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A NOTE ON REAL SUBSETS OF A RECURSIVELY SATURATED MODEL

by ATHANASSIOS TZOUVARAS in Thessaloniki (Greece)

§ 0. Introduction

Let $L = \{+, \cdot, ', <, 0\}$ be the language of Peano Arithmetic (PA) and let M be a countable recursively saturated model of PA. For every $a \in M$ let $\sim_a$ be the equivalence

$$x \sim_a y \leftrightarrow \text{ for all } \varphi(v_0, v_1) \in L, \quad M \models \varphi(x, a) \leftrightarrow \varphi(y, a).$$

The sets $X \subseteqq M$ which consist of whole equivalence classes with respect to some $\sim_a$ are called *real*. The nonreal sets are called *imaginary*.

There is an old and well-known criterion for real sets due to D. KUEKER and G. E. REYES (see [4], Theorem 1.5) which roughly says that real sets are those with few automorphic images.

Theorem 0.1. (KUEKER-REYES). *Let M be countable. If M has uncountably many automorphisms, then $X \subseteqq M$ is real iff the set $\{f''X: f \in \mathrm{Aut}(M)\}$ is countable.*

As far as we know there had been no further interest on the properties of the sets with few and many automorphic images, until this division reappeared, possibly independently, some years later, in the work of members of the Prague School concerning the Alternative Set Theory. In particular the terms "real" and "imaginary" were used by K. ČUDA and P. VOPĚNKA in [3]. However in that paper no mention of the results of KUEKER and REYES is made. It contains various basic results concerning real and imaginary sets as well as philosophical speculations converging to the conclusion that the real sets are "good" and "natural" (whence the name "real"), while the imaginary ones are "artificial", "used mainly for calculations". The results of [3] show indeed that all sets having some kind of definition are real; on the other hand all the examples of imaginary sets are constructed by choice. Besides, the collection of real sets is closed under relative definability, i.e. it satisfies the full comprehension scheme.

We believe the division "real-imaginary" is indeed interesting and natural and it worths to be studied in models not so strong as those of AST and not so weak to simply possess a lot of automorphisms. Recursively saturated models seem to be good for that purpose.

Our further goal is to shed light on the relation between a) sets defined by choice and b) imaginary sets.

The relation is not clear since e.g. one can choose an arbitrary set X and then take

$$\bar{X} = \{y : y \sim_a x : x \in X\}.$$

$\bar{X}$ is real although choice has played a major role in its construction. On the other hand, not all choices lead to irregular (i.e. imaginary) sets.

The problem obviously lies in the notion "sets defined by choice" which is intuitive, while "imaginary sets" is a formal notion.

A first step towards the attempt to bridge the gap is to confirm that all objects defined currently in M, whose existence lies heavily on choice, are imaginary. This is also the spirit of [3]. Here we show that many of the results of that paper can be transferred to a countable recursively saturated model $M \models \mathrm{PA}$. We prove also some new. We could summarize by saying that almost everything which is expected to be imaginary–namely, a) well-orderings, b) automorphisms, c) bijections between "very unequal sets", d) non-trivial ultrafilters–is really such. We have an open question concerning ultrafilters of a certain kind.

§ 1. The collection of real sets

Throughout the paper M is a countable recursively saturated model of PA.

We use standard notation and terminology. If $X \subseteq M$, then L_X is L with parameters from X. Also X is used as a unary predicate which extends L to $L(X) = L \cup \{X\}$ and expands M to the structure (M, X).

$\mathrm{Def}_X(M)$, or simply Def_X, is the collection of subsets of M defined by formulas of L_X. On the other hand $\mathrm{def}\, X$ is the set of elements of M definable in M from parameters in X.

We use the usual coding devices:

$$\langle x_1, \ldots, x_a \rangle = p_1^{x_1+1} \ldots p_a^{x_a+1}, \quad \text{where } p_a \text{ is the } a^{\text{th}} \text{ prime;}$$

$$y \in D_x \quad \text{if} \quad p_y \mid x;$$

$$(x)_y = z \quad \text{if the exponent of the } y^{\text{th}} \text{ prime in the prime factorization of } x \text{ is } z;$$

$$X_{(x)} = \{y : \langle x, y \rangle \in X\}.$$

A *coded sequence* is a set of the form

$$\{(c)_n : n \in \omega\}$$

The second order extension of L is $L^* = L \cup \{\in\}$ plus variables $U_0, U_1, \ldots$ for sets. L^* is interpreted in pairs $(M, \mathfrak{M})$, where $\mathfrak{M} \subseteq P(M)$. $\mathrm{Aut}(M)$ is the set of automorphisms of M. (In our case $\mathrm{Aut}(M)$ is uncountable). For every $\mathfrak{M} \subseteq P(M)$ and every $f \in \mathrm{Aut}(M)$,

$$f''\mathfrak{M} = \{f''X : X \in \mathfrak{M}\}.$$

Lemma 1.1. *Let* $\mathfrak{M} \subseteq P(M)$, $\varphi(v_1, \ldots, v_m, U_1, \ldots, U_n) \in L^*$ *and* $f \in \mathrm{Aut}(M)$. *Then for every* $x_1, \ldots, x_m \in M$ *and* $X_1, \ldots, X_n \in \mathfrak{M}$,

$$(M, \mathfrak{M}) \models \varphi(x_1, \ldots, x_m, X_1, \ldots, X_n) \quad \textit{iff}$$
$$(M, f''\mathfrak{M}) \models \varphi(f(x_1), \ldots, f(x_m), f''X_1, \ldots, f''X_n).$$

Proof. By induction on the length of φ. □

The definition of real sets has been given in the introduction. Since $x \sim_a y$ iff (equivalently) there is an $f \in \mathrm{Aut}(M)$ such that $f(a) = a$ and $f(x) = y$, X is real w.r.t. $\sim_a$ iff $f''X = X$ for every $f \in \mathrm{Aut}(M)$ such that $f(a) = a$.

Let $\mathfrak{R}_M$ be the collection of all real subsets of M.

Proposition 1.2.

a) $\mathfrak{R}_M$ *is uncountable.*

b) $\mathfrak{R}_M$ *is closed under automorphic images.*

c) $\mathfrak{R}_M$ *satisfies the full comprehension scheme.*

d) *If* $\mathfrak{M}$ *is any countable collection closed under automorphic images, then* $\mathfrak{M} \subsetneqq \mathfrak{R}_M$.

e) *Let* $\Sigma^1_\infty(M, X)$ *be the set of all sets definable in* (M, X) *by formulas of* $L^*_M(X)$ *(analytically definable). If* X *is real, then*

$$\Sigma^1_\infty(M, X) \subseteqq \mathfrak{R}_M.$$

In particular,

$$\Sigma^1_\infty(M, \omega) \subseteqq \mathfrak{R}_M.$$

Proof. a) It suffices to see that for any $a \in M$ $\sim_a$ produces an infinity of equivalence classes. Take a partition X_0, X_1 of M such that $X_0, X_1 \in \mathrm{Def}_{\{a\}}$. Let $\mathrm{Mon}_a(x)$ be the equivalence class of x w.r.t. $\sim_a$. Let $x_0 \in X_0$, $x_1 \in X_1$. Then $\mathrm{Mon}_a(x_0) \subseteqq X_0$ and $\mathrm{Mon}_a(x_1) \subseteqq X_1$. Next suppose X_1 is infinite and divide it into $X_2, X_3 \in \mathrm{Def}_{\{a\}}$. If $x_1 \in X_2$, choose $x_2 \in X_3$. Then

$$\mathrm{Mon}_a(x_3) \cap \mathrm{Mon}_a(x_2) = \emptyset.$$

Continuing this process we find infinitely many monads

$$\mathrm{Mon}_a(x_n),\ n \in \omega.$$

b) If X is real w.r.t. $\sim_a$ and $f \in \mathrm{Aut}(M)$, then $f''X$ is real w.r.t. $\sim_{f(a)}$.

c) Let $\varphi(v_1, \ldots, v_m, U_1, \ldots, U_n) \in L^*$, $x_1, \ldots, x_m \in M$ and $X_1, \ldots, X_n \in \mathfrak{R}_M$. Let X_i be real w.r.t. a_i for $i = 1, \ldots, n$. Put $b = \langle x_1, \ldots, x_m, a_1, \ldots, a_n \rangle$. It suffices to see that the set

$$Y = \{x \in M : (M, \mathfrak{R}_M) \models \varphi(x, x_1, \ldots, x_m, X_1, \ldots, X_n)\}$$

is real w.r.t. $\sim_b$ or, equivalently, for every $f \in \mathrm{Aut}(M)$ such that $f(b) = b$, $f''Y \subseteqq Y$. Let $f(b) = b$. Then $f(x_j) = x_j$ for $j = 1, \ldots, m$ and $f(a_i) = a_i$, hence $f''X_i = X_i$, for $i = 1, \ldots, n$. By the foregoing lemma, for every $x \in M$ we have

$$\begin{aligned}&(M, \mathfrak{R}_M) \models \varphi(x, x_1, \ldots, x_m, X_1, \ldots, X_n)\\ &\leftrightarrow (M, f''\mathfrak{R}_M) \models \varphi(f(x), f(x_1), \ldots, f(x_m), f''X_1, \ldots, f''X_n)\\ &\leftrightarrow (M, \mathfrak{R}_M) \models \varphi(f(x), x_1, \ldots, x_m, X_1, \ldots, X_n).\end{aligned}$$

Therefore if $x \in Y$, then $f(x) \in Y$.

d) This is immediate from theorem 0.1.

e) Immediate from (c) above. That ω is real follows from the fact that $\mathrm{Mon}_0(n) = \{n\}$ for every $n \in \omega$. ☐

§ 2. Automorphisms

The next lemma is forklore.

Lemma 2.1. *Let M be countable and recursively saturated. Then*

a) $x \in \mathrm{def}\{a\} \rightarrow \mathrm{Mon}_a(x) = \{x\}$ *and* $x \notin \mathrm{def}\{a\} \rightarrow \mathrm{Mon}_a(x)$ *is infinite.*

b) $x \in \mathrm{def}\{a, b\}$ *iff there is a function* $F \in \mathrm{Def}_{\{b\}}$ *such that* $F(a) = x$.

c) $x \notin \mathrm{def}\{a\} \leftrightarrow (\exists f \in \mathrm{Aut}(M))(f(a) = a \,\&\, f(x) \neq x)$. ☐

The following combinatorial result is well known and used under various forms in various contexts. Originally it is attributed to M. Katětov. For a proof see [2], lemma 9.1. The proof of the present version is similar, so we omit it.

Lemma 2.2. *Let* $X \in Def_{\{a\}}$ *be infinite and let* $F \in Def_{\{a\}}$ *be a function from* X *into* X *such that* $F(x) \neq x$ *for all* $x \in X$. *Then* X *can be partitioned into three sets* $X_0, X_1, X_2 \in \mathrm{Def}_{\{a\}}$ *such that* $X_i \cap F''X_i = \emptyset$ *for* $i = 0, 1, 2$. □

Proposition 2.3. *Let* $F \in \mathrm{Def}_{\{a\}}$ *be a function. Then for every* $x \in \mathrm{dom}(F)$, $F(x) \sim_a x$ *implies* $F(x) = x$.

Proof. Suppose $F(x) \neq x$. Let $Y = \{z: F(z) \neq z\}$. Then $x \in Y$ and $Y \in \mathrm{Def}_{\{a\}}$. If Y is finite, then $x \in \mathrm{def}\{a\}$, hence $\mathrm{Mon}_a(x) = \{x\}$, according to lemma 2.1. Thus $F(x) \nsim_a x$. Assume that Y is infinite. Without loss of generality we may suppose that $F''Y \subseteq Y$. Apply the preceding lemma to find a partition $Y_0, Y_1, Y_2 \in \mathrm{Def}_{\{a\}}$ of Y such that $Y_i \cap F''Y_i = \emptyset$ for $i = 0, 1, 2$. Let $x \in Y_i$. If $F(x) \sim_a x$, then $F(x) \in Y_i$ which is false. Hence $F(x) \nsim_a x$. □

Lemma 2.4. *Let* $f \in \mathrm{Aut}(M)$ *and* $f \neq \mathrm{id}$. *Then for every* $a \in M$ *there is a* $g \in \mathrm{Aut}(M)$ *such that* $g(a) = a$ *and* $g \cdot f \neq f \cdot g$.

Proof.

Case 1. Let $f(a) = a$. Since $f \neq \mathrm{id}$, there is a $b \in M$ such that $f(b) \neq b$. Then $f(b) \sim_a b$. We claim that $f(b) \notin \mathrm{def}\{a, b\}$. In fact, otherwise there would be, by lemma 2.1, a function $F \in \mathrm{Def}_{\{a\}}$ such that $F(b) = f(b)$. Thus $F(b) \sim_a b$ and $F(b) \neq b$, which contradicts proposition 2.3. By lemma 2.1, there is an automorphism g such that $g(a) = a$, $g(b) = b$ and $g(f(b)) \neq f(b)$. If follows that $gf(b) \neq fg(b)$, therefore $g \circ f \neq f \circ g$.

Case 2. Let $f(a) \neq a$. This is treated similarly working with $\sim_0$ instead of $\sim_a$. □

Theorem 2.5. *Every* $f \in \mathrm{Aut}(M)$, $f \neq \mathrm{id}$, *is imaginary.*

Proof. Let $f \neq \mathrm{id}$ and let f be real w.r.t. $\sim_a$. Then for every $g \in \mathrm{Aut}(M)$ such that $g(a) = a$ we would have $g''f = f$. Now

$$g''f = \{\langle g(x), g(y)\rangle : f(x) = y\} = \{\langle x, y\rangle : fg^{-1}(x) = g^{-1}(y)\}$$
$$= \{\langle x, y\rangle : gfg^{-1}(x) = y\} = g \circ f \circ g^{-1}.$$

Hence for every $g \in \mathrm{Aut}(M)$ with $g(a) = a$, $g \circ f \circ g^{-1} = f$ or, equivalently, $g \circ f = f \circ g$. This, however, contradicts the preceding lemma. □

§ 3. Well-orderings

Theorem 3.1. *Every well-ordering of (the whole)* M *is imaginary.*

Proof. Let $\lesssim$ be a well-ordering of M and suppose it is real w.r.t. $\sim_a$. Then, for every $f \in \mathrm{Aut}(M)$ such that $f(a) = a$, $f'' \lesssim\, = \,\lesssim$. It means that for all $x, y \in M$,

$$x \lesssim y \leftrightarrow f(x) \lesssim f(y).$$

Therefore f is an order-isomorphism of $(M, \lesssim)$ onto itself, hence $f = \mathrm{id}$. This contradicts the fact that there are $f \neq \mathrm{id}$ with $f(a) = a$. □

Of course there are real well-orderings $\lesssim$ with $\mathrm{Field}(\lesssim) \neq M$. For example the ordering on ω is real.

We shall strengthen the preceding theorem by showing that all real well-orderings in M are of the type of the forementioned example.

A set $X \subseteq M$ is said to be *thin* if $X \subseteq \operatorname{def}\{a\}$ for some a. This is equivalent to say that

$$\operatorname{Mon}_a(x) = \{x\}$$

for every $x \in X$.

Obviously every thin set is real and there are uncountably many thin sets.

Theorem 3.2. *If the well-ordering $\lesssim$ is real, then $\lesssim$ is thin.*

Proof. Suppose $\lesssim$ is real w.r.t. $\sim_a$ and not thin. Let $X = \operatorname{Field}(\lesssim)$. Then X is also real and not thin. For every $f = \operatorname{Aut}(M)$ such that $f(a) = a$, f is again an order isomorphism of $(X, \lesssim)$ on itself, hence $f(x) = x$ for all $x \in X$. It follows from lemma 2.1, c) that $X \subseteq \operatorname{def}\{a\}$, a contradiction. □

§ 4. Cuts

Theorem 4.1. *The cut I of M is real iff there is a thin set either cofinal in I or coinitial in $M - I$.*

Proof. The condition is clearly sufficient. Suppose, conversely, that I is real w.r.t. $\sim_a$ and suppose there is no thin set either cofinal in I or coinitial in $M - I$. Let

$$X_1 = I \cap \operatorname{def}\{a\} \quad \text{and} \quad X_2 = (M - I) \cap \operatorname{def}\{a\}.$$

Choose a b such that $I < b < X_2$. We claim there is a formula $\varphi(v_0, v_1)$ such that

$$\text{(1)} \qquad M \models \varphi(b, a) \quad \text{and} \quad \{x : M \models \varphi(x, a)\} \subseteq M - I.$$

In fact assume that for every $\varphi(v_0, v_1) \in L$, $M \models \varphi(b, a)$ implies

$$\{x : M \models \varphi(x, a)\} \cap I \neq \emptyset,$$

therefore for all $\varphi(v_0, v_1) \in L$

$$\text{(2)} \qquad \{x : M \models \varphi(x, a) \leftrightarrow \varphi(b, a)\} \cap I \neq \emptyset.$$

Let $t(v) = \{\psi_n(v) : n \in \omega\}$, where

$$\psi_n(v) \equiv (v)_n = (\mu u)\big(\bigwedge_{i \le n} \varphi_i(u, a) \leftrightarrow \varphi_i(b, a)\big),$$

where $\varphi_i(v_0, v_1)$ is a recursive enumeration of all $\varphi(v_0, v_1) \in L$. Then $t(v)$ is a recursive type, hence realizable in M. Let c realize $t(v)$. Then, by (2), for every $n \in \omega$, $(c)_n \in I$ and

$$M \models \bigwedge_{i \le n} (\varphi_i((c)_n, a) \leftrightarrow \varphi_i(b, a)).$$

Therefore there is a $d > \omega$ such that, for all $I < i < d$, $(c)_i \in \operatorname{Mon}_a(b)$. Since $\{(c)_n : n \in \omega\}$ is not cofinal in I, we can assume that $(c)_i \in I$. Thus $I \cap \operatorname{Mon}_a(b) \neq \emptyset$. This is a contradiction because $M - I$ is real w.r.t. $\sim_a$ and $b \in M - I$. This proves the claim (1).

But if e is the least element of $\{x : M \models \varphi(x, a)\}$, then $e \in M - I$ and $e \leq b$. Thus $I < e < X_2$. This is a contradiction because $e \in \operatorname{def}\{a\}$. □

§ 5. Bijections

The following lemma is in the spirit of lemma 1.3 of [5].

Lemma 5.1. *Let I be a cut closed under addition and let $a < I < b$. Then, for every $c \in M$ there is a d such that $a < d < b$ and $d \notin \mathrm{def}\{x, c\}$ for all $x \leqq a$.*

Proof. Let $X = \bigcup\{\mathrm{def}\{x, c\}: x \leqq a\}$.

Claim. *For every coded set $D_y \subseteq X$, there is an $n \in \omega$ such that $|D_y| \leqq n \cdot a$* (where $|D_y|$ is the internal cardinality of D_y).

Proof of the claim. Let $t_n(v_0, v_1)$ be a recursive enumeration of all terms $t(v_0, v_1)$ of L, and for every $n \in \omega$ let $X_n = \{t_i(x, c): i \leqq n \;\&\; x \leqq a\}$. Then $X_n \subseteq X_m$ if $n < m$, each X_n is a coded set and $X = \bigcup_n X_n$. By recursive saturation we can easily see that if $D_y \subseteq X$, then $D_y \subseteq X_n$ for some $n \in \omega$. Since $|X_n| \leqq n \cdot a$, the claim follows.

By hypothesis, for every $n \in \omega$, $n \cdot a \in I < b$, hence $b - a > na$. Therefore $[a, b] \nsubseteq X$ and if we choose $d \in [a, b] - X$, we are done. □

Theorem 5.2. *Let $a < I < b$ such that I is closed under addition. Then every bijection between $[0, a]$ and $[0, b]$ is imaginary.*

Proof. Let $F: [0, a] \to [0, b]$ be a bijection. It suffices to show that $\{f''F: f \in \mathrm{Aut}(M)\}$ is uncountable. For every $f \in \mathrm{Aut}(M)$, $f''F$ is, clearly, a bijection between $[0, f(a)]$, $[0, f(b)]$. Suppose we have countably many functions G_n with $\mathrm{dom}(G_n) = [0, a_n]$ and $\mathrm{rng}(G_n) = [0, b_n]$.It suffices to find an automorphism f such that

$$f''F \neq G_n \text{ for all } n = \omega.$$

We construct f by back and forth. For each n we shall find a pair $\langle x_n, y_n\rangle$ such that

$$\langle x_n, y_n\rangle \in F \quad \text{and} \quad \langle f(x_n), f(y_n)\rangle \notin G_n.$$

Suppose a part of f

$$f_m: (c_1, \ldots, c_m) \cong (c'_1, \ldots, c'_m)$$

and pairs $\langle x_i, y_i\rangle$, $i = 1, \ldots, n$, have been defined so that $x_i \in \mathrm{dom}(f_m)$ and (1) holds. Let $c = \langle c_1, \ldots, c_n\rangle$ and $c' = \langle c'_1, \ldots, c'_m\rangle$. By lemma 5.1 there is an x such that $a < x < b$ and $x \notin \mathrm{def}\{y, c\}$ for all $y \leqq a$. Since F maps $[0, a]$ onto $[0, b]$, there is an $x \in [0, a]$ such that $F(x) \notin \mathrm{def}\{x, c\}$. Find x' such that $f'_m = f_m \cup \{\langle c, c'\rangle, \langle x, x'\rangle\}$ is a partial isomorphism. Let $F(x) = y$. If $x' \notin [0, b_n]$, then for any y' such that $f_{m+1} = f'_m \cup \{\langle y, y'\rangle\}$ is a partial isomorphism, (1) holds if $x_{n+1} = x$ and $y_{n+1} = y$. Let $x' \in [0, b_n]$. Since $y \notin \mathrm{def}\{x, c\}$, the set

$$Y = \{z \in M: M \models \varphi(z, x', c') \leftrightarrow \varphi(y, x, c)\}$$

is infinite. Choose $y' \in Y$ so that $y' \neq G_n(y)$. Then put again $f_{m+1} = f'_m \cup \{\langle y, y'\rangle\}$ and (1) holds for $x_{n+1} = x$ and $y_{n+1} = y$. □

Of course there are coded sets of unequal internal cardinality between which there are real bijections.

For example for every $a > \omega$ and $n \in \omega$, if $I < a$ is a real cut, then the function $F: [0, a] \to [0, a + n]$ such that $F(i) = i$ for $i \in I$ and $F(j) = j + n$ for $j > I$ is a real bijection.

If M is expandable to a model $(M, \mathfrak{M})$ of AST with $FN = \omega$, it follows from [3] that the preceding result is best possible, i.e. for all $a < b$ such that $b < na$ for some $n \in \omega$, there is a real bijection between $[0, a]$ and $[0, b]$. It is open, however, whether this holds in any recursively saturated M.

§ 6. Ultrafilters

In this section we shall deal with sets of coded subsets of M, so we have to replace coded sets by their codes. We shall represent the coded set D_x by its *least code* x_0. And we shall say that "x is set" when $(\forall v)(D_v = D_x \rightarrow x \leqq v)$. For every set x we write $y \in x$ instead of $y \in D_x$. Also the expressions $y \subseteqq x$, $x \cap y$ etc. between sets have the obvious meaning. Observe that $x \subseteqq y \,\&\, y \subseteqq x \rightarrow x = y$. Also $P(x)$ is the set of coded subsets of x and $|x| = |D_x|$. Clearly if $|x| = a$, then $|P(x)| = 2^a$.

Let a be a set. For any $x \subseteqq a$ we write $-x$ for the set $a - x$. An *ultrafilter on* a is any set $E \subseteqq M$ such that:

(i) $x \in E \rightarrow x \subseteqq a$,

(ii) $x, y \in E \rightarrow x \cap y \in E$,

(iii) $x \in E \,\&\, x \subseteqq y \subseteqq a \rightarrow y \in E$,

(iv) $0 \notin E$,

(v) $(\forall x \subseteqq a)(x \subseteqq E \vee -x \in E)$.

E is nonprincipal if all $x \in E$ are infinite sets.

An *ultrafilter base*, or simply a base, is a sequence of sets (x_n), $n \in \omega$, such that

(i) $(\forall n \in \omega)(x_{n+1} \subseteqq x_n)$,

(ii) $(\forall y \subseteqq a)(\exists n)(x_n \subseteqq y \vee x_n \subseteqq -y)$.

Clearly every base generates an ultrafilter and every ultrafilter is generated by a base. To see the latter observe that every ultrafilter E on a is countable, say $E = \{y_n : n \in \omega\}$, and if we put $x_n = \bigcap_{i \leqq n} y_i$ for every $n \in \omega$, then $(x_n)_{n \in \omega}$ is a base for E.

As in [6] we can assign to every ultrafilter E two cuts $\mu(E)$, $\nu(E)$ as follows:

$$\mu(E) = \{b \in M : (\forall p)(p \text{ is a partition of a set in } E \text{ and } |p| = b \rightarrow p \cap E \neq \emptyset\},$$

$$\nu(E) = \{b \in M : (\forall x \in E)(|x| < b)\}.$$

Lemma 6.1. *For every nonprincipal* E

a) $\mu(E) \leqq \nu(E)$;

b) $\mu(E)$ *is closed under multiplication;*

c) $(\forall x \in \mu(E))(\forall y \in \nu(E))(x \cdot y \in \nu(E))$.

Proof. a) Let $\nu(E) < \mu(E)$ and let $\nu(E) < b < \mu(E)$. Then there is $x \in E$ with $|x| = b$. If $p = \{\{n\} : y \in x\}$, then $\{y\} \in E$ for some $y \in x$ since $|p| \in \mu(E)$. Hence E is trivial.

b) Let $b, c \in \mu(E)$ and take a partition p of a into $b \cdot c$ sets $\{x_{ij} : i \leqq b,\ j \leqq c\}$. Let $x_i = \bigcup \{x_{ij} : j \leqq c\}$ for every $i \leqq b$. Then $\bigcup_i x_i = a$, hence for some $i_0 \leqq b$, $x_{i_0} \in E$. Similarly there is a $j_0 \leqq c$ such that $x_{i_0 j_0} \in E$.

c) Let $b \in \mu(E)$ and $c \in \nu(E)$. If $b \cdot c > \nu(E)$ there is a set $x \in E$ with $|x| = b \cdot c$. Partition x into b subsets, each with cardinality c. Then some of them must belong to E, but this contradicts the definition of $\nu(E)$. □

Theorem 6.2. (Sochor-Vopěnka). *Let $I \subsetneqq J$ be cuts in M having the properties of lemma 6.1. Then there is an ultrafilter E with $\mu(E) = I$ and $\nu(E) = J$.* □

The proof is combinatorial and cumbersome and can be found in [6], so we omit it.

To every $X \subseteq M$ there corresponds a pair of cuts defined as follows:

$$i(X) = \{a \in M : (\exists x)(x \subseteq X \;\&\; |x| = a)\},$$

$$o(x) = \{a \in M : (\forall x)(X \subseteq x \to a < |x|)\}.$$

Obviously $i(X) \subseteq o(X)$ and they are proper cuts, unless the set X is definable or X is unbounded (when $o(X) = M$).

We can think of $i(X)$, $o(X)$ as a kind of "inner" and "outer measure" of X respectively.

Lemma 6.3. *If X is real, then for every a such that $i(X) < a < o(X)$ and every $b > \omega$,*

$$a \cdot b > o(X).$$

Proof. Let X be real with respect to $\sim_c$ and let $i(X) < a < o(X)$. Let $\varphi_n(v_0, v_1)$ be a recursive enumeration of $\varphi(v_0, v_1) \in L$. For every $n \in \omega$ and $x \in M$, put

$$S_{n,x} = \{y : M \models \varphi_n(y, n) \leftrightarrow \varphi_n(x, c)\}.$$

Since $\bigcap_n S_{n,x} = \mathrm{Mon}_c(x)$ and the latter does not contain any coded set of cardinality a, it follows that for every $x \in X$, there is an n such that $|S_{n,x}| < a$. Let $F: M \to M$ be such that

$$F(z) = \{S_{z,x} : |S_{z,x}| < a\}.$$

Using an inductive satisfaction class, F is defined for nonstandard z and $F(z)$ contains at most two elements. Moreover for every $x \in X$ there is an n such that $x \in \bigcup F(n)$. Thus $X \subseteq \bigcup\{\bigcup F(n) : n \in \omega\}$. Now $|\bigcup F(n)| < 2a$ for every n. Hence, for every $b > \omega$, $|\bigcup\{\bigcup F(n) : n \in \omega\}| < 2ab$. Therefore for all $b > \omega$, $o(X) < 2ab$ and this proves the claim. □

Lemma 6.4. *Let E be a nonprincipal ultrafilter on a. Then*

$$i(E) < 2^{|a|-1} < o(E) < 2^{|a|}.$$

Proof. For $X \subseteq P(a)$, let $\bar{X} = \{-x : x \in X\}$. Then from the properties of ultrafilters follows that $\bar{X} = -E$. Thus $x \subseteq E \leftrightarrow \bar{x} \subseteq \bar{E} = -E$. Since $|x| = |\bar{x}|$ we get $i(E) = i(-E)$ and $o(E) = o(-E)$. To see that $2^{|a|-1} \not< i(E)$ let $x \subseteq E$ and $|x| = 2^{|a|-1}$. Then $|-x| = 2^{|a|-1}$ and $\bar{x} \subseteq -E \subseteq -x$. Since $|\bar{x}| = |-x|$ it follows that $\bar{x} = -E = -x$ which is a contradiction. Similarly we see that $o(E) \not< 2^{|a|-1}$. □

The above can be strengthened using a combinatorial theorem of A. Brace and D. E. Daykin [1].

Let $1 < k < n$ be natural numbers. We say that a family $\mathcal{F}$ of subsets of $[1, n]$ is a *(k, n)-family* if

α) $\bigcap \mathcal{F} = \varnothing$,

β) $F_1 \cap \ldots \cap F_k \neq \varnothing$ for every $\{F_1, \ldots, F_k\} \subseteq \mathcal{F}$.

Consider in particular the family

$$\mathcal{F}(k, n) = \{F \subseteq [1, n]: |F \cap [1, k+1]| \leq k\}.$$

Evidently $\mathcal{F}(k, n)$ is a (k, n)-family and $|\mathcal{F}(k, n)| = (k+2) \cdot 2^{n-k-1}$. Moreover the following holds:

Theorem 6.5. (Brace-Daykin). *For every (k, n)-family $\mathcal{F}$,*

$$|\mathcal{F}| \leq |\mathcal{F}(k, n)| = (k+2) \cdot 2^{n-k-1}. \quad \square$$

Inspecting the proof of the theorem we see that it can be formalized in PA. Hence the result is true in every model of PA.

Corollary 6.6. *Let E be nonprincipal on a. Then*

$$i(E) \leq \{x: (\forall b \in \mu(E))(x < 2^{|a|-b})\}.$$

If moreover $\mu(E) = \nu(E)$, then

$$i(E) = \{x: (\forall b \in \mu(E))(x < 2^{|a|-b})\}.$$

Proof. Let $x \subseteq E$ and $b \in \mu(E)$. Then $2b \in \mu(E)$ and x is, essentially, a $(2b, |a|)$-family. By theorem 6.5, $|x| < (2b+2) \cdot 2^{|a|-2b-1}$. Since $2b+2 < 2^{b+1}$ it follows that $|x| < 2^{|a|-b}$. This proves the first claim.

On the other hand let $\nu(E) < b < a$ and let $y \in E$ with $|y| = b$. Then

$$\{x: y \subseteq x \subseteq a\} \subseteq E \quad \text{and} \quad |x: y \subseteq x \subseteq a\}| = 2^{|a|-b}.$$

Hence

$$\{x: (\exists b > \nu(E))(x < 2^{|a|-b})\} \leq i(E).$$

If $\mu(E) = \nu(E)$, all these cuts are equal. $\square$

Corollary 6.7. *If E is nonprincipal and $\mu(E) > \omega$, then e is imaginary.*

Proof. Let E be nonprincipal and let $\omega < b < \mu(E)$. Then, according to Corollary 6.6, $i(E) < 2^{|a|-b} < o(E)$, and for c such that $\omega < c < b$, $c \cdot 2^{|a|-b} < 2^{|a|-1} < o(E)$. By Corollary 6.3, E cannot be real. $\square$

The case $\mu(E) = \omega$ is more difficult. There is, however, the following trivial situation: Let us say that E is *on X*, where X is a thin set, if

$$(\forall x)(X \subseteq x \rightarrow x \subseteq E).$$

We can easily check that E is determined by the ultrafilter $E \restriction X = \{x \cap X: x \in E\}$ and since X is real, E is real and $\mu(E) = \nu(E) = \omega$.

But it is not known whether every E with $\nu(E) = \omega$ is on some thin set or whether every real E is on some thin set.

The answers to these questions seem to depend on the following open problem:

Problem. Given a (coded) set x and c such that $x \cap \operatorname{def}\{c\} = \emptyset$ does there exist $f \in \operatorname{Aut}(M)$ such that $f(c) = c$ and $x \cap f(x) = \emptyset$?

If the answer is affirmative we can easily deduce that:

a) The ultrafilter E is real iff it is on a thin set.

b) If $\nu(E) > \omega$, then E is imaginary.

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A. Tzouvaras
Department of Mathematics
University of Thessaloniki
54006 Thessaloniki
Greece

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