(J))$. So (see Wiggins 1988):

• μ is a C^r 2-dimensional normally hyperbolic manifold which has C^r 3-dimensional stable (W^s) and unstable (W^u) manifolds.

Since W^s and W^u are codimension-one, they, generally, intersect transversely. Then the formation of lobes and turnstiles leads to a geometrical structure and chaotic dynamics similar to 2- dimensional symplectic maps (Wiggins 1990). In Figure 1 we demonstrate a 3-dimensional projection of μ and W^u which is formed by the map (1) for $\rho_1 = 1.35, b = 1.5$. So for initial conditions in a domain near the W^s and W^u we get chaotic trajectories. But, from the above geometrical picture, we conclude also the existence of invariant tori outside and inside of this domain (Figure 2a).

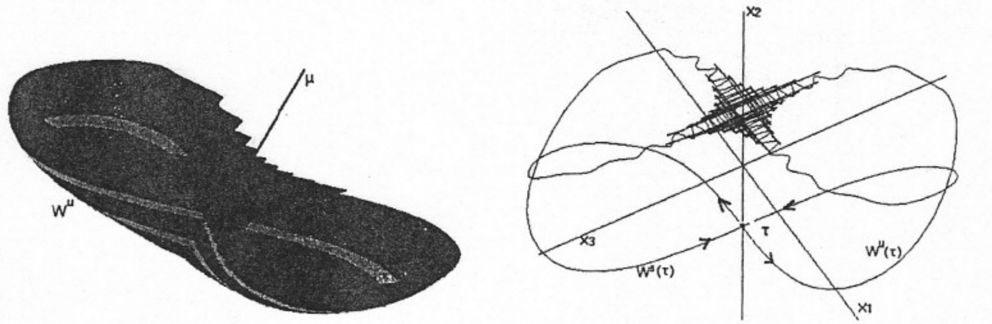

Figure 1: (a) The geometry of W^u in space, (b) A special flow of (1) which shows similar chaotic dynamics to 2-dimensional maps

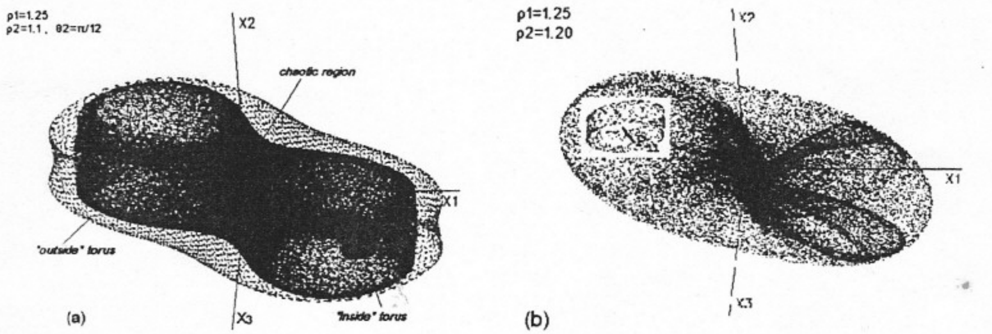

Figure 2: (a) Trajectories around the simple unstable fixed point x_o. (b) Chaotic trajectory for initial conditions very close to the double unstable fixed point x_o and a regular one near P_{11}

2.2 The case of double instability

The fixed point x_o of (1) is double unstable if $(\rho_i, \theta_i) \in D_1^{(u)}$ for $i = 1, 2$. The method which is applied in the previous case, now fails. The stable and unstable manifolds of the fixed point are 2-dimensional in the 4-dimensional space of (1). They may intersect in a homoclinic point which results to a family of other homoclinic points. Bountis et al (1995) showed in some cases the existence of such intersections for a 4-dimensional map. But, there is no proof yet that such intersections create finally a chaotic set. Their numerical experiments show the existence of chaotic trajectories. This situation is also true for the map (1). Trajectories, which correspond to initial conditions near the fixed point are chaotic and they occupy a large region in the space.

The points $P_i = (0, 0)$ are hyperbolic points for the maps $M^{(i)}$ which are associated with the existence of a pair of elliptic fixed points P_{i1} and P_{i2} for each map. By following the previous ideas about the existence of tori or hyperbolic manifolds, we conclude the following:

• Consider a small open domain D_1 in the neighborhood of P_1 and a small domain D_2 in the neighborhood of P_{21} or P_{22}. Then a trajectory with initial conditions $(x_1, x_3) \in D_1$ and $(x_2, x_4) \in D_2$ is quasiperiodic if $\rho_2 > \rho_1$ (existence of invariant tori) and chaotic if $\rho_2 < \rho_1$ (existence of a 2-dimensional normally hyperbolic manifold).

So, in the large chaotic sea around the double unstable fixed point, we can localize regions with regular trajectories (Figure 2b).

3 Conclusions and remarks

A 4-dimensional symplectic map has been introduced. We study the evolution of trajectories around the fixed point $x_o = 0$ by considering a separation of the map in two 2-dimensional maps with a coupling. In every case, it seems that the stability character of the fixed point plays an important role in the topological structure of the phase space. Chaotic trajectories appear when x_o becomes unstable. The transition to chaos is explained only for the case of simple instability by the existence of 2-dimensional normally hyperbolic manifolds. A similar behavior can arise also around periodic orbits of 3 d.o.f Hamiltonian systems.

In the above analysis *we did not take into account resonances*. In the map (1) significant resonances appear only far from the central fixed point where we focus our interest. For, approximately, $\rho_i > 1.3, b = 1.5$, large chaotic regions appear near resonances, and chaotic trajectories, with initial conditions near $x_o = 0$, pass through those regions and escape to infinity.

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