Non-circular, non-well-founded set universes

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Abstract

We show that there are universes of sets which contain descending $\in$-sequences of length α for every ordinal α, though they do not contain any $\in$-cycle. It is also shown that there is no set universe containing a descending $\in$-sequence of length On.

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[Note: In this reproduction of the article we have made some minor corrections to the original published version.]

1 Introduction

As is well-known the foundation (or regularity) axiom (F) is the assertion that the relation $\in$ is well-founded, which (in the presence of some choice principle) amounts to the claim that there is no infinite descending $\in$-sequence

$$\cdots \in x_2 \in x_1 \in x_0. \tag{1}$$

If S is any set theory containing the above statement, we denote by S^- the corresponding theory resulting from S by dropping F. An *anti-foundation* axiom (AF) is any statement refuting F.

There is a fairly rich bibliography about systems of the form S^- and $S^- + AF$. The interest in non-well-founded set universes has been recently renewed by the appearance of P. Aczel's book [1]. There one can also find a short history of the subject.

Concerning now descending $\in$-sequences of the form (1), we should distinguish them into two kinds: the circular and the non-circular ones. The sequence (1) is *circular* if $x_i = x_j$ for some $i \neq j$, and *non-circular* otherwise. Thus the set $x = \{x\}$, if supposed to exist, produces by iterated self-reference the "infinite" progression $\cdots \in x \in x \in x$, though the set x itself seems to be an ordinary finite object. On the other hand, a non-circular sequence

$$\cdots \in x_2 \in x_1 \in x_0, \qquad x_i \neq x_j, \tag{2}$$

expresses a totally different situation: Namely, a really infinite descend through countably many distinct stairs.

In [1] the emphasis is on circularity; the author's aim was to provide a set theory suitable to model circular phenomena occurring in some areas of computer science. Here, on the contrary, we shall be concerned only with universes which contain infinite descending non-circular $\in$-sequences but no circular ones. We can think of possible interpretations of $\in$ with respect to which a sequence of the form (2) has good chances to be true. For example, let x, y range over forms of matter and interpret $\in$ as "consists of", or let x, y range of artificial objects and let $\in$ be the relation "is a proper part of". Concerning the first interpretation we could find many physicists sharing the view that matter is endlessly divisible (at least in principle), thus structurally non-well-founded. Analogously, in the second interpretation, we can imagine a highly complicated artifact, the dismantling of which into atomic parts takes so much time that it practically never comes to an end. Besides, in both these interpretations, there is no room for circular phenomena, since $\in$ now means proper reduction of something to something else which is strictly prior or simpler or more primitive.

We assume familiarity of the reader with Fraenkel-Mostowki permutation models. For details one may consult [2].

2 A non-circular permutation model

Let E be a relation intended to interpret $\in$. An *E-cycle* is any finite sequence $(x_0, x_1, \ldots, x_n)$ such that $x_0 E x_1 E \cdots E x_n E x_0$. The relation E is non-circular if it contains no cycles.

Similarly a universe (A, E) satisfying a set theory S^- is said to be non-circular if its membership relation E is non-circular.

The stimulus for the present notice has been the question of how long descending $\in$-sequences there can be in non-circular universes. Obviously the simplest such sequences are those of length ω. (Longer sequences, after all, have to be defined and this will be done in the next section.)

Theorem 2.1 *(ZFC) There is a non-circular permutation model containing infinite descending $\in$-sequences.*

Proof. Let V be the universe of ZFC. Choose a countable set of infinite subsets of ω and enumerate them by means of the integers. Let $(y_n)_{n\in\mathbb{Z}}$ be this enumeration. Put $x_n = \{y_n\}$, $n \in \mathbb{Z}$, and define the mapping $\pi : V \to V$ as follows:

$$\pi(x_{n+1}) = y_n, \quad \pi(y_{n+1}) = x_n, \quad \pi(x) = x \text{ if } x \neq x_n, y_n \text{ for } n \in \mathbb{Z}.$$

Clearly π is a permutation of V. Let $\in_\pi$ be the usual relation defined by

$$x \in_\pi y \text{ iff } \pi(x) \in y.$$

Then we have $\pi(x_{n+1}) = y_n \in x_n$ for all $n \in \mathbb{Z}$, hence

$$\cdots \in_\pi x_2 \in_\pi x_1 \in_\pi x_0 \in_\pi x_{-1} \in_\pi \cdots$$

Thus it remains to show that $\in_\pi$ is non-circular. Let us denote by $z_1, z_2, \ldots$ the elements of $V - \{x_n, y_n : n \in \mathbb{Z}\}$, while $w_1, w_2, \ldots$ range over arbitrary elements of V. Assume there is a cycle

$$w \in_\pi w_1 \in_\pi \cdots \in_\pi w_n \in_\pi w. \tag{3}$$

We shall reach a contradiction.

Claim 1. The sequence (3) does not contain both x_i's and y_j's.

Proof. Suppose it does. Then there will be x_i and y_j separated by z_k's only. Thus we have the following two cases:

Case (a). $x_i \in_\pi z_1 \in_\pi \cdots \in_\pi z_k \in_\pi y_j$. This is equivalent to $\pi(x_i) \in z_1 \in \cdots \in z_k \in y_j$, or $y_{i-1} \in z_1 \in \cdots \in z_k \in y_j$. It follows that $y_{i-1} \in TC(y_j)$ (the transitive closure of y_j), which is impossible since $TC(y_j)$ contains just natural numbers.

Case (b). $y_j \in_\pi z_1 \in_\pi \cdots \in_\pi z_k \in_\pi x_i$. This is equivalent to $x_{j-1} \in z_1 \in \cdots \in z_k \in x_i$, which is impossible too.

Claim 2. The sequence (3) contains at most one y_j.

Proof. Suppose (3) contains y_i and y_j where $i \neq j$. Since by Claim 1, there are no x_k's in (3), y_i and y_j are separated by z_k's, i.e., we have $y_i \in_\pi z_1 \in_\pi \cdots \in_\pi z_k \in_\pi y_j$. This is equivalent to $x_{i-1} \in z_1 \in \cdots \in z_k \in y_j$, a contradiction.

From the above claims it follows that (3) contains either a single y_j and no x_i or no y_j. In the first case (3) is of the form

$$z \in z_1 \in \cdots \in y_j \in_\pi \cdots \in_\pi z_k \in_\pi z.$$

This is equivalent to the conjunction $z \in z_1 \in \cdots \in y_j$ and $x_{j-1} \in \cdots \in z_k \in z$, which is false. Therefore (3) contains x_i's and z_k's only.

Claim 3. If x_i, x_j are contained in (3), then there can be no z_k between them. Moreover, if $x_i \in_\pi x_j$, then $i = j + 1$.

Proof. Suppose $x_i \in_\pi z \in x_j$. Then, equivalently, $y_{i-1} \in z \in \{y_j\}$, a contradiction. Further let $x_i \in_\pi x_j$. Then $y_{i-1} \in \{y_j\}$, hence $i - 1 = j$.

We conclude that the only possible form of (3) were

$$w \in_\pi x_{i+k} \in_\pi x_{i+k-1} \in_\pi \cdots \in_\pi x_i \in_\pi w,$$

for some $k > 0$. But in this case $x_i \in_\pi w \in_\pi x_{i+k}$. Then, by Claim 3, $w = x_{i-1}$ and $i + k = i - 2$, whence $k = -2$, again a contradiction. This completes the proof that there is no cycle of the form (3). $\dashv$

3 Paths and graphs

A *relation* is any class X whose elements are ordered pairs. As usual we define $dom(X)$, $rng(X)$ and $fld(X)$. The elements of $fld(X)$ are called *nodes* of X, while the elements of X are the *edges* of X. If $(x, y) \in X$, then the nodes x, y are said to be *adjacent*. For any x, let $X_{(x)} = \{y : (x, y) \in X\}$. The relation X is called a *graph* if $X_{(x)}$ is a set for every $x \in dom(X)$.

Given a relation X, by *path* in X one usually understands a tour along a finite or countable set of adjacent nodes. That is, the sequence $\{x_0, x_1, \ldots\} \subseteq fld(X)$ is a path if $(x_n, x_{n+1}) \in X$ for every n. Thus the length of a path is usually either finite or countable. Can there be meaningful transfinite paths? Yes, provided we shall specify what crossing a limit stage really means. The conditions, for instance, $(x_n, x_{n+1}) \in X$ and $(y_n, y_{n+1}) \in X$ do not suffice for the $\omega \cdot 2$-sequence $Y = \{x_0, x_1, \ldots, y_0, y_1, \ldots\} \subseteq fld(X)$ to be considered as a path in the relation X. Because in such a case the concatenation of any two arbitrary paths would do the job. I think that the right extra condition we need to impose is that each limit node x should be connected with every node of some final segment preceding x. These connections make the passage to the limit nodes not by arbitrary jumps but by smooth, continuous progression.

Definition 3.1 Let X be a relation. A *path in X,* or an *X-path,* is a class $Y \subseteq fld(X)$ well-ordered by an ordering $\prec$ such that:

(a) for any $x \in Y$, $(x, x^+) \in X$, where x^+ is the immediate successor of x in $(Y, \prec)$.

(b) if x is a limit point of $(Y, \prec)$, then there is a $y_0 \in Y$ such that $y_0 \prec x$ and $(y, x) \in X$ for all $y \in Y$ such that $y_0 \prec y \prec x$.

If Y is a path and $ord(Y, \prec) = \alpha \in On$, then Y is said to be an *α-path* and we call α the *length* of Y.

A path has always a first element, but need not have a last one. If it does, and x, y are these elements, respectively, then we say that the path *joins* x and y. Given an X-path $(Y, \prec)$ we call the set $Tr(Y) = \prec \cap X$ of the pairs of $\prec$ contained on X the *trace* of Y. If Y is an X-path joining the nodes x and y (i.e., Y starts with x and ends with y), and if it happens that $(y, x) \in X$, then we call the class $Tr(Y) \cup \{(y, x)\}$ a (minimal) *cycle.* More general (i.e. non-minimal) cycles can be defined in the obvious way. A relation X is said to be circular if it contains cycles.

Lemma 3.2 *Let X be a relation. If the nodes x, y are joined by a path, then they are joined by a finite path. In particular, if X contains cycles, then it contains finite cycles.*

Proof. Any infinite path joining x, y has an order type $\alpha + n$ where α is a limit ordinal and $n \in \omega$, $n > 0$. Thus we can use induction on α to prove the claim. Suppose that for every limit $\beta < \alpha$ and any $n \in \omega$, if the nodes x, y are joined by a $\beta + n$-path, then they are joined by a finite path. It suffices to show that it holds also for α. Let $Y = \{z_\nu : \nu < \alpha + n\}$ be an $\alpha + n$-path joining x, y. By Definition 3.1 there is a $\gamma < \alpha$ such that $(z_\gamma, z_\alpha) \in X$. Consider the path $Y' = \{z_\nu : \nu \leq \gamma\}$. By the inductive hypothesis, there is a finite path Z joining x and z_γ. Then the path $Z \cup \{z_\alpha, \ldots, z_{\alpha+n-1}\}$ is finite and joins x and y. $\dashv$

Thus for $\alpha \geq \omega$ the only nontrivial paths are those whose length α is a limit ordinal. Concerning now the possible length of a path, we first show the following.

Lemma 3.3 *The following are equivalent.*

(a) There is a graph containing a path of length On.

(b) There is a function $F : LOn \to On$, where LOn is the class of limit ordinals, such that:

(i) $(\forall \alpha \in LOn)(F(\alpha) < \alpha)$,

(ii) $(\forall \gamma \in On)(\{\alpha \in LOn : F(\alpha) \leq \gamma < \alpha\}$ *is bounded in* $On)$.

Proof. (a)$\Rightarrow$(b). Let Y be a path in the graph X of length On. We can identify the nodes of Y with the elements of On and the path ordering with that of the ordinals. Then for every $\alpha \in LOn$ there is a $\beta < \alpha$ such that $(\gamma, \alpha) \in Y$ for all γ with $\beta \leq \gamma < \alpha$. Choose for every $\alpha \in On$ such a β and put $F(\alpha) = \beta$. Then the condition (i) for F is satisfied. Furthermore,

$$(\forall \alpha \in LOn)(\forall \gamma \in On)(F(a) \leq \gamma < \alpha \Rightarrow (\gamma, \alpha) \in Y).$$

Since X is a graph, the class $X_{(x)}$ is a set for every $x \in fld(X)$. Consequently $\{\alpha \in LOn : F(\alpha) \leq \gamma < \alpha\} \subseteq X_{(\gamma)}$ is also a set, hence condition (ii) for F is satisfied.

(b)$\Rightarrow$(a). Given $F : LOn \to On$ with properties (i) and (ii), then clearly

$$Y = \{(\gamma, \alpha) : F(\alpha) \leq \gamma < \alpha\} \cup \{(\gamma, \gamma + 1) : \gamma \in On\}$$

is a graph and a graph-path of length On in itself. $\dashv$

Lemma 3.4 *There is no function F of the kind described in clause (b) of the preceding lemma.*

Proof. First it is easy to check that property (ii) for F is equivalent to the following condition:

(iii) $(\forall \gamma \in On)(\exists \beta \in On)(\forall \alpha \in LOn)(\alpha \geq \beta \Rightarrow F(\alpha) > \gamma)$.

Let $H : On \to$ On be defined by

$$H(\gamma) = \min\{\beta \in On : (\forall \alpha \in LOn)(\alpha \geq \beta \Rightarrow F(\alpha) > \gamma\},$$

and define $\overline{H} : On \to On$ as follows:

$$\overline{H}(0) = 0, \quad \overline{H}(\nu + 1) = \max\{\overline{H}(\nu) + 1, \overline{H}(\nu + 1)\}, \text{ and}$$

$$\overline{H}(\lambda) = \sup\{\overline{H}(\nu) : \nu < \lambda\}, \text{ if } \lambda \text{ is a limit ordinal.}$$

Clearly $\overline{H}$ is strictly increasing, continuous and, since $\overline{H}(\gamma) \geq H(\gamma)$, for every $\gamma \in On$,

$$(\forall \alpha \in LOn)(\alpha \geq \overline{H}(\gamma) \Rightarrow F(\alpha) > \gamma). \tag{4}$$

Therefore, $\overline{H}$ contains fixed points. Let γ_0 be such a fixed point. Then γ_0 is a limit ordinal and since $\overline{H}(\gamma_0) = \gamma_0$, it follows from (4) that $F(\gamma_0) > \gamma_0$. This contradicts property (i) of F and completes the proof. $\dashv$

As an immediate consequence of Lemmas 3.3 and 3.4 we obtain:

Theorem 3.5 *Let S^-+AF be any ordinary set theory with an anti-foundation axiom. Then there is no universe of that set theory containing a non-circular descending $\in$-sequence of length On.*

4 Non-circular models containing paths of every ordinal length

Let GBC^- be the usual Gödel-Bernays theory of classes with the global axiom of choice

$$\text{(GC)} \qquad V \approx On.$$

We fix a wellordering $<$ of V of order type On. Then every proper subclass of V is order-isomorphic to On, too. Moreover, we can partition V into a proper class of disjoint proper classes X_α, $\alpha \in On$.

We shall define a cumulative hierarchy G_α, $\alpha \in On$, of graphs analogous to the familiar hierarchy V_α, starting now not simply from $\emptyset$ but from $1 = \{\emptyset\}$ together with a proper class of sets x_α, $\alpha \in On$, where x_α is of "depth" α.

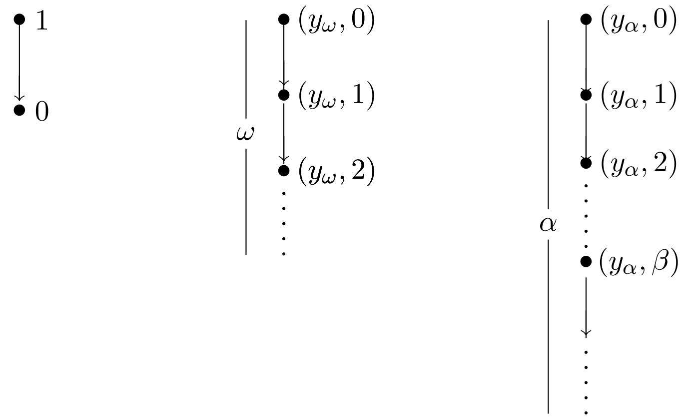
Let $V = \{y_\alpha : \alpha \in On\}$ be an enumeration of V yielded by the ordering $<$ and let

$$X_0 = \{0, 1\} \cup \{(y_\alpha, \beta) : \alpha \in LOn \;\&\; \beta < \alpha\}.$$

Let furthermore $(X_\alpha)_{\alpha>0}$ be a partition of $V - X_0$ into disjoint proper classes. Put

$$G_0 = \{(1, 0)\} \cup \{((y_\alpha, \beta), (y_\alpha, \gamma)) : \beta < \gamma < \alpha\}.$$

G_0 is the union of the disjoint paths shown in the diagram below. (The reason that we consider $1 = \{\emptyset\}$ instead of just $\emptyset$ is that the node $\emptyset$ does not have any children, hence it would not be able to be in $fld(G_0)$ had we started with $\emptyset$. Now however $\emptyset \in fld(G_0)$.)



For each limit α the path starting at the node $(y_\alpha, 0)$ has length α.

Suppose the graph G_α has been defined. Then $G_{\alpha+1}$ is G_α plus a class of new nodes and edges representing those subsets of $fld(G_\alpha)$ which are not already represented in G_α, where for any graph G, a subset $u \subseteq fld(G)$ is said to be *represented in G by the node x* if $G_{(x)} = u$. (For example, in the graph G_0 above the empty set is represented by the node 0). Let

$$P(G_\alpha) = \{u \subseteq fld(G_\alpha) : u \text{ is not represented in } G_\alpha\}.$$

Of course, $P(G_\alpha)$ is a subclass of V well-ordered by $<$. Put

$$G_{\alpha+1} = G_\alpha \cup \{(x, y) : x \text{ is the } \beta\text{-th element of } X_{\alpha+1} \text{ with respect to } <$$

$$\text{and } y \text{ belongs to the } \beta\text{-th element of } P(G_\alpha) \text{ with respect to that ordering}\},$$

and for a limit α, let $G_\alpha = \bigcup\{G_\beta : \beta < \alpha\}$. Finally let

$$G = \bigcup\{G_\alpha : \alpha \in On\}.$$

(The elements of X_α, for a limit α, are not in fact employed in the preceding construction, but this is no harm.)

A graph X is said to be *extensional* if

$$(\forall x, y \in fld(X))(X_{(x)} = X_{(y)} \rightarrow x = y).$$

X is called *full* if

$$(\forall x \subseteq fld(X))(\exists y \in fld(X))(X_{(y)} = x).$$

From the definition of G it follows:

Lemma 4.1 *G is a non-circular, extensional and full graph.*

G interprets the language of ZF set theory in the obvious way, i.e., the quantifiers range over the class $fld(G)$ and

$$G \models x \in y \text{ iff } (y, x) \in G.$$

To interpret further the class variables of GB we can modify slightly the construction of G so that $A = V - fld(G)$ is a proper class. Then we extend G to G^* as follows: Consider the family $\mathfrak{G}$ of proper subclasses of G which are definable over G by a predicative formula $\phi(x, <^G)$ containing as parameter the wellordering $<^G$ of G in the sense of G, i.e.,

$$x \in <^G \Leftrightarrow (\exists y, z \in fld(G))(y < z \;\&\; G \models x = (y, z)).$$

Clearly $\mathfrak{G}$ can be enumerated by On in the form $\mathfrak{G} = \{Z_\alpha : \alpha \in On\}$. Enumerate also the class $A = V - fld(G)$ by $A = \{z_\alpha : \alpha \in On\}$ and add to G the extra nodes z_α having as children the elements of Z_α, respectively, in symbols:

$$G^* = G \cup \{(z_\alpha, x) : \alpha \in On \;\&\; x \in Z_\alpha\}.$$

Obviously, the z_α's are maximal nodes of G^* interpreting the class variables of GB.

Consider the following anti-foundation axiom:

$$\text{(AF)} \;\; (\forall \alpha \in LOn)(\exists f)(dom(f) = \alpha \;\&\; (\forall \beta, \gamma < \alpha)(\beta < \gamma \Rightarrow f(\gamma) \in f(\beta))).$$

Then the following holds:

Theorem 4.2 *(a) $G \models ZFC^- + AF$. (b) $G^* \models GBC^- + AF$.*

Proof. (a) By Rieger's theorem (see [1], Appendix B), $G \models \text{ZFC}^-$, since, by Lemma 4.1, G is extensional and full. Also $G \models AF$ as a consequence of the construction of G_0. (b) The class $\mathfrak{G}$ is closed under relative definability and contains $<^G$. Therefore $V \approx On$ and existence of predicatively defined classes holds in G^*. ⊣

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