Large transitive models in local ZFC

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Abstract

This paper is a sequel to [7], where a local version of ZFC, LZFC, was introduced and examined and transitive models of ZFC with properties that resemble large cardinal properties, namely Mahlo and Π^1_1-indescribable models, were considered. By analogy we refer to such models as "large models", and the properties in question as "large model properties". Continuing here in the same spirit we consider further large model properties, that resemble stronger large cardinals, namely, "elementarily embeddable", "extendible" and "strongly extendible", "critical" and "strongly critical", "self-critical" and "strongly self-critical", the definitions of which involve elementary embeddings. Each large model property ϕ gives rise to a localization axiom Loc^{ϕ}(ZFC) saying that every set belongs to a transitive model of ZFC satisfying ϕ. The theories LZFC$^{\phi}$ = LZFC+Loc^{ϕ}(ZFC) are local analogues of the theories ZFC+"there is a proper class of large cardinals ψ", where ψ is a large cardinal property. If $sext(x)$ is the property of strong extendibility, it is shown that LZFC^{sext} proves Powerset and Σ_1-Collection. In order to refute $V = L$ over LZFC, we combine the existence of strongly critical models with an axiom of different flavor, the Tall Model Axiom (TMA). $V = L$ can also be refuted by TMA plus the axiom GC saying that "there is a greatest cardinal", although it is not known if $TMA + GC$ is consistent over LZFC. Finally Vopěnka's Principle (VP) and its impact on LZFC are examined. It is shown that LZFC^{sext} + VP proves Powerset and Replacement, i.e., ZFC is fully recovered. The same is true for some weaker variants of LZFC^{sext}. Moreover the theories LZFC^{sext}+VP and ZFC+VP are shown to be identical.

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1 Introduction

In [7] we initiated the study of a local version of ZFC called LZFC, whose main axiom, denoted *Loc*(ZFC), says that "every set belongs to a transitive model of ZFC". Namely, this is the statement:

$$(Loc(\mathrm{ZFC})) \qquad \forall x \exists y (x \in y \wedge Tr(y) \wedge (y, \in) \models \mathrm{ZFC}). \tag{1}$$

LZFC is the theory BST+*Loc*(ZFC), where BST (which stands for "basic set theory") is the set of the following elementary axioms: Empty set, Extensionality, Pair, Union, Cartesian Product, "ω exists", Δ_0-Separation. Actually BST is needed only for the formalization of the notion "model of ZFC" involved in *Loc*(ZFC) (see Remark 2.6 of [7]).

LZFC lacks the Powerset and Replacement axioms, as well as $\in$-induction.[1] These principles hold only locally in transitive set models which, in compensation, exist everywhere across the universe. In view of this fact, LZFC redirects our interest from absolute infinite cardinals to transitive models of ZFC which are now construed as analogues of inaccessible cardinals. Additional or stronger properties can make models look like stronger large cardinals. For instance in [7] we defined and studied Mahlo and Π^1_1-indescribable models of ZFC, as analogues of Mahlo and weakly compact cardinals, respectively. By analogy we refer to such models as *large models* of ZFC, and the corresponding properties as *large model properties.* In this paper we continue the search for new kinds of large transitive models of ZFC. So let us state from the outset the following:

Convention. Throughout the paper "model" means "transitive model of ZFC".

Of course not all large cardinal properties are expected to have sensible analogues for models. For example such are the properties formulated in terms of ultrafilters. In the absence of the Powerset axiom non-principal ultrafilters cannot be shown to exist even as proper classes. On the other hand, some properties formulated in terms of ultrafilters are equivalently formulated over ZFC in terms of elementary embeddings, a notion particularly fitted to a world with a plethora of local models. So the properties formulated in the next section concern mostly existence of embeddings. The difference is that the embeddings

[1] The usual Regularity axiom holds in LZFC. $\in$-induction and, equivalently, *On*-induction is missing because full Separation is missing.

employed in large cardinal definitions are mappings $j : V \to W$ (where W is an inner model), which are *internal*, i.e., definable in the universe V, while here we deal with embeddings $j : M \to N$, between models M, N of ZFC, which are in general *external* to both M and N.

The main results of the paper are contained in section 3 (Theorem 3.10), section 5 (Theorem 5.4 and Proposition 5.9) and section 6 (Theorem 6.3).

The paper is organized as follows. In section 2 we briefly review Π^1_1-indescribable models, which have been introduced in [7], and give a simpler characterization of them in Lemma 2.2.

In section 3 we introduce new stronger large model properties. To every such property $\phi(x)$ there naturally corresponds the localization principle Loc^ϕ(ZFC) saying that "every set belongs to a model satisfying ϕ". The theories

$$\text{LZFC}^\phi = \text{LZFC} + Loc^\phi(\text{ZFC})$$

extend LZFC in the same spirit as the theories

$$\text{ZFC} + \text{"there is a proper class of cardinals } \psi\text{"},$$

where ψ is a large cardinal property, extend ZFC.

Firstly we consider "(elementarily) extendible" models and show that every such model is Π^1_1-indescribable. Further, "strongly extendible" models are defined and shown to be Σ_2-elementary submodels of the universe V. In particular if $sext(x)$ is the property of strong extendibility, the theory LZFC^{sext} proves Powerset and Σ_1-Collection. This theory, as well as variants of it, are also largely used in section 6.

In section 4 we consider elementary embeddings $j : M \to N$ between models and their "critical models". Stronger notions like "strongly critical", as well as "self-critical" and "strongly self-critical models", are also introduced and some consequences of ZFC+Loc^ϕ(ZFC), for various properties ϕ, are proved. Apparently critical models are the analogues of measurable cardinals. However, due to the fact that the embeddings $j : M \to N$ are external, the consequences of their existence are much weaker than those of measurable cardinals. For example the existence of strongly critical models alone does not seem to yield $V \neq L$ over LZFC.

For that purpose in section 5 we introduce the Tall Model Axiom (TMA) saying, roughly, that for every ordinal κ there is some ordinal $\alpha > \kappa$ such that there are models of arbitrarily big height which do not collapse α to κ. One of the main results of this section (Theorem 5.4) says that the theory LZFC+TMA+"there is a strongly critical model" proves $V \neq L$. Further, the axiom of Greatest Cardinality (GC), saying that there is a (set of) greatest cardinality, is introduced. In the presence of $V = L$, this is equivalent to $\neg(TMA)$ over LZFC. Therefore LZFC+TMA+GC yields $V \neq L$. However it is open whether LZFC+$TMA + GC$ is consistent.

In section 6 we examine the implications of Vopěnka's Principle (VP) when added to LZFC. The key fact underlying these implications is an old ZFC result of P. Vopěnka, A. Pultr and Z. Hedrlín (abbreviated V-P-H) saying that for every set A there is a relation $R \subset A \times A$ such that (A, R) has no non-trivial endomorphism. The main result of section 6 is Theorem 6.3 saying that if T is a theory such that LZFC $\subseteq T$ and $T \vdash$ V-P-H, then ZFC $\subseteq T + VP$, i.e., T fully restores ZFC. An example of such a theory is LZFC^{sext}, as well as some weaker variants of it. Moreover the theories LZFC^{sext}+VP and ZFC+VP are identical. It is not known whether LZFC+VP alone restores ZFC. However if LZFC+VP proves either V-P-H or Loc^{sext}(ZFC), this is indeed the case. In connection to this it is shown that LZFC+$VP \vdash Loc^{ext}$(ZFC).

2 Π^1_1 indescribable models

Let us recall from [7] the definition of Π^1_1-indescribability. Let $\mathcal{L} = \{\in\}$, let $\mathcal{L}_2$ be $\mathcal{L}$ augmented with second order variables, and let $\mathcal{L}_2 \cup \{\mathbf{S}, \mathbf{c}_i\}$ be $\mathcal{L}_2$ augmented with a unary predicate $\mathbf{S}(\cdot)$ intended to be interpreted as a set U, and sufficient amount of first-order constants $\mathbf{c}_i$. When M is a model of ZFC, then the constants $\mathbf{c}_i$ are chosen to be names of elements c_i of M. $Def(M)$ denotes the set of subsets of M definable by formulas of $\mathcal{L}$. $Def(M)$ can be proved to exist in LZFC because it is absolute and can be constructed inside any model N of ZFC that contains M as a member.

Definition 2.1 (LZFC) A model $M \models$ ZFC is said to be Π^1_1-*indescribable* if for every $U \in Def(M)$ and every Π^1_1 sentence ϕ of $\mathcal{L}_2(\mathbf{S}, \mathbf{c}_i)$, if $(M, \in, U, Def(M)) \models \phi$, then there is a model $N \in M$ such that $U \cap N \in Def(N)$ and $(N, \in, U \cap N, Def(N)) \models \phi$.

Actually in the above definition the set $U \cap N$ can be taken to be defined in N by the same formula as U in M, which simplifies things considerably. Indeed, given a first order formula $\theta(x, \bar{y})$ without parameters, a model M of ZFC, and $\bar{c} \in M$, let $\theta[M, \bar{\mathbf{c}}]$, or just $\theta[M]$ denote the set $\{x \in M : M \models \theta(x, \bar{\mathbf{c}})\}$. Then we have the following characterization of Π^1_1-indescribability (not contained in [7]).

Lemma 2.2 *A model $M \models$ ZFC is Π^1_1-indescribable iff for every first-order formula $\theta(x, \bar{y})$ without parameters and every Π^1_1 sentence ϕ, if $(M, \in, \theta[M], Def(M)) \models \phi$, and $\bar{c} \in M$, then there is a model $N \in M$*

such that $\bar{c} \in N$, $(N, \in, \theta[N], Def(N)) \models \phi$, and $\theta(x, \bar{\mathbf{c}})$ is absolute between M and N (i.e., $M \models \theta(x, \bar{\mathbf{c}}) \Leftrightarrow N \models \theta(x, \bar{\mathbf{c}})$, for every $x \in N$).

Proof. Suppose M is Π^1_1-indescribable. Let $\theta(x, \bar{y})$ be a first order formula, $\bar{c} \in M$, $U = \theta[M]$, and let ϕ be a Π^1_1-formula of $\mathcal{L}_2(\mathbf{S}, \mathbf{c}_i)$ such that $(M, \in, U, Def(M)) \models \phi$. Let $\sigma := \forall x(\mathbf{S}(x) \leftrightarrow \theta(x, \bar{\mathbf{c}}))$. Then $(M, \in, U, Def(M)) \models \phi \wedge \sigma$, and $\phi \wedge \sigma$ is Π^1_1. By Π^1_1-indescribability there is a model $N \in M$ such that $U \cap N \in Def(N)$ and $(N, \in, U \cap N, Def(N)) \models \phi \wedge \sigma$. But $(N, \in, U \cap N, Def(N)) \models \sigma$ means that $U \cap N = \{x : N \models \theta(x, \bar{\mathbf{c}})\} = \theta[N]$, thus $(N, \in, \theta[N], Def(N)) \models \phi$. Moreover, since $\theta[N] = \theta[M] \cap N$, it follows that for every $x \in N$, $M \models \theta(x, \bar{\mathbf{c}}) \Leftrightarrow N \models \theta(x, \bar{\mathbf{c}})$, i.e., $\theta(x, \bar{\mathbf{c}})$ is absolute between M and N.

Conversely, suppose M is a model for which the assumption of the lemma holds. We show that M is Π^1_1-indescribable. Let $U \in Def(M)$ and let ϕ be a Π^1_1 sentence such that $(M, \in, U, Def(M)) \models \phi$. Let $U = \theta[M]$ for some θ. By our assumption there is a model $N \in M$ such that $(N, \in, \theta[N], Def(N)) \models \phi$, and θ is absolute between M and N. By the last condition it follows that $\theta[N] = \theta[M] \cap N$. Hence $(N, \in, U \cap N, Def(N)) \models \phi$. So M is Π^1_1-indescribable. $\dashv$

It was shown in [7, Proposition 5.5] that Π^1_1-indescribability implies α-Mahloness for every model M of ZFC. α-Mahloness is an absolute (i.e., Δ_1) property in our formal language $\mathcal{L} = \{\in\}$ (see [7]). The same is true for the property of Π^1_1-indescribability. It can be formalized by the following absolute formula $\pi^1_1 ind(x)$:

$$\pi^1_1 ind(x) := [x \models \text{ZFC} \ \wedge \ (\forall y \in Def(x))(\forall \phi \in \Pi^1_1)(\exists z \in x)$$

$$(z \models \text{ZFC} \wedge (x, \in, y, Def(x)) \models \phi \Rightarrow y \cap z \in Def(z) \wedge (z, \in, y \cap z, Def(z)) \models \phi)],$$

where Π^1_1 is the set of (codes of) Π^1_1-formulas of $\mathcal{L}_2(\mathbf{S}, \mathbf{c}_i)$. Then for any models M, N such that $M \in N$, "M is Π^1_1-indescribable" (in V) iff $N \models \pi^1_1 ind(M)$.

3 Extendible and strongly extendible models

In this section we introduce properties for models of ZFC stronger than Π^1_1-indescribability.

Definition 3.1 A model M is said to be *elementarily extendible,* or just *extendible,* if there is a model N such that $M \in N$ and $M \prec N$.

The next Lemma says that in the above definition the condition "$M \in N$" is redundant. This is a consequence of the fact that for models of ZFC, $M \prec N$ implies that N is also an *end-extension* of M, i.e., the ordinals of N extend those of M. Details are left to the reader.

Lemma 3.2 *M is extendible iff there is a model N such that $M \prec N$. In fact, $M \prec N \Rightarrow M \in N$.*

The following induction scheme for the class On is not part of LZFC.[2]

(Found$_{On}$) $\qquad \exists\alpha \in On\, \phi(\alpha) \rightarrow \exists\alpha \in On[\phi(\alpha) \wedge \forall\beta < \alpha \neg\phi(\beta)]$

Proposition 3.3 *(i) In* LZFC*: If M is extendible, then M is Π^1_1-indescribable.*

(ii) In $\mathrm{LZFC} + \mathrm{Found}_{On}$*: The converse of (i) is false. I.e., if there is a Π^1_1-indescribable model, then there is one which is not extendible.*

Proof. (i) Suppose M is not Π^1_1-indescribable. Then, by Lemma 2.2, there are a Π^1_1 formula ϕ and a first-order formula θ such that $(M, \in, \theta[M], Def(M)) \models \phi$, and for every model $x \models$ ZFC such that $x \in M$, either θ is not absolute between M and x, or $(x, \in, \theta[x], Def(x)) \not\models \phi$. This means that $(M, \in, \theta[M], Def(M)) \models \phi$ and $M \models \psi_{\phi,\theta}$, where $\psi_{\phi,\theta}$ is the first order formula

$$\psi_{\phi,\theta} := \forall x[x \models \mathrm{ZFC} \wedge (\forall y \in x)(\theta^x(y) \leftrightarrow \theta(y)) \rightarrow (x, \in, \theta[x], Def(x)) \not\models \phi].$$

Suppose that M is extendible, i.e., there is N such that $M \in N$ and $M \prec N$. Then $N \models \psi_{\phi,\theta}$, i.e.,

$$N \models \forall x[x \models \mathrm{ZFC} \wedge (\forall y \in x)(\theta^x(y) \leftrightarrow \theta(y)) \rightarrow (x, \in, \theta[x], Def(x)) \not\models \phi]. \quad (2)$$

But $M \in N$, $M \models$ ZFC, $(\forall y \in M)(\theta^M(y) \leftrightarrow \theta^N(y))$, since $M \prec N$, and $(M, \in, \theta[M], Def(M)) \models \phi$. This contradicts (2) and proves the claim.

(ii) Suppose (in $\mathrm{LZFC} + \mathrm{Found}_{On}$) that there are Π^1_1-indescribable models. By Found_{On} there is one such model M of least height. Obviously M cannot contain a Π^1_1-indescribable model. Let $\pi^1_1 ind(x)$ be the formalization of the Π^1_1-indescribability property. $\pi^1_1 ind(x)$ is absolute as noticed above. Then M is not extendible. Assume

[2] See [7, Lemma 2.11], where Found_{On} is shown to be equivalent to the $\in$-induction scheme

(Found*) $\qquad \exists x\phi(x) \rightarrow \exists x[\phi(x) \wedge \forall y \in x \neg\phi(y)]$

over LZFC.

the contrary. Then there is a model N such that $M \in N$ and $M \prec N$. Since $\pi^1_1 ind(M)$ and $\pi^1_1 ind(x)$ is absolute, it follows that $N \models \pi^1_1 ind(M)$. Therefore $N \models \exists x(x \models \text{ZFC} \wedge \pi^1_1 ind(x))$. But then $M \models \exists x(x \models \text{ZFC} \wedge \pi^1_1 ind(x))$, which contradicts the fact that M does not contain Π^1_1-indescribable models. $\dashv$

As an immediate generalization of 3.3 (ii) we have the following:

Proposition 3.4 (LZFC + Found_{On}) *If $\phi(x)$ is an absolute property about models and M is a model of least height such that $\phi(M)$, then M is not extendible.*

In contrast to Mahloness and Π^1_1-indescribability, which are absolute properties, the formalization of extendibility is given by the following predicate:

$$ext(x) := [(x, \in) \models \text{ZFC} \wedge (\exists y)((x, \in) \prec (y, \in))].$$

By analogy to large cardinal properties we may refer to properties like $ext(x)$, $mahlo(x)$, $\pi^1_1 in(x)$ (as well as to those that will be introduced later), as *large model properties.* Analogously we may refer to existence axioms $\exists x\ ext(x)$, $\exists x\ mahlo(x)$, etc, as *large model axioms.* Also for every large model property ϕ let

$$Loc^{\phi}(\text{ZFC}) := \forall x \exists y(x \in y \wedge y \text{ transitive} \wedge \phi(y) \wedge y \models \text{ZFC}).$$

Namely, Loc^{ϕ}(ZFC) says that every set x belongs to a model satisfying the large model property ϕ. So we shall refer to Loc^{ϕ}(ZFC) as *strong localization axioms,* since they strengthen *Loc*(ZFC). Moreover, for any such ϕ as above, we shall denote by LZFC^{ϕ} the theory resulting from LZFC if we replace the axiom *Loc*(ZFC) by Loc^{ϕ}(ZFC). That is:

$$\text{LZFC}^{\phi} := \text{LZFC} + Loc^{\phi}(\text{ZFC}) = \text{BST} + Loc^{\phi}(\text{ZFC}).$$

The theories LZFC^{ϕ} are extensions of LZFC, pretty analogous to the extensions of ZFC of the form

$$\text{ZFC} + \text{“there is a proper class of cardinals } \psi\text{”},$$

for some large cardinal property ψ. Later in this section as well as in section 6, we shall largely work in the theory LZFC^{sext}, where $sext(x)$ is the property of strong extendibility (see Definition 3.8 below), and some variants of it.

The next result of ZFC is a slight generalization of [6, Theorem 8.1]) and will be needed in sections 5 and 6. The proof is similar to the standard one and left to the reader.

Lemma 3.5 (ZFC) *If there are ordinals $\alpha < \beta$ such that $V_\alpha \prec V_\beta$, then V_α, V_β are models of* ZFC. *More generally, if $\alpha > \omega$, $(V_\alpha, \in) \prec (A, \in)$ and $V_\alpha \in A$, for some transitive set A, then $V_\alpha \models$* ZFC.

For every axiomatized set theory T (where usually $T \supseteq$ ZFC), let $Loc(T)$ denote the corresponding localization principle for T, i.e.,

$$Loc(T) := \forall x \exists y (x \in y \ \wedge \ Tr(y) \ \wedge \ (y, \in) \models T).$$

We have seen in [7] that every Mahlo model of ZFC satisfies $Loc_n(\mathrm{ZFC})$, for every $n \in \omega$, where these properties are defined inductively by the clauses: $Loc_0(\mathrm{ZFC}) = Loc(\mathrm{ZFC})$ and $Loc_{n+1}(\mathrm{ZFC}) = Loc(\mathrm{ZFC} + Loc_n(\mathrm{ZFC}))$. More generally, in [7] we considered strengthenings of $Loc(\mathrm{ZFC})$, of the form $Loc(\mathrm{ZFC} + \phi)$, where ϕ is a new axiom added to ZFC.

The sentences $Loc_n(\mathrm{ZFC})$ are estimates of how much stronger a Mahlo model is compared to an ordinary model of ZFC. The question is whether there are analogous estimates for Π^1_1-indescribable and elementarily extendible models. We see below that the answer is yes.

We have seen that the properties of Mahloness, Π^1_1-indescribability and elementary extendibility are in increasing strength (for the last two, when working in LZFC+$Found_{On}$). But a natural question is: In what sense are the models of one of these classes *internally stronger* than those of another. For instance, what properties does a Π^1_1-indescribable model satisfy which a Mahlo model does not? The next two lemmas reasonably justify this internal increasing strength. Recall that $mahlo_\alpha(x)$ abbreviates the property $mahlo(\alpha, x)$ defined in [7].

Lemma 3.6 *If M is Π^1_1-indescribable, then $M \models Loc^{mahlo_\alpha}(\mathrm{ZFC})$, for all $\alpha \in On \cap M$.*

Proof. Let M be Π^1_1-indescribable. Fix some $a \in M$ and let $\alpha \in On^M$. We have to show that the set $X = \{x \in M : a \in x \wedge x \models \mathrm{ZFC} \wedge mahlo(\alpha, x)\} \neq \emptyset$. Let $Y = \{x \in M : a \in x\}$. Obviously, Y is a club of M. By Proposition 5.5 of [7], since M is Π^1_1-indescribable, M is $(\alpha+1)$-Mahlo. This means that the set $Z = \{x \in M : x \models \mathrm{ZFC} \wedge mahlo(\alpha, x)\}$ is stationary in M. Hence $Z \cap Y \neq \emptyset$. But $Z \cap Y = X$, therefore $X \neq \emptyset$. ⊣

Lemma 3.7 *If M is extendible then $M \models Loc^{\pi^1_1 ind}(\mathrm{ZFC})$.*

Proof. Let M be extendible and let $a \in M$. We have to show that $X = \{x \in M : a \in x \wedge x \models \mathrm{ZFC} \wedge \pi^1_1 ind(x)\} \neq \emptyset$. Suppose that $X = \emptyset$. Since $\pi^1_1 ind(x)$ and "$x \models$ ZFC" are absolute, this is equivalent to

$$M \models \forall x (a \in x \wedge x \models \mathrm{ZFC} \rightarrow \neg \pi^1_1 ind(x)). \tag{3}$$

By our assumption there is a model N such $M \in M$ and $M \prec N$. Then, by (3),

$$N \models \forall x(a \in x \wedge x \models \text{ZFC} \rightarrow \neg\pi^1_1 ind(x)). \tag{4}$$

But by Proposition 3.3, M is Π^1_1-indescribable, i.e., $\pi^1_1 ind(M)$, the latter property being absolute. So $a \in M$, $M \in N$, $M \models \text{ZFC}$ and $N \models \pi^1_1 ind(M)$, which contradicts (4). $\dashv$

The following strengthening of extendibility turns out to be an interesting and powerful property.

Definition 3.8 A model M is *strongly extendible* if for every x there is a model N such that $x \in N$ and $M \prec N$.

Formally the property of strong extendibility is written:

$$sext(x) := [(x, \in) \models \text{ZFC} \wedge (\forall y)(\exists z)(y \in z \wedge (x, \in) \prec (z, \in))].$$

Given a model M let us write $M \prec_{\Sigma_n} V$ if every Σ_n sentence ϕ with parameters from M is absolute for M, i.e., $M \models \phi$ iff $V \models \phi$. [A Σ_n formula is one of the form $(\exists \overline{x}_1)(\forall \overline{x}_2)\cdots(Q\overline{x}_n)\psi$, with ψ bounded, where $\overline{x}_i$ are tuples of variables. Similarly for a Π_n formula.] Obviously $M \prec_{\Sigma_n} V$ iff $M \prec_{\Pi_n} V$.

The important feature of strongly extendible models is the following.

Lemma 3.9 *Let M be strongly extendible. Then:*

(i) $M \prec_{\Sigma_1} V$. Therefore, in LZFC^{sext} *every set belongs to some model $M \prec_{\Sigma_1} V$.*

(ii) Actually $M \prec_{\Sigma_2} V$.

Proof. Fix a strongly extendible model M.

(i) Let $\phi = (\exists \overline{x})\psi(\overline{x}, \overline{c})$ be a Σ_1 sentence, with ψ bounded and $\overline{c} \in M$. Then clearly $M \models (\exists \overline{x})\psi(\overline{x}, \overline{c})$ implies $V \models (\exists \overline{x})\psi(\overline{x}, \overline{c})$. Conversely, assume $V \models (\exists \overline{x})\psi(\overline{x}, \overline{c})$, and let $V \models \psi(\overline{a}, \overline{c})$ for some tuple $\overline{a}$. By strong extendibility there is a model N such that $M \prec N$ and $\overline{a} \in N$. By the absoluteness of ψ, $N \models \psi(\overline{a}, \overline{c})$, which implies $N \models (\exists \overline{x})\psi(\overline{x}, \overline{c})$. Since $M \prec N$, it follows that $M \models (\exists \overline{x})\psi(\overline{x}, \overline{c})$.[3]

(ii) Let $\phi = (\exists \overline{x})(\forall \overline{y})\psi(\overline{x}, \overline{y}, \overline{c})$ be a Σ_2 sentence with ψ bounded and $\overline{c} \in M$, and let $M \models (\exists \overline{x})(\forall \overline{y})\psi(\overline{x}, \overline{y}, \overline{c})$. Then for some $\overline{a} \in M$, $M \models (\forall \overline{y})\psi(\overline{a}, \overline{y}, \overline{c})$. The sentence $(\forall \overline{y})\psi(\overline{a}, \overline{y}, \overline{c})$ is Π_1, so by (i) above, $V \models$

[3] Actually the proof of this clause does not require the full strength of the condition of strong extendibility. It is easy to see that the following weaker condition for M, called "Σ_1-strong extendibility", suffices: For every x there is a model N such that $x \in N$ and $M \prec_{\Sigma_1} N$. See section 6 for more on this property.

$(\forall\overline{y})\psi(\overline{a},\overline{y},\overline{c})$. Therefore $V\models(\exists\overline{x})(\forall\overline{y})\psi(\overline{x},\overline{y},\overline{c})$. Conversely, assume $V\models(\exists\overline{x})(\forall\overline{y})\psi(\overline{x},\overline{y},\overline{c})$. Then $V\models(\forall\overline{y})\psi(\overline{a},\overline{y},\overline{c})$ for some $\overline{a}$. By the strong extendibility of M there is a model N such that $\overline{a}\in N$ and $M\prec N$. Since N is a transitive submodel of V, $(\forall\overline{y})\psi(\overline{a},\overline{y},\overline{c})$ is Π_1, and the latter is true in V, it follows that $N\models(\forall\overline{y})\psi(\overline{a},\overline{y},\overline{c})$. Consequently, $N\models(\exists\overline{x})(\forall\overline{y})\psi(\overline{x},\overline{y},\overline{c})$. It follows that $M\models(\exists\overline{x})(\forall\overline{y})\psi(\overline{x},\overline{y},\overline{c})$. ⊣

Already Loc^{sext}(ZFC) partly restores ZFC. Namely:

Theorem 3.10 *(i)* $\mathrm{LZFC}^{sext}\vdash$ Powerset.
(ii) $\mathrm{LZFC}^{sext}\vdash\Sigma_1$-Collection.

Proof. (i) Let a be a set and let M be a strongly extendible model such that $a\in M$. Let $b=\mathcal{P}^M(a)$ be the powerset of a in M, i.e., $M\models b=\mathcal{P}(a)$. The predicate $y=\mathcal{P}(x)$ is Π_1, so by 3.9 (i), $V\models b=\mathcal{P}(a)$, so b is the absolute powerset of a.

(ii) Let $\phi(x,y,\overline{c})=(\exists\overline{z})\psi(x,y,\overline{z},\overline{c})$ be a Σ_1 formula with parameters $\overline{c}$, let a be a set and let $(\forall x\in a)(\exists y)\phi(x,y,\overline{c})$ be true (in V). We have to show that there is a set b such that $(\forall x\in a)(\exists y\in b)\phi(x,y,\overline{c})$. In LZFC^{sext} we can pick a model M such that $a,\overline{c}\in M$ and M is strongly extendible. By 3.9 (ii), $M\prec_{\Pi_2}V$ and by assumption $V\models(\forall x\in a)(\exists y)\phi(x,y,\overline{c})$, or

$$V\models(\forall x\in a)(\exists y)(\exists\overline{z})\psi(x,y,\overline{z},\overline{c}).$$

The last formula is Π_2, so $M\models(\forall x\in a)(\exists y)(\exists\overline{z})\psi(x,y,\overline{z},\overline{c})$. Since M satisfies Collection, there is a $b\in M$ such that

$$M\models(\forall x\in a)(\exists y\in b)(\exists\overline{z})\psi(x,y,\overline{z},\overline{c}).$$

By $M\prec_{\Pi_2}V$ again,

$$V\models(\forall x\in a)(\exists y\in b)(\exists\overline{z})\psi(x,y,\overline{z},\overline{c}),$$

i.e., $V\models(\forall x\in a)(\exists y\in b)\phi(x,y,\overline{c})$, as required. ⊣

Further, sets $V_\alpha=\mathcal{P}^\alpha(\emptyset)$ can be defined in LZFC^{sext} for every $\alpha\in On$ without the help of induction along On, i.e., Found_{On}. Moreover, as in the case of ZFC, V_α are set approximations of V and include all strongly extendible models.

Proposition 3.11 *In* LZFC^{sext} *the following hold:*

(i) Given any specific ordinal α, for all $\beta\leq\alpha$, there are transitive sets $V_\beta=\mathcal{P}^\beta(\emptyset)$ with the usual ZFC *properties, namely, $V_{\beta+1}=\mathcal{P}(V_\beta)$ and $V_\beta=\bigcup_{\gamma<\beta}V_\gamma$, for limit β. Moreover, for every strongly extendible model M such that $\alpha\in M$, $M_\beta:=V_\beta^M=V_\beta$, for all $\beta\leq\alpha$.*

(ii) The predicate "$x = V_\alpha$" is definable, therefore so is the class $\{V_\alpha : \alpha \in On\}$.

(iii) For every strongly extendible M, there is α such that $M = V_\alpha$.

(iv) $V = \bigcup_\alpha V_\alpha$. Moreover there are arbitrarily large α such that $V_\alpha \prec_{\Sigma_2} V$.

Proof. (i) Given an ordinal α pick, by Loc^{sext}(ZFC), a strongly extendible M such that $\alpha \in M$. By Theorem 3.10 (i) and induction inside M,

$$M_\beta = V_\beta^M = (\mathcal{P}^\beta(\emptyset))^M = \mathcal{P}^\beta(\emptyset) = V_\beta$$

for all $\beta \leq \alpha$.

(ii) V_α can be defined by induction up to α inside any strongly extendible model containing α, the steps of which will be absolute because of 3.10 (i). Namely "$x = V_\alpha$" is written:

$$(\exists f)[dom(f) = \alpha + 1 \ \wedge \ (\forall \beta < \alpha)(f(\beta + 1) = \mathcal{P}(f(\beta)) \wedge$$

$$(\forall \beta \leq \alpha)(\beta \text{ limit } \Rightarrow f(\beta) = \bigcup_{\gamma < \beta} f(\gamma)) \ \wedge \ f(\alpha) = x].$$

(iii) Let M be strongly extendible. We can show that $\alpha = \min\{\beta : \beta \notin M\}$, the height of M, exists without invoking Found_{On}. Indeed, given M just take a model N, by Loc(ZFC), such that $M \in N$. Then clearly $\alpha = \min\{\beta : \beta \notin M\}$ exists in N. By (i) above V_α exists too and for every $\beta \in M$, $V_\beta = M_\beta$, therefore

$$V_\alpha = \bigcup_{\beta < \alpha} V_\beta = \bigcup_{\beta < \alpha} M_\beta = M.$$

(iv) Given x, by Loc^{sext}(ZFC) there is a strongly extendible M such that $x \in M$. But by (iii) above, $M = V_\alpha$ for some α, thus $\bigcup_\alpha V_\alpha = V$. Also, by 3.9 (ii), $V_\alpha \prec_{\Sigma_2} V$ for every strongly extendible V_α. ⊣

What is the consistency strength of Loc^{sext}(ZFC)? Surprisingly enough, this is no higher than the existence of a strongly inaccessible cardinal. Namely the following holds.

Proposition 3.12 (ZFC) If κ is strongly inaccessible, then $V_\kappa \models Loc^{sext}$(ZFC).

Proof. It is well-known that if κ is strongly inaccessible, then the set $C = \{\alpha < \kappa : (V_\alpha, \in) \prec (V_\kappa, \in)\}$ is closed unbounded (see [4, p. 171]). Given $\alpha < \beta$ in C, we have $V_\alpha \prec V_\kappa$ and $V_\beta \prec V_\kappa$, and therefore $V_\alpha \prec V_\beta$. Since C is cofinal in κ, it follows immediately that for every $\alpha \in C$, V_α is strongly extendible in V_κ. Also every $x \in V_\kappa$ belongs to V_α, for some $\alpha \in C$, so $V_\kappa \models Loc^{sext}$(ZFC). ⊣

One further question: For which cardinals κ of ZFC is V_κ strongly extendible? Note that if in the proof of 3.12 κ is the *least* strongly inaccessible cardinal of the universe, then for each $\alpha \in C$, V_α is a strongly extendible model, and moreover $V_\alpha \prec V_\kappa$, without α being strongly inaccessible. Therefore strong extendibility of V_κ, in the context of ZFC, does not presume even the inaccessibility of κ. However if we need a *sufficient* condition about κ in order for V_κ to be strongly extendible, we must go up to strong cardinals. Recall the following.

Definition 3.13 (ZFC) A cardinal κ is *α-strong* if there is an elementary embedding $j : V \to W$ such that $\kappa = \mathrm{crit}(j)$, $j(\kappa) > \alpha$ and $V_{\kappa+\alpha} \subseteq W$. κ is *strong* if it is α-strong for every α.

Lemma 3.14 (ZFC) *If κ is a strong cardinal then V_κ is a strongly extendible model.*

Proof. Let κ be strong and x be a set. We have to find a model M such that $V_\kappa \prec M$ and $x \in M$. Let $\alpha > \kappa$ be such that $x \in V_\alpha$ and $\kappa + \alpha = \alpha$. Since κ is α-strong, there is $j : V \to W$ with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \alpha$ and $V_\alpha = V_{\kappa+\alpha} \subseteq W$. Therefore $V_\beta = W_\beta$ for every $\beta \leq \alpha$. Since $\mathrm{crit}(j) = \kappa$, $V_\kappa \prec j(V_\kappa) = W_{j(\kappa)}$. Also $V_\alpha = W_\alpha \subset W_{j(\kappa)}$, thus $x \in W_{j(\kappa)}$. So for $M = W_{j(\kappa)}$, M is as required. $\dashv$

4 Elementary embeddings and critical models

4.1 Elementary embeddings

In this section we define models of ZFC with properties analogous to those of some large cardinals, employing elementary embeddings. Let M, N be models of ZFC and let $j : M \to N$ be an elementary embedding. If j is non-trivial, then there is a critical ordinal $\mathrm{crit}(j)$ for j, i.e., a least ordinal $\alpha \in M$ such that $j(\alpha) > \alpha$. Moreover we can see that $\mathrm{crit}(j)$ is inaccessible in M. (This is essentially proved in [2].) More generally, let us call a set x *critical* for j, if $j{\restriction}x = id$ while $j(x) \neq x$. Let $\mathrm{Crit}(j)$ be the set of critical sets of j. In particular $\mathrm{crit}(j) \in \mathrm{Crit}(j)$. In view of the axiom *Loc*(ZFC), given M, N and $j : M \to N$, $\mathrm{Crit}(j)$ is a set in LZFC because it can be (absolutely) defined in every model K of ZFC such that $\{M, N, j\} \subset K$. The following two lemmas contain some basic facts about elementary embeddings between arbitrary models of ZFC. The proofs are mostly standard and scattered (usually in the form of exercises) in [4], [5], etc. For the reader's convenience we outline here some of them.

Lemma 4.1 (LZFC) *Let $j : M \to N$ be a nontrivial elementary embedding between models of* ZFC*. Then:*

(i) $\mathrm{crit}(j)$ *exists.*

(ii) If $\kappa = \mathrm{crit}(j)$*, then* $j{\restriction}M_\kappa = id$ *(where as usual* $M_\kappa = V_\kappa^M$*). Therefore* $M_\kappa \in \mathrm{Crit}(j)$*. In particular, for every* $A \in M_\kappa$*,* $\mathcal{P}(A)^M = \mathcal{P}(A)^N$*. Conversely, if* $K \in \mathrm{Crit}(j)$ *is a model, then* $ht(K) = \kappa$*.*

(iii) If $\kappa = \mathrm{crit}(j)$*, then* M_κ *is the greatest transitive critical set of* M*, in fact if* x *is a transitive set of* M *and* $j{\restriction}x = id$*, then* $x \subseteq M_\kappa$*.*

(iv) If $\kappa = \mathrm{crit}(j)$*, then* $\mathcal{P}(\kappa)^M \subseteq \mathcal{P}(\kappa)^N$*.*

(v) If $\kappa = \mathrm{crit}(j)$*, then* $M \models$ *"*κ *is strongly inaccessible". Therefore* $M_\kappa \models \mathrm{ZFC}$*. If in addition* $N = M$*, then* $M \models$ *"*κ *is* n*-ineffable", for every* $n \in \omega$*.*[4]

(vi) For every first-order structure $A \in M$*,* $j{\restriction}A : A \to j(A)$ *is an elementary embedding. If in addition,* $A \in \mathrm{Crit}(j)$*, then* $A \prec j(A)$*. In particular, for the critical* κ*,* $M_\kappa \prec j(M_\kappa)$*.*

Proof. (i) For ZFC, the proof that j has a critical point is given in [5, Prop. 5.1 (b)] for the case $M \models \mathrm{AC}$. To show this in LZFC, given M, N, j, simply pick by *Loc*(ZFC) a model $K \models \mathrm{ZFC}$ such that $M, N, j \in K$ and work in K.

Clauses (ii)-(iv) are left to the reader.

(v) Let $\kappa = \mathrm{crit}(j)$. In [2, p. 338] it is proved (in ZFC) that if V is a model of ZFC and $j : V \to V$ is an elementary embedding, in general external with respect to V, and $\mathrm{crit}(j) = \kappa$, then κ is inaccessible. The same proof actually works for general elementary embeddings $j : M \to N$. Indeed, let $j : M \to N$ be a nontrivial elementary embedding and let $\kappa = \mathrm{crit}(j)$. We show first that $M \models$ "κ is regular". Obviously κ is a limit ordinal. Assume on the contrary that there is $\alpha < \kappa$ and a cofinal $f : \alpha \to \kappa$, $f \in M$. Then $dom(f) = \alpha$, therefore $dom(j(f)) = j(\alpha) = \alpha$. Moreover, for every $\beta < \alpha$, if $f(\beta) = \gamma$, then $\gamma < \kappa$, so $j(\beta) = \beta$ and $j(\gamma) = \gamma$. Therefore, $j(f)(\beta) = j(f)(j(\beta)) = j(f(\beta) = j(\gamma) = \gamma = f(\beta)$. Thus $dom(j(f)) = dom(f)$ and $j(f)(\beta) = f(\beta)$ for every $\beta \in dom(f)$, so $j(f) = f$. But since $f''\alpha$ is cofinal in κ, $j(f)''\alpha = f''\alpha$ should be cofinal in $j(\kappa) > \kappa$, a contradiction.

Next suppose κ is not a strong limit in M, i.e., there is $\alpha < \kappa$ such that $M \models |\mathcal{P}(\alpha)| \geq \kappa$. Now since $\alpha < \kappa$, by (ii) above, $\mathcal{P}(\alpha)^M = \mathcal{P}(\alpha)^N$. Let $g \in M$ be a surjection $g : \mathcal{P}(\alpha)^M \to \kappa$. Since $j(\mathcal{P}(\alpha)^M) = \mathcal{P}(\alpha)^N = \mathcal{P}(\alpha)^M$, it follows as before that $j(g) = g$. But then $j(g) = g$ must be a surjection of $\mathcal{P}(\alpha)^N$ onto $j(\kappa) > \kappa$, a contradiction. This completes the proof that κ is strongly inaccessible in M. Therefore $M \models$ "V_κ is a model of ZFC", which means that $M_\kappa \models \mathrm{ZFC}$. Finally, if $N = M$, then the proof of Theorem 2.12 of [2] works also here, showing that the critical κ is n-ineffable for every $n \in \omega$.

[4] A cardinal λ is *n-ineffable* if for every partition $f : [\lambda]^{n+1} \to \{0, 1\}$ there is a stationary homogeneous set $H \subseteq \lambda$.

(vi) For every formula $\phi(x_1,\dots,x_n)$ of the language of A, and every $a_1,\dots,a_n \in A$, in view of the absoluteness of satisfaction we have:

$$A \models \phi(a_1,\dots,a_n) \Leftrightarrow M \models (A \models \phi(a_1,\dots,a_n)) \Leftrightarrow$$

$$N \models (j(A) \models \phi(j(a_1),\dots,j(a_n)) \Leftrightarrow j(A) \models \phi(j(a_1),\dots,j(a_n)),$$

which means that $j{\restriction}A : A \to j(A)$ is an elementary embedding. If $A \in \mathrm{Crit}(j)$, then $j(a_i) = a_i$ for $a_i \in A$, while $j(A) \neq A$. Thus $A \models \phi(a_1,\dots,a_n) \Leftrightarrow j(A) \models \phi(a_1,\dots,a_n)$, i.e., $A \prec j(A)$. $\dashv$

Definition 4.2 A model M of ZFC is said to be *elementarily embeddable* or just *embeddable,* if there is a non-trivial elementary embedding $j : M \to N$ for some N. M is said to be *inner-embeddable* if there is an elementary embedding $j : M \to N \subseteq M$. Finally, M is *self-embeddable* if there is a (non-trivial) elementary embedding $j : M \to M$.

The relationship between elementary embeddability and elementary extendibility (considered in the previous section) is not quite clear. One might say that extendibility is stronger since if M is extendible and $M \prec N$, then M is also embeddable in N with respect to the trivial embedding *id*. But the essence of embeddability is exactly the existence of a *non-trivial* embedding which gives rise to critical points. The two notions are probably incomparable.

As we have seen in 3.7, if M is elementarily extendible then $M \models Loc^{\pi^1_1 ind}(\mathrm{ZFC})$. But if M is embeddable or even self-embeddable, M is unlikely to satisfy even $Loc(\mathrm{ZFC})$. However the following holds.

Lemma 4.3 *(i) If M is embeddable, then $M \models$"there is a model x such that $x \models \mathrm{ZFC} + Loc(\mathrm{ZFC})$".*

(ii) If M is self-embeddable and $j : M \to M$ is an embedding with $\kappa = \mathrm{crit}(j)$, then the cardinals $j^n(\kappa)$ are n-ineffable in M.

Proof. Let $j : M \to N$ be an elementary embedding. By 4.1 (v), if $\kappa = \mathrm{crit}(j)$, $M \models$ "κ is strongly inaccessible". But then $M \models (V_\kappa \models \mathrm{ZFC} + Loc(\mathrm{ZFC}))$. That is, the model $x \in M$ in question such that $x \models \mathrm{ZFC} + Loc(\mathrm{ZFC})$ is M_κ.

(ii) By 4.1 (v). $\dashv$

Elementary embeddability is formalized by the following predicate:

$$emb(x) := [x \models \mathrm{ZFC} \ \wedge \ (\exists y)(\exists j)(y \models \mathrm{ZFC} \ \wedge \ j : x \to y \text{ is el. emb.})].$$

4.2 Critical and strongly critical models

Definition 4.4 M is said to be *critical* if there are models N, K such that $M \in N$ and an elementary embedding $j : N \to K$ such that $M \in \mathrm{Crit}(j)$.

The property is formalized also be the Σ_1^{ZFC} predicate:

$$crit(x) := [x \models \text{ZFC} \ \wedge \ (\exists y)(\exists z)(\exists j)$$

$$(x \in y \ \wedge \ y \models \text{ZFC} \ \wedge \ z \models \text{ZFC} \ \wedge \ j : y \to z \text{ is el. emb} \wedge \ x \in \text{Crit}(j))].$$

Critical models are roughly analogues of measurable cardinals of ZFC. And their consistency strength is no greater than that of the latter.

Lemma 4.5 (ZFC) *If κ is a measurable cardinal, then V_κ is critical.*

Proof. Let κ be a measurable cardinal in ZFC, and let $j : V \to W$ be an elementary embedding (where $W \subseteq V$)), with $\text{crit}(j) = \kappa$. Then clearly $j{\restriction}V_\kappa = id$, while $j(V_\kappa) \neq V_\kappa$. So it suffices to find models M, N such that $j{\restriction}M : M \to N$ and $V_\kappa \in M$. Set $M = j(V_\kappa) = W_{j(\kappa)}$, and $N = j(M)$. Then, clearly M, N are models of ZFC since V_κ is so, $V_\kappa = W_\kappa \in W_{j(\kappa)} = M$, and if $j' = j{\restriction}M$, then $j' : M \to N$ is an elementary embedding with $V_\kappa \in \text{Crit}(j')$. ⊣

Lemma 4.6 (LZFC) *If M is critical, then M is extendible. Therefore over* LZFC*, Loc^{crit}*(ZFC) $\Rightarrow$ *Loc^{ext}*(ZFC).

Proof. Let M be critical in LZFC. Then there are N, K, j such that $M \in N$, $j : N \to K$ is an elementary embedding and $M \in \text{Crit}(j)$. By Lemma 4.1 (vi), $M \prec j(M)$. Therefore M is extendible. ⊣

The following strengthening of criticalness is reasonable:

Definition 4.7 Let M be a model and x be a set. M is said to be x-*strongly critical* if there are models N, K and an elementary embedding $j : N \to K$, such that $\{M, x\} \subset N$ and $M \in \text{Crit}(j)$. M is said to be *strongly critical* if it is x-strongly critical for every x.

Here is a simpler characterization of strong criticalness.

Lemma 4.8 (LZFC) *M is strongly critical iff for every model N such that $M \in N$, there is a model K and an elementary embedding $j : N \to K$ such that $M \in \text{Crit}(j)$.*

Proof. The condition is necessary. Let M be strongly critical and let N be a model such $M \in N$. By definition M is N-critical, i.e., there are models R, S with $\{M, N\} \subset R$ and an elementary embedding $j : R \to S$ such that $M \in \text{Crit}(j)$. Let $j' = j{\restriction}N$ and $K = j(N)$. Then, according to Lemma 4.1 (vi), $j' : N \to K$ is an elementary embedding with $M \in \text{Crit}(j')$. Conversely, suppose the condition holds for M and let x be a set. By *Loc*(ZFC) there is a model N of ZFC such that $\{M, x\} \subset N$. By our condition there are K and an elementary embedding $j : N \to K$ with $M \in \mathrm{Crit}(j)$. Thus M is x-strongly critical for every x, and therefore is strongly critical. $\dashv$.

Proposition 4.9 (LZFC) *Let M be a strongly critical model of height κ. Then:*

(i) For every ordinal $\alpha > \kappa$, L_α is embeddable into some L_β with critical point κ. In particular there is a cofinal class $C \subseteq On$, such that for every $\alpha \in C$, L_α is a model of ZFC *and L_α is embeddable into some L_β with critical point κ.*

(ii) If in addition $V = L$, then there is a cofinal class $C \subseteq On$, such that for every $\alpha \in C$, L_α is a model of ZFC*, $M \in L_\alpha$ and L_α is embeddable into some L_β with critical model M.*

Proof. (i) Suppose M is a strongly critical model of height κ. Let $\alpha > \kappa$. By strong criticalness, there are models N, K such that $\{\alpha, M\} \subset N$ and an elementary embedding $j : N \to K$ such that $\mathrm{Crit}(j) = M$; hence $\mathrm{crit}(j) = \kappa$. Then $L_\alpha \in N$. If $j' = j \restriction L_\alpha$, then by Lemma 4.1 (vi), $j' : L_\alpha \to L_{j(\alpha)}$ is a nontrivial elementary embedding with $\mathrm{crit}(j) = \kappa$. As for the second claim, let

$$C = \{\alpha \in On : \alpha > \kappa \wedge L_\alpha \models \mathrm{ZFC}\}.$$

By *Loc*(ZFC), C is cofinal in On. By the preceding argument, for every $\alpha \in C$, there is a model N such that $\alpha \in N$ and there is an embedding $j : N \to K$ such that $\kappa = \mathrm{crit}(j)$. Then $j \restriction L_\alpha : L_\alpha \to L_{j(\alpha)}$ has also critical point κ.

(ii) If $V = L$, then $M \in L$ and if we set

$$C = \{\alpha \in On : M \in L_\alpha \wedge L_\alpha \models \mathrm{ZFC}\},$$

then by the same argument as above, C is a required. The difference now is that if $\alpha \in C$ and $j \restriction L_\alpha : L_\alpha \to L_{j(\alpha)}$, then $M \in \mathrm{Crit}(j)$ instead of just $\kappa = \mathrm{crit}(j)$. $\dashv$

The assumption that there is a strongly critical model is no stronger (over ZFC) than *Loc*(ZFC)+ "there is a measurable cardinal".

Lemma 4.10 ZFC+*Loc*(ZFC)+"κ is a measurable cardinal" *proves that V_κ is strongly critical. In particular the same is proved in* ZFC+"there is a proper class of inaccessibles"+ "κ is a measurable cardinal"*, as well as in the theory* ZFC+"there is a proper class of V_α such that $V_\alpha \models$ ZFC+"κ is a measurable cardinal".[5]

[5] In [7, Prop. 2.23] it is observed that *Loc*(ZFC) is no stronger than "there is a proper class of inaccessibles", where the last principle is denoted by IC^∞. Also in [7, footnote 6]

Proof. Let κ be measurable and let x be a set. Let $j : V \to W$ be an elementary embedding of the universe with $\kappa = \mathrm{crit}(j)$. By *Loc*(ZFC) there is a model $N \models$ ZFC such that $\{V_\kappa, x\} \subset N$. Let $j' = j{\restriction}N$. Then $j(N)$ is also a model of ZFC and $j' : N \to j(N)$ is an elementary embedding with $V_\kappa \in \mathrm{Crit}(j)$. Thus V_κ is x-strongly critical for every x. ⊣

It seems unlikely that one can prove the inference of 4.10 without the assumption *Loc*(ZFC), i.e., to prove that strongly critical models exist in ZFC+"there is a measurable cardinal" alone. This however can be done in ZFC+"there is a strong cardinal" (see definition 3.13).

Lemma 4.11 (ZFC) *If κ is a strong cardinal then V_κ is strongly critical.*

Proof. Let κ be strong and let x be a set. It suffices to show that V_κ is x-strongly critical. Let V_α be such that $\{x, V_\kappa\} \subset V_\alpha$. Since κ is α-strong, there is a $j : V \to W$, where W is an inner model, such that $j(\kappa) > \alpha$ and $V_{\kappa+\alpha} \subseteq W$. Then $j(\kappa)$ is inaccessible in W, therefore $W_{j(\kappa)} \models$ ZFC. Also for every $\beta \leq \kappa + \alpha$, $V_\beta = W_\beta$, therefore $V_\alpha = W_\alpha \subseteq W_{j(\kappa)}$. Let $K = W_{j(\kappa)}$. Then $x, V_\kappa \in K$. If $j' = j{\restriction}K$ and $N = W_{j^2(\kappa)}$, then the models K, N and the elementary embedding $j' : K \to N$ witness the fact that V_κ is x-strongly critical. ⊣

The notions of critical and strongly critical model can be strengthened even further by demanding the elementary embedding to be an inner or self-embedding.

Definition 4.12 Let M be a model and x be a set. M is said to be *x-self-critical* (resp. *x-inner critical*) if there is a model N and an elementary embedding $j : N \to N$ (resp. if there are models N, K such that $K \subseteq N$ and an elementary embedding $j : N \to K$), such that $\{M, x\} \subset N$ and $M \in \mathrm{Crit}(j)$. M is said to be *strongly self-critical* (resp. *strongly inner critical*) if it is x-self-critical (resp. x-inner critical) for every x.

The consistency strength of the existence of strongly inner-critical models is not very high. Namely the following holds.

Lemma 4.13 (ZFC) *If κ is a measurable cardinal and there exists a proper class of V_α's such that $V_\alpha \models$ ZFC (i.e., if NM^∞ holds), then V_κ is a strongly inner critical model.*

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we denoted by NM the axiom "there is a natural model" (i.e., a V_α such that $V_\alpha \models$ ZFC). Let us denote by NM^∞ the axiom "there is a proper class of natural models". It is clear that over ZFC we have $IC^\infty \Rightarrow NM^\infty \Rightarrow$ *Loc*(ZFC). Under mild large cardinal assumptions these arrows cannot be reversed.

Proof. We have to show that for every x there is a model M such that $\{V_\kappa, x\} \subseteq M$ and a $j : M \to N \subseteq M$ with $V_\kappa \in \mathrm{Crit}(j)$. Let U be a κ-complete ultrafilter on κ. By NM^∞ there is a V_α such that $\{V_\kappa, x, U\} \subseteq V_\alpha$ and $V_\alpha \models \mathrm{ZFC}$. Then κ is measurable in V_α, so there is an elementary embedding $j : V_\alpha \to V_\alpha^\kappa/U \subseteq V_\alpha$ with $\mathrm{crit}(j) = \kappa$. Therefore $V_\kappa \in \mathrm{Crit}(j)$. ⊣

However the consistency strength of the existence of strongly self-critical models seems to be much higher. We shall prove their consistency assuming the existence (in ZFC) of some kind of rank-to-rank elementary embeddings. Recall that a cardinal κ is I_3, denoted $\mathrm{I}_3(\kappa)$, if there is an elementary embedding $j : V_\lambda \to V_\lambda$ with $\mathrm{crit}(j) = \kappa$ (see [5, p. 325]). I_3 cardinals are n-huge for every $n < \omega$. Yet we need something stronger.

Definition 4.14 (ZFC) We say that a cardinal κ is *strongly* I_3*,* denoted by $\mathrm{SI}_3(\kappa)$, if for every $\alpha > \kappa$ there is a $\lambda \geq \alpha$ and an elementary embedding $j : V_\lambda \to V_\lambda$ with $\mathrm{crit}(j) = \kappa$.

Lemma 4.15 (ZFC) *If* $\mathrm{SI}_3(\kappa)$*, then* V_κ *is strongly self-critical.*

Proof. Suppose $\mathrm{SI}_3(\kappa)$. Then, given x, there is a $\lambda > \kappa$, $rank(x)$, and an elementary embedding $j : V_\lambda \to V_\lambda$ with $\mathrm{crit}(j) = \kappa$. But then $V_\kappa \in \mathrm{Crit}(j)$. Hence V_κ is x-self-critical for every x, thus V_κ is strongly self-critical. ⊣

4.3 $\mathrm{ZFC} + Loc(\mathrm{ZFC}) + \cdots$

As already pointed out in [7, p. 584], $\mathrm{ZFC} + Loc(\mathrm{ZFC})$ is a mild substitute of the theory ZFC + "there is a proper class of inaccessible cardinals". Accordingly, $\mathrm{ZFC} + Loc^{\pi^1_1 ind}(\mathrm{ZFC})$ is a mild substitute of ZFC + "there is a proper class of weakly compact cardinals". Mild here means "local", i.e., with no reference to large cardinals. More generally, for $\phi(x)$ a large model property, the theories $\mathrm{ZFC} + Loc^\phi(\mathrm{ZFC})$ and $\mathrm{ZFC} + Loc(\mathrm{ZFC}) + \exists x\phi(x)$, or $\mathrm{ZFC} + Loc(\mathrm{ZFC})$ together with a combination of the axioms $Loc^\phi(\mathrm{ZFC})$ and $\exists x\phi(x)$, and possibly augmented with Found_{On} or some piece of Replacement, seem to be worth studying. What would one expect to prove in such theories? The following could be among the expected results:

(a) Existence of models of ZFC with special closure properties, e.g. natural models.

(b) Existence of "internally strong" models, i.e., satisfying ZFC+ a large cardinal property. (As it follows from Lemma 4.1 (v), $\mathrm{ZFC} + Loc(\mathrm{ZFC})$+"there is an embeddable model" proves that there is a model $M \models$ ZFC+"there is a strongly inaccessible cardinal".)

(c) Existence (i.e., restoration) of large cardinals.

(d) Results having some impact on everyday mathematics. (Such results are least likely to be proved if we judge by the analogous capabilities of the classical large cardinal axioms.)

The result below belongs to group (a) above.

Proposition 4.16 ZFC + *Loc*(ZFC)+"there is a V_α-strongly critical model M"*, for some* V_α *such that* $M \subseteq V_\alpha$*, proves that* "there is a natural model of ZFC"*.*

Proof. Let M be V_α-critical such that $M \subseteq V_\alpha$ in ZFC+*Loc*(ZFC), for some α. Then there are models N, K and an elementary embedding $j : N \to K$ such that $M \in \mathrm{Crit}(j)$ and $V_\alpha \in N$. It follows that $N_\xi = V_\xi$ for every $\xi \leq \alpha$. Let $ht(M) = \beta$. Since $M \subseteq V_\alpha$, we have that $\beta \leq \alpha$, therefore $N_\beta = V_\beta$. Moreover $\mathrm{crit}(j) = \beta$ and $N_\beta \in \mathrm{Crit}(j)$. By Lemma 4.1 (v), $N_\beta \models$ ZFC. It follows that $V_\beta \models$ ZFC, i.e., ZFC has a natural model. [Alternatively: Since $V_\beta \in \mathrm{Crit}(j)$, we have $V_\beta = N_\beta = K_\beta$. It follows that $j(V_\beta) = K_{j(\beta)}$ and $V_\beta = K_\beta \in K_{j(\beta)}$, since $j(\beta) > \beta$. Thus $V_\beta \prec K_{j(\beta)}$ and $V_\beta \in K_{j(\beta)}$. So by Lemma 3.5, $V_\beta \models$ ZFC.] ⊣

The next result strengthens the previous one and belongs to group (c).

Proposition 4.17 ZFC+*Loc*(ZFC)+"there is a strongly critical model" *proves that there is a strongly inaccessible cardinal.*

Proof. Let κ be a cardinal. By the Reflection principle of ZFC, the class of ordinals α such that for every $\kappa < \alpha$, $V_\alpha \models$ "κ is strongly inaccessible" implies κ is strongly inaccessible, is a proper class. So given a strongly critical model M, there is α such that $M \in V_\alpha$ and for every κ,

$$V_\alpha \models \text{"}\kappa \text{ is strongly inaccessible"} \;\Rightarrow\; \kappa \text{ is strongly inaccessible.} \quad (5)$$

Fix such a V_α. By strong criticalness of M, there are models N, K and an elementary embedding $j : N \to K$ such that $V_\alpha \in N$ (so $\{M, V_\alpha\} \subset N$) and $M \in \mathrm{Crit}(j)$. Let $\kappa = ht(M)$. Then $\kappa = \mathrm{crit}(j)$ and by Lemma 4.1 (v), $N \models$ "κ is strongly inaccessible". Since $V_\alpha \in N$, a fortiori $V_\alpha \models$ "κ is strongly inaccessible". So by (5), κ is strongly inaccessible. ⊣

Let $scrit(x)$ be the formula expressing "x is a strongly critical model". A rather immediate consequence of the proof of 4.17 is the following.

Corollary 4.18 ZFC+Loc^{scrit}(ZFC) *proves "there is a proper class of inaccessible cardinals".*

Proposition 4.19 ZFC+Loc(ZFC)+ *"there is a strongly self-critical model" proves that there is a cardinal κ which is n-ineffable for every $n \in \omega$.*

Proof. The proof is similar to that of 4.17. Given a strongly self-critical M, let $\kappa = ht(M)$. By Reflection there is V_α such that $M \in V_\alpha$, V_α contains all mappings $f : [\kappa]^n \to \{0,1\}$, for every n, and

$$V_\alpha \models \text{"}(\forall n)(\kappa \text{ is } n\text{-ineffable})\text{"} \Rightarrow (\forall n)(\ \kappa \text{ is } n\text{-ineffable}). \qquad (6)$$

It is clear that if some function $f : [\kappa]^n \to \{0,1\}$ has a stationary homogeneous subset $X \subseteq \kappa$, then $X \in V_\alpha$. Let N be a model such that $V_\alpha \in N$. By strong self-criticalness there is an elementary embedding $j : N \to N$ such that $M \in \mathrm{Crit}(j)$. Since $\kappa = ht(M)$, it follows that $\kappa \in \mathrm{crit}(j)$. By Lemma 4.1 (v), $N \models$ "κ is n-ineffable" for every $n \in \omega$. Then, by the way we chose N it is clear that $V_\alpha \models$ "κ is n-ineffable" for every $n \in \omega$. So, by (6), κ is n-ineffable for every $n \in \omega$. $\dashv$

Let $sscrit(x)$ be the formula expressing "x is a strongly self-critical model". As a consequence of the proof of Proposition 4.19 we have:

Corollary 4.20 ZFC+Loc^{sscrit}(ZFC) *proves "there is a proper class of n-ineffable cardinals", for every $n \in \omega$.*

5 LZFC and $V = L$. The Tall Model Axiom

In this section we examine the relationship of LZFC with $V = L$ and especially with $V \neq L$. We shall need below the following well-known facts of ZFC (see e.g. [4, Th.18.20], [3, Th. 4.3] and [5, Th. 9.12, 9.17]).

Theorem 5.1 (ZFC) *(i) The following are equivalent:*

(a) $0^\#$ exists.

(b) There is an elementary embedding $j : L \to L$.

(c) There is an elementary embedding $j : L_\alpha \to L_\beta$, where α, β are limit ordinals, with $\mathrm{crit}(j) = \kappa < |\alpha|$.

(ii) If $0^\#$ exists, then there is a proper class of ordinals α such that $L_\alpha \prec L$.

(iii) If $0^\#$ exists, then for every infinite $x \in L$, $|\mathcal{P}(x)^L| = |x|$.

First concerning the consistency of LZFC + $V = L$, let us observe the following:

Lemma 5.2 *(i) If* ZFC+"there is an inaccessible" *is consistent, then so is* ZFC+Loc(ZFC) $+ V = L$.

(ii) If ZFC+"$0^{\#}$ exists" *is consistent, then so is* ZFC+Loc^{sext}(ZFC) $+ V = L$.

Proof. (i) It was seen in [7, Pr. 2.23 (ii)] that if κ is a strongly inaccessible cardinal in the ZFC universe V, then $V_\kappa \models \mathrm{ZFC} + Loc(\mathrm{ZFC})$. From this it is straightforward that $L_\kappa \models \mathrm{ZFC} + Loc(\mathrm{ZFC}) + V = L$.

(ii) Let M be a model of ZFC+"$0^{\#}$ exists". It suffices to see that $L^M \models Loc^{sext}(\mathrm{ZFC})$. This follows from the fact that, by 5.1 (ii), there is in M a cofinal class of models L_α such that $L_\alpha \prec L^M$. Then, as in the proof of Proposition 3.12, all these models are strongly extendible, so $L^M \models Loc^{sext}(\mathrm{ZFC})$. ⊣

In view of the fact that existence of $0^{\#}$ is the standard means to refute $V = L$ over ZFC, the question is what "large model axioms" are needed in order to refute $V = L$ over LZFC. By 5.2 (ii), $V = L$ is consistent even with strengthenings of Loc(ZFC), like Loc^{sext}(ZFC).

So below we shall try some new axioms giving information about the internal truths of local models, e.g., about how they see the cardinalities of certain sets. For instance suppose a set x is given. It is a straightforward consequence of Loc(ZFC) that there is a set y and a model M of ZFC such that $\{x, y\} \subset M$ and $M \models |x| < |y|$. For simplicity, and without loss of generality we can deal just with ordinals instead of arbitrary sets. Thus the statement

$$(\forall\kappa)(\exists\alpha > \kappa)(\exists M)(\alpha \in M \wedge M \models |\kappa| < |\alpha|)$$

is a theorem of LZFC. Now let us make the question a bit harder. Let κ be given. Does there exist an $\alpha > \kappa$ such that for every $\delta \geq \alpha$ there is a model M such that $\delta \in M$ (equivalently $ht(M) > \delta$) and $M \models |\alpha| > |\kappa|$? This is the statement:

$$(\forall\kappa)(\exists\alpha > \kappa)(\forall\delta \geq \alpha)(\exists M)(\delta \in M \wedge M \models |\kappa| < |\alpha|). \qquad (7)$$

(7) says that for every κ there is an $\alpha > \kappa$ such that there are arbitrarily "tall" models which do not collapse α to κ. We shall refer to (7) as the *Tall Model Axiom*, or TMA for short. TMA is a principle of mild consistency strength as is seen by the following:

Lemma 5.3 *(i)* $\mathrm{ZFC} + Loc(\mathrm{ZFC}) \vdash TMA$.

(ii) If λ is a limit cardinal in $\mathrm{ZFC} + Loc(\mathrm{ZFC})$*, then* $H(\lambda) \models \mathrm{LZFC} + TMA$. *More generally, if* $N \models \mathrm{LZFC}$ *and* N *does not have a greatest cardinality, then* $N \models TMA$.

Proof. (i) Given κ, just pick an (absolute) cardinal α such that $\kappa < \alpha$. For every $\delta \geq \alpha$ pick, by Loc(ZFC), a model M such that $\delta \in M$. Then obviously $M \models |\kappa| < |\alpha|$.

(ii) Let λ be a limit cardinal in the theory ZFC + *Loc*(ZFC). That $H(\lambda)$ satisfies *Loc*(ZFC) follows from *Loc*(ZFC). Indeed, let $x \in H(\lambda)$, i.e., $|TC(x)| < \lambda$. By *Loc*(ZFC) there is a model M of ZFC such that $x \in M$. Then, by Löwenheim-Skolem there is an elementary submodel N of M containing $TC(x)$ and $|N| < \lambda$. If $\bar{N}$ is the transitive collapse of N, then $x \in \bar{N}$, $\bar{N} \models$ ZFC and $\bar{N} \in H(\lambda)$. Now let $\kappa_0 \in H(\lambda)$. Since λ is a limit cardinal, there is a cardinal $\mu \in H(\lambda)$ such that $\kappa_0 < \mu$. For every ordinal $\delta \geq \mu$ pick as before, by *Loc*(ZFC), a model K such that $\delta \in K$. Then obviously $K \models |\kappa_0| < |\mu|$. By Löwenheim-Skolem K can be chosen so that $|K| < \lambda$. Therefore $H(\lambda) \models TMA$. The proof of the more general statement is the same. ⊣

We can now see that if TMA is added to the theory LZFC+"there exists a strongly critical model", then $V = L$ fails.

Theorem 5.4 LZFC+TMA+"there exists a strongly critical model" *proves* $V \neq L$.

Proof. We assume LZFC+TMA+"there exists a strongly critical model"$+V = L$. It suffices to reach a contradiction. Let M be strongly critical with $ht(M) = \kappa_0$. By TMA there is an $\alpha_0 > \kappa_0$ such that

$$(\forall \delta \geq \alpha_0)(\exists N)(\delta \in N \wedge N \models |\kappa_0| < |\alpha_0|). \quad (8)$$

Since $V = L$, $M \in L$, so by proposition 4.9 (ii) there are limit ordinals β_0, γ_0 such that $\alpha_0 < \beta_0 \leq \gamma_0$ and an elementary embedding $j : L_{\beta_0} \to L_{\gamma_0}$ with $M \in \text{Crit}(j)$. Hence $\text{crit}(j) = \kappa_0$. By $V = L$, $j \in L$. Therefore $j \in L_\delta$ for some $\delta > \gamma_0$. Then it follows from (8) that there is a model N of ZFC which contains δ and $N \models |\kappa_0| < |\alpha_0|$, so $N \models |\kappa_0| < |\beta_0|$. A fortiori, $L^N \models |\kappa_0| < |\beta_0|$. Moreover $j \in L^N$ since $j \in L_\delta \in L^N$. Thus L^N is a model of ZFC that contains an elementary embedding $j : L_{\beta_0} \to L_{\gamma_0}$ with critical point κ_0 such that $L^N \models |\kappa_0| < |\beta_0|$. But then also $L^N \models \kappa_0 < |\beta_0|$, since the ordinals κ_0 and $|\kappa_0|^{L^N}$ are equipollent in L^N. By Theorem 5.1 (i), $L^N \models$ "$0^\#$ exists". This contradicts the fact that $L^N \models V = L$. ⊣

Question 5.5 *Can we remove TMA from the assumptions of the previous theorem, i.e., does* LZFC+"there exists a strongly critical model" *prove $V \neq L$?*

We have already seen in Lemma 5.3 (ii) that TMA holds in any universe of LZFC without greatest cardinality. So let us focus on this last property. Recall that an ordinal α is said to be a cardinal in LZFC, if (exactly as in ZFC) there is no $\beta < \alpha$ such that $\alpha \sim \beta$. Moreover, the (absolute) cardinal of x, denoted $|x|$, is the least α (if there exists) such that $x \sim \alpha$. However, due to the lack of $\in$-induction, we cannot ensure that for any given set x $|x|$ exists. But the symbol $|x|$ can keep being used in the expressions $|x| = |y|$, $|x| \leq |y|$, $|x| < |y|$ which have the ordinary meaning (that is, existence of bijections or injections between x, y, etc.) Below we shall denote by GC the statement "there is a greatest cardinality", i.e.,

$$(GC) \qquad (\exists\kappa)(\forall y)(|y| \leq |\kappa|).$$

A particular case of GC is the axiom $(\forall x)(|x| \leq \omega)$, which we often refer to as "all sets are countable".

Lemma 5.6 *The following are equivalent over* LZFC*:*
(i) GC *(:=* $(\exists\kappa)(\forall y)(|y| \leq |\kappa|)$*).*
(ii) $(\exists\kappa)(\forall\alpha)(|\alpha| \leq |\kappa|)$*.*
(iii) $(\exists x)(\forall\alpha)(|\alpha| \leq |x|)$*.*
(iv) $(\exists x)(\forall y)(|y| \leq |x|)$*.*

Proof. (i)$\Rightarrow$ (ii) and (ii)$\Rightarrow$ (iii) are trivial.

(iii)$\Rightarrow$(iv): Assume x_0 is a set satisfying (iii) and let y be any given set. We have to show that $|y| \leq |x_0|$. Working in a model M containing y, we can find an ordinal α such that $|\alpha| = |y|$. Then by (iii) $|\alpha| \leq |x_0|$. Therefore $|y| \leq |x_0|$.

(iv)$\Rightarrow$(i): Assume x_0 is a set satisfying (iv) and find an ordinal κ as before such that $|x_0| = |\kappa|$. Let y be any given set. By (iv) $|y| \leq |x_0|$, therefore $|y| \leq |\kappa|$. $\dashv$

Lemma 5.7 LZFC $+ \neg(GC) \vdash TMA$.

Proof. This has essentially been shown in Lemma 5.3 (ii). Suppose in LZFC there is no greatest cardinal, and let κ be an ordinal. Then there is an ordinal $\alpha > \kappa$ such that $|\kappa| < |\alpha|$. Then for every $\delta \geq \alpha$ and every model M such that $\delta \in M$, obviously $M \models |\kappa| < |\alpha|$. Therefore TMA is true. $\dashv$

The converse of 5.7 is open.

Question 5.8 *Is* LZFC $+ GC + TMA$ *consistent?*

What we can only prove here is the following:

Proposition 5.9 LZFC $+ GC + V = L \vdash \neg(TMA)$.
In particular,
LZFC $+$ "every set is countable" $+ V = L \vdash \neg(TMA)$.

Proof. Assume LZFC $+ GC + V = L$. We have to show $\neg(TMA)$, i.e., that

$$(\exists\kappa)(\forall\alpha > \kappa)(\exists\delta)(\forall M)(\delta \in M \Rightarrow M \models |\kappa| = |\alpha|) \qquad (9)$$

is true. By assumption there is an ordinal κ such that $\forall\alpha(|\alpha| \leq |\kappa|)$. Fix such a κ_0. It suffices to show that

$$(\forall\alpha > \kappa_0)(\exists\delta)(\forall M)(\delta \in M \Rightarrow M \models |\kappa_0| = |\alpha|). \tag{10}$$

Pick an $\alpha > \kappa_0$. Since $|\alpha| \leq |\kappa_0|$, there is (in V) a bijection $f : \kappa_0 \to \alpha$. Since $V = L$, $f \in L$. Hence there is a δ such that $f \in L_\delta$. Then δ witnesses the truth of (10), since if M is a model of ZFC such that $\delta \in M$, then $L_\delta \in M$, and so $f \in M$. Therefore $M \models |\alpha| = |\kappa_0|$. ⊣

It follows immediately from 5.7 and 5.9 that $\neg(TMA)$ and GC are equivalent over LZFC+$V = L$, i.e.,

Corollary 5.10 LZFC $+ V = L \vdash \neg(TMA) \Leftrightarrow GC$.

Corollary 5.11 *(i)* LZFC $+ \neg(GC) + \neg(TMA)$ *is inconsistent.*

(ii) LZFC$+\neg(GC)+TMA$ *is consistent, provided* ZFC+Loc(ZFC) *is so.*

(iii) LZFC $+ GC + \neg(TMA)$ *is consistent, provided* ZFC $+$ Loc(ZFC) $+ V = L$ *is so.*

Proof. (i) follows from Lemma 5.7. (ii) follows from Lemma 5.3 (ii). (iii) follows from Proposition 5.9. ⊣

Before closing this section let us consider the following reasonable variants of TMA:

(TMA_1) $\quad (\forall x)(\exists\alpha)(\forall\delta \geq \alpha)(\exists M)(\{x, \delta\} \subset M \wedge M \models |x| < |\alpha|)$.

(TMA_2) $\quad (\forall x)(\exists y)(\forall\delta)(\exists M)(\{x, y, \delta\} \subset M \wedge M \models |x| < |y|)$.

Concerning the relative strength of TMA, TMA_1 and TMA_2, it is immediate that $TMA_1 \Rightarrow TMA$ and $TMA_1 \Rightarrow TMA_2$, over LZFC.

Lemma 5.12 LZFC $+ \neg(GC) \vdash TMA_1$, *therefore* LZFC $+ \neg(GC) \vdash TMA_2$.

Proof. Similar to that of Lemma 5.7. ⊣

Proposition 5.13 *(i)* LZFC $+ GC + V = L \vdash \neg(TMA_2)$.

(ii) LZFC $+ GC + V = L \vdash \neg(TMA_1)$.

Proof. Similar to that of Proposition 5.9. ⊣

Corollary 5.14 TMA, TMA_1, TMA_2 *and* $\neg(GC)$ *are all equivalent over* LZFC $+ V = L$.

Proof. It follows immediately from 5.12 and 5.13. $\dashv$

Finally we include here a result of independent interest and potential applicability, saying that *Loc*(ZFC) is preserved under forcing extensions.[6]

Proposition 5.15 *Any generic extension of a model of* ZFC + *Loc*(ZFC) *is also a model of Loc*(ZFC).

Proof. Let $M \models \mathrm{ZFC} + Loc(\mathrm{ZFC})$, $\mathbb{P} \in M$ be a forcing notion, and $G \subseteq \mathbb{P}$ be M-generic. To show that $M[G] \models Loc(\mathrm{ZFC})$, pick $a \in M[G]$. It suffices to find $M_1 \in M[G]$ such that $M_1 \models \mathrm{ZFC}$ and $a \in M_1$. Let $\dot{a} \in M$ be a $\mathbb{P}$-name of a. Since $M \models Loc(\mathrm{ZFC})$ there is $N \in M$ such that $\{\mathbb{P}, \dot{a}\} \subset N$ and N is a model of ZFC in the sense of M. Now $N \subseteq M$ and G is M-generic, therefore G is also N-generic, so $N[G] \models \mathrm{ZFC}$. Moreover, $\dot{a}$ is a $\mathbb{P}$-name in the sense of N, so $a \in N[G]$ and $N[G] \in M[G]$. Thus $N[G]$ is the required model. $\dashv$

6 Vopěnka's Principle

Among strong large cardinal properties Vopěnka's Principle (VP) seems to be particularly fitting to the context of LZFC. In this section we shall show that if we add VP to a strengthened version of LZFC, then ZFC is recovered.

Recall that VP is a scheme rather than a single axiom, defined as follows: Given a formula $\phi(x)$ in one free variable, let X_ϕ denote the extension $\{x : \phi(x)\}$ of ϕ. Then clearly "X_ϕ is a proper class" is a first-order statement. In particular the following expression is a first-order statement:

(VP_ϕ) If X_ϕ is a proper class of structures (of some fixed first-order language), then there are distinct $x, y \in X_\phi$ and an elementary embedding $j : x \to y$,

(where $j : x \to y$ may be trivial, i.e., $x \prec y$). Then $VP = \{VP_\phi : \phi \in \mathcal{L}\}$. That VP is an appealing scheme for LZFC follows from the fact that, as a consequence of *Loc*(ZFC), every set belongs to a proper class of models of ZFC. Indeed, for every set x, let $\mathcal{M}(x)$ be the class of models of ZFC containing x. That is,

$$\mathcal{M}(x) = \{y : x \in y \ \wedge Tr(y) \ \wedge \ (y, \in) \models \mathrm{ZFC}\}.$$

Lemma 6.1 (LZFC) *For every x, $\mathcal{M}(x)$ is a proper class.*

[6] Generic extensions preserve also the related principles NM^∞ and IC^∞ mentioned in footnote 5 above.

Proof. Loc(ZFC) implies that for every x, $\bigcup \mathcal{M}(x) = V$. Thus if $\mathcal{M}(x)$ is a set, so is V. But then, by Δ_0-Separation, so is also $R = \{x \in V : x \notin x\}$, that is, Russell's paradox reappears. ⊣

The consequences of VP established below stem from two sources:

Firstly, the fact that in a proper-class context VP works as a set existence principle. Specifically, VP is an implication of the form "if X_ϕ is a proper class, then such and such is the case". Taking the contrapositive we have equivalently, "if such and such is not the case, then X_ϕ is a set".

Secondly, an old theorem of P. Vopěnka, A. Pultr and Z. Hedrlín concerning rigid graphs. Given a set A and a binary relation $R \subset A \times A$, we refer to (A, R) as a *graph.* Given a graph (A, R), a mapping $f : A \to A$ is an *endomorphism* if for all $x, y \in A$, $(x, y) \in R \Rightarrow (f(x), f(y)) \in R$. The graph (A, R) is said to be *rigid* if the only endomorphism of (A, R) is the identity. The Vopěnka-Pultr-Hedrlín (V-P-H) result [8] is the following:

Theorem 6.2 (V-P-H [8]) (ZFC) *For any infinite set A, there is a binary relation $R \subset A \times A$ such that the graph (A, R) is rigid.*[7]

Henceforth we refer to Theorem 6.2 as "V-P-H". V-P-H was originally proved in ZFC. It is open whether it is provable in LZFC, but as we shall see there are reasonable "local style" extensions of LZFC that prove it. The next theorem is the main result of this section.

Theorem 6.3 *If T is a theory such that* LZFC $\subseteq T$ *and* $T \vdash$ *V-P-H, then* ZFC $\subseteq T + VP$.

Proof. Let T be as stated. It suffices to show that $T + VP$ proves Replacement and Powerset. So the theorem is an immediate consequence of the next two Lemmas. ⊣

Lemma 6.4 *Let T be a theory such that* LZFC $\subseteq T$ *and T proves V-P-H. Then:*

(i) $T + VP \vdash \Delta_0$-Repl.

(ii) $T + VP + \Delta_0$-Repl $\vdash$ Repl.

(iii) $T + VP \vdash$ Repl.

[7] Theorem 6.2 provides a proper class whose members are *individually* rigid structures. As reported in [1], pp. 278-279, after the proof of this result P. Vopěnka believed that not only such a proper class of individually rigid structures, but also a proper class of structures which is *rigid in itself* could be constructed. That is to say, he believed that there exists a proper class of structures such that no non-trivial homomorphism exists between any two distinct elements of it (which is, roughly, the negation of what we call today VP). He then playfully set forth the *negation* of this conjecture, i.e., what we now call VP, just in order to "tease" other mathematicians, and make them finally disprove something he believed was definitely false.

Proof. Let T be as stated and let us work in $T+VP$. To prove Δ_0-Replacement, let $\phi(x,y)$ be a Δ_0 formula such that $(\forall x)(\exists! y)\phi(x,y)$. As usual this defines a class mapping $F_\phi : V \to V$ such that $F_\phi(x) = y$ iff $\phi(x,y)$. Fix a set A. We have to show that the class $B = F_\phi'' A = \{F_\phi(x) : x \in A\}$ is a set. By assumption V-P-H is true in T, so fix a binary relation $R \subset A \times A$ such that (A,R) is rigid. For every $b \in B$ consider the structure $K_b = (A \times \{b\}, R_b, E_b)$, where R_b is the binary relation on $A \times \{b\}$ induced by R, i.e.,

$$((x,b),(y,b)) \in R_b \iff (x,y) \in R,$$

and E_b is the unary relation on $A \times \{b\}$ defined by:

$$(x,b) \in E_b \iff F_\phi(x) = b.$$

Let $\mathcal{Z} = \{K_b : b \in B\}$. We claim that it suffices to show that $\mathcal{Z}$ is a set. For suppose $\mathcal{Z}$ is a set. Then clearly for some $n \in \mathbb{N}$, $B \subseteq \bigcup^n \mathcal{Z} = s$ and s is a set. Therefore,

$$B = \{y \in s : (\exists x \in A)(F_\phi(x) = y)\} = \{y \in s : (\exists x \in A)\phi(x,y)\}.$$

Since the formula $(\exists x \in A)\phi(x,y)$ is Δ_0 and Δ_0-Separation holds in T (because LZFC $\subseteq T$ and Δ_0-Separation holds in LZFC), it follows that B is set.

So let us verify that $\mathcal{Z}$ is a set. Towards reaching a contradiction assume that $\mathcal{Z}$ is a proper class. Then by VP there are $b,c \in B$, $b \neq c$, and an elementary embedding $f : K_b \to K_c$. f induces the mapping $g : A \to A$ such that for every $x \in A$, $f(x,b) = (g(x),c)$. It is easy to see that $g : (A,R) \to (A,R)$ is an endomorphism. Indeed, by the elementarity of f we have,

$$(x,y) \in R \Leftrightarrow ((x,b),(y,b)) \in R_b \Leftrightarrow (f(x,b), f(y,b)) \in R_c \Leftrightarrow$$

$$((g(x),c),(g(y),c)) \in R_c \Leftrightarrow (g(x),g(y)) \in R.$$

By the rigidity of (A,R), $g = id_A$, so for every $x \in A$, $f(x,b) = (x,c)$. But then, for every $x \in A$ we have:

$$F_\phi(x) = b \Leftrightarrow (x,b) \in E_b \Leftrightarrow f(x,b) \in E_c \Leftrightarrow (x,c) \in E_c \Leftrightarrow F_\phi(x) = c,$$

a contradiction, since $b \neq c$.

(ii) Now we work in $T + VP$+Δ_0-Repl, and prove that full Replacement holds. The proof is to a great extent similar to that of clause (i) above. Let $\phi(x,y)$ be a formula such that $(\forall x)(\exists! y)\phi(x,y)$, and let $F_\phi(x) = y$ iff $\phi(x,y)$. We fix again a set A and show that if $B = F_\phi'' A$, then B is a set. We define the structures K_b as before and set $\mathcal{Z} = \{K_b : b \in B\}$. The only departure from the proof of (i) is on how we infer that B is a set from $\mathcal{Z}$ being a set. Since ϕ now need not be Δ_0 and full Separation is not available in LZFC, the previous argument does not work. But we can invoke Δ_0-Replacement. In view of the latter, in order to show that B is a set it suffices to show that $\mathcal{Z}$ is a set, since the mapping $\mathcal{Z} \ni K_b \mapsto b \in B$ is Δ_0-definable and onto. Then we continue exactly as in (i). That is, we assume that $\mathcal{Z}$ is a proper class and reach the same contradiction.

(iii) Immediate from (i) and (ii). ⊣

Lemma 6.5 *Let T be a theory such that* LZFC $\subseteq T$ *and T proves V-P-H. Then $T + VP + \Delta_0$-*Repl $\vdash$ Powerset*. Therefore, by 6.4 (i),* $T + VP \vdash$ Powerset*.*

Proof. Let T be as stated and assume the contrary, i.e., in $T+VP+\Delta_0$-Repl there is an infinite set A such that $\mathcal{P}(A)$ is a proper class. It suffices to reach a contradiction. Fix such a set A. Fix also, by theorem V-P-H, a binary relation $R \subset A\times A$ such that (A, R) is rigid. For every set $X \in \mathcal{P}(A)$ consider the first-order structure $S_X = (A, R, X)$, where X is interpreted as a unary predicate. Let $\mathcal{X} = \{S_X : X \in \mathcal{P}(A)\}$. The mapping $\mathcal{X} \ni S_X \mapsto X \in \mathcal{P}(A)$ is Δ_0-definable and onto, so since $\mathcal{P}(A)$ is a proper class, $\mathcal{X}$ is a proper class too. By VP, there are $X, Y \in \mathcal{P}(A)$, $X \neq Y$, and an elementary embedding $f : S_X \to S_Y$, i.e., $f : (A, R, X) \to (A, R, Y)$. Then in particular $f : (A, R) \to (A, R)$ is an endomorphism, so $f = id_A$ since (A, R) is rigid. Now for every $x \in A$, $S_X \models X(x) \Leftrightarrow S_Y \models Y(f(x))$, or $x \in X \Leftrightarrow f(x) \in Y$. Since $f = id_A$, it follows that $X = Y$, a contradiction. ⊣

Proof of Theorem 6.3. In view of Lemmas 6.4 and 6.5, the proof of Theorem 6.3 is complete. ⊣

Remark 6.6 Note that if instead of the general Powerset axiom one wants only to show that $\mathcal{P}(\omega)$ is a set, one can prove this simply in LZFC $+ VP + \Delta_0$-Repl, skipping Theorem V-P-H and using just the rigidity of ω. Namely, it suffices to consider for every $X \subseteq \omega$, the structure $K_X = (\omega, S, 0, X)$, where S is the successor operation of ω and X is a unary predicate. Then working exactly as in the proof of 6.5, and using the fact that $(\omega, S, 0)$ is rigid, one concludes that $\mathcal{P}(\omega)$ is a set.

Let us next turn to the question: Is LZFC sufficient for the proof of V-P-H? This requires an inspection of the ZFC proof (see also [1, Lemma 2.64] for a more concise version of it). The idea of the proof is, given a set A, first to identify A with an ordinal of the form $\lambda + 2$ (using AC), and then to define R by cleverly employing the ordinals $\alpha < \lambda+2$ with *countable cofinality.* Then the rigidity of $(\lambda+2, R)$, i.e., the fact that for every endomorphism f, $f(\beta) = \beta$, for every $\beta < \lambda+2$, is shown by induction on $\lambda + 2$. Thus the property "$\mathrm{cf}(\alpha) = \omega$" plays a crucial role in the construction of R. This property is expressed by a Σ_1 sentence, so the construction is not absolute. It means that if we work in LZFC and given a set A, we pick a model M such that $A \in M$ and define a relation R on A such that $M \models$ "(A, R) is rigid", we cannot be sure that (A, R) is actually rigid. So LZFC does not seem adequate to prove V-P-H, at least *in the way* the latter is proved in ZFC.[8] But if given a set A, we can pick a model M such that $A \in M$ and $M \models \mathrm{cf}(\alpha) = \omega$ iff $\mathrm{cf}(\alpha) = \omega$, then the construction of R in M guarantees the absolute rigidity of (A, R).

Thus we arrive at the following definition which looks rather *ad hoc* but gives precisely the minimum requirements for an extension of LZFC in order to prove V-P-H.

Definition 6.7 (LZFC) A model M is said to be *special* if for every ordinal $\alpha \in M$, $\mathrm{cf}^M(\alpha) = \omega \Leftrightarrow \mathrm{cf}(\alpha) = \omega$.[9]

Let *spec*(x) be the formal expression of the property "x is a special model". Let as usual $\mathrm{LZFC}^{spec} = \mathrm{LZFC} + Loc^{spec}(\mathrm{ZFC})$.

Theorem 6.8 $\mathrm{LZFC}^{spec} \vdash$ *V-P-H. Therefore* $\mathrm{ZFC} \subseteq \mathrm{LZFC}^{spec} + VP$.

Proof. Let A be a set. By $Loc^{spec}(\mathrm{ZFC})$ we can pick a special model M such that $A \in M$. Applying V-P-H inside M we construct an $R \subset A \times A$ as in the standard proof so that (A, R) is rigid in M. Namely we identify A with an ordinal $\lambda + 2$ of M and define R as in V-P-H. We claim that $(\lambda + 2, R)$ is absolutely rigid. Indeed, let f be an endomorphism of $(\lambda + 2, R)$ outside M. By $Loc(\mathrm{ZFC})$ there is a model N such that $M \subset N$ and $f \in N$. Since M is special, for every $\alpha < \lambda + 2$, $N \models \mathrm{cf}(\alpha) = \omega$ iff $M \models \mathrm{cf}(\alpha) = \omega$ iff $\mathrm{cf}(\alpha) = \omega$. Thus in N one can show by induction that $f(\beta) = \beta$ for every $\beta < \lambda + 2$, exactly as this is proved in M for the endomorphisms of M. It follows that $f = id$. The other claim follows from Theorem 6.3. $\dashv$

Now special models are related to strongly extendible models considered in section 3, and in particular their property to be Σ_1-elementary submodels of V (see Lemma 3.9 (i)). As already remarked in footnote 3, not the full strength of strong extendibility is needed for that property but only the Σ_1 part of it. So the following is an intermediate property between being special and strongly extendible.

Definition 6.9 A model M is said to be Σ_1-*strongly extendible* if for every x there is a model N such that $x \in N$ and $M \prec_{\Sigma_1} N$.

[8] Of course the possibility of a *new* proof of V-P-H in LZFC, essentially different from that of [8], cannot be excluded.

[9] This particular kind of "special models" is not connected with special models as used in classical model theory.

Let $\Sigma_1 sext(x)$ denote the formula expressing "x is Σ_1-strongly extendible".

Lemma 6.10 *(i) If M is Σ_1-strongly extendible, then $M \prec_{\Sigma_1} V$.*
(ii) For every M,

$$sext(M) \Rightarrow \Sigma_1 sext(M) \Rightarrow spec(M).$$

(iii) $\mathrm{LZFC}^{spec} \subseteq \mathrm{LZFC}^{\Sigma_1 sext} \subseteq \mathrm{LZFC}^{sext}$.

Proof. (i) Let M be Σ_1-strongly extendible, and let $\exists \overline{x} \phi(\overline{x}, \overline{c})$ be a Σ_1 sentence with $\overline{c} \in M$ such that $V \models \exists \overline{x} \phi(\overline{x}, \overline{c})$. Then $V \models \phi(\overline{a}, \overline{c})$ for some $\overline{a}$. By Σ_1-strong extendibility there is a model N such that $\overline{a}, \overline{c} \in N$, and $M \prec_{\Sigma_1} N$. Then $N \models \phi(\overline{a}, \overline{c})$ since ϕ is bounded. Thus $N \models \exists \overline{x} \phi(\overline{x}, \overline{c})$, so $M \models \exists \overline{x} \phi(\overline{x}, \overline{c})$ too.

(ii) The first implication is trivial. The other one follows from (i) and the fact that the predicate "$\mathrm{cf}(\alpha) = \omega$" is Σ_1.

(iii) Immediate from (ii). ⊣

Corollary 6.11 $\mathrm{LZFC}^{\Sigma_1 sext} \vdash$ *V-P-H. Therefore*

$$\mathrm{ZFC} \subseteq \mathrm{LZFC}^{\Sigma_1 sext} + VP \subseteq \mathrm{LZFC}^{sext} + VP.$$

Proof. By 6.10 (iii) $\mathrm{LZFC}^{\Sigma_1 sext}$ is stronger than LZFC^{prec}. So the claim follows from 6.8. [Alternatively, the proof follows from 6.10 (i). Observe that the statement "(A, R) is rigid" itself is Π_1. So if $(A, R) \in M$, M is Σ_1-strongly extendible and $M \models$"(A, R) is rigid", then (A, R) is absolutely rigid.] ⊣

Corollary 6.12 *(i)* $\mathrm{LZFC}^{\Sigma_1 sext} \vdash$ Powerset.
(ii) In $\mathrm{LZFC}^{\Sigma_1 sext}$*, we have $V = \bigcup_\alpha V_\alpha$, where $V_\alpha = \mathcal{P}^\alpha(\emptyset)$. If M is Σ_1-strongly extendible, $M = V_\alpha$ for some ordinal α.*

Proof. As in the proofs of 3.10 (i) and 3.11, both claims just need the fact that every set belongs to a model M such that $M \prec_{\Sigma_1} V$, which holds in $\mathrm{LZFC}^{\Sigma_1 sext}$ in view of 6.10 (i). ⊣

We see that the theories LZFC^{spec}, $\mathrm{LZFC}^{\Sigma_1 sext}$ and LZFC^{sext} are reasonable examples of the theory T of 6.3 which, when augmented with VP, restores ZFC. Moreover, as shown in Proposition 3.12, the consistency strength of LZFC^{sext} (and thus of the rest of the preceding theories) is no higher than that of the existence of a strongly inaccessible cardinal.

We can prove further that for T being some of the preceding theories $T + VP$ and $\mathrm{ZFC} + VP$ are identical. Actually it suffices to show that $\mathrm{ZFC} + VP \vdash Loc^{sext}(\mathrm{ZFC})$. For this we need to employ *extendible* cardinals the definition of which we recall (see [4, p. 379f]).

Definition 6.13 A cardinal κ is *extendible* if for every $\alpha > \kappa$ there is a β and an elementary embedding $j : V_\alpha \to V_\beta$ such that $\kappa = \mathrm{crit}(j)$.

Extendible cardinals are very large. Every extendible cardinal is supercompact, therefore it is strong. So by Lemma 3.14, if κ is extendible then V_κ is strongly extendible.

Lemma 6.14 *In* ZFC + VP *there is a proper class of extendible cardinals.*

Proof. See Lemma 20.25 of [4] and the remark immediately after its proof. ⊣

Theorem 6.15 ZFC + $VP \vdash Loc^{sext}(\mathrm{ZFC})$. *Hence*

$$\mathrm{LZFC}^{sext} + VP \subseteq \mathrm{ZFC} + VP.$$

Proof. Assume ZFC + VP and let x be a set. By 6.14 there is an extendible cardinal κ such that $x \in V_\kappa$. Every extendible cardinal is strong; therefore, by Lemma 3.14, V_κ is a strongly extendible model. Thus every x belongs to a strongly extendible model and so $Loc^{sext}(\mathrm{ZFC})$ is true. ⊣

Theorem 6.16 *The theories (a)* ZFC + VP*, (b)* $\mathrm{LZFC}^{sext} + VP$*, (c)* $\mathrm{LZFC}^{\Sigma_1 sext} + VP$*, (d)* $\mathrm{LZFC}^{spec} + VP$ *are identical.*

Proof. By 6.8, 6.10 (iii) and 6.15 we have

$$\mathrm{ZFC} + VP \subseteq \mathrm{LZFC}^{spec} + VP \subseteq \mathrm{LZFC}^{\Sigma_1 sext} + VP \subseteq$$

$$\mathrm{LZFC}^{sext} + VP \subseteq \mathrm{ZFC} + VP.$$

⊣

It follows from 6.14 and 6.15 that VP has strong consequences when combined with ZFC. But what when combined with LZFC? In particular the following are open.

Question 6.17 *Does* LZFC + VP *prove V-P-H?*

Question 6.18 *Does* LZFC+VP *prove* $Loc^{sext}(\mathrm{ZFC})$ *(or some of the weaker principles* $Loc^{\Sigma_1 sext}(\mathrm{ZFC})$*,* $Loc^{spec}(\mathrm{ZFC})$*)?*

If the answer to either of the preceding questions is affirmative, then LZFC+VP also restores ZFC, so the results of this section can be considerably improved. However what we were able to prove is only the following.

Proposition 6.19 LZFC + $VP \vdash Loc^{ext}(\mathrm{ZFC})$.

Proof. Let a be a set in the universe of LZFC+VP. We have to show that there is an extendible model M of ZFC such that $a \in M$. Let $\mathcal{U} = \{(M, \in, \{x\}_{x \in TC(\{a\})}) : M \in \mathcal{M}(a)\}$. In view of 6.1 above it is easy to see that $\mathcal{U}$ is a proper class. By VP there are $M \neq N$ in $\mathcal{M}(a)$ and an elementary embedding

$$j : (M, \in, \{x\}_{x \in TC(\{a\})}) \to (N, \in, \{x\}_{x \in TC(\{a\})}).$$

If $j = id$, then $M \prec N$; thus M is extendible. Let $j \neq id$. By the definition of j, $j \restriction TC(\{a\}) = id$. Let $\kappa = \text{crit}(j)$. By Lemma 4.1 (iii), M_κ is the greatest transitive subset of M fixed by j, so $TC(\{a\}) \subseteq M_\kappa$. Thus $a \in M_\kappa$. Also, by 4.1 (v), $M_\kappa \models$ ZFC and, by 4.1 (vi), $M_\kappa \prec j(M_\kappa) = N_{j(\kappa)}$. Thus M_κ is an extendible model and $a \in M_\kappa$. $\dashv$

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