DISCRETE ORDERINGS AND COMMUTATIVE MONOIDS

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Abstract

We show that a countable totally and discretely ordered set with first element inherently carries the structure of an ordered commutative euclidean monoid, provided its order type is of a certain kind. As an application we specify the order types of all discretely ordered sets which can be expanded to ordered commutative euclidean monoids.

1 Motivation and Introduction

The motivation behind this paper has been the problem of expanding a (commutative) discretely ordered group $(G, +, <, 0)$ to a (commutative) discretely ordered ring $R = (G, \cdot)$, possibly with extra properties that make it look more and more like $\mathbb{Z}$. This is a problem either of logic (model theory, see for example [1], or of algebra (cf. [2]) or of both (cf. [3]), depending on the extra properties the ring expansion is required to satisfy. If for instance we want (the positive part of) $(G, \cdot)$ to be a model of Peano arithmetic, then we are clearly in the area of logic, but if we want $(G, \cdot)$ to be, say, euclidean, the problem is algebraic. While almost all logical expandability questions of the preceding kind have been settled (at least for countable G) in a terms of uniform model theoretic characterization ("recursive saturation", [1]), the corresponding algebraic ones seem to be subtler and harder. For most of them only necessary conditions are known. The present paper just deals with one of those conditions.

Specifically, the problem of expanding G to a euclidean ring $(G, \cdot)$, partly reduces to the problem of expanding a discretely or densely ordered set $(A, <, 1)$ with least element 1 (the quotient of G under a certain equivalence relation), to a commutative ordered euclidean monoid $(A, *, <, 1)$, where 1 is also the identity. Similarly in order for G to be expandable to a ring $(G, \cdot)$ satisfying open induction, $(A, <, 1)$ must be densely ordered and expandable to a monoid $(A, *, <, 1)$ which is euclidean and radically closed. Thus we come to the following definitions.

A totally ordered commutative monoid (henceforth *ordered monoid* or just *monoid*) is a structure $A = (A, *, <, 1)$ such that:

i) $(A, *, 1)$ is a commutative monoid with identity 1,

ii) $(A, <, 1)$ is a (totally) ordered set with least element 1, and

iii) for all $x, y, z \in A$, $x < y \;\Rightarrow\; x * z < y * z$.

The monoid A above is said to be *euclidean* if for all $x, y \in A$,

$x < y \;\Rightarrow\; (\exists z)(y = x * z)$.

A is said to be *radically closed* if for every $x \in A$ and $n \in \mathbb{N}$, $n > 0$, there is

$n > 0 \;\Rightarrow\; (\exists y)(y^n = x$.

The monoid $A = (A, *, <, 1)$ looks multiplicative, although this is inessential (one might think of $*$ as addition and 1 as zero). So if there is no danger of confusion we write x^n for $x * \cdots * x$, n times.

Let us first characterize the property of a monoid to be euclidean. Given the monoid $(A, *, <, 1)$, let $\overline{A}$ be the group generated by A, i.e., $\overline{A}$ is the symmetric extension of A obtained by adding an inverse x^{-1} for each x and extending the operations on them. $<$ also extends on $\overline{A}$ in the obvious way and $\overline{A}$ becomes an *ordered group*. However the "positive part" $\overline{A}^+ = \{x \in \overline{A} : x \geq 1\}$ of $\overline{A}$ does not in general coincide with A. The euclidean property guarantees exactly this fact. Namely:

Lemma 1.1 *The monoid* $(A, *, <, 1)$ *is euclidean iff* $A = \overline{A}^+$.

Proof. Suppose A is euclidean. It suffices to show that $\overline{A}^+ \subseteq A$. Let $x \in \overline{A}^+$. Then clearly x has the form $x = a_1^{n_1} * \cdots * a_s^{n_s} * b_1^{-m_1} * \cdots * b_t^{-m_t}$, where $a_i, b_j \in A$ and $n_i, m_j > 0$. Since $x \geq 1$, by the properties of the order, $a_1^{n_1} * \cdots * a_s^{n_s} \geq b_1^{m_1} * \cdots * b_t^{m_t}$. If $a = a_1^{n_1} * \cdots * a_s^{n_s}$ and $b = b_1^{m_1} * \cdots * b_t^{m_t}$, then $a, b \in A$ and $a \geq b$. By the euclidean property there is $c \in A$ such that $a = b * c$. Thus $x = a * b^{-1} = c \in A$.

For the converse, suppose $A = \overline{A}^{+}$ and let $a, b \in A$ such that $a \leq b$. Then $b * a^{-1} \geq 1$, i.e., $b * a^{-1} \in \overline{A}^{+} = A$. Therefore if $c = b * a^{-1}$, then $c \in A$ and $c * a = b$, hence A is euclidean. □

The following example shows that a radically closed monoid need not be euclidean.

Example. Let $B = \{m^{1/n} : m, n \in \mathbb{N}\backslash\{0\}\}$. If $\cdot, <$ are the multiplication and ordering of the reals, then clearly $(B, \cdot, <, 1)$ is a radically closed monoid. However B is not euclidean. For instance $2 < 3 \in B$ but there is no $x \in B$ such that $3 = 2 \cdot x$.

Let us say that the ordered set $(A, <, 1)$ *expands* if there exists a multiplication $*$ turning $(A, <, 1)$ into a monoid $(A, *, <, 1)$.

The questions we shall address here are the following: Given an ordered set $(A, <, 1)$, under what conditions is it (a) expandable, (b) expandable to a euclidean monoid, (c) expandable to a radically closed monoid? Since the only property that $(A, <, 1)$ possesses is its order type, the above questions clearly ask for the order type of A satisfying (a), (b) or (c) above.

Now if $(A, <, 1)$ expands, then, clearly A is infinite, so the order type of $(A, <, 1)$ can be arbitrarily complicated and question (a) seems to be intractable. On the other hand, we can see that the requirements of (b) and (c) restrict drastically the type of the ordering of A. $(A, <, 1)$ is said to be *discrete* if every $x \neq 1$ has an immediate successor and an immediate predecessor.

Lemma 1.2 *i) If $(A, <, 1)$ expands to a euclidean monoid, then $(A, <, 1)$ is either discrete or dense. If it is discrete, then for every $*$ expanding $(A, <, 1)$, the successor of every $x \in A$ is $a * x$, where a is the successor of 1.*

ii) If $(A, <, 1)$ expands to a radically closed monoid, then $(A, <, 1)$ is dense.

Proof. i) Suppose $(A, <, 1)$ expands to a euclidean monoid $(A, *, <, 1)$. Then either A contains an immediate successor a of 1 or not. Assume the first. Then for every x, $x < a * x$. We claim that $a * x$ is the immediate successor of x. Suppose not and let y such that $a \leq x < y < a * x$. By the euclidean property, there is z such that $y = z * x$. Now $z * x < a * x$ implies $z < a$. But then $z = 1$, or $y = x$, a contradiction. Thus $z \geq a$, whence $y = x * z \geq x * a$,

Assume now that there is no immediate successor of 1 and let $x < y$. Let $y = x * z$. Then $z > 1$ hence there is $1 < u < z$. Then $x < x * u < x * z = y$, that is we can always find an element strictly between x, y.

ii) Let A expand to a radically closed $(A, *, <, 1)$. By the proof of (i) it suffices to show that there is no immediate successor of 1. Let $1 < x$. Then there is y such that $y^2 = x$. By the monotonicity of $*$, $1 < y < y^2 = x$, hence there is no immediate successor of 1. □

Let $otp(A, <, 1)$ denote the *order type* of $(A, <, 1)$. Let also $\omega, \eta, \eta_0, \omega$ be the order types of the nonnegative integers, the rationals and the nonnegative rationals, respectively. For countable A the following is a partial converse to the preceding lemma.

Lemma 1.3 *Let A be countable. Then*

i) If $otp(A, <, 1) = \eta_0$, then $(A, <, 1)$ expands to a euclidean and radically closed ordered monoid.

ii) If $otp(A, <, 1) = \omega$, then $(A, <, 1)$ expands to a euclidean monoid of the form $\{1, a, a^2 \ldots\}$.

Proof. i) Suppose $(A, <, 1)$ is dense. Let $\mathbb{A}$ be the set of real algebraic numbers and let $\mathbb{A}_1 = \{x \in \mathbb{A} : x \geq 1\}$. Take any order-preserving bijection $f : A \to \mathbb{A}_1$ such that $f(1) = 1$ and define $x * y = z \iff f(x) \cdot f(y) = f(z)$ (where $\cdot$ is of course the real multiplication). Then clearly $(A, *, <, 1)$ is euclidean and radically closed.

ii) Let $otp(A, <) = \omega$. Assume 1 has the 0-th place in the ordering. Define $x * y$ as follows: If x is the m-th element and y is the n-th element of A, let $x * y =$ be the $(m + n)$-th element of A. Clearly if a is the successor of 1, then the n-th element is a^n, that is $A = \{1, a, a^2, \ldots\}$, hence A is euclidean.
□

In view of 1.2 and 1.3, it remains to consider the case when $(A, <, 1)$ is countable discrete and nonstandard. We shall treat the problem in the next section.

2 Discrete nonstandard orderings

In this section we deal with countable discretely ordered sets with first element. It will be shown that most of them carry a natural monoidal structure generated by the ordering. It follows that a discretely ordered set $(A, <, 1)$ expands to a euclidean monoid iff it has one of certain concrete order types. Although we finally focus on countable structures, most of the facts shown below hold also for the uncountable. So unless otherwise stated, the sets A considered below will be of any infinite cardinality.

Let ω^* be the reverse order type of ω, hence $otp(\mathbb{Z}) = \omega^* + \omega$. (The symbols $+$, $\cdot$ are used also for addition and multiplication of order types.)

Let $(A, <, 1)$ be a discretely ordered set with least element 1. For every $x \in A$ and $0 < n \in \mathbb{N}$, $x^{(n)}$ denotes the n-th successor of x and $x^{(-n)}$ its n-th predecessor (provided of course $x \geq 1^{(n)}$). We let also $x^{(0)} = x$. x is said to be *nonstandard* if $x > 1^{(n)}$ for every n. Thus for nonstandard x, $x^{(k)}$ is defined for every $k \in \mathbb{Z}$. Moreover for all $x, y \in A$,

$$x = y^{(k)} \iff y = x^{(-k)}, \quad x^{(k)} < x^{(l)} \iff x^{(-k)} > x^{(-l)}. \tag{1}$$

Let $\sim$ be the equivalence on A: $x \sim y$ if $y = x^{(k)}$ for some $k \in \mathbb{Z}$. Define inductively the nested sequence of equivalences $\sim_\alpha$, $\alpha \in On$, on A, their equivalence classes $[x]_\alpha$, and the sets $A_\alpha = A/\sim_\alpha$, as follows:

a) $\sim_0 =$ equality.

b) If α is a limit ordinal and for all $\beta < \alpha$, $\sim_\beta$ are defined, let $\sim_\alpha = \cup_{\beta<\alpha} \sim_\beta$.

c) If $\sim_\alpha$ is defined and $(A_\alpha, <)$ is an infinite discretely ordered set (where $<$ is the obvious ordering of the convex sets $[x]_\alpha$), then

$$x \sim_{\alpha+1} y \iff [x]_\alpha \sim [y]_\alpha.$$

Otherwise $\sim_{\alpha+1}$ is not defined and the definition terminates.

Since $[x]_0 = \{x\}$, we can identify $[x]_0$ with x. Clearly for any limit $\alpha > 0$,

$$[x]_\alpha = \cup\{[x]_\beta : \beta < \alpha\},$$

while for successor $\alpha + 1$,

$$[x]_{\alpha+1} = \cup\{[y]_\alpha : [y]_\alpha \sim [x]_\alpha\} = \cup\{[x]_\alpha^{(k)} : k \in \mathbb{Z}\}, \tag{2}$$

(where $[x]_\alpha^{(k)}$ is the k-th successor of $[x]_\alpha$ in the discrete $(A_\alpha, <)$. Thus $\beta < \alpha \Rightarrow [x]_\beta \subset [x]_\alpha$, so for cardinality reasons the above definition will terminate at some α. Since for limit α, $\sim_\alpha$ is defined provided all $\sim_\beta$ are defined for $\beta < \alpha$, there will be a first α such that $\sim_{\alpha+1}$ is not defined. This will happen because $(A_\alpha, <)$ is either finite or infinite but a non-discrete. Let $\rho = otp(A_\alpha, <)$. In that case α is said to be the *closure* ordinal of A, and the pair (α, ρ) is said to be the *index* of A, and write $ind(A) = (\alpha, \rho)$.

It is not hard to compute the order type of A in terms of its index (α, ρ). Note first that $(A_\alpha, <)$ has always a least element $[1]_\alpha$, so ρ is a type of the form $\rho_0 = 1 + \rho$, where ρ is either finite or infinite non-discrete. Thus

$$A = [1]_\alpha \cup (\cup_{d \in D} [d]_\alpha), \tag{3}$$

where D is a choice set for the $\sim_\alpha$-classes of A, of order type ρ. So

$$otp(A) = otp([1]_\alpha, <) + otp([d]_\alpha, <) \cdot \rho, \tag{4}$$

provided (as we shall see) that for all $d_1, d_2 \in D$, $([d_1]_\alpha, <) \cong ([d_2]_\alpha, <)$. We shall prove this in a roundabout way which gives us much more information about the discrete ordering and especially reveals its monoidal structure. This is the main construction of the paper.

Types and type representations of elements. Let $(A, <, 1)$ be discretely ordered with closure ordinal α. Fix some $x \in A$ and the class $[x]_\alpha$. Fix also a choice function F which for every $y \in [x]_\alpha$, every $\beta < \alpha$ and every $k \in \mathbb{Z}$ picks an element $F(y, \beta, k) \in [y]_\beta^{(k)}$, subject only to the condition that for all y and β, $F(y, \beta, 0) = y$. Consider now finite sets (or even multisets) of pairs of the form

$$\{(\beta_1, k_1), \cdots, (\beta_m, k_m)\},$$

where $\beta_i < \alpha$ and $k_i \in \mathbb{Z}$. We shall refer to such sets as *types*. For every type $\{(\beta_1, k_1), \cdots, (\beta_m, k_m)\}$, and for the fixed x, we shall define (with respect to F) an element

$$x_{\beta_1 \cdots \beta_m}^{k_1 \cdots k_m} \in [x]_\alpha.$$

Keeping x fixed, $x_{\beta_1 \cdots \beta_m}^{k_1 \cdots k_m}$ is the element *represented* by the type $\{(\beta_1, k_1), \cdots, (\beta_m, k_m)\}$. Note that in the above notation β_i and k_i are just sub- and superscripts. Soon however it will become clear that the integers k_i behave like real exponents.

The definition of $x^{k_1\cdots k_m}_{\beta_1\cdots\beta_m}$ is by induction on the cardinality m of the type.

1) $m=1$. For every type $\{(\beta,k)\}$, $x^k_\beta := F(x,\beta,k)$.

2) $m=2$. For every type $\{(\beta,k),(\gamma,l)\}$, let

$$x^{kl}_{\beta\gamma} := \begin{cases} (x^k_\beta)^l_\gamma \text{ if } \beta>\gamma, \\ (x^l_\gamma)^k_\beta \text{ if } \beta<\gamma, \\ x^{k+l}_\beta \text{ if } \beta=\gamma \end{cases}$$

3) In general define $x^{k_1\cdots k_m}_{\beta_1\cdots\beta_m}$ as follows: Let $X=\{(\beta_1,k_1),\cdots,(\beta_m,k_m)\}$ be a type of cardinality $m>1$. For every β occurring in pairs of X let $X{\restriction}\beta=\{(\gamma,k)\in X:\gamma=\beta\}$ and $X(\beta)=\{k:(\beta,k)\in X\}$. Transform X along the following two steps: (a) For every β occurring in pairs of X, replace the subset $X{\restriction}\beta$ of X with the pair $(\beta,\Sigma_{k\in X(\beta)}k)$. (b) Delete from X all pairs of the form $(\beta,0)$. Let $Y=\{(\gamma_1,l_1),\ldots,(\gamma_n,l_n)\}$ be the resulting type with $\gamma_1>\cdots>\gamma_n$. Then let

$$x^{k_1\cdots k_m}_{\beta_1\cdots\beta_m} := (\cdots((x^{l_1}_{\gamma_1})^{l_2}_{\gamma_2})^{l_3}_{\gamma_3}\cdots)^{l_n}_{\gamma_n}.$$

This completes the definition.

Let Γ_x be the set of elements $x^{k_1\cdots k_m}_{\beta_1\cdots\beta_m}$ constructed above. We shall refer to Γ_x as a set of *x-representations.* Also the set $X=\{(\beta_1,k_1),\cdots,(\beta_m,k_m)\}$ is said to be the *type* of the element $y=x^{k_1\cdots k_m}_{\beta_1\cdots\beta_m}$ with respect to x and write

$$tp_x(y)=\{(\beta_1,k_1),\cdots,(\beta_m,k_m)\}.$$

Note. In the above definition one could work as well with any class $[x]_\beta$, instead of $[x]_\alpha$ (α=the closure ordinal). We considered $[x]_\alpha$ simply because of its maximality.

The following are immediate from the definition.

Lemma 2.1 *i) For every permutation s of* $\{1,\ldots,m\}$,

$$x^{k_1\cdots k_m}_{\beta_1\cdots\beta_m} = x^{k_{s(1)}\cdots k_{s(m)}}_{\beta_{s(1)}\cdots\beta_{s(m)}}.$$

ii)

$$x^{k_1\cdots k_m kl}_{\beta_1\cdots\beta_m\gamma\gamma} = x^{k_1\cdots k_m(k+l)}_{\beta_1\cdots\beta_m\gamma}.$$

iii)

$$x^{k_1\cdots k_m 0}_{\beta_1\cdots\beta_m\gamma} = x^{k_1\cdots k_m}_{\beta_1\cdots\beta_m}.$$

For $x, y \in A$ such that $x \in [y]_\alpha$, let $r(x,y) = \text{least}\{\beta : y \in [x]_\beta\}$ be the *rank* of x with respect to y. Clearly $r(x,y) = 0$ iff $x = y$ and if $x \neq y$ and $x \in [y]_\alpha$, $r(x,y)$ is a successor ordinal (since for a limit β, $[y]_\beta = \cup_{\gamma<\beta}[y]_\gamma$).

The main fact concerning type representations of elements is the following.

Proposition 2.2 *Let α be the closure ordinal of A, let $y \in A$ and let Γ_y be a set of y-representations. Then for every $x \in [y]_\alpha$ such that $r(x,y) = \beta + 1$, there are unique ordinals $\beta = \beta_1 > \cdots > \beta_m \neq 0$ and unique integers $k_1, \ldots, k_m \neq 0$, such that $x = y^{k_1 \cdots k_m}_{\beta_1 \cdots \beta_m}$. Moreover $x < y$ iff $k_1 < 0$.*

In particular, for $x \in [1]_\alpha$ and $x \neq 1$, there are unique sequences $\beta_1 > \cdots > \beta_m \neq 0$ and $k_1, \ldots, k_m \neq 0$, such that $x = 1^{k_1 \cdots k_m}_{\beta_1 \cdots \beta_m}$, with $k_1 > 0$.

Proof. Suppose x, y are as stated. Then $x \in [y]_{\beta+1}$, i.e., $[x]_{\beta+1} = [y]_{\beta+1}$, or $[x]_\beta \sim [y]_\beta$, or $[x]_\beta = [y]^{(k_1)}_\beta$ for some unique $k_1 \in Z$. If $k_1 = 0$, then $[x]_\beta = [y]_\beta$, that is, $x \in [y]_\beta$, contrary to the fact that $r(x,y) = \beta + 1$. Moreover $k_1 < 0 \iff [x]_\beta < [y]_\beta$ therefore $k_1 < 0 \iff x < y$. Now we employ the representation $y^{k_1}_\beta$. By definition, $y^{k_1}_\beta = F(y, \beta, k_1) \in [y]^{(k_1)}_\beta$, hence $[y^{k_1}_\beta]_\beta = [y]^{(k_1)}_\beta$. Putting $\beta_1 = \beta$ we have

$$[x]_\beta = [y]^{(k_1)}_{\beta_1} = [y^{k_1}_{\beta_1}]_{\beta_1},$$

or $x \in [y^{k_1}_{\beta_1}]_{\beta_1}$.

If $r(x, y^{k_1}_{\beta_1}) = 0$, it means that $x = y^{k_1}_{\beta_1}$, and the claim is proved. Otherwise $r(x, y^{k_1}_{\beta_1}) = \beta_2 + 1$, with $\beta_2 + 1 \leq \beta_1$, hence $\beta_2 < \beta_1$. Then, using the representation $y^{k_1 k_2}_{\beta_1 \beta_2}$, there is $k_2 \neq 0$ such that

$$[x]_{\beta_2} = [y^{k_1}_{\beta_1}]^{(k_2)}_{\beta_2} = [y^{k_1 k_2}_{\beta_1 \beta_2}]_{\beta_2}.$$

Continuing this process we find ordinals $\beta = \beta_1 < \cdots < \beta_i$ and non-zero integers $k_1, \ldots, k_i$ such that $x \in [y^{k_1 \cdots k_i}_{\beta_1 \cdots \beta_i}]_{\beta_i}$. Since we cannot have an infinite regression of ordinals, the process will terminate at some step m, which means that $r(x, y^{k_1 \cdots k_m}_{\beta_1 \cdots \beta_m}) = 0$, i.e., $x = y^{k_1 \cdots k_m}_{\beta_1 \cdots \beta_m}$.

The second claim follows immediately for $y = 1$. Since for every $x \neq 1$, $x > 1$, it follows that if $x = 1^{k_1 \cdots k_m}_{\beta_1 \cdots \beta_m}$, k_1 must be positive. This completes the proof. □

In clause 3 of the construction of representations we used a reduction of the type X to the type Y. Let us call this reduction *normalization* and the resulting Y the *normal form* of X. Namely a type $X = \{(\beta_1, k_1), \cdots, (\beta_m, k_m)\}$ is said to be *normal* if it satisfies the conditions:

a) $\beta_i = \beta_j \Rightarrow k_i = k_j$, and

b) $k_i \neq 0$.

Then the normal form of X, denoted $n(X)$, is the normal type obtained from X by the normalization procedure described above. Given a representation $x = y^{k_1 \cdots k_m}_{\beta_1 \cdots \beta_m}$, let the *normal type* of x with respect to y be the normal form of $tp_y(x)$, denoted $ntp_y(x)$, i.e.,

$$ntp_y(x) = n(tp_y(x)).$$

The preceding result says that every element of $[y]_\alpha$ has a y-representation. Moreover this representation is *unique* with respect to normal types (i.e., types like those employed in the last proposition). As an immediate corollary of the preceding proposition we have that for any $x_1, x_2 \in [y]_\alpha$ (y fixed), $ntp_y(x_1) = ntp_y(x_2) \Rightarrow x_1 = x_2$. However much more can be said. In fact the normal types of elements of $[y]_\alpha$ determine completely their ordering.

Pairs (β, n) such that $n > 0$ are said to be *positive*. A *positive normal type* is a normal type containing only positive pairs.

Let $<_1$ be the lexicographic ordering of all positive pairs. Using $<_1$, every positive normal type can be identified with a decreasing sequence of positive pairs, so let $<_2$ be the lexicographic ordering of normal positive types.

Let NT be the set of normal types. Clearly every element of NT is of the form

$$\{(\beta_1, n_1), \cdots, (\beta_m, n_m), (\gamma_1, -l_1), \cdots, (\gamma_t, -l_t)\},$$

where $n_i, l_j > 0$. Define the ordering $<_3$ of NT as follows:

$$\{(\beta_1, n_1), \cdots, (\beta_m, n_m), (\gamma_1, -l_1), \cdots, (\gamma_t, -l_t)\} <_3 \\ \{(\delta_1, p_1), \cdots, (\delta_q, p_q), (\epsilon_1, -r_1), \cdots, (\varepsilon_s, -r_s)\}$$

iff

$$\{(\beta_1, n_1), \cdots, (\beta_m, n_m), (\varepsilon_1, r_1), \cdots, (\varepsilon_s, r_s)\} <_2 \\ \{(\delta_1, p_1), \cdots, (\delta_q, p_q), (\gamma_1, l_1), \cdots, (\gamma_t, l_t)\},$$

where in the last inequality the types are positive normal (after normalization if needed).

Proposition 2.3 *For all $x_1 \neq x_2 \in [y]_\alpha$,*

$$x_1 < x_2 \iff ntp_y(x_1) <_3 ntp_y(x_2).$$

Proof. The proof is easy though tedious and details are left to the reader. We just sketch the following two steps. For every x, β, γ, and $m, n > 0$,

i) $x_\beta^m < x_\gamma^n \iff (\beta, m) <_1 (\gamma, n)$,

and

ii) $x_\beta^{-m} < x_\gamma^{-n} \iff x_\beta^m > x_\gamma^n$.

(i): It suffices to show that $(\beta, m) <_1 (\gamma, n) \Rightarrow x_\beta^m < x_\gamma^n$. Recall that $x_\beta^m = F(x, \beta, m) \in [x]_\beta^{(m)}$. Let $(\beta, m) <_1 (\gamma, n)$. Then either $\beta < \gamma$ or $\beta = \gamma$ and $m < n$. Assume the first. Then $x_\beta^m \in [x]_\beta^{(m)} \subset [x]_{\beta+1} \subseteq [x]_\gamma$. Since $n > 0$, $[x]_\gamma < [x]_\gamma^{(n)}$, the last two sets are disjoint and x_γ^n belongs to the last one, so $x_\beta^m < x_\gamma^n$.

Let now $\beta = \gamma$ and $m < n$. Again the claim follows from the fact that $[x]_\beta^{(m)}$, $[x]_\gamma^{(n)}$ are disjoint and $[x]_\beta^{(m)} < [x]_\gamma^{(n)}$.

ii): Immediate from the definition of x_β^k and the fact that

$$[x]_\beta^{(-m)} < [x]_\beta^{(-n)} \iff [x]_\beta^{(m)} > [x]_\beta^{(n)}.$$

□

Various corollaries follow from 2.2 and 2.3. A first is the specification of the order type of every discretely ordered set $(A, <, 1)$ of any cardinality.

Corollary 2.4 *i) Let $(A, <_A, 1_A)$, $(B, <_B, 1_B)$ be any discretely ordered sets with closure ordinals α^A, α^B respectively, and let $\alpha^A \leq \alpha^B$. Then for any $\beta \leq \alpha^A$, $([1_A]_\beta, <_A) \cong ([1_B]_\beta, <_B)$, and for every $x \in A$, $y \in B$ such that $[x]_\beta > [1_A]_\beta$ and $[y]_\beta > [1_B]_\beta$, $([x]_\beta, <_A) \cong ([y]_\beta, <_B)$.*

ii) Also if $[x]_\beta > [1]_\beta$ and we set $[x^+]_\beta = \{y \in [x]_\beta : y \geq x\}$, then $([x^+]_\beta) \cong ([1]_\beta, <)$.

iii) If $ind(A) = (\alpha, \rho_0)$, where $\rho_0 = 1 + \rho$, then

$$otp(A) = otp([1]_\alpha) + otp([x]_\alpha) \cdot \rho,$$

for any $[x]_\alpha > [1]_\alpha$.

Proof. i) Consider the sets of representations Γ_x and Γ_y respectively, and for every $z \in [x]_\beta$ let $h(z) \in [y]_\beta$ be such that $ntp_x(z) = ntp_y(h(z))$. By 2.2, h is a bijection, and by 2.3

$$z_1 <_A z_2 \iff ntp_x(z_1) <_3 ntp_x(x_2) \iff$$

$$ntp_y(h(z_1)) <_3 ntp_y(h(z_2)) \iff h(z_1) <_B h(z_2).$$

ii) Recall from 2.2 that $z \in [x^+]_\beta$ iff $ntp_x(z) = \{(\beta_1, n_1), \ldots\}$, with $n_1 \geq 0$, and these are also exactly the types of the elements of $[1]_\beta$. Thus the similarity is obtained in the obvious way.

iii) Immediate from (i) and (4). □

By 2.4 (i), the order types of the sets $[1]_\beta$ and $[x]_\beta$, are independent of the x and the particular set A. So let us denote

$$\zeta_\beta := otp([x]_\beta, <), \text{and } \zeta_\beta/2 := otp([1]_\beta, <). \tag{5}$$

The notation $\zeta_\beta/2$ is to suggest that, as follows from 2.3 (ii), $[1]_\beta$ is (the right) half of the set $[x]_\beta$, when $[x]_\beta > [1]_\beta$.

Now clearly for finite n as well as for successor ordinals $\beta + n$ we have

$$\zeta_n = (\omega^* + \omega)^n, \text{ and } \zeta_{\beta+n} = (\omega^* + \omega)^n \cdot \zeta_\beta,$$

where of course $(\omega^* + \omega)^n$ is the order type of the set $\mathbb{Z}^n$ ordered antilexicographically. For limit β, however, ζ_β are primitive order types, like η, λ (the order type of the reals), etc. (The attempt to define ζ_β as the order type of $\mathbb{Z}^\beta$ does not work, since there is no antilexicographic ordering of this set, except for finite β.) Thus by 2.4 (iii) we finally have the following expressions for the order types of discretely order sets with least element (note that $\zeta_1/2 = \omega$):

Corollary 2.5 *Let $ind(A) = (\alpha, \rho_0)$, where $\rho_0 = 1 + \rho$. Then*

$$otp(A) = \zeta_\alpha/2 + \zeta_\alpha \cdot \rho.$$

The type representation of the elements of any class $[x]_\alpha$ obviously suggests a monoidal (partial) operation defined as follows.

We shall use the 1-representations 1^1_β, $\beta < \alpha$, as "generators". Let us first rename them setting

$$c_\beta := 1^1_\beta.$$

We call c_β, $\beta < \alpha$, *generators* of the set $[1]_\beta$. Define the operation "$\circ$" between any finite number of such generators with integer exponents (when this makes sense) and a particular x by setting

$$x \circ c^{k_1}_{\beta_1} \circ \cdots \circ c^{k_m}_{\beta_m} := x^{k_1 \cdots k_m}_{\beta_1 \cdots \beta_m}. \tag{6}$$

In view of this operation we can easily establish the following.

Proposition 2.6 *Let $(A, <, 1)$ be a discretely ordered set with $ind(A) = (\alpha, \rho_0)$, where $\rho_0 = 1 + \rho$. If ρ is the order type of a set carrying the structure of a monoid (resp.euclidean monoid) (including the trivial one $\{1\}$), then $(A, <, 1)$ expands to a monoid (resp.euclidean monoid).*

Proof. Since $ind(A) = (\alpha, \rho_0)$, A is the disjoint union of the classes $[x]_\alpha$, whose ordering is of type ρ. Choose an element d from each class, with $d = 1$ for the class $[1]_\alpha$, and form the set D. Then $otp(D, 1, <) = \rho_0$, and by the assumption, D may expand to a monoid $(D, 1, \bullet, <)$. Now for each $x \in A$, there is a unique $d \in D$ such that $x \in [d]_\alpha$, hence, by 2.2, $x = d^{k_1 \cdots k_m}_{\beta_1 \cdots \beta_m}$, or, by (6), $x = d \circ c^{k_1}_{\beta_1} \circ \cdots \circ c^{k_m}_{\beta_m}$.

Define the operation $*$ on A as follows: If $x = d_1 \circ c^{k_1}_{\beta_1} \circ \cdots \circ c^{k_m}_{\beta_m}$ and $y = d_2 \circ c^{l_1}_{\gamma_1} \circ \cdots \circ c^{l_n}_{\gamma_n}$, let

$$x * y := (d_1 \bullet d_2) \circ c^{k_1}_{\beta_1} \circ \cdots \circ c^{k_m}_{\beta_m} \circ c^{l_1}_{\gamma_1} \circ \cdots \circ c^{l_n}_{\gamma_n}.$$

It is easy to verify that $(A, *, <, 1)$ is a monoid, and further, if D is euclidean, then so is A. $\square$

We show the converse of proposition 2.6 for the euclidean case.

Lemma 2.7 *Let A be a discrete euclidean monoid (of any cardinality) with $ind(A) = (\alpha, \rho_0)$. Then for every $\beta \leq \alpha$, A_β has a euclidean expansion. Consequently $\rho_0 = otp(A_\alpha)$ is the order type of some euclidean monoid.*

Proof. We just transfer $*$ to A_β setting $[x]_\beta * [y]_\beta = [x*y]_\beta$. The euclidean property of A is needed only to prove that this operation is well defined. E.g. let $\beta = 1$, and let us write $[x]$ instead of $[x]_1$. Then, by lemma 1.2 (i), if a is the successor of 1, the successor of every $x \in A$ is $a * x$, hence $x \sim y$ iff $x = a^k * y$ for some $k \in \mathbb{Z}$. Thus if $x_1 \sim x_2$ and $y_1 \sim y_2$, then $x_1 = a^k * x_2$, $y_1 = a^l * y_2$, hence $[x_1 * y_1] = [a^k * x_2 * a^l * y_2] = [x_2 * y_2 * a^{k+l}] = [x_2 * y_2]$. So $*$ is well defined. Then use induction on β. At limit β we just take unions. Details are easy and left to the reader. □

Note that without the euclidean property the shifting of $*$ to the classes is not possible. For instance, let N be a model of PA. Then the monoid $(N, \cdot, <, 1)$ is non-euclidean, that is, $2x$ is not the successor of x. Here obviously $x_1 \sim x_2$ and $y_1 \sim y_2$ does not imply $[x_1 \cdot y_1] = [x_2 \cdot y_2]$.

Let us summarize the expandability conditions obtained above.

Corollary 2.8 *Let $(A, <, 1)$ be a countable ordered set. Then*

i) A expands to a euclidean monoid iff $otp(A)$ is one of the following: η_0, ω, $\zeta_\alpha/2$, $\zeta_\alpha/2 + \zeta_0 \cdot \eta$.

ii) A expands to a radically closed monoid iff $otp(A) = \eta_0$.

Proof. i) Suppose A expands to a euclidean monoid. Then, by lemma 1.2 (i), A is either dense or discrete. In the first case, since A is countable, $otp(A) = \eta_0$. In the second case either $otp(A) = \omega$, or, A is nonstandard with index (α, ρ_0), where $\rho_0 = otp(A_\alpha)$, is either 1, or a non-discrete order type. But A_α, by 2.7, has a euclidean expansion, therefore it is dense, so in the latter case $\rho_0 = \eta_0$. Thus either $\rho_0 = 1$ or $\rho_0 = \eta_0$. By 2.5, in the first case $otp(A) = \zeta_\alpha/2$, and in the second case $otp(A) = \zeta_\alpha/2 + \zeta_\alpha \cdot \eta$. Thus the possible order types of A are η_0, ω, $\zeta_\alpha/2$, $\zeta_\alpha/2 + \zeta_0 \cdot \eta$.

Conversely. If $otp(A) = \eta_0$ or ω, then, by 1.3, A expands to a euclidean monoid. If $otp(A) = \zeta_\alpha/2$, or $\zeta_\alpha/2 + \zeta_\alpha \cdot \eta$, then $\rho_0 = 1$ or η_0, which are the order types of the euclidean monoids $\{1\}$ and $\mathbb{Q}_1 = \{x \in \mathbb{Q} : x \geq 1\}$. Hence by 2.6, A expands to a euclidean monoid.

ii) By lemmas 1.2 and 1.3. □

In view of lemma 1.1 and the fact that we may have $+$ for $*$ and 0 for 1, we get the following.

Corollary 2.9 *Let $(A, <, 0)$ be a countable discretely ordered set with least element* 0*. Then there is a discretely ordered group G such that $A = G^+$, iff $otp(A) = \omega$, $\zeta_\alpha/2$, or $\zeta_\alpha/2 + \zeta_\alpha \cdot \eta$.*

Proof. By 1.1 and taking $* = +$, there is a discretely ordered group G such that $A = G^+$, iff $(A, <, 0)$ expands to a euclidean monoid $(A, +, <, 0)$. Thus the claim follows from 2.8. □

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