THE RUDIN-KEISLER RELATION ON OBJECTS OTHER THAN ULTRAFILTERS

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We prove that the Rudin-Keisler relation is an ordering for finite partitions and finite intersections of ultrafilters. However it fails to be an ordering if infinite partitions and countable intersections of ultrafilters are considered.

1. **Preliminaries.** Let X be a set, $P^1(X) = P(X)$ (the power-set of X) and $P^{n+1}(X) = P(P^n(X))$. Any mapping $f : X \to X$ induces a sequence of mappings $(f^{(n)}, n < \omega)$, where $f^{(0)} = f$ and

$$f^{(n)} : P^n(X) \to P^n(X)$$

such that $f^{(n+1)}(x) = \{f^{(n)}(y) : y \in X\}$.

The Rudin-Keisler $(R\text{-}K)$ relation $\leqslant_{RK}$ on $P^n(X)$ is defined as follows : For any two elements F, G of $P^n(X)$, $G \leqslant_{RK} F$ if there is some surjection $f : X \to X$ such that $f^{(n)}(F) = G$.

F, G are said to be *isomorphic* if there is a permutation π of X such that $\pi^{(n)}(F) = G$. We then write $F \cong G$.

The relation $\leqslant_{RK}$ is reflexive and transitive. If in addition, in some subset of $P^n(X)$, $F \leqslant_{RK} G$ and $G \leqslant_{RK} F$ imply that $F \cong G$, $\leqslant_{RK}$ is an order relation on the isomorphism classes of this subset.

We recall the following well-known facts :

1.1. In $P^2(X)$ and on the set of ultrafilters (u.f.) βX, $\leqslant_{RK}$ is an order relation (for a proof of this very important result see for example [2], Th. 3.3).

1.2. In $P(X)$ $A \leqslant_{RK} B$ and $B \leqslant_{RK} A$ if and only if $|A| = |B|$ and $|X - A| = |X - B|$ or, equivalently, $A \cong B$.

1.3. In X trivially $x \leqslant_{RK} y$ and $x \cong y$ for any $x, y \in X$.

It is natural to ask for which other objects of $P^2(X)$ or, more generally, of $P^n(X)$ the $R\text{-}K$ relation is an ordering. The question in this general setting is quite difficult, but we may concentrate to some familiar categories of elements of $P^2(X)$, such as partitions, filters, topologies etc.

In this paper we give two positive and two negative results concerning partitions and filters.

All mappings used in the sequel are surjections.

For convenience we write $\leqslant$ instead of $\leqslant_{RK}$ and denote by the same symbol f each one of the induced mappings $f^{(n)}$.

2. **Partitions.** Let $\mathfrak{A} = \{V_i | i \in I\}$ be a partition of X. $\mathfrak{A}$ is said to be finite if $|\mathfrak{A}| < \omega$. Let us set $c(\mathfrak{A}) = \{k | \exists\, V \in \mathfrak{A} : |V| = k\}$ and $\mathfrak{A}^k = \{V \in \mathfrak{A} : |V| = k\}$.

LEMMA 2.1. *Let* $\mathfrak{A}_1$, $\mathfrak{A}_2$ *be two partitions of* X. *Then* $\mathfrak{A}_1 \cong \mathfrak{A}_2$ *if and only if* $c(\mathfrak{A}_1) = c(\mathfrak{A}_2)$ *and* $|\mathfrak{A}_1^k| = |\mathfrak{A}_2^k|$ *for all* $k \in c(\mathfrak{A}_1)$.

Proof. If $\mathfrak{A}_1 \cong \mathfrak{A}_2$ the conditions are satisfied.

Let $c(\mathfrak{A}_1) = c(\mathfrak{A}_2)$ and $|\mathfrak{A}_1^k| = |\mathfrak{A}_2^k|$. For each $k \in c(\mathfrak{A}_1)$ we choose a bijection $e_k : \mathfrak{A}_1^k \to \mathfrak{A}_2^k$ and for each $V \in \mathfrak{A}_1^k$ we choose a bijection $f_V : V \to e_k(V)$. Then the function $f : X \to X$ such that $f(x) = f_V(x)$ if $x \in V$, is a permutation of X such that $f(\mathfrak{A}_1) = \mathfrak{A}_2$.

THEOREM 2.2. *If $\mathfrak{A}_1$, $\mathfrak{A}_2$ are finite and $\mathfrak{A}_1 \leqslant \mathfrak{A}_2$, $\mathfrak{A}_2 \leqslant \mathfrak{A}_1$ then $\mathfrak{A}_1 \cong \mathfrak{A}_2$.*

Proof. Let $f(\mathfrak{A}_1) = \mathfrak{A}_2$ and $g(\mathfrak{A}_2) = \mathfrak{A}_1$. Since $\mathfrak{A}_1$, $\mathfrak{A}_2$ are finite f, g are bijections on $\mathfrak{A}_1$, $\mathfrak{A}_2$ respectively and, consequently, on $\mathfrak{A}_1^k$ $\mathfrak{A}_2^\lambda$ for each $k \in c(\mathfrak{A}_1)$ and $\lambda \in c(\mathfrak{A}_2)$. Suppose $c(\mathfrak{A}_1) = \{k_1, \ldots, k_n\}$ with $k_1 < \ldots$ $\ldots < k_n$. If $V \in \mathfrak{A}_1$ and $|V| = k_1$ then $k_1 \leqslant |gf(V)| \leqslant |f(V)| \leqslant |V| = k_1$, hence $\min c(\mathfrak{A}_2) = k_1$ and $f(\mathfrak{A}_1^{k_1}) = \mathfrak{A}_2^{k_1}$, $g(\mathfrak{A}_2^{k_1}) = \mathfrak{A}_1^{k_1}$, thus $|\mathfrak{A}_1^{k_1}| = |\mathfrak{A}_2^{k_1}|$

Suppose now that $k \in c(\mathfrak{A}_2)$, $f(\mathfrak{A}_1^k) = \mathfrak{A}_2^k$, $g(\mathfrak{A}_2^k) = \mathfrak{A}_1^k$ for all $k \in c(\mathfrak{A}_1)$ such that $k < k_i$.

If $V \in \mathfrak{A}_i^{k_i}$ then $|f(V)| = k_i$. For if $|f(V)| = \lambda < k_i$ then either λ equals someone of the cardinals $k_1, \ldots, k_{i-1}$, which is impossible since f is 1-1, or $k_j < \lambda < k_{j+1}$ for some $j \leqslant i$. It follows that $|gf(V)| \leqslant k_i$ which is also impossible since g is 1-1.

Therefore $k_i \in c(\mathfrak{A}_2)$, $f(\mathfrak{A}_1^{k_i}) = \mathfrak{A}_2^{k_i}$ and $g(\mathfrak{A}_2^{k_i}) = \mathfrak{A}_1^{k_i}$. We have shown that $c(\mathfrak{A}_1) \subseteq c(\mathfrak{A}_2)$ and $|\mathfrak{A}_1^k| = |\mathfrak{A}_2^k|$ for all $k \in c(\mathfrak{A}_1)$. Since g is 1-1 and the images $g(\mathfrak{A}_2^{k_i})$, $i = 1, 2, \ldots, n$, cover the whole $\mathfrak{A}_1$, $c(\mathfrak{A}_2)$ contains no other cardinals than those in $c(\mathfrak{A}_1)$.

This, in view of Lemma 1.1, completes the proof.

Remark. A partition $\mathfrak{A}$ is *uniform* if $|c(\mathfrak{A})| = 1$. An immediate corollary of Lemma 1.1 is the following : On the set of uniform partitions of X the R-K relation is an ordering.

We'll show now by a counterexample that $\leqslant_{RK}$ is not an order relation for partitions in general.

Example 1.3. On the set of natural numbers ω consider the partitions : $\mathfrak{A}_1 = \{A_n : n < \omega\} \cup \{B_n : n < \omega\} \cup \{C_n : n < \omega\}$ where $A_n = \{6n\}$, $B_n = \{6n+1, 6n+2\}$, $C_n = \{6n+3, 6n+4, 6n+5\}$ and $\mathfrak{A}_2 = \{D_n : n < \omega\} \cup \{E_n : n < \omega\}$ where $D_n = \{4n\}$, $E_n = \{4n+1, 4n+2, 4n+3\}$.

Define $f : \omega \to \omega$ as follows :

$$f(6n) = f(6n+1) = f(6n+2) = 4n,\ f(6n+3) = 4n+1,$$
$$f(6n+4) = 4n+2,\ f(6n+5) = 4n+3.$$

Then $f(A_n) = f(B_n) = D_n$, $f(C_n) = E_n$. Hence $f(\mathfrak{A}_1) = \mathfrak{A}_2$. Define also $g : \omega \to \omega$ as follows :

$$g(4n) = 6n,\ g(8n+1) = 6n+1,\ g(8n+2) = g(8n+3) = 6n+2$$
$$g(8n+5) = 6n+3,\ g(8n+6) = 6n+4,\ g(8n+7) = 6n+5.$$

Then $g(D_n) = A_n$, $g(E_{2n}) = B_n$, $g(E_{2n+1}) = C_n$. Hence $g(\mathfrak{A}_2) = \mathfrak{A}_1$. However $\mathfrak{A}_1 \not\cong \mathfrak{A}_2$ since $c(\mathfrak{A}_1) = \{1, 2, 3\}$ while $c(\mathfrak{A}_2) = \{1, 3\}$.

3. **Filters.** Call a filter F *quasi-ultrafilter* (*q.u.f*) *of order* n, $n < \omega$, if it is the intersection of n distinct u.f.'s. If $p_1, \ldots, p_n$ are u.f.'s and $F = \bigcap_{i=1}^{n} p_i$, the only u.f.s extending F are p_i. So q.u.f.s of order n are exactly

the filters extended by precisely n u.f'.s. (This is not the case, however, for infinite intersections of u.f'.s., because if $F = \cap \{p_i : i \in I\}$ and I infinite, there are in general more u.f'.s than p_i extending F).

THEOREM 3.1. *If* F, G *are* q.u.f.s. and $f(F) = G$, $g(G) = F$, *then* $F \cong G$.

Proof. If $F = \bigcap_i p_i$ then $f(F) = \bigcap_i f(p_i)$, so the order of $f(F)$ is $\leqslant$ the order of F. This means that F, G are of the same order. Let A, B be the sets of u.f'.s extending F, G respectively. A, B need not be disjoint, so A, B are of the form

$$A = \{p_1, \ldots, p_n, r_1, \ldots, r_m\}, \quad B = \{q_1, \ldots, q_n, r_1, \ldots, r_m\}$$

Then f maps A onto B and g maps B onto A. We claim that for any $x \in A$, $f(x) \cong x$. Let $f(x) = y$. Consider the mappings $h_v, v < \omega$, where $h_1 = g$ and $h_{v+1} = gfh_v$ and the set $I = \{h_v(y) : v < \omega\}$. Each h_v maps B onto A, therefore I is finite and there is v_0 such that

$$v_0 = \max\{v : h_i(y) \neq h_j(y), \text{ for all } i, j \leqslant v\}$$

It follows that $h_{v_0+1}(y) = h_1(y)$ and finally $h_{v_0}(y) = x$. The u.f'.s x, y satisfy the relations $x \leqslant y$ and $y \leqslant x$, therefore $x \cong y$. Let $A_0 = \{p_1, \ldots$ $\ldots, p_n\}$, $B_0 = \{q_1, \ldots, q_n\}$. We construct a mapping φ of A_0 onto B_0 such that $\varphi(x) \cong x$. Let $x \in A_0$.

Define $f^1(x) = f(x)$ and $f^{k+1}(x) = f(f^k(x))$ if $f^k(x) \in A \cap B$. If $f^k(x) \notin$ $\notin A \cap B$, $f^{k+1}(x)$ is not defined. Clearly $f^k(x) \neq f^\lambda(x)$ when defined, so there is a first index k such that $f^k(x)$ is not defined, i.e. $f^k(x) \in B_0$. Set then

$$\varphi(x) = f^k(x).$$

We can easily verify that φ maps A_0 onto B_0. Since $f(x) \cong x$, it follows that $f^k(x) \cong x$, i.e $\varphi(x) \cong x$.

Changing, if needed, the indexing of B_0 we may write

$$p_1 \cong q_1, \ldots, p_n \cong q_n.$$

Find disjoint sets $V_i \in p_i$, $U_i \in q_i$ $D_j \in r_j$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ and permutations π_i of X such that $\pi_i(p_i) = q_i$. We may suppose that $\pi_i(V_i) = U_i$ (otherwise we can refine V_i and U_i by the sets $U_i' = \pi_i(V_i)$ $\cap U_i$ and $V_i' = \pi_i^{-1}(U_i')$). Now the function

$$\pi(x) = \begin{cases} \pi_i(x) & \text{if } x \in V_i \\ \pi_i^{-1}(x) & \text{if } x \in U_i \\ x & \text{otherwise} \end{cases}$$

is a permutation of X sending p_i to q_i, q_i to p_i and leaving r_i unchanged. Therefore $\pi(F) = G$.

Next we show a counterexample that the preceding theorem fails if one considers countably infinite intersections of u.f'.s.

Example 3.2. In ω consider P-points p, r, q_0 such that $p < r < q_0$. (For the definition of P-points see [3] and for their R-K ordering see [1]). $p < q$ means of course $p \leqslant q$ and $p \not\cong q$. Then $h(q_0) = r$ and $e(r) = p$

for some mappings $h, e : \omega \to \omega$. Choose disjoint sets $A \in p$, $D \in r$, $B_0 \in q_0$ such that $|\omega - A \cup D \cup B_0| = \omega$ We may suppose that $h(B_0) = D$, otherwise we refine suitably D and B_0. Let $X = \omega - A \cup D \cup B_0$. Take a countable partition of X into countable sets B_i, $i = 1, 2, \ldots$ and permutations π_i of ω such that $\pi_i(B_i) = B_{i-1}$. Such permutations exist since $B_0 \cong B_1 \cong \ldots$ Let $\pi_i^{-1}(q_0) = q_1$ and $\pi_i^{-1}(q_{i-1}) = q_i$. The u.f.'s q_i, $i = 0, 1, 2, \ldots$ are isomorphic P-points. Let $S = \{p, q_i : i = 0, 1, 2, \ldots\}$ and $T = \{p, r, q_i : i = 0, 1, 2, \ldots)$

We define $f : \omega \to \omega$ as follows :

$$f(x) = \begin{cases} h(x) & \text{if } x \in B_0 \\ \pi_i(x) & \text{if } x \in B_i, \ i \geqslant 1 \\ x & \text{otherwise.} \end{cases}$$

Clearly f is a surjection such that $f(p) = p$, $f(q_0) = r$, $f(q_{i+1}) = q_i$ for $i = 0, 1, \ldots$, i.e. $f(S) = T$.

Suppose again that $e(D) = A$. Divide A into two disjoint countable subsets A_1, A_2, with $A_1 \in p$ and map A_2 onto D by a bijection σ. Then the function

$$g(x) = \begin{cases} e(x) & \text{if } x \in D \\ \sigma(x) & \text{if } x \in A_2 \\ x & \text{otherwise} \end{cases}$$

is a surjection such that $g(r) = g(p) = p$ and $g(q_i) = q_i$, $i = 0, 1, \ldots$, i.e. $g(T) = S$.

If $F = \cap S$, $G = \cap T$ it is clear that $f(F) = G$ and $g(G) = F$. Now the u.f.'s extending F (resp. G) are the elements of $\mathrm{cl}_{\beta\omega} S$ (resp. $\mathrm{cl}_{\beta\omega} T$), where $\beta\omega$ is the Stone-Cech compactification of the discrete space ω, and the only P-points of $\mathrm{cl}_{\beta\omega} S$ (resp. $\mathrm{cl}_{\beta\omega} T$) are those of S (resp. T), because a P-point cannot belong to $\mathrm{cl}_{\beta\omega} A - A$ if A is a countable subset of $\beta\omega$. Since there is a P-point extending G, namely r, which is not isomorphic to any P-point extending F, it follows that $F \not\cong G$.

It is well known that filters on X correspond to closed subsets of βX. A direct consequence of Theorem 3.1 is the following :

COROLLARY 3.3. *In $P^3(X)$ and on the set of finite subsets of βX, the R-K relation is an ordering.*

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