ΛΟΓΙΣΜΟΣ Ι, - Ασκήσεις 10, Λυσεις

1. (α) Βρισκομε τις παραγωγους

$$f(x) = \sin(x^2), \quad f'(x) = 2x\cos(x^2), \quad f''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$$
$$f'''(x) = -4x\sin(x^2) - 8x\sin(x^2) - 8x^3\cos(x^2)$$

και τις τιμες τους στο 0

$$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 2, \quad f'''(0) = 0.$$

Αρα

$$\begin{aligned} P_{3,0}(x) =& f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 \\ =& 0 + 0(x-0)^2 + \frac{2}{2}(x-0)^2 + \frac{0}{6}(x-0)^3 = x^2. \end{aligned}$$

Ομοια βρισκομε
(β) $P_{3,0}(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{48}x^3$
(γ) $P_{3,0}(x) = 1 - x^2$
(δ) $P_{6,0}(x) = 1-2(x-1)+3(x-1)^2-4(x-1)^3+5(x-1)^4-6(x-1)^5+7(x-1)^7$

2.

$$\lim_{x\to 0}\frac{\arcsin(x)}{x} = \lim_{x\to 0}\frac{\frac{1}{\sqrt{1-x^2}}}{1} = 1.$$

$$\begin{aligned} \lim_{x\to 0}\frac{\sin(x) - x + \frac{x^3}{6}}{x^5} &= \lim_{x\to 0}\frac{\cos(x) - 1 + \frac{1}{2}x^2}{5x^4} = \lim_{x\to 0}\frac{-\sin(x) + x}{20x^3} \\ &= \lim_{x\to 0}\frac{-\cos(x)+1}{60x^2} = \lim_{x\to 0}\frac{\sin(x)}{120x} = \frac{1}{120}. \end{aligned}$$

$$\lim_{x\to 0}\frac{\arctan(x^2) - x^2}{x^2} = \lim_{x\to 0}\frac{\frac{2x}{1+x^4} - 2x}{2x} = \lim_{x\to 0}\frac{\frac{1}{1+x^4} - 1}{1} = 0.$$

$$\lim_{x\to 0}\frac{e^{\sin(x)} - 1}{x} = \lim_{x\to 0}\frac{e^{\sin(x)}\cos(x)}{1} = \frac{e^0\cdot 1}{1} = 1.$$

3. Ετσω $y \in \mathbb{R}$ σταθεροποιημενο. Θεωρουμε την συναρτηση

$$f(x) = \cos(x+y).$$

παραγωγιζοντας δυο φορες βρισκομε

$$f''(x) = -\cos(x+y) = -f(x), \qquad \text{δηλ.} \quad f + f'' = 0.$$

Απο το Θεωρημα 4 (σελ. 264)

$$f(x) = b\sin(x) + a\cos(x),$$

οπου

$$a = f(0) = \cos(0+y) = \cos(y), \qquad b = f'(0) = -\sin(0+y) = -\sin(y).$$

Δηλ.

$$\cos(x+y) = f(x) = -\sin(y)\sin(x) + \cos(y)\cos(x).$$