Totally non-immune sets

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Abstract

Let $\mathcal{L}$ be a countable first-order language and $\mathcal{M} = (M, \ldots)$ be an $\mathcal{L}$-structure. "Definable set" means a subset of M which is $\mathcal{L}$-definable in $\mathcal{M}$ with parameters. A set $X \subseteq M$ is said to be immune if it is infinite and does not contain any infinite definable subset. X is said to be partially immune if for some definable A, $A \cap X$ is immune. X is said to be totally non-immune if for every definable A, $A \cap X$ and $A \cap (M \backslash X)$ are not immune. Clearly every definable set is totally non-immune. Here we ask whether the converse is true and prove that it is false for every countable structure $\mathcal{M}$ whose class of definable sets satisfies a mild condition. We investigate further the possibility of an alternative construction of totally non-immune non-definable sets with the help of a subclass of immune sets, the class of cohesive sets, as well as with the help of a generalization of definable sets, the semi-definable ones (the latter being naturally defined in models of arithmetic). Finally connections are found between totally non-immune sets and generic classes in nonstandard models of arithmetic.

Keywords. Immune set, partially immune, totally non-immune, cohesive set, partially cohesive, totally non-cohesive, semi-definable set, generic class.

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1 Immunity and partial immunity

This paper arose from an attempt to approximate the class of definable sets (initially of the structure $(\omega, +, \cdot, S, 0)$ and later of any countable first-order structure) by *eliminating* some important aspects of non-definability to the greatest possible extent. Specifically, at the opposite side of definable sets lie the "immune" sets. We adopt the term "immune" from computability theory, where it means a set $X \subset \omega$ that does not contain any r.e. subset. Here we use the term with "definable" in place of "r.e.". Namely, given a first-order structure $\mathcal{M} = (M, \dots)$, a set $X \subset M$ is said to be immune (in $\mathcal{M}$) if it is infinite and does not contain any infinite subset definable in $\mathcal{M}$ with parameters. So an immune set has *zero definable content.* On the other hand there exist some natural *degrees of immunity,* that is, natural intermediate properties between non-immunity and definability. For example, if X is immune and A is an infinite definable set such that $A \cap X = \emptyset$, then $X \cup A$ is neither immune nor definable. Rather it is something between, it has the property that we call "partial immunity". In general, X is partially immune if there is a definable A such that $A \cap X$ is immune. An even weaker property is what could be called "very partial immunity" (although the term will not be used below, to avoid terminological inflation). X is very partially immune if either X or its complement $M \backslash X$ is partially immune. We feel that these two intermediate properties, partial immunity and very partial immunity, seem to exhaust the reasonable weak degrees of immunity. (Of course any suggestions for even finer and weaker kinds of immunity are welcome.) Thus we call a set X totally non-immune (t.n.i.) if neither X nor $M \backslash X$ is partially immune.

There are also other significant properties that cause non-definability. Such are *cohesiveness* and *genericity,* but both of them are stronger than immunity. That is, every cohesive or generic set is immune. So it is obviously more likely to reach definable sets by "killing" immunity (and its weaker degrees) rather than killing cohesiveness or/and genericity.

Thus, intuitively, total non-immunity is pretty close to definability. But how close indeed? Strictly speaking, is it *identical* to definability? The main result of the paper is that the answer is no. Namely we show (Theorem 2.1 below) that almost every countable structure contains t.n.i. sets which are non-definable. Such sets are constructed as generic subsets sets of a suitable partially ordered set. In particular this is true for all expansions $\mathcal{M} = (\omega, +, \cdot, S, 0, \dots)$ of the standard model of Peano Arithmetic (PA).

In section 3 we try to find alternative constructions of totally non-immune sets by means of the class of cohesive sets mentioned above, which is also a notion adapted from computable sets, and which has an interest in its own. The goal is not fully obtained. We only show that the complement of a cohesive is not partially immune, a property weaker than total non-immunity.

In the last section we focus on models of PA. There one can define a natural notion of semi-definability, with the purpose to reach t.n.i. sets through a different route. What is shown however is that there exist semi-definable sets with any degree of immunity, depending on the set's "support", but semi-definability itself does not provide an alternative method for constructing t.n.i. non-definable sets. In a different vein, we examine the relationship of t.n.i. sets with another approximation of definable sets occurring in some *nonstandard* models, discovered and used long ago, the so called "classes" studied in a series of papers like [3], [5], [6]. Not all non-standard models of PA contain *proper* (i.e., non-definable) classes. That is, in some cases the classes "collapse" to just the definable sets. We show however that whenever they exist (in which case they are constructed as *generic* classes), they are t.n.i. sets.

For the rest of the paper and unless otherwise stated, $\mathcal{L}$ is a fixed countable first-order language and $\mathcal{M} = (M, \dots)$ is a countable $\mathcal{L}$-structure. By "set" we always mean a subset of M. "Definable set" means a set $\mathcal{L}$-definable in $\mathcal{M}$ with parameters. Let us first fix some notation.

• Capital letters X, Y, Z, with subscripts, will denote arbitrary subsets of M.

• Capital letters A, B, C, with subscripts, will denote definable subsets of M.

• Lowercase letters a, b, c, with subscripts, will denote elements of M.

Definition 1.1 A set $X \subseteq M$ is said to be *immune* if it is infinite and does not contain any infinite definable subset of M. X is said to be *partially immune* if for some definable A, $A \cap X$ is immune. X is *totally non-immune* (t.n.i. in brief) if neither X nor $M \backslash X$ is partially immune.

Let Def, IM, PIM, TNI be the classes of definable, immune, partially immune and t.n.i. subsets of M, respectively. Sometimes we denote by Def^{∞} the class of infinite definable sets. The following relations follow immediately from the definitions:

(1) $X \in TNI \Leftrightarrow X \notin PIM \wedge (M \backslash X) \notin PIM$,

(2) $IM \subseteq PIM$,

(3) $Def \subseteq TNI$.

Immune sets lie at the antipodes of definable sets, as they have zero definable content. Also any infinite subset of an immune set is immune. Yet the following holds:

Proposition 1.2 *Every infinite set splits into two immune subsets.*

Proof. Let $X \subseteq M$ be infinite and let $\mathcal{K} = \{A \in Def^\infty : A \subseteq X\}$. If $\mathcal{K} = \emptyset$, then X is immune so if X_1, X_2 form an arbitrary partition of X into infinite sets, X_1, X_2 are immune. Let $\mathcal{K} \neq \emptyset$. Then $\mathcal{K}$ is infinite and let $\mathcal{K} = \{A_n : n \in \omega\}$ be an enumeration of $\mathcal{K}$. We construct inductively sequences $(a_n)_{n\in\omega}$, $(b_n)_{n\in\omega}$ of elements of X such that for every n:

(a) $a_n, b_n \in A_n$,

(b) $\{a_1, \ldots, a_n\} \cap \{b_1, \ldots, b_n\} = \emptyset$.

Since each A_n is infinite and $A_n \subseteq X$, this construction can be carried out for every n. Pick $X_1, X_2 \subseteq X$ such that $\{a_n : n \in \omega\} \subseteq X_1$, $\{b_n : n \in \omega\} \subseteq X_2$, and $X_1 \cap X_2 = \emptyset$. Without loss of generality we may assume $X_1 \cup X_2 = X$ (otherwise we extend each one of them properly in order to form a partition of X). The only definable sets that could be contained in X_1, X_2 are those of $\mathcal{K}$. But for every $A_n \in \mathcal{K}$, $A_n \cap X_1 \neq \emptyset$, so $A_n \not\subseteq X_2$, and $A_n \cap X_2 \neq \emptyset$, so $A_n \not\subseteq X_1$. It follows that X_1, X_2 are immune. $\dashv$

Concerning the question whether the inclusion $IM \subseteq PIM$ is proper, we have the following:

Proposition 1.3 *If Def contains at least one infinite and co-infinite set, then $IM \subsetneq PIM$.*

Proof. Pick such an infinite and co-infinite $A \in Def$. By the previous proposition A splits into two immune sets X_1, X_2. Set $Y = X_1 \cup (M \backslash A)$. Then $Y \in PIM \backslash IM$. Indeed, $Y \notin IM$ since it contains the infinite definable $M \backslash A$. On the other hand $Y \cap A = X_1 \in IM$, so $Y \in PIM$. $\dashv$

Next we come to the question whether the inclusion $Def \subseteq TNI$ is proper. One can find easy and natural examples of sets $X \in TNI \backslash Def$ in structures with simple classes of definable sets, specifically certain o-minimal ones. (A totally ordered structure $\mathcal{M} = (M, <, \ldots)$ is o-minimal if every $X \in Def(\mathcal{M})$ is a finite union of open intervals, with end-points in M, and singletons of M, see e.g. [1, p. 31].)

Example 1. In $(\mathbf{Q}, <)$ or in $(\mathbf{R}, <)$ consider two infinite sequences of points $(a_n)_{n\in\omega}$, $(b_n)_{n\in\omega}$ such that

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < \cdots.$$

Let $X = \bigcup_{n=1}^{\infty}(a_n, b_n)$. Then $\mathbf{Q} \backslash X = (-\infty, a_1] \cup \bigcup_{n=1}^{\infty}[b_n, a_{n+1}]$. Clearly both X and $\mathbf{Q} \backslash X$ are non-definable in $(\mathbf{Q}, <)$. But for every interval A (thus for every definable A), if $A \cap X$ is infinite, then it contains some interval, and similarly for $A \cap (\mathbf{Q} \backslash X)$. Therefore $X \in TNI \backslash Def$.

EXAMPLE 2. Let ρ be an irrational, and let $(-\infty, \rho)$, (ρ, ∞) denote the corresponding intervals of $(\mathbf{Q}, <)$, i.e., $(-\infty, \rho) \cap \mathbf{Q}$, and $(\rho, \infty) \cap \mathbf{Q}$, respectively. Then $(-\infty, \rho)$ and (ρ, ∞) are non-definable in $(\mathbf{Q}, <)$. Let $\{a_1, a_2, \ldots\} \subset \mathbf{Q}$ be a strictly increasing sequence converging to ρ and let $X = \{a_1, a_2, \ldots\} \cup (\rho, \infty)$. It is easy to see that $X \in TNI \backslash Def$. Indeed, $X \notin Def$ and it is not hard to see that $X \notin PIM$, i.e., for every $A \in Def$, if $A \cap X$ is infinite, then it includes some interval. (Notice that $\{a_1, a_2, \ldots\}$ is immune, but since $(-\infty, \rho)$ is not definable in $(\mathbf{Q}, <)$, there is no way to separate $\{a_1, a_2, \ldots\}$ alone from X by a definable set.) Also $\mathbf{Q} \backslash X = (-\infty, a_1) \cup (\bigcup_{n=1}^{\infty} (a_n, a_{n+1}))$ is clearly not partially immune. Therefore $X \in TNI$.

This example shows further that the class TNI is not in general closed under intersections (while, as follows from the definitions, is closed under complements). Indeed, let $\{b_1, b_2, \ldots\} \subset \mathbf{Q}$ be a strictly decreasing sequence converging to the above irrational ρ and let $Y = (-\infty, \rho) \cup \{b_1, b_2, \ldots\}$. For the same reasons as before $Y \in TNI \backslash Def$. Now $X \cap Y = \{a_1, a_2, \ldots\} \cup \{b_1, b_2, \ldots\}$. But the latter is immune so $X \cap Y \notin TNI$.

The relationship between definable and immune sets reminds roughly the relationship between open and nowhere dense sets. Indeed, if we think of definable sets as "open", then we should think of immune sets as "nowhere dense", since they have zero definable content, i.e., zero open interior. Specifically, given any structure $\mathcal{M} = (M, \ldots)$, the elements of Def^{∞} (infinite definable subsets of M) can be seen as producing the basis of a pseudo-topology on M. It is not a real topology because Def^{∞} is not closed under intersections. However it is convenient to use this term as a terminological convention and give the following.

Definition 1.4 A set $X \subseteq M$ is said to be *open* if $X = \bigcup S$ for some $S \subseteq Def^{\infty}$. X is *clopen* if both X and $M \backslash X$ are open.

This is a reasonable generalization of definable sets. Let OP, CLO be the classes of open and clopen subsets of M, respectively. Obviously

$$Def \subseteq CLO \subseteq OP.$$

It is easy to see that non-immunity is identical to openness.

Proposition 1.5 *Let $X \subset M$ be infinite. Then $X \in OP$ iff $X \notin IM$.*

Proof. Obviously every open set is non-immune. Conversely, let X be infinite and non-immune. Then there is $A \in Def^{\infty}$ such that $A \subseteq X$. Let $S = \{B \in Def^{\infty} : B \subseteq X\}$. It suffices to show that $X = \bigcup S$. Clearly $\bigcup S \subseteq X$. On the other hand, if $a \in X$ then $A \cup \{a\} \in S$, so $X = \bigcup S$. $\dashv$

Corollary 1.6 *(i) $TNI \subseteq CLO$.*
(ii) $TNI \subsetneq CLO$ even in some o-minimal structures.

Proof. (i) If $X \in TNI$, then $X \notin IM$ and $(M \backslash X) \notin IM$. Thus, by 1.5, $X \in OP$ and $(M \backslash X) \in OP$.

(ii) Consider Example 2 above, but with $\mathbf{R}$ in place of $\mathbf{Q}$, i.e., let $X = \{a_1, a_2, \ldots\} \cup (\rho, \infty)$, with $\rho = \lim_n a_n$. Both X and $\mathbf{R} \backslash X$ are not immune, thus open, so $X \in CLO$. But now X is partially immune, because $(-\infty, \rho) \in Def$ and $(-\infty, \rho) \cap X = \{a_1, a_2, \ldots\}$ is immune. Therefore $X \notin TNI$. $\dashv$

In section 4.1 we shall return to a particular subclass of clopen sets, the semi-definable sets, which are naturally defined in models of arithmetic. Meanwhile we examine the existence of t.n.i. sets in general structures.

2 Existence of non-definable t.n.i. sets in general structures

Apart from the preceding examples with o-minimal structures, it is by no means clear what happens in structures with complicated classes of definable sets, like models of arithmetic or expansions of them. Below we show that indeed $Def \subsetneq TNI$ in all structures satisfying a mild condition. Namely we show that $Def \subsetneq TNI$ holds in any structure $\mathcal{M}$ satisfying the condition:

(*) *Every infinite definable set splits into two infinite definable subsets.*

(Equivalently, (*) says that the Boolean algebra of definable sets, modulo the ideal of finite ones, is atomless. In more technical model-theoretic terms, this is equivalent to say that no infinite definable subset is strongly minimal, or that the Morley rank of every infinite definable set is not ordinal-valued (see [1, pp. 240–244]).)

We shall construct $X \in TNI \backslash Def$ as a generic subset of a partially ordered set.

Given the set Def over $\mathcal{M}$, let $\mathbb{P}$ be the set consisting of pairs $p = (p_0, p_1)$ such that:

(i) $p_0, p_1 \in Def^\infty$,
(ii) $p_0 \cap p_1 = \emptyset$,
(iii) $M \backslash (p_0 \cup p_1)$ is infinite.

We order $\mathbb{P}$ by the relation $p \le q := (p_0 \supseteq q_0)$ & $(p_1 \supseteq q_1)$. Thus $\mathbb{P} \subseteq Def \times Def$. Let S be a transitive set (in ZFC) which contains the structure $\mathcal{M}$, together with the sets Def, $\mathbb{P}$ and ω, and rich enough so that the predicate "v is infinite" is absolutely defined in $(S, \in)$. That is, for every $x \in S$, x is infinite iff there is in S an injection $f : \omega \to x$. Let

$$\mathcal{S} = (S, \in, M, Def, \mathbb{P})$$

be the structure $(S, \in)$ augmented with M, Def, $\mathbb{P}$ construed as unary predicates, and let $\mathcal{L}_2 = \{\in, M(\cdot), Def(\cdot), \mathbb{P}(\cdot)\}$ be the language of the structure $\mathcal{S}$, in which M, Def, $\mathbb{P}$ are treated as unary predicate symbols (sorts). Let:

• $\mathcal{D}$ = the set of all dense subsets of $\mathbb{P}$ which are definable in $\mathcal{S}$ by formulas of $\mathcal{L}_2$ with parameters from $M \cup Def \cup \mathbb{P}$.

Since $\mathcal{L}$ and M are countable, $Def \cup M \cup \mathbb{P}$ is countable too, and thus so is $\mathcal{D}$. Therefore there is a $\mathcal{D}$-generic $G \subset \mathbb{P}$ in the usual sense, i.e., G is closed upward, any two elements of G are compatible and $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$. Fix such a G and let $G_0 = \bigcup\{p_0 : p \in G\}$ and $G_1 = \bigcup\{p_1 : p \in G\}$. The following is the main result of the paper.

Theorem 2.1 *Let $\mathcal{M}$ be a structure that satisfies condition (*) above, and let G be a $\mathcal{D}$-generic subset of $\mathbb{P}$. Then:*
(i) $G_0 \cap G_1 = \emptyset$ and $G_0 \cup G_1 = M$.
(ii) $G_0 \notin Def$. Hence also $G_1 \notin Def$.
(iii) $G_0, G_1 \notin PIM$, thus $G_0, G_1 \in TNI$.
(iv) Therefore $Def \subsetneq TNI$.

Proof. (i) We show $G_0 \cap G_1 = \emptyset$. Assume the contrary. Then there are $p, q \in G$ such that $p_0 \cap q_1 \neq \emptyset$. Since $p, q \in G$, p, q are compatible, i.e., there is $r \in G$ such that $r \le p, q$. Hence $r_0 \supseteq p_0$ and $r_1 \supseteq q_1$. Therefore $r_0 \cap r_1 \neq \emptyset$, which contradicts the definition of conditions above. Next we show that $G_0 \cup G_1 = M$. For each $a \in M$ let

$$D_a = \{p \in \mathbb{P} : a \in p_0 \vee a \in p_1\}.$$

Clearly each D_a is definable in $\mathcal{S}$ and dense in $\mathbb{P}$. Indeed let $q \in \mathbb{P}$. If $a \in q_0$ or $a \in q_1$ we are done. If $a \notin q_0 \cup q_1$, it suffices to set $p_0 = q_0 \cup \{a\}$ and $p_1 = q_1$, or $p_0 = q_0$ and $p_1 = q_1 \cup \{a\}$. In both cases the pair $p = (p_0, q_0)$ is in D_a and extends q. Thus for each $a \in M$, $D_a \in \mathcal{D}$, so $G \cap D_a \neq \emptyset$. It follows that there is $p \in G$ such that $a \in p_0 \cup p_1$, therefore $a \in G_0 \cup G_1$.

(ii) To show that $G_0 \notin Def$, we have to show that $A \neq G_0$ for every infinite and co-infinite set $A \in Def$. Fix such an A. It suffices to prove that for some $p \in G$, either $p_0 \not\subseteq A$ or $p_1 \not\subseteq M \backslash A$. For in such a case either $G_0 \not\subseteq A$ or $G_1 \not\subseteq M \backslash A$, so in either case $A \neq G_0$, since G_0, G_1 form a partition of M, according to (i) above. For every infinite and co-infinite $A \in Def$ set

$$D_A = \{p \in \mathbb{P} : p_0 \not\subseteq A \vee p_1 \not\subseteq M \backslash A\}.$$

D_A is again definable in $\mathcal{S}$. So it suffices to show that D_A is dense in $\mathbb{P}$. In such a case $D_A \in \mathcal{D}$ and thus $G \cap D_A \neq \emptyset$, as required.

Indeed, let $q = (q_0, q_1) \in \mathbb{P}$. If $q \in D_A$ we are done. Suppose $q \notin D_A$. Then $q_0 \subseteq A$ and $q_1 \subseteq M \backslash A$. However either $q_0 \subsetneq A$ or $q_1 \subsetneq M \backslash A$. Because otherwise $q_0 = M \backslash q_1$ which contradicts the definition of the conditions in $\mathbb{P}$ (property (iii) above). Suppose $q_0 \subsetneq A$. Then pick an element $a \in A \backslash q_0$ and set $p_1 = q_1 \cup \{a\}$ and $p_0 = q_0$. Then clearly $p = (p_0, p_1) \in \mathbb{P}$, $p \leq q$ and $p \in D_A$, because $p_1 \not\subseteq M \backslash A$. Similarly if $q_1 \subsetneq M \backslash A$, pick $a \in M \backslash (A \cup q_1)$ and set $p_0 = q_0 \cup \{a\}$ and $p_1 = q_1$. Then $p \leq q$ and $p \in D_A$ because $p_0 \not\subseteq A$.

(iii) Finally we show that G_0 and G_1 are not partially immune. We show it first for G_0. Fix an $A \in Def$ such that $A \cap G_0$ is infinite. Since for every $p \in \mathbb{P}$, p_0, p_1 are definable, and for every $p \in G$, $A \cap p_0 \subset A \cap G_0$, it suffices to show that for some $p \in G$, $A \cap p_0$ is infinite. Consider the set

$$E_A = \{p \in \mathbb{P} : A \cap p_0 \text{ is infinite, or } A \backslash p_1 \text{ is finite}\}.$$

We claim that for this particular A, E_A is dense in $\mathbb{P}$. Indeed, let $q = (q_0, q_1) \in \mathbb{P}$. If $A \cap q_0$ is infinite or $A \backslash q_1$ is finite, then $q \in \mathbb{P}$ and we are done. Assume $A \cap q_0$ is finite and $A \backslash q_1$ is infinite. Since $A \cap q_0$ is finite, $A \backslash (q_0 \cup q_1)$ is infinite too. Here is the point where we need condition (*). $A \backslash (q_0 \cup q_1)$ is infinite definable, so using (*) we split it into two infinite definable parts B_0, B_1 and we set $p_0 = q_0 \cup B_0$ and $p_1 = q_1$. Then $p_0 \cap p_1 = \emptyset$ and $B_1 \subseteq M \backslash (p_0 \cup p_1)$. Thus $p = (p_0, p_1) \in \mathbb{P}$ since B_1 is infinite. Also $B_0 \subseteq A \cap p_0$ and B_0 infinite, therefore $p \in E_A$ and $p \leq q$. So E_A is dense. Now clearly $E_A \in \mathcal{D}$, therefore $G \cap E_A \neq \emptyset$. Let $p \in G \cap E_A$. Then $A \cap p_0$ is infinite, or $A \backslash p_1$ is finite. But $A \backslash p_1$ cannot be finite because $p_1 \subseteq G_1$ and $A \cap G_0 = A \backslash G_1 \subset A \backslash p_1$. Thus, since by assumption $A \cap G_0$ is infinite, so is $A \backslash p_1$. It follows that $A \cap p_0$ is infinite. That is, $A \cap p_0$ is an infinite definable set contained in $A \cap G_0$. Therefore $A \cap G_0$ is not immune. A similar argument which interchanges the roles of p_0 and p_1 in the definition of E_A shows that $A \cap G_1$ is not immune.

(iv) It follows from (i)-(iii) above that $G_0, G_1 \in TNI \backslash Def$. $\dashv$

Corollary 2.2 *Let $\mathcal{L}$ be a language that extends the language of Peano Arithmetic and let $\mathcal{A} = (\omega, \ldots)$ be an $\mathcal{L}$-structure over ω. Then in $\mathcal{A}$, $Def \subsetneq TNI$.*

Proof. Clearly $\mathcal{A}$ satisfies condition (*), so the claim follows from Theorem 2.1. $\dashv$

Let us refer to sets G_0 or G_1 of theorem 2.1 as *generic t.n.i-sets.* A further property of these sets is given below.

Lemma 2.3 *Let G_0 be a generic t.n.i. set and $A \in Def^\infty$ such that $A \subset G_0$. Then there is $p \in G$ such that $A \subseteq p_0$.*

Proof. Given any $A \in Def^\infty$, let

$$\Gamma_A = \{p \in \mathbb{P} : A \subseteq p_0 \vee A \cap p_1 \neq \emptyset\}.$$

Γ_A is dense in $\mathbb{P}$. Indeed let $q \notin \Gamma_A$. Then $A \not\subseteq p_0$ and $A \cap p_1 = \emptyset$. Pick $a \in A \backslash p_0$ and set $p_0 = q_0$ and $p_1 = q_1 \cup \{a\}$. Then $p \leq q$ and $p \in \Gamma_A$ since $A \cap p_1 \neq \emptyset$. Moreover $\Gamma_A \in \mathcal{D}$. Now let $A \in Def^\infty$ and $A \subset G_0$. Take $p \in G \cap \Gamma_A$. Since $A \subset G_0$, $A \cap p_1 = \emptyset$, so $A \subseteq p_0$. $\dashv$

The question is: Is it possible to find examples of non-definable t.n.i. sets without the use of generics? To investigate this question, we consider in the next section an interesting subclass of immune sets, the class of *cohesive* sets (defined modulo Def).[1] We shall show that if $X \subseteq M$ is cohesive, then $M \backslash X \notin PIM$. So if both X and $M \backslash X$ were cohesive, then $X \in TNI \backslash Def$. However for no X can both X and $M \backslash X$ be cohesive. So cohesiveness only partially answers the above question.

3 Cohesiveness and partial cohesiveness

Just like immune sets, cohesive sets too originate in computability theory (see [4, p. 72]). Originally, a set $X \subseteq \omega$ is said to be cohesive if it is infinite and for every r.e. set A, either $A \cap X$ or $(\omega \backslash A) \cap X$ is finite. Here we replace r.e. sets with $\mathcal{L}$-definable ones. We remind that we still refer to a countable language $\mathcal{L}$ and a countable $\mathcal{L}$-structure $\mathcal{M} = (M, \ldots)$.

[1] If we restrict ourselves to the structure $(\omega, +, \cdot, S, 0)$, or expansions of it, another interesting class of non-definable sets occurs, besides cohesive sets, namely *generic* sets (defined modulo Def). Every generic subset of ω is also immune in the sense of this paper. However generics require a rather special treatment, as elements of the Cantor space 2^ω, so we will not deal with them here.

Definition 3.1 A set $X \subseteq M$ is said to be *cohesive,* if it is infinite and for every infinite $A \in Def$, either $A \cap X$ or $(M \backslash A) \cap X$ is finite. X is *partially cohesive* if there exists $A \in Def$ such that $A \cap X$ is cohesive. X is *totally non-cohesive* if neither X nor $M \backslash X$ is partially cohesive.

We denote by CO, PC and TNC the classes of cohesive, partially cohesive and totally non-cohesive subsets of M, respectively.

As an immediate consequence of the definition, if X is cohesive then all infinite subsets of X are cohesive. This fact, together with the non-effectiveness of the proof of existence of cohesive sets, implies that the latter sets are fairly far from the definable ones.

Simple examples of cohesive sets are easily available again in o-minimal structures. Namely, either in $(\mathbf{Q}, <)$ or in $(\mathbf{R}, <)$ every *unbounded* set of the form $X = \{a_1 < a_2 < \cdots\}$ is cohesive (and not just immune as we said in the previous section). Indeed, let A be a definable set in either of these structures. Since A is the union of only finitely many intervals, it is clear that $A \cap X$ is infinite if and only if A contains an interval of the form (b, ∞). Moreover in this case (b, ∞) contains almost all elements of X (that is, except finitely many), which belong to $(\mathbf{Q} \backslash A) \cap X$. Thus for any definable A, either $A \cap X$ or $(\mathbf{Q} \backslash A) \cap X$ is finite.

However, existence of cohesive sets in general countable structures can be shown by imitating the standard proof of Dekker and Myhill for the existence of cohesive subsets of ω (modulo r.e. sets) (see [4, Prop. III.4.16]).

Proposition 3.2 $CO \neq \emptyset$.

Proof. Def is countable so let $(A_n)_{n \in \omega}$ be an enumeration of its elements. $\mathcal{M}$ in general is not linearly ordered. Nevertheless this is not a problem. We can just use an external ω-ordering $<$ of M, e.g. write $M = \{c_0 < c_1 < \cdots\}$. Then one can follow essentially the proof of Dekker and Myhill with the sets A_n in place of W_n and with $<$ in place of the ordering of integers. For the reader's convenience I shall sketch the proof here. We construct a cohesive set X as the intersection $\bigcap_m X_m$ of a nested family of infinite sets $X_m = \{a_0^m < a_1^m < \cdots\}$, where $X_{m+1} \subseteq X_m$. That is, each sequence $\{a_0^{m+1} < a_1^{m+1} < \cdots\}$ is a subsequence of $\{a_0^m < a_1^m < \cdots\}$. We construct the sets X_m by induction on m as follows:

1) $X_0 = M$, i.e., $a_n^0 = c_n$.

2) Suppose $X_m = \{a_0^m < a_1^m < \cdots\}$ is defined and is infinite. We set $Y_m = X_m \cap A_m$ if $X_m \cap A_m$ is infinite, and $Y_m = X_m \cap (M \backslash A_m)$ otherwise. Thus in any case Y_m is infinite and is contained either in the definable set

A_m or in its complement. Then we define the elements $\{a_0^{m+1} < a_1^{m+1} < \cdots\}$ of X_{m+1} by induction on n as follows.

- $a_n^{m+1} = a_n^m$, if $n \leq m$,
- $a_{n+1}^{m+1} =$ the $<$-smallest element of Y_m greater than a_n^{m+1}, if $n > m$.

It follows that almost all elements of X_{m+1}, i.e., except possibly the finitely many elements a_n^{m+1}, for $n \leq m$, are in Y_m, and therefore almost all elements of X_{m+1} are either in A_m or in $M \backslash A_m$. A fortiori, for every m, almost all elements of $X = \bigcap_m X_m$ are either in A_m or in $M \backslash A_m$. That is, for every m, either $A_m \cap X$ or $(M \backslash A_m) \cap X$ is finite. Moreover, for each n, the sequence $(a_n^m)_{m \in \omega}$ becomes eventually stable. So for each n, $a_n = \max\{a_n^m : m \in \omega\}$ exists and is the n-th element of X. Therefore X is cohesive. $\dashv$

As P. Odifreddi points out in [4, p. 289], the proof of (the r.e. analogue of) 3.2 is highly noneffective, first because we ask whether the sets Y_m are infinite, and second because we ask membership questions about Y_m. In the case of r.e. subsets of ω, an "effectivization" of the preceding proof leads to *maximal r.e.* sets (modulo finite sets), and these are exactly the r.e. sets whose complement is cohesive ([4, p. 290]). However such sets do not seem to make sense for cohesiveness as defined here. (See also Remark 3.9 (ii) below.)

Let $A \in Def$ and $X \subseteq A$. Let us say that X is *cohesive relative to A*, notation $X \in CO(A)$, if for every $B \in Def$ such that $B \subseteq A$, either $X \cap B$ or $X \cap (A \backslash B)$ is finite. In fact every relative cohesive set is cohesive. The proof of the following is easy and left to the reader.

Lemma 3.3 *For every $A \in Def^\infty$ and every $X \subseteq A$,*

$$X \in CO(A) \Leftrightarrow X \in CO.$$

Corollary 3.4 *For every $A \in Def^\infty$ there is $X \subseteq A$ such that $X \in CO$.*

Proof. Let $A \in Def^\infty$. Taking an enumeration $(B_n)_{n \in \omega}$ of all definable subsets of A and an ω-ordering of A we can relativize the construction of Proposition 3.2 in A to obtain a set $X \in CO(A)$. But by Lemma 3.3, $X \in CO$. $\dashv$

Proposition 3.5 *(i) $CO \subseteq IM$.*
(ii) $PCO \subseteq PIM$.
(iii) $TNI \subseteq TNC$.

Proof. (i) Let $X \notin IM$. If X is finite then $X \notin CO$. Let X be infinite. Then there is an infinite definable $A \subseteq X$. If $X \backslash A$ is finite, then $X \in Def$, so $X \notin CO$ too. If $X \backslash A$ is infinite, then A splits X into two infinite parts, so again $X \notin CO$.

(ii) Let $X \in PCO$. Then there is $A \in Def$ such that $A \cap X \in CO$. By (i), $A \cap X \in IM$, so $X \in PIM$.

(iii) Let $X \in TNI$. Then $X \notin PIM$ and $(M \backslash X) \notin PIM$. Thus $X \notin PCO$ and $(M \backslash X) \notin PCO$. Therefore $X \in TNC$. ⊣

That all the inclusions of the previous proposition are proper will be a consequence of the next theorem.

Theorem 3.6 *Let $\mathcal{M}$ satisfy condition (*). Then $IM \cap TNC \neq \emptyset$.*

Proof. We shall construct $X \subseteq M$ such that $X \in IM \cap TNC$. Def is countable so let $(A_n)_{n \in \omega}$ be an enumeration of all infinite elements of Def. Then $X \in IM \cap TNC$ iff it satisfies the properties:

(a) $A_n \not\subseteq X$, for every $n \in \omega$.

(b) $X \cap A_n \notin CO$ and $(M \backslash X) \cap A_n \notin CO$, for every $n \in \omega$.

In order to cope with (b) let us pick for every n a splitting of A_n into two infinite subsets $A_n^0, A_n^1 \in Def$. Then it suffices that the following subconditions hold:

(b_1) $X \cap A_n^0$ is infinite,

(b_2) $X \cap A_n^1$ is infinite,

(b_3) $(A_n \backslash X) \cap A_n^0$ is infinite,

(b_4) $(A_n \backslash X) \cap A_n^1$ is infinite.

Clearly condition (a) guarantees that $X \in IM$. Conditions (b_1) and (b_2) will guarantee that $X \cap A_n \notin CO$ for all n, while (b_3) and (b_4) will guarantee that $(M \backslash X) \cap A_n \notin CO$ for all n. Therefore (b_1)-(b_4) together guarantee that $X \in TNC$. Also either of the conditions (b_3) and (b_4) entails property (a) above (since if $A_n \subseteq X$, then $A_n \backslash X = \emptyset$). So it suffices to construct X so that the conditions (b_1)-(b_4) are met. We define inductively and simultaneously sequences $(a_0^n)_{n \in \omega}$, $(a_1^n)_{n \in \omega}$, $(b_0^n)_{n \in \omega}$ $(b_1^n)_{n \in \omega}$, such that:

(i) $a_n^0, b_n^0 \in A_n^0$,

(ii) $a_n^1, b_n^1 \in A_n^1$,

(iii) $(\{a_0^0, \ldots, a_n^0\} \cup \{a_0^1, \ldots, a_n^1\}) \cap (\{b_0^0, \ldots, b_n^0\} \cup \{b_0^1, \ldots, b_n^1\}) = \emptyset$.

Suppose the sequences have been defined up to n so that (i)-(iii) are true and let K be the totality of the points chosen so far. K is finite. Since A_{n+1}^0, A_{n+1}^1 are infinite we pick $a_{n+1}^0 \neq b_{n+1}^0 \in A_{n+1}^0 \backslash K$, and $a_{n+1}^1 \neq b_{n+1}^1 \in A_{n+1}^1 \backslash (K \cup \{a_{n+1}^0, b_{n+1}^0\})$. Then clearly (i)-(iii) hold for the sequences up to

$n+1$. Let $X = \{a_n^0 : n \geq 0\} \cup \{a_n^1 : n \geq 0\}$. Then $X \cap (\{b_n^0 : n \geq 0\} \cup \{b_n^1 : n \geq 0\}) = \emptyset$, so conditions ($b_1$)-($b_4$) hold true and $X \in IM \cap TNC$. ⊣

The properties of immunity, cohesiveness and definability are not affected by finite changes of sets. To be precise, given sets X, Y let $X \subseteq' Y$ denote the fact that $X \backslash Y$ is finite, and $X =' Y$ the fact that $X \subseteq' Y$ and $X \subseteq' Y$. Then for sets X, Y such that $X =' Y$, we have trivially: (a) X is definable iff Y is so, (b) X is immune (partially immune) iff Y is so, (c) X is cohesive (partially cohesive) iff Y is so. In contrast to Proposition 1.2 the following holds.

Proposition 3.7 *(i) No $A \in Def^\infty$ splits into an immune and a cohesive set. (A fortiori, A doesn't split into two cohesive sets.)*

(ii) Stronger, no $A \in Def^\infty$ splits into an immune and a partially cohesive set. (A fortiori, A doesn't split into a cohesive and a partially cohesive set.)

(iii) No $A \in Def^\infty$ splits into a cohesive and a partially immune set.

Proof. (i) Let $A \in Def^\infty$, and let $A = X_1 \cup X_2$ be a partition of A such that $X_1 \in IM$ and $X_2 \in CO$. Let A_1, A_2 be a partition of A into two infinite definable sets. Then either $X_2 \cap A_1$ or $X_2 \cap A_2$ is finite. If $X_2 \cap A_1 = u$ is finite, then $(A_1 \backslash u) \subseteq X_1$. If $X_2 \cap A_2$ is finite, then $(A_2 \backslash u) \subseteq X_1$. But $A_1 \backslash u$ and $A_2 \backslash u$ are infinite definable sets contrary to the assumption that X_1 is immune.

(ii) Let now $A \in Def^\infty$, and let $A = X_1 \cup X_2$ be a partition of A such that $X_1 \in IM$ and $X_2 \in PCO$. Then there is an infinite $B \in Def$ such that $X_2 \cap B$ is cohesive. Then $A \cap B = (X_1 \cap B) \cup (X_2 \cap B)$, and $A \cap B$ is infinite since $X_2 \cap B$ is so. If $X_1 \cap B$ were finite, then $A \cap B =' X_2 \cap B$, which is impossible since one of them is definable and the other cohesive. Thus $X_1 \cap B$ is infinite, so it is immune since X_1 is so. But then the definable $A \cap B$ splits into an immune set $X_1 \cap B$ and a cohesive set $X_2 \cap B$, contrary to (i) above.

(iii) Let $A \in Def$ and let $X \subseteq A$ be cohesive. Let also $A \backslash X \in PIM$, i.e., there is a $B \in Def$ such that $B \subseteq A$ and $(A \backslash X) \cap B$ is immune. Suppose $A \backslash B$ is finite. Then $B =' A$, therefore $(A \backslash X) \cap B =' (A \backslash X) \cap A = A \backslash X$. It follows that $A \backslash X \in IM$, thus A splits into a cohesive and an immune set which contradicts clause (i) above. Therefore B and $A \backslash B$ are both infinite. Then $X \in CO$ implies that either $X \cap B$ or $X \cap (A \backslash B)$ is finite. In the first case $B \subseteq' (A \backslash X)$, so $(A \backslash X) \cap B =' B$, a contradiction since $(A \backslash X) \cap B \in IM$. In the second case $(A \backslash B) \subseteq' (A \backslash X)$, thus $X \subseteq' B$.

Therefore $B =' X \cup ((A \backslash X) \cap B)$. But X is cohesive while $(A \backslash X) \cap B$ is immune and they partition the definable B, contrary again to (i) above. $\dashv$

In contrast to Proposition 3.7 above, we have the following:

Proposition 3.8 *If $\mathcal{M}$ satisfies (*), then every $A \in Def^\infty$ splits into two partially cohesive sets.*

Proof. (iv) Let $A \in Def^\infty$. By (*) there is a partition of A into $A_1, A_2 \in Def^\infty$. By Corollary 3.4, there are $X_1, X_2 \in CO$ such that $X_1 \subset A_1$ and $X_2 \subset A_2$. Set $X = X_1 \cup (A_2 \backslash X_2)$ and $Y = X_2 \cup (A_1 \backslash X_1)$. Then X, Y form a partition of A, and $X \cap A_1 = X_1$ and $Y \cap A_2 = X_2$ are both cohesive, thus $X, Y \in PCO$. $\dashv$

Remarks 3.9 (i) By 3.7 (iii), for every $X \subseteq M$,

$$X \in CO \ \Rightarrow (M \backslash X) \notin PIM. \tag{1}$$

However, since by 3.7 (i), X and $M \backslash X$ cannot both be cohesive, we cannot use this fact to obtain non-definable sets such that $X, M \backslash X \notin PIM$, i.e., $X \in TNI \backslash Def$.

(ii) Immediately after the proof of Proposition 3.2 we mentioned the well-known fact that if $X \subseteq \omega$ is r.e. and $\omega \backslash X$ is cohesive, then X is a *maximal r.e.* set (modulo finite sets). There is no analogous fact in our case. Namely, if X is non-partially immune and $M \backslash X$ is cohesive, it doesn't follow that X is a maximal non-definable and non-partially immune (modulo finite sets). Indeed, let us split the cohesive $M \backslash X$ into two infinite and co-infinite subsets Y_1, Y_2. Then both Y_1, Y_2 are cohesive too. Hence according to (i) above, $M \backslash Y_1$ is non-partially immune, non-definable and $X = M \backslash (Y_1 \cup Y_2) \subset M \backslash Y_1$. Also, since Y_2 is infinite, $X \neq' (M \backslash Y_1)$. Therefore X in not maximal with respect to the properties in question.

(iii) The implication (1) is best possible, in the sense that it does not hold if we replace cohesive with partial cohesive. Thats is, the implication

$$X \in PCO \ \Rightarrow M \backslash X \notin PIM$$

is false. Indeed, by 3.8 pick $X \subset M$ such that both X and $M \backslash X$ are partially cohesive. Then by 3.5 (ii), $X \in PCO$ and $M \backslash X \in PIM$.

4 The case of models of Arithmetic

In this section we focus on models of PA, standard or nonstandard in the first subsection and exclusively nonstandard in the second one.

4.1 Semi-definable sets

In section 1 we saw that the infinite definable sets give rise to a pseudo-topology of open sets, where the latter coincide with the non-immune sets. In models of arithmetic this can be taken a bit further because of the special property of a model $\mathcal{M} \models \text{PA}$ to code infinite partitions of M. Throughout this section $\mathcal{M}$ is a countable model of PA, standard or nonstandard. When $\mathcal{M}$ is non-standard, the natural analogue of "finite subset" is "$\mathcal{M}$-finite", or "definable and bounded". This is the reason that in most cases below we use "bounded" rather than "finite".

Definition 4.1 Let $\mathcal{M}$ be a model of PA and $A \subseteq M$ be definable. A family $P = \{P_a : a \in A\}$ of subsets of M is a *uniformly definable partition,* or a *udp* for short, if it is disjoint and there is a formula $\psi(v_0, v_1, \vec{c})$, with parameters, such that for each $a \in A$, $P_a = \{b \in M : \mathcal{M} \models \psi(a, b, \vec{c})\}$ and P_a is unbounded.

Clearly if P is a udp, then $\bigcup P$ is an infinite definable set and P is a partition of $\bigcup P$. Most often $\bigcup P = M$, i.e., P is a partition of the entire M, but not necessarily.

A standard example of a udp is the family $R = \{R_a : a \in M\}$, where

$$R_a = \{J(a, b) : b \in M\},$$

and $J : M \times M \to M$ is the usual pairing bijection that codes pairs by single elements, defined by $J(a, b) = \frac{(a+b)(a+b+1)}{2} + a$. Clearly there are countably many udp's.

Lemma 4.2 *For every unbounded definable $A \subseteq M$ there is a udp P such that $A = \bigcup P$.*

Proof. If A is unbounded, there is a definable bijection $f : M \to A$. If R is the standard udp of M mentioned above, and we set $f[R_a] = P_{f(a)}$, then $P = \{P_b : b \in A\}$ is a udp such that $\bigcup P = A$. ⊣

Definition 4.3 A $X \subseteq M$ is said to be *semi-definable,* if there is a udp $P = \{P_a : a \in A\}$ of M, where A is an unbounded definable set and $I \subset A$, such that $X = \bigcup_{i \in I} P_i$. I is said to be the *support of X with respect to P.*

For simplicity, and without any serious loss of generality, henceforth we assume that the index set A of every udp $P = \{P_a : a \in A\}$ is identical to the entire M.

Clearly if X is semi-definable with respect to a udp P and with support I, then $(M\backslash X) \in SDef$ with support $M\backslash I$. So it is straightforward that $SDef \subseteq CLO$. We shall show that the inclusion is proper. First we need the following definition.

Definition 4.4 Given $A \in Def^\infty$ and a udp of M $P = \{P_a : a \in M\}$, we say that P is *A-coarse,* if for every $a \in M$, $P_a\backslash A$ is infinite.

Lemma 4.5 *Let $A \in Def^\infty$ and let $A \subseteq X$ such that $X \in SDef\backslash Def$. Then there is an A-coarse udp Q such that $X = \bigcup_{j\in J} Q_j$ for some $J \subseteq M$.*

Proof. Let $A \subseteq X$ and $X \in SDef\backslash Def$. Then there is a udp $P = \{P_a : a \in M\}$ such that $X = \bigcup_{i\in I} P_i$ for some I. Since $X \notin Def$, $I \notin Def$. Let

$$K = \{a \in M : P_a\backslash A \text{ is bounded}\}.$$

K is definable and $K \subseteq I$ because for every $i \in K$, $P_i \cap A \neq \emptyset$. Moreover $K \subsetneq I$ because I is not definable. If K were cofinite, then so would be I, so I would be definable which is false. Thus $M\backslash I$ and $M\backslash K$ are infinite. Let $B = \bigcup_{i\in K} P_i$. B is definable. Suppose first that $B\backslash A$ is unbounded. Then we set $Q_0 = B = \bigcup_{i\in K} P_i$ and let Q_a, $a \in M$, be a re-enumeration of P_a, for $a \in M\backslash K$, so that Q forms a udp of M (which is coarser than P). Then clearly $Q_a\backslash A$ is unbounded for every a, so Q is A-coarse. Moreover there is a J such that $X = \bigcup_{j\in J} Q_j$. Suppose next that $B\backslash A$ is bounded. Then we pick an $i_0 \in I\backslash K$ and set $Q_0 = P_{i_0} \cup B$. Since $P_{i_0}\backslash A$ is unbounded, so is $Q_0\backslash A$. Further it suffices to let Q_a, $a > 0$, be a re-enumeration of Q_a for $a \in M\backslash K \cup \{i_0\}$. Again Q is A-coarse, and there is J such that $X = \bigcup_{j\in J} Q_j$. ⊣

Proposition 4.6 *$SDef \subsetneq CLO$.*

Proof. As we already mentioned above it is straightforward that $SDef \subseteq CLO$. We construct a $X \in CLO\backslash SDef$. Pick $A, B \in Def^\infty$ such that $A \cap B = \emptyset$ and $M\backslash(A \cup B)$ is infinite. Let $(P^n)_{n\in\omega}$ be an enumeration of all A-coarse uniform partitions of M. Each such partition contains countably many sets so let $R = \{\Gamma_n : n \in \omega\}$ be an enumeration of $\bigcup_n P^n$, i.e., of the set $\{P^n_a : a \in M, n \in \omega\}$. For all n, $\Gamma_n\backslash A$ is infinite. We define by ω-induction a sequence $(a_n)_{n\in\omega}$ of elements of M such that $a_n \in \Gamma_n\backslash A$, as follows: Suppose $k_0, \ldots, k_n$ have been defined. Since $\Gamma_{n+1}\backslash A$ is infinite, so is $\Gamma_{n+1}\backslash(A \cup \{k_0, \ldots, k_n\})$, so we can pick $k_{n+1} \in \Gamma_{n+1}\backslash(A \cup \{k_0, \ldots, k_n\})$. Set

$$X = A \cup (M\backslash(B \cup \{k_n : n \in \omega\})).$$

We claim that $X \in CLO \backslash SDef$. That $X \in CLO$ follows from the fact that $A \subseteq X$ and $B \subseteq M \backslash X$, Thus by 1.5, X and $M \backslash X$ are open. On the other hand assume $X \in SDef$. Since $A \subseteq X$, by Lemma 4.5 there is a A-coarse uniform partition $P = \{P_a : a \in M\}$ such that $X = \bigcup_{i \in I} P_i$. Then $P = P_m$ for some m, i.e., $P_i = \Gamma_{j_i}$ for some j_i. Thus $k_{j_i} \in \Gamma_{j_i} \subseteq X$, which contradicts the definition of X. $\dashv$

The reason for considering semi-definable sets was to examine their probable relationship with t.n.i. sets. Namely, whether we can give examples of t.n.i sets in the form of semi-definable ones. In the next proposition we see that if $X = \bigcup_{i \in I} P_i$ is a semi-definable set with support I, then each one of the sought properties for X is reflected to its support I and vice-versa.

Proposition 4.7 *Let $X \in SDef$ and let I be the support of X with respect to some udp P, i.e., $X = \bigcup_{i \in I} P_i$. Then:*
(i) $X \in IM$ iff $I \in IM$.
(ii) $X \in PIM$ iff $I \in PIM$.
(iii) $X \in TNI$ iff $I \in TNI$.

Proof. Let X, P, I be as stated above, and $X = \bigcup_{i \in I} P_i$. Note that (iii) follows immediately from (ii). Because $X \in TNI$ iff $X \notin PIM$ and $(M \backslash X) \notin PIM$, so by (ii) (noting that I is the support of X iff $M \backslash I$ is the support of $M \backslash X$), $I \notin PIM$ and $M \backslash I \notin PIM$, i.e., iff $I \notin TNI$. Further (i) is a special case of (ii). Namely, it is the case of (ii) where in the intersections $A \cap X$ or $A \cap I$, for arbitrary $A \in Def$, we take $A = M$. So it suffices to prove (ii).

"$\Rightarrow$": Assume $X \in PIM$. Then there is $A \in Def$ such that $X \cap A = \bigcup_{i \in I}(P_i \cap A) \in IM$. It follows by immunity that $|P_i \cap A| < \omega$ for every $i \in I$. Let $W = \{a \in M : P_a \cap A \neq \emptyset\}$. Clearly W is infinite and definable, so there is a definable choice function f for the sets $P_i \cap A$, for $i \in W$, i.e., $f(i) \in P_i \cap A$ for every $i \in W$. Also $W \cap I$ is infinite, otherwise $X \cap A$ would be finite. Moreover for every $i \in W \cap I$, $f(i) \in X \cap A$, so $f[W \cap I] \subseteq X \cap A$. Since $X \cap A \in IM$, it follows that $f[W \cap I] \in IM$ too, so $W \cap I \in IM$ since f is a definable injection. Since W is definable, $I \in PIM$.

"$\Leftarrow$": Assume $I \in PIM$. Let $A \in Def$ such that $A \cap I \in IM$. Using induction inside $\mathcal{M}$ we can find a definable choice function for the udp family P. Then clearly $f[A \cap I] = f[A] \cap f[I] = f[A] \cap X$, since I is the support of X. Also $f[A \cap I]$ is immune since $A \cap I$ is so and f is a definable injection. Thus $f[A] \cap X$ is immune. Since $f[A]$ is definable, $X \in PIM$. $\dashv$

It follows from Proposition 4.7 that in trying to construct t.n.i. sets in the form of semi-definable ones, one falls into the regression "X is t.n.i. iff its support is t.n.i.", that one can hardly see how it could be broken. The net outcome of 4.7 is that *having already* at hand t.n.i. sets (as well as sets from the other classes), one can construct semi-definable t.n.i. sets.

Corollary 4.8 *Let $\mathcal{M} \models$ PA. Then the class $SDef$ meets each one of the classes IM, PIM, TNI and their complements.*

Proof. Pick a udp $P = \{P_a : a \in M\}$ of M. Then picking $I \in IM$, or $I \in PIM$, or $I \in TNI \backslash Def$, or I in the complement of any of preceding classes, and setting $X = \bigcup_{i \in I} P_i$, it follows from Proposition 4.7 that $X \in SDef$ and also $X \in IM$, or $X \in PIM$, or $X \in TNI$, etc, respectively. $\dashv$

4.2 Nonstandard models and generic classes

In this section we restrict ourselves to countable *nonstandard* models of PA. The reason is that in such models particular examples of non-definable t.n.i. sets have been constructed and used by J. Schmerl [5], M. Kaufmann [3], and others, under the name "class" and "generic class".

Recall that if $\mathcal{M} \models$ PA is a nonstandard model, then a set $X \subset M$ is *M-finite* if it is definable and bounded. There is a definable enumeration C_a, $a \in M$, of all these sets, in the sense that there is a formula $\phi(x, y)$ such that for every $a \in M$ (or for some unbounded definable $K \subset M$), $C_a = \{b \in M : \mathcal{M} \models \phi(a, b)\}$. A set $X \subset M$ is said to be a *class* if it is not M-finite and for every M-finite C_a, $X \cap C_a$ is again M-finite. Clearly, in order for an unbounded X to be a class, it suffices that $X \cap [0, c]$ is M-finite for every $c \in M$. Obviously every unbounded definable set is a class. Moreover, there exist models of PA in which every class is a definable set. Such models are called "rather classless" and [3], [6] deal mainly with the proof of their existence under various conditions.

Some non-definable classes however may have trivial character when the model $\mathcal{M}$ has countable cofinality, i.e., when there exists a sequence $(c_n)_{n \in \omega}$ cofinal in M. Such a set $X = \{c_n : n \geq 0\}$ is clearly non-definable, while for every M-finite C_a, $C_a \cap X$ is ω-finite, thus also M-finite. Therefore it is a class, but one that is remote from definable sets: Actually X is *immune*. Similarly if we take the set $C_a \cup X$, for some M-finite C_a. Actually $C_a \cup X$ is partially immune.

In order to get classes which are t.n.i sets we have to appeal to *generic classes* defined below. Given a bounded set X and an arbitrary $Y \subseteq M$, let

$X \preccurlyeq Y$ denote the fact that X is an initial segment Y, with respect to the natural ordering of M. In M we consider the partial ordering $\trianglelefteq$ defined as follows: $a \trianglelefteq b$ iff $C_a \preccurlyeq C_b$. A definable $A \subseteq M$ is *dense* in $(M, \trianglelefteq)$ if for every $a \in M$ there is a $b \in A$ such that $a \trianglelefteq b$. Below by "dense" we shall always mean "dense in $(M, \trianglelefteq)$".

Definition 4.9 (Schmerl [5]) Let $\mathcal{M} \models \mathrm{PA}$ be nonstandard. A set $X \subset M$ is said to be a *generic class* if for every definable dense $A \subset M$, there is $a \in A$ such that $C_a \prec X$.

For countable $\mathcal{M}$ the existence of generic classes is shown as usual, due to the fact that there are only countably many definable dense subsets of $(M, \trianglelefteq)$. (Actually Schmerl shows in [5] a much stronger result: If M has countable cofinality then there exists a class $\mathcal{X}$ of generic classes such that $|\mathcal{X}| = |M|$.)

Lemma 4.10 ([5]) *If $X \subset M$ is a generic class, then X is non-definable and for every M-finite C_a, $X \cap C_a$ is M-finite. Moreover X contains for every $c \in M$ an initial segment of internal cardinality $> c$.*[2]

Proof. Let $A = \{a \in M : C_a \not\prec X\}$. A is dense in $(M, \trianglelefteq)$. If X were definable, A would be definable too, so A would meet X in the sense that for some $a \in A$, $C_a \prec X$, a contradiction. To show that X is a class, it suffices to show that for every $c \in M$, $X \cap [0, c]$ is definable. For every M-finite C_a, let $|C_a|$ denote the internal cardinality of C_a. Given $c \in M$ let

$$B_c = \{a \in M : |C_a| > c\}.$$

Clearly B_c is definable and dense, so there is $a \in B_c$ such that $C_a \prec X$. Since $|C_a| > c$ it follows that $X \cap [0, c] = C_a \cap [0, c]$, thus $X \cap [0, c]$ is definable. The fact that $C_a \prec X$ and $|C_a| > c$ proves also other claim. $\dashv$

Given any definable unbounded set $A \subset M$, the above partial order $\trianglelefteq$ relativizes to A in the obvious way. Namely let A^* be the set of codes of the bounded subsets of A, i.e., $A^* = \{a : C_a \subset A\}$. Then we work as before with $(A^*, \trianglelefteq)$ rather than $(M, \trianglelefteq)$. A set $X \subseteq A$ is a *generic class relative to A,* or *A-generic,* if definition 4.9 applies to X with A in place of M. Obviously M-generic classes coincide with generic classes.

[2]For a bounded definable set $C_a \subset M$, the internal cardinality of C_a is the unique $b \in M$ for which there is a definable bijection $f : C_a \to [0, b-1]$. Then we write $|C_a| = b$.

Lemma 4.11 *If $A \subset M$ is a definable unbounded set and $X \subset M$ is a generic class, then $A \cap X$ is an A-generic class.*

Proof. (Sketch) First it can be easily seen that if D is a dense subset of $(M, \trianglelefteq)$ and we set $D^A = \{b \in A^* : (\exists a \in D)(C_b = A \cap C_a)\}$, then D^A is a dense subset of $(A^*, \trianglelefteq)$. Moreover, every dense subset of $(A^*, \trianglelefteq)$ is of this form. Thus to show that $A \cap X$ is A-generic it suffices to show that for every dense D^A of $(A^*, \trianglelefteq)$, there is a $b \in D^A$ such that $C_b \prec A \cap X$. But given a D which is dense in $(M, \trianglelefteq)$, there is $a \in D$ such that $C_a \prec X$. Therefore $A \cap C_a \prec A \cap X$. So if $C_b = A \cap C_a$, then $C_b \prec A \cap X$ and $b \in D^A$. ⊣

Let $GCL(M)$, or just GCL, be the family of all generic classes of M.

Proposition 4.12 *Let $\mathcal{M} \models$ PA be a nonstandard model containing generic classes, i.e., $GCL \neq \emptyset$. Then $GCL \subseteq TNI$.*

Proof. Let $X \in GCL$. It is easy to see that $(M \backslash X) \in GCL$, so it suffices to show that $X \notin PIM$, i.e., for every $A \in Def^{\infty}$, if A is infinite then $A \cap X$ contains a $B \in Def^{\infty}$. Let $A \in Def^{\infty}$ and $A \cap X$ be infinite. If A is bounded, then $A \cap X$ is definable, since X is a class, so we are done. Let A be unbounded. Then by Lemma 4.11, $A \cap X$ is A-generic. According to the second claim of Lemma 4.10, $A \cap X$ contains, for every $c \in M$, an initial segment of internal cardinality $> c$. Thus in any case $A \cap X$ contains an infinite definable set. Hence $X \notin PIM$. ⊣

A most interesting fact is that Schmerl shows the existence of generic classes not just in countable models of PA, but also in models with *countable cofinality* (using simultaneously external induction along ω and internal induction along M). In view of Proposition 4.12, this means that every such model contains also t.n.i. non-definable sets. Thus:

Corollary 4.13 *If $\mathcal{M}$ is a model of* PA *with countable cofinality, then $TNI \backslash Def \neq \emptyset$.*

By 4.12 above, in every nonstandard model of PA with generic classes $GCL \subseteq TNI$. However the following is open.

Question 4.14 *Is $GCL \subsetneq TNI$ in every nonstandard model containing generic classes?*

Concerning the distinction between bounded and unbounded definable sets in models of PA, we can see that the construction of section 2 and theorem 2.1 can be slightly modified in order to involve bounded sets only. Specifically, let $\mathbb{P}_b$ be the poset defined just like $\mathbb{P}$ in section 2, except that for every $p = (p_0, p_1) \in \mathbb{P}_b$, p_0, p_1 are infinite *bounded* definable subsets of M (rather than arbitrary infinite definable sets), such that $p_0 \cap p_1 = \emptyset$. Then condition (iii) of the definition of $\mathbb{P}$ is automatically satisfied. Let $\mathcal{D}$ be again the (adapted) set of $\mathcal{L}_2$-definable dense subsets of $\mathbb{P}_b$. Then by some obvious adjustments of the proof of Theorem 2.1 we can show the following:

Theorem 4.15 *Let $\mathcal{M}$ be a countable nonstandard model of PA. If G is a $\mathcal{D}$-generic subset of $\mathbb{P}_b$, then G_0 and G_1 are non-definable t.n.i. subsets of M.*

5 Concluding remarks and possibilities for future work

The referee expressed certain concerns about the restricted list of references of this paper, that is, about its not being related to other current research, although some of the notions involved, like immunity and cohesiveness, have been a subject of considerable research in the last few decades. This is indeed the case and is due to the fact that the existing literature on immunity and cohesiveness concerns exclusively the investigation of these notions as standard notions of computability theory, while here they are used with a different meaning, as notions of definability theory. So both the departure point as well as the aim and tools for studying them are different in the two areas. To give an example, in computability theory immune sets were originally related to simple sets, while the latter were connected with the solution of Post's problem. A set $X \subset \omega$ is simple if it is r.e. and $\omega \backslash X$ is immune, i.e., has no r.e. subset. However, in definability theory there is no analogue of simple set (that is, a corresponding notion resulting from the replacement of r.e. with definable), since the complement of a definable set is definable, so it cannot be immune. Hence the whole literature around simple sets in computability does not make any sense in the area of definability.

As I said in the introduction, the aim of this paper was to approximate the class of definable sets of a first-order structure from outside, i.e., from the area of undefinable sets, by eliminating aspects and degrees of the latter along a number of steps. In doing so the aim of approximating the definable sets was inevitably coupled with the "dual" aim of isolating notions of "randomness" for subsets of a structure, where randomness is now construed not as a property at the antipodes of computable sets but as a property at the antipodes of definable sets. So the various notions of immunity dealt with in this paper are notions of randomness. Actually, while working on the paper, my hope was that by eliminating all reasonable forms of such randomness (=immunity) one would reach the definable sets. In other words I expected that a totally non-immune set should be definable. The proof that this is not the case came rather as a surprise.

A similar idea on the relationship between immunity and randomness is explicitly stated in [2], although the latter is a paper on computability, so it treats immunity in its classical meaning. More precisely, [2] deals with bi-immunity: A set $X \subset \omega$ is bi-immune if both X and $\omega \backslash X$ are immune. The authors believe that Martin-Löf randomness (or 1-randomness) is the "correct" implementation of intuitive randomness. But they consider also weaker notions. As they say (p. 977):

> We shall also consider notions which have some of the spirit of randomness but are too weak to be considered true notions of randomness.

Such weak notions are Kurtz-randomness and bi-immunity. Concerning the latter:

> This can be thought of as a very weak kind of randomness since it says that it is impossible to correctly predict for infinitely many n whether or not n belongs to X.

Bi-immunity is immediately carried over to definability. Fixing a structure $\mathcal{M} = (M, \ldots)$, let us say that $X \subset M$ is bi-immune if neither X nor $M \backslash X$ contain any infinite definable set. Obviously X is bi-immune iff its complement is so. Bi-immune sets exist in abundance. For example, by Proposition 1.2, every set splits in a pair of bi-immune sets. Let BIM denote the class of bi-immune subsets of M. Recall also that IM and CO denote the classes of immune and cohesive subsets of M, respectively.

Fact 5.1 *(i)* $BIM \subsetneq IM$.
(ii) If $X \in BIM$*, then for every infinite definable* A*,* $A \cap X$ *is infinite.*
(iii) Therefore $BIM \cap CO = \emptyset$*.*

Proof. (i) Obviously $BIM \subseteq IM$. On the other hand, if A is an infinite and co-infinite definable subset of M, clearly we can find an immune $X \subset A$. Then $X \in IM \backslash BIM$.

(ii) Let A be infinite and definable. If $A \cap X$ were finite, then A minus a finite part would be contained in $M \backslash X$, contrary to the fact that $M \backslash X$ is immune.

(iii) Let $X \in BIM$. Pick an infinite and co-infinite definable A. Then by (ii), $A \cap X$ and $(M \backslash A) \cap X$ are infinite. So X is not cohesive. $\dashv$

Recall that by 3.5 and 3.6 also $CO \subsetneq IM$. Thus the classes BIM and CO are proper and disjoint subclasses of IM, offering (mutually incompatible) notions of randomness stronger than that of IM. There are still further notions of randomness stronger than immunity. In footnote 1 above we already mentioned one: genericity (defined modulo the class of definable sets). In addition, for the particular structure $(\omega, +, \cdot, S, 0)$, *hyperimmunity* (modulo definability), the analogue of the corresponding notion from computability, also makes sense: $X \subset M$ is called hyperimmune if there is no definable $f : \omega \to \omega$ such that $p_X(n) \leq f(n)$ for almost all $n \in \omega$ (i.e., except finitely many), where $p_X(n)$ is the n-th element of X in its natural ordering.

Concerning future research, I would see roughly two directions: One towards isolating and studying strong notions of randomness/immunity, like genericity and hyperimmunity in the above sense. And an opposite direction, towards finding weak such notions, namely even weaker than total non-immunity, i.e., finding natural classes of sets $\mathcal{X}$ such that $Def \subsetneq \mathcal{X} \subsetneq TNI$. This is actually a generalization of Question 4.14 already asked above, i.e., whether $GCL \subsetneq TNI$ in nonstandard models containing generic classes, where GCL is the class of generic classes.

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