Uncountable cardinals have the same monadic $\forall_1^1$ positive theory over large sets

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Abstract

We show that uncountable cardinals are indistinguishable by sentences of the monadic second-order language of order of the form $(\forall X)\phi(X)$ and $(\exists X)\phi(X)$, for ϕ positive in X and containing no set-quantifiers, when the set variables range over large (=cofinal) subsets of the cardinals. This strengthens the result of Doner-Mostowski-Tarski [3] that $(\kappa, \in)$, $(\lambda, \in)$ are elementarily equivalent when κ, λ are uncountable. It follows that we can consistently postulate that the structures $(2^\kappa, [2^\kappa]^{>\kappa}, <)$, $(2^\lambda, [2^\lambda]^{>\lambda}, <)$ are indistinguishable with respect to $\forall_1^1$ positive sentences. A consequence of this postulate is that $2^\kappa = \kappa^+$ iff $2^\lambda = \lambda^+$ for all infinite κ, λ. Moreover, if large cardinals do not exist, GCH is true.

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1 Preliminaries

Let $\mathcal{L} = \{\prec\}$ be the first-order language of order with equality. The following is a special case of a much more general result proved in [3]:

Theorem 1.1 *For any uncountable cardinals κ, λ $(\kappa, <) \equiv_{\mathcal{L}} (\lambda, <)$, where $<$ is the natural ordering of ordinals (i.e., $<=\in$).*

Proof. The reader is warned that there is a mistake in the definition of *congruence modulo* ω^ω given in [3], p. 51, as Professor Doner kindly informed me. The correct definition, that can be found e.g. in [2], is as follows: The ordinals α, β are congruent modulo ω^ω, if there are ξ, η and $\delta < \omega^\omega$ such that $\alpha = \omega^\omega \cdot \xi + \delta$, $\beta = \omega^\omega \cdot \eta + \delta$ and either $\xi = \eta = 0$ or both $\xi \neq 0$ and $\eta \neq 0$. Since for all cardinals $\kappa, \lambda > \omega$, $\kappa = \omega^\omega \cdot \kappa$ and $\lambda = \omega^\omega \cdot \lambda$, it follows that κ, λ are congruent modulo ω^ω. So the claim follows from Corollary 44 of [3]. $\dashv$

Henceforth we write $(\kappa, <) \equiv (\lambda, <)$ instead of $(\kappa, <) \equiv_{\mathcal{L}} (\lambda, <)$. Let x, y, $z, \ldots$ range over the individual variables of $\mathcal{L}$. The *monadic (second-order)* extension of $\mathcal{L}$, denoted by $\mathcal{L}_{mon}$, is $\mathcal{L}$ augmented with $\in$ and set variables $X, Y, Z, \ldots$. Hence $\mathcal{L}_{mon}$ has $x \in X$ as additional atoms. The standard interpretations of $\mathcal{L}_{mon}$ are the structures $(A, \mathcal{P}(A), <)$ where $(A, <)$ is an ordered set and $\mathcal{P}(A)$ is the set of its subsets. The expressive power of $\mathcal{L}_{mon}$ is huge compared to that of $\mathcal{L}$. For instance it is shown in [1] that each of the structures $(\omega_n, <)$, $n \geq 0$, is finitely axiomatizable and categorical with respect to $\mathcal{L}_{mon}$ (under the standard interpretation). That is, for every cardinal ω_n, the ordered set $(\omega_n, <)$ is characterized, up to isomorphism, by a sentence of $\mathcal{L}_{mon}$.

However there are more general interpretations of $\mathcal{L}_{mon}$ besides the standard one. These are of the form $(A, \mathcal{A}, <)$, where $\mathcal{A} \subset \mathcal{P}(A)$. $\mathcal{A}$ may be some class of "small" sets, i.e., an ideal of $\mathcal{P}(A)$, or, in the opposite direction, a class of "large" sets. Given cardinals $\kappa < \lambda$, let us fix the following notation:

$$[\lambda]^{<\kappa} = \{X \subseteq \lambda : |X| < \kappa\}, \quad [\lambda]^{\kappa} = \{X \subseteq \lambda : |X| = \kappa\},$$

$$[\lambda]^{>\kappa} = \{X \subseteq \lambda : |X| > \kappa\}, \quad Cof(\lambda) = \{X \subseteq \lambda : X \text{ is cofinal in } \lambda\}.$$

Clearly if κ is regular $[\kappa]^\kappa = Cof(\kappa)$. We often call the elements of $[\kappa]^{<\kappa}$ *small* subsets of κ. Accordingly we call the sets of $[\kappa]^\kappa$ and $Cof(\kappa)$ *large* and *cofinal*, respectively. Then X is *co-small*, i.e., $-X \in [\kappa]^{<\kappa}$, iff $X \cap Y \neq \emptyset$ for every $Y \in [\kappa]^\kappa$.

The aim of this paper is to strengthen theorem 1.1 above as follows: For any uncountable κ, λ, the second-order structures $(\kappa, Cof(\kappa), <)$ and $(\lambda, Cof(\lambda), <)$ are indistinguishable with respect to $\forall_1^1$ and $\exists_1^1$ positive formulas (defined below).

Both positive and $\forall_1^1$ are natural classes of formulas. As an application, observe that the instance $2^\kappa = \kappa^+$ of the GCH can be expressed by the truth of the $\forall_1^1$ positive formula $\forall X(X$ is cofinal) in the structure $(2^\kappa, [2^\kappa]^{>\kappa}, <)$. By the main result of the paper, we can consistently postulate (see section 4) that for all infinite κ, λ, the structures $(2^\kappa, [2^\kappa]^{>\kappa}, <)$, $(2^\lambda, [2^\lambda]^{>\lambda}, <)$ are indistinguishable with respect to $\forall_1^1$ positive formulas.

Concerning the optimality of the result: We do not know whether $(\kappa, Cof(\kappa), <)$ and $(\lambda, Cof(\lambda), <)$ are indistinguishable with respect to all $\forall_1^1$ and $\exists_1^1$ formulas. Also we do not know if the structures $(\kappa, \mathcal{P}(\kappa), <)$ and $(\lambda, \mathcal{P}(\lambda), <)$ are indistinguishable either with respect to all $\forall_1^1$ and $\exists_1^1$ formulas, or with respect to positive $\forall_1^1$ and $\exists_1^1$ formulas. What we do know is that the structures $(\omega_1, Cof(\omega_1), <)$ and $(\omega_2, Cof(\omega_2), <)$, are distinguishable by a formula of the form $(\forall x)(\exists X)\psi$, where ψ has no second-order quantifiers but is not positive. And analogously for the structures concerning ω_m, ω_n (see the last section).

We come to the definition of positive formulas.

Definition 1.2 Let $\phi(X)$ be a formula $\mathcal{L}_{mon}$ with at most one set variable X. $\phi(X)$ is *normal* if it has no set quantifiers. A normal formula ϕ is *positive in the set variable X* (or just *positive*) if it belongs to the smallest class of formulas which (i) contains all formulas of $\mathcal{L}$, (ii) contains the atomic formulas $x \in X$ and (iii) is closed under the logical operations $\wedge$, $\vee$, and the first-order quantifiers $\exists$ and $\forall$.

The following result is due to Moschovakis [6]:

Lemma 1.3 (Moschovakis) *Let $\phi(X)$ be a positive in X formula of $\mathcal{L}_{mon}$. Then there is a quantifier-free and X-free formula $\theta(\overline{w}, u)$, where $\overline{w} = (w_1, \ldots, w_n)$, and a string of quantifiers $\overline{Q} = (Q_1, \ldots, Q_n)$ such that, for every structure $(A, \mathcal{P}(A), <)$ and every $Z \neq A$,*

$$\phi(Z) \iff (\overline{Q}\overline{w})(\forall u)(\theta(\overline{w}, u) \Rightarrow u \in Z)$$

holds in $(A, \mathcal{P}(A), <)$.

Proof. See [6], p. 59. ⊣

In view of the above we may assume that every positive in X formula has the form

$$\phi(X) \equiv (\overline{Q}\overline{w})(\forall u)(\theta(\overline{w}, u) \Rightarrow u \in X). \tag{1}$$

We have only to make sure that $X \neq A$ when using the above form in a structure $(A, <)$. We shall refer to (1) as the *canonical form* of ϕ.

We shall further restrict ourselves mainly to $\forall^1_1$ *positive formulas* of $\mathcal{L}_{mon}$, i.e., those (equivalent to one) with prenex form $(\forall X)\phi$, where ϕ is positive in X.

2 The main theorem

From now on we shall be dealing with second-order structures of the form $(\kappa, Cof(\kappa), <)$, where κ is an uncountable cardinal. If κ is regular this structure is identical to $(\kappa, [\kappa]^\kappa, <)$. Otherwise $[\kappa]^\kappa \subseteq Cof(\kappa)$. So for economy let us abbreviate $(\kappa, <)$ by $[[\kappa]]$ and $(\kappa, Cof(\kappa), < \ )$ by $[[\kappa^2]]$, i.e., let

$$[[\kappa]] := (\kappa, <), \qquad [[\kappa^2]] := (\kappa, Cof(\kappa), <).$$

This section contains the proof the following:

Theorem 2.1 (Main Theorem) *For all uncountable cardinals* κ, λ,

$$[[\kappa^2]] \equiv^{\forall^1_1}_{pos} [[\lambda^2]],$$

i.e.,

$$[[\kappa^2]] \models (\forall X)\phi(X) \iff [[\lambda^2]] \models (\forall X)\phi(X),$$

for every positive $\phi(X)$.

Plan of the proof. To help the reader let us describe briefly the general plan of the prof. Let

$$\phi(X) \equiv (\overline{Q}\overline{w})(\forall u)(\theta(\overline{w}, u) \Rightarrow u \in X)$$

be a positive formula in its canonical form. Our aim is to show that $(\forall X)\phi(X)$ is absolute between $[[\kappa^2]]$ and $[[\lambda^2]]$. The proof of this fact will be split into several cases corresponding to various syntactic forms of $\phi(X)$, namely forms of $\overline{Q}\overline{w}$ and θ. The totality of cases of course constitute a partition of truth. In each case we reduce the truth of $(\forall X)\phi(X)$ in $[[\kappa^2]]$ to that of a first-order formula in $[[\kappa]]$, which, in view of Theorem 1.1, means that $(\forall X)\phi(X)$ is absolute between $[[\kappa^2]]$ and $[[\lambda^2]]$. Some of the cases in question are treated as Lemmas 2.2, 2.8 and 2.9 below (the rest of the Lemmas provide technical tools for the treatment of θ). After Lemma 2.9, we concentrate on three remaining cases which are treated explicitly as Cases 1, 2 and 3 with several subcases. After exhausting them all, we give the final proof in which we simply summarize the line of thought we previously followed through the lemmas and cases.

To start with, given κ, it is convenient to rewrite $[[\kappa^2]] \models \phi(X)$ as follows: We set, for every $\overline{w} \in \kappa$,

$$R_\theta^\kappa(\overline{w}) = \{u \in \kappa : [[\kappa]] \models \theta(\overline{w}, u)\}.$$

Then

$$(\forall u)(\theta(\overline{w}, u) \Rightarrow u \in X) \iff R_\theta^\kappa(\overline{w}) \subseteq X,$$

so, over κ, the following identification will often be used in the sequel:

$$\phi(X) \equiv (\overline{Q}\overline{w})[R_\theta^\kappa(\overline{w}) \subseteq X]. \tag{2}$$

We consider first the following easy case:

Lemma 2.2 *Let $\overline{Q}$ contain no existential quantifier, i.e., $\overline{Q} = \overline{\forall}$. Then for all uncountable κ, λ,*

$$[[\kappa^2]] \models (\forall X)\phi(X) \iff [[\lambda^2]] \models (\forall X)\phi(X).$$

Proof. Suppose $\overline{Q} = \overline{\forall}$. Then, by (2) above,

$$[[\kappa^2]] \models (\forall X)\phi(X) \iff [[\kappa^2]] \models (\forall X)(\overline{\forall}\overline{w})[R_\theta^\kappa(\overline{w}) \subseteq X].$$

Now it is easy to see that

$$[[\kappa^2]] \models (\forall X)(\overline{\forall}\overline{w})[R_\theta^\kappa(\overline{w}) \subseteq X] \iff [[\kappa]] \models (\overline{\forall}\overline{w})[R_\theta^\kappa(\overline{w}) = \emptyset]. \tag{3}$$

Direction "$\Leftarrow$" of (3) is immediate: If $(\overline{\forall}\overline{w})[R_\theta^\kappa(\overline{w}) = \emptyset]$ holds in $[[\kappa]]$, then for every cofinal X, $(\overline{\forall}\overline{w})[R_\theta^\kappa(\overline{w}) \subseteq X]$ holds in $[[\kappa^2]]$, hence so does $(\forall X)(\overline{\forall}\overline{w})[R_\theta^\kappa(\overline{w}) \subseteq X]$. Conversely, suppose $[[\kappa^2]] \models (\forall X)(\overline{\forall}\overline{w})[R_\theta^\kappa(\overline{w}) \subseteq X]$. Pick any two disjoint cofinal Z_1, Z_2. Then for any tuple $\overline{a} \in \kappa$, we have $R_\theta^\kappa(\overline{a}) \subseteq Z_1$ and $R_\theta^\kappa(\overline{a}) \subseteq Z_2$. Since $Z_1 \cap Z_2 = \emptyset$, it follows that $R_\theta^\kappa(\overline{a}) = \emptyset$. Since this holds for every $\overline{a} \in \kappa$, $[[\kappa]] \models (\overline{\forall}\overline{w})[R_\theta^\kappa(\overline{w}) = \emptyset]$ and (3) is proved. Now by (3) and theorem 1.1, for uncountable κ, λ we have:

$$[[\kappa^2]] \models (\forall X)\phi(X) \iff [[\kappa]] \models (\overline{\forall}\overline{w})[R_\theta^\kappa(\overline{w}) = \emptyset] \iff$$

$$[[\lambda]] \models (\overline{\forall}\overline{w})[R_\theta^\lambda(\overline{w}) = \emptyset] \iff [[\lambda^2]] \models (\forall X)\phi(X).$$

$\dashv$

If $\overline{Q}$ in the above canonical form of $\phi(X)$ contains existential quantifiers, then a more thorough analysis of the structure of θ is needed. $\theta(\overline{w}, u)$ is a Boolean combination of atomic formulas of $\mathcal{L} = \{\prec\}$. Therefore θ is a Boolean combination of atoms $t \prec s$ and $t = s$, where $t, s \in \{u, w_1, \ldots, w_n\}$. But over a linearly ordered set the formulas $t \not\prec s$ and $t \neq s$ are equivalent to $s \prec t \ \vee \ s = t$ and $t \prec s \ \vee \ s \prec t$, respectively, therefore θ is a $\{\vee, \wedge\}$-combination of atoms $t \prec s$ and $t = s$. So $\theta(\overline{w}, u)$ has the following disjunctive normal form

$$\theta(\overline{w}, u) = \bigvee_{1 \leq p \leq k} \sigma_p(\overline{w}, u), \tag{4}$$

where each σ_p is a conjunction of atoms $t \prec t$, $t = s$. We call σ_p's *clauses*. A clause is consistent if it does not contain contradictory atoms, e.g. of the form $t \prec s \ \wedge \ s \preceq t$, or $t \prec s \ \wedge \ t = s$. We may assume that each σ_p is consistent (otherwise we drop it).

Definition 2.3 A clause σ_p is said to be *full* if for any two variables t, s occurring in it, σ_p proves one of the formulas $t \prec s$, $s \prec t$, $t = s$. A *filling* of σ_p is a formula σ'_p which is consistent, full, contains the same variables as σ_p, and $\sigma'_p \vdash \sigma_p$. (That is, σ'_p contains some extra atoms that guarantee completeness.)

If σ_p is a full clause containing the variables $t_1, \ldots, t_m \in \{u, w_1, \ldots, w_n\}$, then it is clear that (by renaming t_i if necessary) σ_p entails (and is equivalent to) a total ordering of t_i of the form $t_1 \preceq t_2 \preceq \cdots \preceq t_m$ (where of course $\preceq$ means $\prec$ or $=$). More precisely the preceding relation is written $t_1 \ R \ t_2 \ R \cdots R \ t_m$, where R is either $\prec$ or $=$. Thus we have shown clause (i) of the following (clause (ii) is straightforward).

Lemma 2.4 *(i) If σ_p is complete, then it is equivalent to a conjunction of the form*

$$t_1 \ R \ t_2 \ R \cdots R \ t_m,$$

where R is either $\prec$ or $=$ and all t_i are distinct variables of $\{u, w_1, \ldots, w_n\}$.

(ii) Every (consistent) σ_p has a filling, and if σ'_{pi}, $i \leq l$, are all the distinct completions of σ_p, then $\sigma_p = \bigvee_{i \leq l} \sigma_{pi}$.

In view of 2.4 (ii), we may assume that every clause σ_p in the analysis (4) of θ is full, hence, by 2.4 (i), it is of the form

$$t_1 \; R \; t_2 \; R \cdots R \; t_m.$$

σ_p is said to be a *u-clause* if it contains the variable u. Otherwise it is called *u-free*. If δ denotes the disjunction of all u-free clauses, then $\neg\theta$ is written

$$\theta(\overline{w}, u) \equiv \bigvee_{1 \le p \le k} \sigma_p(\overline{w}, u) \;\vee\; \delta(\overline{w}), \tag{5}$$

where each σ_p is a u-clause. Further, each u-clause σ_p consists of u- and u-free atoms. Writing ζ_p and η_p for the conjunction of u- and u-free atoms of σ_p respectively, we have

$$\sigma_p(\overline{w}, u) \equiv \zeta_p(\overline{w}, u) \wedge \eta_p(\overline{w}).$$

A u-clause $\sigma_p(\overline{w}, u)$ is said to be *equational in u* (or just *equational*) if it contains an identity $u = t$.

Now if σ_p is equational we may assume that ζ_p consists of a single equation $u = t$, i.e., $\zeta_p \equiv (u = t_{k_p})$. Indeed, first we may assume that σ_p contains only one u-equation. Otherwise we shall have the sequence

$$t_1 \; R \; t_2 \; R \cdots t_i = u = t_j \; R \cdots R \; t_m,$$

which can be simply rewritten as

$$t_1 \; R \; t_2 \; R \cdots R \; t_i = t_j = u \; R \cdots R \; t_m,$$

or

$$t_1 \; R \; t_2 \; R \cdots R \; u = t_i = t_j \; R \cdots R \; t_m.$$

If σ_p contains besides a u-literal which is not an equation, say $u = t_1 \wedge u \prec t_2$, then $(u = t_1 \wedge u \prec t_2) \equiv (u = t_1 \wedge t_1 \prec t_2)$. So ζ_p can be taken to be a single equation and thus an equational σ_p has the form

$$\sigma_p \equiv (u = t_{k_p}) \wedge \eta_p. \tag{6}$$

On the other hand, if a u-clause σ_p is non-equational, then u occurs only in inequalities, so σ_p will be of the form

$$t_1 \; R \; t_2 \; R \cdots R \; t_i \prec u \prec t_j \; R \cdots R \; t_m,$$

or

$$u \prec t_1 \ R \ t_2 \ R \cdots R \ t_m,$$

or

$$t_1 \ R \ t_2 \ R \cdots R \ t_m \prec u.$$

Therefore every non-equational σ_p has the form

$$\sigma_p \equiv (t_{i_p} \prec u \prec t_{j_p}) \wedge \eta_p, \tag{7}$$

where one of the t_{i_p}, t_{j_p} may be missing. For every equational $\sigma_p(\overline{w}, u)$ and for a given cardinal κ, let

$$E_p^\kappa(\overline{w}) = \{u : [[\kappa]] \models \sigma_p(\overline{w}, u)\}$$

(E from "equation"). Similarly for non-equational $\sigma_p(\overline{w}, u)$ and κ, let

$$I_p^\kappa(\overline{w}) = \{u : [[\kappa]] \models \sigma_p(\overline{w}, u)\}$$

(I from "interval"). Finally, let

$$D^\kappa(\overline{w}) = \{u : [[\kappa]] \models \delta(\overline{w})\}.$$

By the above analysis we get immediately the following:

Lemma 2.5 *(i) If σ_p is equational, then $E_p^\kappa(\overline{w})$, is a singleton or $\emptyset$, namely,*

$$E_p^\kappa(\overline{w}) = \{u : u = t_{k_p} \ \wedge \eta_p(\overline{w})\},$$

where $t_{k_p} \in \{w_1, \ldots, w_n\}$.

(ii) If σ_p is non-equational, then $I_p^\kappa(\overline{w})$ is an interval (maybe empty), namely

$$I_p^\kappa(\overline{w}) = \{u : t_{i_p} \prec u \prec t_{i_p} \ \wedge \eta_p(\overline{w})\},$$

where and t_{i_p}, $t_{j_p} \in \{w_1, \ldots, w_n\}$ and one of the t_{i_p}, t_{j_p} may be missing.

(iii) $D^\kappa(\overline{w})$ is either κ or $\emptyset$.

(iv) $R_\theta^\kappa(\overline{w}) = \bigcup_{1 \le p \le r} E_p^\kappa(\overline{w}) \cup \bigcup_{r < p \le k} I_p^\kappa(\overline{w}) \cup D^\kappa(\overline{w})$, for some $r \le k$.

Lemma 2.6 *Let $[[\kappa^2]] \models (\forall X)\phi(X)$. Then*

$[[\kappa^2]] \models (\forall X)(\overline{Q}\overline{w})[R_\theta^\kappa(\overline{w})$ *contains no non-trivial interval* $\wedge \ R_\theta^\kappa(\overline{w}) \subseteq X]$.

Proof. Note first that "$R^\kappa_\theta(\overline{w})$ contains no non-trivial interval" is the informal formulation of a formula of $\mathcal{L}$. Assume the contrary, i.e., that there is a cofinal set Z such that

$$[[\kappa^2]] \models (\overline{Q}'\overline{w})[R^\kappa_\theta(\overline{w}) \text{ contains a non-trivial interval } \vee\ R^\kappa_\theta(\overline{w}) \not\subseteq Z],$$

where $\overline{Q}'$ is the dual string of $\overline{Q}$. But then we can get a cofinal $Y \subseteq Z$ that contains no non-trivial interval (e.g. the limit points of the ordering $(Z,<)$). So, clearly, the sentence "$R^\kappa_\theta(\overline{w})$ contains a non-trivial interval" implies $R^\kappa_\theta(\overline{w}) \not\subseteq Y$ and also $R^\kappa_\theta(\overline{w}) \not\subseteq Z$ implies $R^\kappa_\theta(\overline{w}) \not\subseteq Y$. Therefore the above disjunction implies

$$[[\kappa^2]] \models (\overline{Q}'\overline{w})[R^\kappa_\theta(\overline{w}) \not\subseteq Y].$$

It follows that $[[\kappa^2]] \models (\exists X)(\overline{Q}'\overline{w})[R^\kappa_\theta(\overline{w}) \not\subseteq X]$. Since

$$\phi(X) \equiv (\overline{Q}\overline{w})[R^\kappa_\theta(\overline{w}) \subseteq X],$$

we get $[[\kappa^2]] \models (\exists X)\neg\phi(X)$, which contradicts our assumption. ⊣

Lemma 2.7 *Let* $\phi(X) \equiv (\overline{Q}\overline{w})(\forall u)(\theta(\overline{w},u) \Rightarrow u \in X)$*, and let* $[[\kappa^2]] \models (\forall X)\phi(X)$*. Then there is a quantifier free* $\theta^*(\overline{w},\overline{z},u)$*, and a string of quantifiers* $\overline{Q}^*$ *such that if* $\phi^*(X) \equiv (\overline{Q}^*\overline{w}\overline{z})(\forall u)(\theta^*(\overline{w},u) \Rightarrow u \in X)$*, then* $\phi(X)$ *and* $\phi^*(X)$ *are equivalent over* $[[\kappa^2]]$ *and*

$$[[\kappa^2]] \models (\forall X)(\overline{Q}^*\overline{w}\overline{z})[R^\kappa_{\theta^*}(\overline{w},\overline{z}) \textit{ contains no non-trivial interval} \wedge$$

$$R^\kappa_{\theta*}(\overline{w},\overline{z}) \subseteq X],$$

and $\theta^*(\overline{w},\overline{z},u) = \bigvee_q \tau_q(\overline{w},\overline{z},u)$*, where all* τ_q *are equational.*

Proof. By the assumption and lemma 2.6,

$$[[\kappa^2]] \models (\forall X)(\overline{Q}\overline{w})[R^\kappa_\theta(\overline{w}) \text{ contains no non-trivial interval} \wedge R^\kappa_\theta(\overline{w}) \subseteq X]. \tag{8}$$

Recall from lemma 2.5 (iv), that $R^\kappa_\theta(\overline{w})$ can be written in the form

$$R^\kappa_\theta(\overline{w}) = \bigcup_{1\le p\le r} E^\kappa_p(\overline{w}) \cup \bigcup_{r<p\le k} I^\kappa_p(\overline{w}) \cup D^\kappa(\overline{w}).$$

Since the set $D^\kappa(\overline{w})$ is either $\emptyset$ or κ, it follows from (8) that $D^\kappa(\overline{w}) = \emptyset$ (otherwise $D^\kappa(\overline{w}) = R^\kappa_\theta(\overline{w}) = \kappa$ which contains non-trivial intervals). Therefore

$$R^\kappa_\theta(\overline{w}) = \bigcup_{1\le p\le r} E^\kappa_p(\overline{w}) \cup \bigcup_{r<p\le k} I^\kappa_p(\overline{w}).$$

Let I_p^κ be some of the intervals contained above, defined by a non-equational clause (7). By (8),

$$[[\kappa]] \models (\overline{Q}\overline{w})(I_p^\kappa = \emptyset \text{ or a singleton}).$$

But the fact that $t_{i_p} \prec u \prec t_{j_p} \ \wedge \ \eta_p$ defines at most a singleton implies that the latter formula can be equivalently replaced by the formula:

$$(t'_{i_p} = u \ \wedge \ u' = t'_{j_p}) \ \wedge \eta_p,$$

where $t' = s$ abbreviates the formula "s is the immediate successor of t". The above formula is analytically written:

$$(\exists z_1)[t_{i_p} \prec z_1 \prec t_{j_p} \ \wedge (\forall z_2)(z_2 \preceq t_{i_p} \ \vee z_2 = u \ \vee t_{j_p} \preceq z_2)] \ \wedge \ \eta_p. \quad (9)$$

But (9) is equational in u, at the cost that it contains some extra bound variables.

Concerning intervals of the form (t_{i_p}, ∞), or $[0, t_{j_p})$, in view of (8), if σ_p is of the form $(t_{i_p} \prec u) \ \wedge \ \eta_p$, then the defined interval must be always $\emptyset$, therefore we can replace $(t_{i_p} \prec u) \ \wedge \ \eta_p$ by $\bot$. If σ_p is of the form $(u \prec t_{j_p}) \ \wedge \ \eta_p$, then, in order for $[0, t_{j_p})$ to be trivial, we must have either $\neg\eta$, or $t_{j_p} = 0$, or $t_{j_p} = 0'$. Therefore, $(u \prec t_{j_p}) \ \wedge \ \eta_p$ can be replaced by the formula $(t_{j_p} = 0 \ \vee \ t_{j_p} = 0') \ \wedge \eta$.

Now if we replace each non-equational clause σ_p by a formula of this kind and bring the resulting formula in prenex normal form, we shall get a formula $\theta^*(\overline{w}, \overline{z}, u)$, whose disjunctive normal form contains only u-equational clauses. Moreover, if $\phi^*(X)$ is the positive formula resulting from θ^* in the obvious way, then $\phi(X)$ and $\phi^*(X)$ are equivalent over $[[\kappa^2]]$ and

$$[[\kappa^2]] \models (\forall X)(\overline{Q}^*\overline{wz})[R^\kappa_{\theta*}(\overline{w}, \overline{z}) \text{ contains no non-trivial interval} \wedge$$

$$R^\kappa_{\theta*}(\overline{w}, \overline{z}) \subseteq X].$$

$\dashv$

Let $\overline{Q}$ contain existential quantifiers and let $[[\kappa^2]] \models (\forall X)\phi(X)$. By Lemma 2.7, we may assume that if $\theta = \bigvee_p \sigma_p$, then all σ_p are *equational* clauses. This simplifies things considerably and we now turn to examine the form of the string $\overline{Q}$. Without essential loss of generality, and for the sake of illustrating the argument, assume that

$\overline{Q}$ consists of simple alternations of $\forall$ and $\exists$, starting with $\forall$. Let henceforth $\overline{w} = (w_1, \ldots, w_n)$ denote the string of variables bound by $\forall$ and let $\overline{v} = (v_1, \ldots, w_v)$ denote the string of variables bound by $\exists$. So instead of writing $\overline{Q}\overline{w}$ we henceforth write $\overline{Q}\overline{w}\overline{v}$, where

$$\overline{Q}\overline{w}\overline{v} = (\forall w_1)(\exists v_1)(\forall w_2) \cdots (\forall w_n)(\exists v_n).$$

Then $[[\kappa^2]] \models (\forall X)\phi(X)$ is written

$$[[\kappa^2]] \models (\forall X)(\forall w_1)(\exists v_1) \cdots (\forall w_n)(\exists v_n)[R_\theta^\kappa(\overline{w}, \overline{v}) \subseteq X], \tag{10}$$

where $R_\theta^\kappa(\overline{w}, \overline{v})$ is a finite union of singletons $\{w_i\}$ or $\{v_j\}$. We distinguish the equational clauses of θ into $(u = w)$-*clauses*, if the u-equation contained is of the form $u = w_i$ (independent variable), and $(u = v)$-*clauses*, if the u-equation contained is of the form $u = v_j$ (dependent variable). Also we refer to the independent and dependent variables as w-*variables* and v-*variables* respectively.

Lemma 2.8 *If θ consists of $(u = v)$-clauses only, then $[[\kappa^2]] \models (\forall X)\phi(X)$, hence for all κ, λ,*

$$[[\kappa^2]] \models (\forall X)\phi(X) \iff [[\lambda^2]] \models (\forall X)\phi(X).$$

Proof. Let $\theta \equiv \bigvee_p \sigma_p$, where all σ_p are $(u = v)$-clauses. We have to show that

$$[[\kappa^2]] \models (\forall X)(\overline{Q}\overline{w}\overline{v})(\forall u)(\bigvee_p \sigma_p \Rightarrow u \in X).$$

Now given any cofinal X, it suffices to pick arbitrary $\overline{v} \in X$ no matter what the choice of $\overline{w}$ is. Then, each σ_p will define either a singleton $\{v_j\} \subseteq X$, if η_p is satisfied, or $\emptyset$ otherwise. Thus in any case $R_\theta^\kappa(\overline{w}, \overline{v}) \subseteq X$, and we are done. $\dashv$

Lemma 2.9 *If some $(u = w)$-clause contains only w-variables, then for any κ, $[[\kappa^2]] \not\models (\forall X)\phi(X)$, hence for all κ, λ,*

$$[[\kappa^2]] \models (\forall X)\phi(X) \iff [[\lambda^2]] \models (\forall X)\phi(X).$$

Proof. Let σ_p be a clause containing only variables of type w_i. Then it has the form:

$$w_{i_1}\ R\ \cdots\ R\ u = w_{i_p}\ R\ \cdots\ R\ w_{i_l},$$

where R is either $\prec$ or $=$. Let $\overline{Q}'$ be the dual string of the string $\overline{Q}$ of ϕ. $\overline{Q}'$ contains $(\exists w_{i_1})\cdots(\exists w_{i_p})\cdots(\exists w_{i_l})$ as a substring. And since the rest variables of $(\overline{w},\overline{v})$ does not affect σ_p,

$$[[\kappa]] \models (\overline{Q}'\overline{wv})(\exists u)\sigma_p$$

is equivalent to

$$[[\kappa]] \models (\exists w_{i_1})\cdots(\exists w_{i_p})\cdots(\exists w_{i_l})(\exists u)[w_{i_1}\ R\ \cdots\ R\, u = w_{i_p}\ R\ \cdots\ R\, w_{i_l}].$$

Now it is clear that, whatever the particular form of the clause

$$w_{i_1}\ R\ \cdots\ R\ u = w_{i_p}\ R\ \cdots\ R\ w_{i_l}$$

would be, there is a cofinal set Z such that

$$[[\kappa]] \models (\exists w_{i_1})\cdots(\exists w_{i_p})\cdots(\exists w_{i_l})(\exists u)$$

$$[(w_{i_1}\ R\ \cdots\ R\ u = w_{i_p}\ R\ \cdots\ R\ w_{i_l})\ \wedge\ u \notin Z].$$

Equivalently,

$$[[\kappa]] \models (\overline{Q}'\overline{wv})(\exists u)(\sigma_p\ \wedge u \notin Z).$$

Therefore

$$[[\kappa]] \models (\overline{Q}'\overline{wv})(R^{\kappa}_{\theta}(\overline{w},\overline{v}) \not\subseteq Z),$$

whence $[[\kappa^2]] \not\models (\forall X)\phi(X)$. ⊣

In view of Lemmas 2.8 and 2.9, where the truth of $(\forall X)\phi(X)$ in every $[[\kappa^2]]$ is settled under the conditions stated, we assume now that these conditions fail, i.e., that θ contains $(u = w)$-clauses and every $(u = w)$-clause contains v-variables. Then we have to examine three main cases 1, 2 and 3, where cases 1 and 2 contain several subcases, sub-subcases etc. We call them all "Cases" and enumerate them by sequences of numbers. For instance, 1.2.1 and 1.2.2 are subcases of 1.2. We designate a case by $\rightarrow$ if it is a subcase of the previous one, by $\downarrow$ if it as a case of equal depth as the previous one, and by $\leftarrow$ if we return to a case of smaller depth than the previous one. The cases concern either syntactic properties of the clauses of θ or the truth in $[[\kappa]]$ of some (first-order) subformula of θ. It turns out that these characteristics fully determine the truth of falsehood of $(\forall X)\phi(X)$ in $[[\kappa^2]]$. Therefore, in view of theorem 1.1, the same syntactic and semantic conditions determine the truth or falsehood, respectively, of $(\forall X)\phi(X)$ in $[[\lambda^2]]$, for any uncountable cardinal λ. So, roughly, in the cases below we reduce the truth of the second-order formula $(\forall X)\phi(X)$, to the truth of some first-order formula plus some syntactic conditions.

Case 1. There are w-clauses in which all v-variables precede all w-variables (in the enumeration of variables given in (10)), i.e., for all variables w_i, and v_j occurring in these clauses, we have $j < i$. Since every v-variable is bound by $\exists$ and every w-variable is bound by $\forall$, it follows that every value of the v-variables of the clause is independent of any value of a w-variable contained there.

$\rightarrow$ **Case 1.1.** Suppose there is a w-clause $\sigma_p \equiv (u = w_i) \wedge \eta_p$, containing a unique v-variable, say v_j.

$\rightarrow$ **Case 1.1.1.** σ_p contains formulas of the form $w_k \prec v_j$, and specifically let σ_p define m elements below v_j, say by the inequalities $w_{i_1} \prec w_{i_2} \prec \cdots \prec w_{i_m} \prec v_j$.

$\rightarrow$ **Case 1.1.1.1.** There is no clause containing $u = v_j$. Then, choosing $v_j = 0$, we falsify σ_p for any choice of the independent variables, so $[[\kappa^2]] \models (\forall X)\phi(X)$.

$\downarrow$ **Case 1.1.1.2.** There are clauses containing $u = v_j$, and let us enumerate them as $\sigma_{p_k} \equiv (u = v_j) \wedge \eta_{p_k}$, for $k \leq s$.

$\rightarrow$**Case 1.1.1.2.1.** $[[\kappa]] \models (\overline{Q}'\overline{w}\overline{v})(\bigvee_{k\leq s} \eta_{p_k})$, where in the string $\overline{Q}'\overline{w}\overline{v}$, we have $(\forall v_j \in \{0, \ldots, m\})$ instead of $(\forall v_j)$. Then $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

Proof. Indeed, recall from the above that $w_{i_1} \prec w_{i_2} \prec \cdots \prec w_{i_m} \prec v_j$, and $j < i_1, \ldots, i_m$, so in order to show that there is Z such that

$$[[\kappa^2]] \models (\overline{Q}')[R^\kappa_\theta(\overline{w}, \overline{v}) \not\subseteq Z],$$

it suffices to show that there is Z such that

$$[[\kappa^2]] \models (\forall v_j)(\exists w_{i_1}) \cdots (\exists w_{i_m})(\exists u)$$

$$[(w_{i_1} \prec \cdots \prec w_{i_m} \prec v_j \cdots u = w_i \cdots\ ) \vee \bigvee_{k\leq s} \eta_{p_k}(\overline{w}, \overline{v})) \wedge u \notin Z].$$

Take Z to be a cofinal set such that $\{0, \cdots, m\} \cap Z = \emptyset$ and $-Z$ is cofinal. Then it is easy to check that the preceding formula holds in $[[\kappa^2]]$.

$\downarrow$**Case 1.1.1.2.2.** $[[\kappa]] \not\models (\overline{Q}'\overline{wv})\bigvee_{k\leq s}\eta_{p_k}$, where in the string $\overline{Q}'\overline{wv}$, we have $(\forall v_j \in \{0,\ldots,m\})$ instead of $(\forall v_j)$. That is, $[[\kappa]] \models (\overline{Q}\overline{wv})(\bigwedge_{k\leq s}\neg\eta_{p_k})$, where in the string $\overline{Q}\overline{wv}$, we have $(\exists v_j \in \{0,\ldots,m\})$ instead of $(\exists v_j)$. Then, choosing $v_i \in \{0,\ldots,m\}$, we simultaneously falsify both $w_{i_1} \prec w_{i_2} \prec \cdots \prec w_{i_m} \prec v_j \cdots u = w_i \cdots$ and all σ_{p_k}. Hence $(\forall X)\phi(X)$ holds in $[[\kappa^2]]$.

$\leftarrow$ **Case 1.1.2.** σ_p contains no formula of the form $w_k \prec v_j$, i.e., σ_p is of the form

$$v_j \ R \ w_{k_1} \ R \cdots u = w_i \ R \ \cdots \ R \ w_{k_l}.$$

Then $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

Proof. Taking a cofinal set Z whose complement is also cofinal, we have

$$[[\kappa^2]] \models (\forall v_j)(\exists w_{k_1})\cdots(\exists w_{k_l})(\exists u)$$
$$[(v_j \ R \ w_{k_1} \ R \cdots R \ u = w_i \ R \ \cdots \ R \ w_{k_l}) \ \wedge \ u \notin Z].$$

Consequently

$$[[\kappa^2]] \models (\overline{Q}')[R^{\kappa}_{\theta}(\overline{w},\overline{v}) \not\subseteq Z],$$

i.e., $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

$\leftarrow$**Case 1.2.** There is a $(u = w)$-clause $\sigma_p \equiv (u = w_i \ \wedge \ \eta_p)$, containing at least two v-variables. Let $v_{j_1},\ldots,v_{j_t}$ be the list of all v-variables of σ_p. We distinguish the $(u = w)$-clauses into two groups: (a) Those defining a two-end interval with respect to u, i.e. σ_p contains a subformula of the form $v_{j_a} \preceq u = w_i \preceq v_{j_b}$, and (b) those defining a one-end interval, i.e. in which all v_{j_a} are strictly before or after the equation $u = w_i$. We denote the clauses of the first kind by σ^2_p and the second kind by σ^1_r, and similarly their corresponding subformulas by η^2_p and η^1_r respectively. Further, let $\sigma_q \equiv (u = v_{j_q} \wedge \eta_q)$ be the clauses of θ (if any) containing an $(u = v)$-equation for the above mentioned variables $v_{j_1},\ldots,v_{j_t}$.

So the relevant subformula of $\phi(X)$ is written

$$(\exists v_{j_1})\cdots(\exists v_{j_t})(\forall w_{i_1})\cdots(\forall w_{i_s})(\forall u)$$
$$[(\bigvee_p \sigma^2_p \vee \bigvee_r \sigma^1_r \vee \bigvee_q \sigma_q) \Rightarrow \ u \in X]. \qquad (11)$$

→**Case 1.2.1.** In (11) above there are no formulas of type σ_r^1. Then $[[\kappa^2]] \models (\forall X)\phi(X)$.

Proof. We have to show that

$$[[\kappa^2]] \models (\forall X)(\exists v_{j_1})\cdots(\exists v_{j_t})(\forall w_{i_1})\cdots(\forall w_{i_s})(\forall u)[(\bigvee_p \sigma_p^2 \vee \bigvee_q \sigma_q) \Rightarrow\ u \in X].$$

Each of the formulas σ_p^2 contains a pair of variables v_{j_a}, v_{j_b} that bound the equation $u = w_i$ from above and below. So given X it suffices to choose $v_{j_a} = v_{j_b} \in X$. Then the intervals they define are null, independently of the choice of w-variables. Moreover if this choice satisfies η_q, then σ_q defines a singleton whose element belongs to X. Otherwise σ_q defines $\emptyset$ and we are done. So the above formula holds. Observe also that this choice of v_{j_a}, v_{j_b} can be taken arbitrarily high.

↓**Case 1.2.2.** In (11) above there are no formulas of type σ_r^2. Then $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

Proof. We have to show that

$$[[\kappa^2]] \models (\exists X)(\forall v_{j_1})\cdots(\forall v_{j_t})(\exists w_{i_1})\cdots(\exists w_{i_s})(\exists u)[(\bigvee_r \sigma_r^1 \vee \bigvee_q \sigma_q) \wedge u \notin X].$$

Now in every formula σ_r, the equation $u = w_i$ is left or right to all v-variables, so the situation is quite similar to that of Case 1.1.

↓**Case 1.2.3.** In (11) we have formulas of both types σ_r^1 and σ_r^2. Then we distinguish two subcases.

→**Case 1.2.3.1.**

$$[[\kappa]] \models (\exists v_{j_1})\cdots(\exists v_{j_t})(\forall w_{i_1})\cdots(\forall w_{i_s})[\bigvee_p \eta_p^2 \vee (\bigwedge_r \neg\eta_r^1 \wedge \bigwedge_q \neg\eta_q)].$$

Then $[[\kappa^2]] \models (\forall X)\phi(X)$.

Proof. By assumption there is a valuation of the v-variables that make all η_r^1 formulas defining one-end intervals false. So $[[\kappa^2]] \models (\forall X)\phi(X)$ is shown by the argument of Case 1.2.1 above.

↓**Case 1.2.3.2.**

$$[[\kappa]] \models (\forall v_{j_1})\cdots(\forall v_{j_t})(\exists w_{i_1})\cdots(\exists w_{i_s})[\bigwedge_p \neg\eta_p^2 \wedge (\bigvee_r \eta_r^1 \vee \bigvee_q \eta_q)].$$

Then $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

Proof. By assumption for every valuation of the v-variables there is a valuation of the w-variables that make all η_r^2 formulas defining two-end intervals false. So $[[\kappa^2]] \models (\exists X)\neg\phi(X)$ is shown by the argument of Case 1.2.2 above.

This exhausts the examination of Case 1. It is important to note that whenever in the above subcases $[[\kappa^2]] \models (\exists X)\neg\phi(X)$, then for every X, we can choose each v_j arbitrarily high in X, unless the equation $u = v_i$ does not occur in θ, or the choice of v_j falsifies simultaneously all clauses containing $u = v_j$.

←**Case 2.** There is a $(u = w_i)$-clause σ_p containing a formula $w_k \prec v_j$ with $k \leq j$.

→**Case 2.1.** θ does not contain any $(u = v_j)$-clause. Then $[[\kappa^2]] \models (\forall X)\phi(X)$.

Proof. The relevant subformula of ϕ is

$$(\forall w_k)(\exists v_j)(\forall u)[(u = w_i \ \wedge \ w_k \prec v_j) \Rightarrow u \in X].$$

Now since $(u = v_j)$ does not occur in θ, given any X and w_k, we can pick, say, $v_j = w_k$, falsifying thus the hypothesis of the above implication, i.e., σ_p, and so $[[\kappa^2]] \models (\forall X)\phi(X)$.

↓**Case 2.2.** θ contains $(u = v_j)$-clauses and let $(\sigma_q)_q$ be an enumeration of them. Let η_p, η_q be the corresponding subformulas of them.

→**Case 2.2.1.**

$$[[\kappa]] \models (\forall w_k)(\exists v_j)[\neg\eta_p \ \wedge \ \bigwedge_q \neg\eta_q].$$

Then clearly

$$[[\kappa^2]] \models (\forall X)(\forall w_k)(\exists v_j)(\forall u)[(\neg\eta_p \ \wedge \ \bigwedge_q \neg\eta_q) \Rightarrow u \in X],$$

that is $[[\kappa^2]] \models (\forall X)\phi(X)$.

↓**Case 2.2.2.**

$$[[\kappa]] \models (\exists w_k)(\forall v_j)[\eta_p \ \vee \ \bigvee_q \eta_q].$$

We claim that $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

Proof. We examine the relative positions of w_i, w_k, v_j in σ_p. Their possible relative positions are: (a) $u = w_i \preceq w_k \prec v_j$, (b) $w_k \prec u = w_i \preceq v_j$, and (c) $w_k \prec v_j \preceq u = w_i$.

(a) Let the relative positions of w_i, w_k, v_j in σ_p be $u = w_i \preceq w_k \prec v_j$. Then

$$[[\kappa]] \models (\exists w_k)(\forall v_j)[w_i \preceq w_k \prec v_j \ \vee \ \bigvee_q \eta_q].$$

Let $c \in \kappa$ be a value for w_k satisfying the last formula. Take a cofinal Z such that $Z \cap [0, c] = \emptyset$. Then it is easy to verify that

$$[[\kappa^2]] \models (\forall v_j)(\exists u)[(u = w_i \preceq c \prec v_j \ \vee \ \bigvee_q \eta_q) \ \wedge \ u \notin Z].$$

Indeed, for v_j such that $c \prec v_j$, take $u = c$ for which the first clause holds and $c \notin X$. If $v_j \leq c$, then v_j satisfies $\bigvee_q \eta_q)$, where each η_q contains the equation $u = v_j$. So taking $u = v_j$, $u \notin X$ and we are done. It follows that $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

(b) Let the relative positions of w_i, w_k, v_j in σ_p be $w_k \prec u = w_i \preceq v_j$. Then

$$[[\kappa]] \models (\exists w_k)(\forall v_j)[w_k \prec u = w_i \preceq v_j \ \vee \ \bigvee_q \eta_q].$$

Let $c \in \kappa$ be a value for w_k satisfying the last formula. Take a cofinal Z such that $Z \cap [0, c+1] = \emptyset$ and Z contains no two consecutive elements. Then it is easy to verify again that

$$[[\kappa^2]] \models (\forall v_j)(\exists u)[(c \prec u = w_i \preceq v_j \ \vee \ \bigvee_q \eta_q) \ \wedge \ u \notin Z].$$

Indeed for $v_j \leq c+1$, we pick $u = v_j$ which satisfies $\bigvee_q \eta_q$ and $v_j \notin Z$. For $v_j > c+1$, we pick any u such $c < u < v_j$ and $u \notin Z$. So in any case the above holds. Thus again $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

(c) Let the relative positions of w_i, w_k, v_j in σ_p be $w_k \prec v_j \preceq u = w_i$. Then

$$[[\kappa]] \models (\exists w_k)(\forall v_j)[w_k \prec v_j \preceq u = w_i \ \vee \ \bigvee_q \eta_q].$$

Take c for w_k satisfying this formula and a cofinal Z as in (b) above. Then we easily see again that

$$[[\kappa^2]] \models (\forall v_j)(\exists u)[(c \prec v_j \preceq u = w_i \ \vee \ \bigvee_q \eta_q) \ \wedge \ u \notin Z].$$

Hence $[[\kappa^2]] \models (\exists X)\neg\phi(X)$.

This exhausts the examination of Case 2. Note again that whenever in the above subcases $[[\kappa^2]] \models (\exists X)\neg\phi(X)$, then for every X, we can choose each v_j arbitrarily high in X, unless the equation $u = v_j$ does not occur in θ, or the choice of v_j falsifies simultaneously all clauses containing $u = v_j$.

$\leftarrow$**Case 3.** Let Cases 1 and 2 be false. That means that there is no $(u = w)$-clause of θ in which for every pair of variables w_i, v_j, either $j < i$ (Case 1) or $i \leq j$ and $w_i \prec v_j$ is in the clause (Case 2). It follows that every $(u = w)$-clause of θ contains a formula of the form $v_j \prec w_i$ or $v_j = w_i$ such that $i \leq j$. Then we claim that $[[\kappa^2]] \models (\forall X)\phi(X)$.

Proof. In order for $[[\kappa^2]] \models (\forall X)\phi(X)$ to hold, it suffices, for every X, to pick all v_j so that (a) $v_j \in X$ and (b) these v_j falsify all $(u = w)$-clauses. We do not need to bother about the truth or falsity of $(u = v)$-clauses. Because for every such choice, they will define either $\emptyset$ or singletons contained in X, hence $R^\kappa_\theta \subseteq X$ is guaranteed. Now since each $(u = w)$-clause contains a formula of the form $v_j \prec w_i$ or $v_j = w_i$ with $i \leq j$ (i.e., v_j depending on w_i) it suffices, given X and w_i, to pick $v_j \in X$ such that $w_i < v_j$. The same dependent variable v_j may dominate several $w_{i_1}, \ldots, w_{i_m}$. For every such finite sequence $w_{i_1}, \ldots, w_{i_m}$, we can pick $v_j > w_{i_1}, \ldots, w_{i_m}$ and $v_j \in X$. This is always possible because every X is cofinal. (This the first and last place where the cofinality of the sets X is needed in a decisive way.) $\dashv$

Proof of the MAIN THEOREM.
Let

$$\phi(X) \equiv (\overline{Q}\overline{w})(\forall u)(\theta(\overline{w}, u) \Rightarrow u \in X)$$

be the canonical form of ϕ, and let $[[\kappa^2]] \models (\forall X)\phi(X)$. Let also λ be another uncountable cardinal. It suffices to show that $[[\lambda^2]] \models (\forall X)\phi(X)$.

If $\overline{Q} = \overline{\forall}$, the claim follows from lemma 2.2. So let $\overline{Q}$ contain existential quantifiers. In view of lemmas 2.6 and 2.7, we may assume that $\theta = \bigvee_p \sigma_p$, where all σ_p are equational. If θ contains only $(u = v)$-clauses, the theorem follows from lemma 2.8. Hence let $\neg\theta$ contain $(u = w)$-clauses. If some $(u = w)$-clause has no v-variables, by lemma 2.9, $[[\kappa^2]] \models (\exists X)\neg\phi(X)$, which contradicts our assumption. So let every $(u = w)$-clause of θ contain v-variables.

Then we come to the cases 1, 2, 3 treated above. We see that $[[\kappa^2]] \models (\forall X)\phi(X)$ is compatible only with the cases 1.1.1.1, 1.1.1.2.2, 1.2.1, 1.2.3.1, 2.1, 2.2.1, and 3. That is, every $(u = w)$-clause of θ, satisfies the conditions of some of these cases. Moreover, as explained above, the choice of the value for dependent variables v_j can be done so that all conditions are met simultaneously. Since these conditions are first-order, they guarantee the truth of $(\forall X)\phi(X)$ in any other structure $[[\lambda^2]]$ for uncountable λ. Therefore $[[\lambda^2]] \models (\forall X)\phi(X)$. This completes the proof of the Main Theorem. $\dashv$

3 $\exists^1_1$ positive formulas

Theorem 2.1 holds also for $\exists^1_1$ positive formulas, but the proof is much simpler. This is already suggested by the canonical form $(\overline{Q}\overline{w})(R^\kappa_\theta(\overline{w}) \subseteq X)$ of $\phi(X)$. Since the latter obviously holds in κ for $X = \kappa$, it follows that $(\exists X)\phi(X)$ holds in κ and similarly in every uncountable cardinal. However this argument is fallacious because, as remarked after lemma 1.3 of the first section, the canonical form holds only for X different from the universal set.

Proposition 3.1 *For every uncountable cardinals κ, λ and every normal positive $\phi(X)$*

$$[[\kappa^2]] \models (\exists X)\phi(X) \iff [[\lambda^2]] \models (\exists X)\phi(X).$$

Proof. Let κ, $\phi(X)$ be given. Observe that

$$[[\kappa^2]] \models (\exists X)\phi(X) \iff [[\kappa^2]] \models \phi(\kappa). \tag{12}$$

$\Leftarrow$ of (12) is straightforward, while $\Rightarrow$ follows from monotonicity of positive formulas: If $\phi(X)$ is true and $X \subseteq Y$, then $\phi(Y)$ is true. Now it is easy to check, by induction on the construction steps, that for every positive $\phi(X)$

$$\phi(\kappa) \equiv \chi \text{ or } \phi(\kappa) \equiv \top, \tag{13}$$

where χ is a first-order formula. Combining (12) and (13), and the fact that $[[\kappa]] \equiv [[\lambda]]$, we have

$$[[\kappa^2]] \models (\exists X)\phi(X) \iff [[\kappa^2]] \models \phi(\kappa) \iff$$

$$[[\kappa]] \models \chi \text{ (or } \top) \iff [[\lambda]] \models \chi \text{ (or } \top) \iff$$
$$[[\lambda^2]] \models \phi(\lambda) \iff [[\lambda^2]] \models (\exists X)\phi(X).$$

$\dashv$

4 Some consequences of the main theorem

Given an infinite cardinal κ, observe that the principle $2^\kappa = \kappa^+$ can be formulated as a $\forall^1_1$ positive monadic sentence holding in a certain subclass of $\mathcal{P}(2^\kappa)$. Namely,

$$2^\kappa = \kappa^+ \iff (2^\kappa, [2^\kappa]^{>\kappa}, <) \models (\forall X)(X \text{ is cofinal}). \tag{14}$$

Indeed, if $2^\kappa = \kappa^+$ then every $X \in [2^\kappa]^{>\kappa}$ has cardinality κ^+ and therefore is cofinal to κ^+. If, conversely, $2^\kappa > \kappa^+$, then there is set $X \in [2^\kappa]^{>\kappa}$, e.g. κ^+, which is not cofinal to 2^κ.

"X is cofinal" is the formula $(\forall x)(\exists y)(x \prec y \ \wedge \ y \in X)$, which is positive in X. Hence $(\forall X)(X \text{ is cofinal})$ is $\forall^1_1$ positive monadic.

As $2^\kappa = \kappa^+$ expresses a certain relationship between 2^κ and κ, one might reasonably argue that a wider class of such simple relationships could be common to all pairs of $\kappa, 2^\kappa$. A specific implementation of this general idea would be the claim that for all infinite κ, λ, the structures $(2^\kappa, [2^\kappa]^{>\kappa}, <)$ and $(2^\lambda, [2^\lambda]^{>\lambda}, <)$ satisfy the same $\forall^1_1$ positive monadic sentences. Call this principle $\forall^1_1$ *Positive Equity Principle* ($\forall^1_1-$PEP). That is,

$$(\forall^1_1\text{−PEP}) \quad (\forall \kappa, \lambda \geq \omega)[(2^\kappa, [2^\kappa]^{>\kappa}, <) \equiv^{\forall^1_1}_{pos} (2^\lambda, [2^\lambda]^{>\lambda}, <)]. \tag{15}$$

The restriction to cofinal sets and to positive $\forall^1_1$ formulas makes the above principle minimalistic. Of course it would be desirable to have stronger versions for PEP, i.e., $\Gamma-$PEP, for Γ a class of formulas larger than $\forall^1_1$, provided its consistency with ZFC can be established. For the time being, the main theorem of the previous section implies the following.

Theorem 4.1 ZFC $+ \forall^1_1-$PEP *is consistent.*

Proof. It suffices to show that ZFC+GCH$\vdash \forall_1^1-$PEP. In the presence of GCH, $\forall_1^1-$PEP becomes

$$(\kappa^+, [\kappa^+]^{>\kappa}, <) \equiv_{pos}^{\forall_1^1} (\lambda^+, [\lambda^+]^{>\lambda}, <).$$

But

$$[\kappa^+]^{>\kappa} = [\kappa^+]^{\kappa^+} = Cof(\kappa^+)$$

and similarly $[\lambda^+]^{>\lambda} = Cof(\lambda^+)$. So the above reduces to

$$[[(\kappa^+)^2]] \equiv_{pos}^{\forall_1^1} [(\lambda^+)^2]],$$

which follows from the main theorem. ⊣

Corollary 4.2 ZFC $+ \forall_1^1-$PEP $\vdash (\forall\kappa, \lambda \geq \omega)(2^\kappa = \kappa^+ \Leftrightarrow 2^\lambda = \lambda^+)$.

Proof. Immediate from (14). ⊣

Does $\forall_1^1-$PEP settle GCH? The answer is yes, unless a large cardinal assumption is made. Namely, suppose ZFC $+ \forall_1^1 -$ PEP $\nvdash$ GCH. Then the consistency of $\forall_1^1 -$ PEP $+ \neg$GCH implies the consistency of $(\forall\kappa)(2^\kappa > \kappa^+)$, and hence the consistency of $2^\kappa > \kappa^+$ for a strong limit singular cardinal κ. But, by [4] (see also [5], Theorem 36.1), this is equivalent to the consistency of the existence of a measurable cardinal with Mitchell order κ^{++}.

5 Separating ω_m from ω_n

In this section we use (essentially) the formulas employed in page 19 of [1], to distinguish between the structures $(\omega_1, Cof(\omega_1), <)$ and $(\omega_2, Cof(\omega_2), <)$. The formula that makes the distinction is of the form $(\forall x)(\exists X)\psi$, where ψ is normal but not positive.

Consider the formulas:

$succ(x):\ (\exists y \prec x)(\forall z)(z \preceq y \vee x \preceq z)$.
$type_\omega(X, x):\ (\forall y)(y \in X\ \&\ y \prec x\ \Rightarrow y = 0 \vee succ(y))$.
$cof(X, x):\ x = 0 \vee succ(x) \vee (\forall y \prec x)(\exists z \in X)(y \prec z \prec x)$.
$acc_\omega(x):\ (\exists X)[cof(X, x) \wedge type_\omega(X, x)]$.
$\phi_{\omega_1}:\ (\forall x) acc_\omega(x)$.

It is easy to check that ϕ_{ω_1} says that every element has cofinality ω, hence $(\omega_1, \mathcal{P}(\omega_1), <) \models \phi_{\omega_1}$, while $(\omega_2, \mathcal{P}(\omega_2), <) \models \neg\phi_{\omega_1}$. Now

$$\phi_{\omega_1} \equiv (\forall x) acc_\omega(x) \equiv (\forall x)(\exists X)\psi,$$

where $\psi \equiv cof(X, x)$ & $type_\omega(X, x)$. Since $type_\omega(X, x)$ is not positive, ψ also is not positive.

Now, by the same token, the formula $(\forall x)(\exists X)\psi$ distinguishes also the structures $(\omega_1, Cof(\omega_1), <)$ and $(\omega_2, Cof(\omega_2), <)$, since the quantifier $\exists X$ can range only over cofinal sets. Analogous separation formulas can be found for the other cardinals ω_m, ω_n, $m, n \in \omega$.

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