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Integral Equations and Operator Theory

Semigroups of Composition Operators in BMOA and the Extension of a Theorem of Sarason

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Abstract. In this paper we deal with the maximal subspace in $BMOA$ where a general semigroup of analytic functions on the unit disk generates a strongly continuous semigroup of composition operators. Particular cases of this question are related to a well-known theorem of Sarason about $VMOA$. Our results describe analytically that maximal subspace and provide a condition which is sufficient for the maximal subspace to be exactly $VMOA$. A related necessary condition is also proved in the case when the semigroup has an inner Denjoy-Wolff point. As a byproduct we provide a generalization of the theorem of Sarason.

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1. Introduction

Let $\mathcal{H}(\mathbb{D})$ be the Fréchet space of all analytic functions in the unit disk endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$.

A (one-parameter) semigroup of analytic functions is any continuous homomorphism $\Phi : t \mapsto \Phi(t) = \varphi_t$ from the additive semigroup of nonnegative real numbers into the composition semigroup of all analytic functions which map $\mathbb{D}$ into $\mathbb{D}$. In other words, $\Phi = (\varphi_t)$ consists of analytic functions on $\mathbb{D}$ with $\varphi_t(\mathbb{D}) \subset \mathbb{D}$ and for which the following three conditions hold:

1. φ_0 is the identity in $\mathbb{D}$,

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2. $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for all $t, s \geq 0$,
3. $\varphi_t \to \varphi_0$, as $t \to 0$, uniformly on compact subsets of $\mathbb{D}$.

It is well known that condition (3) above can be replaced by

(3′) For each $z \in \mathbb{D}$, $\varphi_t(z) \to z$, as $t \to 0$.

Each such semigroup gives rise to a semigroup (C_t) consisting of composition operators on $\mathcal{H}(\mathbb{D})$,

$$C_t(f) := f \circ \varphi_t, \qquad f \in \mathcal{H}(\mathbb{D}).$$

We are going to be interested in the restriction of (C_t) to certain linear subspaces $\mathcal{H}(\mathbb{D})$. Given a Banach space X consisting of functions of $\mathcal{H}(\mathbb{D})$ and a semigroup (φ_t), we say that (φ_t) generates a semigroup of operators on X if (C_t) is a well-defined strongly continuous semigroup of bounded operators in X. This exactly means that for every $f \in X$, we have $C_t(f) \in X$ for all $t \geq 0$ and

$$\lim_{t \to 0^+} \|C_t(f) - f\|_X = 0.$$

Thus the crucial step to showing that (φ_t) generates a semigroup of operators in X is to pass from the pointwise convergence $\lim_{t\to 0^+} f \circ \varphi_t(z) = f(z)$ on $\mathbb{D}$ to the convergence in the norm of X.

This connection between composition operators and semigroups opens the possibility of studying spectral properties, operator ideal properties or dynamical properties of the semigroup of operators (C_t) in terms of the theory of functions. Papers [1] and [12] can be considered as the starting points in this direction.

Classical choices of X treated in the literature are the Hardy spaces H^p, the disk algebra $A(\mathbb{D})$, the Bergman spaces A^p, the Dirichlet space $\mathcal{D}$ and the chain of spaces Q_p and $Q_{p,0}$ which have been introduced recently and which include the spaces $BMOA$, Bloch as well as their "little oh" analogues. See [21] and [22] for definitions and basic facts of the spaces and [17], [18], and [20] for composition semigroups on these spaces.

Very briefly, the state of the art is the following: (i) Every semigroup of analytic functions generates a semigroup of operators on the Hardy spaces H^p $(1 \leq p < \infty)$, the Bergman spaces A^p $(1 \leq p < \infty)$, the Dirichlet space, and on the spaces VMOA and little Bloch. (ii) No non-trivial semigroup generates a semigroup of operators in the space H^∞ of bounded analytic functions. (iii) There are plenty of semigroups (but not all) which generate semigroups of operators in the disk algebra. Indeed, they can be well characterized in several analytical terms [4].

In this paper we concentrate on the space $BMOA$. As we will see, the strong continuity behavior differs notably from other known cases, since it depends heavily on the particular semigroup.

This has led us to introduce the following notation: Given a semigroup (φ_t) we denote by $[\varphi_t, BMOA]$ the maximal closed linear subspace of $BMOA$ such that (φ_t) generates a semigroup of operators on it. The existence of such a maximal subspace, as well as analytical descriptions of it will be discussed in section two. In that section, we also present an alternative self-contained proof of the fact that every semigroup generates a semigroup of operators on $VMOA$. This in particular means that in our notation

$$VMOA \subseteq [\varphi_t, BMOA]$$

for every semigroup (φ_t). It is important to underline that, in general, this inclusion can be proper.

The chain of inclusions $VMOA \subseteq [\varphi_t, BMOA] \subseteq BMOA$ leads us to wonder about those semigroups with an extreme character, that is, those giving equality

$$VMOA = [\varphi_t, BMOA] \qquad \text{or} \qquad [\varphi_t, BMOA] = BMOA.$$

In section three we deal with the left hand equality $VMOA = [\varphi_t, BMOA]$, and present a sufficient condition on the semigroup for this equality to hold. A different but closely related condition is shown to be necessary for semigroups with inner Denjoy-Wolff point. The conditions are in terms of the growth of the infinitesimal generator of (φ_t) near the boundary of $\mathbb{D}$.

There is an important connection of the above results with a well-known theorem of D. Sarason which characterizes the space $VMOA$. Namely, Sarason [14], [15] proved

Theorem A. (Sarason [14]) *Suppose $f \in BMOA$; then the following are equivalent:*

1. $f \in VMOA$.
2. $\lim_{t\to 0^+} \|f(e^{it}z) - f\|_{BMOA} = 0$.
3. $\lim_{t\to 0^+} \|f(e^{-t}z) - f\|_{BMOA} = 0$.

In our notation this theorem can be written as

$$VMOA = [e^{it}z, BMOA] = [e^{-t}z, BMOA].$$

In other words, the semigroups $\varphi_t(z) = e^{it}z$ of rotations and $\varphi_t(z) = e^{-t}z$ of dilatations with respect to the origin are left-extreme in the above sense. Our results provide many more nontrivial examples of semigroups of this type.

We end this introduction by presenting a quick review of basic facts about a general semigroup of analytic functions (see [18]). The basic material about operator semigroups on Banach spaces can be found in [9, Chapter VIII].

If (φ_t) is a semigroup, then each map φ_t is univalent. The infinitesimal generator of (φ_t) is the function

$$G(z) := \lim_{t\to 0^+} \frac{\varphi_t(z) - z}{t}, \; z \in \mathbb{D}.$$

This convergence holds uniformly on compact subsets of $\mathbb{D}$ so $G \in \mathcal{H}(\mathbb{D})$. Moreover G satisfies

$$G(\varphi_t(z)) = \frac{\partial \varphi_t(z)}{\partial t} = G(z)\frac{\partial \varphi_t(z)}{\partial z}, \; z \in \mathbb{D}, \; t \geq 0. \tag{1.1}$$

Further G has a unique representation

$$G(z) = (\overline{b}z - 1)(z - b)P(z), \; z \in \mathbb{D}$$

where $b \in \overline{\mathbb{D}}$ and $P \in \mathcal{H}(\mathbb{D})$ with $\operatorname{Re} P(z) \geq 0$ for all $z \in \mathbb{D}$. If G is not identically null, the couple (b, P) is uniquely determined from (φ_t) and the point b is called the Denjoy-Wolff point of the semigroup. We want to mention that this point plays a crucial role in the dynamical behavior of the semigroup (see [18], [5]).

Note that for r sufficiently near to one, it is clear from the above representation of G that G has no zeros in the annulus $r < |z| < 1$, so $1/G$ is analytic on that annulus. This remark will be implicitly used throughout the paper.

2. Semigroups in BMOA

For the sake of completeness and to fix notations, we begin with a quick review of basic properties of $VMOA$ and $BMOA$.

$BMOA$ is the Banach space of all analytic functions in the Hardy space H^2 whose boundary values have bounded mean oscillation. There are many characterizations of this space but we will use the one in terms of Carleson measures (see [22, 11]). Namely, a function $f \in H^2$ belongs to $BMOA$ if and only if there exists a constant $C > 0$ such that

$$\int_{R(I)} |f'(z)|^2 (1-|z|^2) dA(z) \leq C|I|,$$

for any arc $I \subset \partial\mathbb{D}$, where $R(I)$ is the Carleson rectangle determined by I, that is,

$$R(I) := \left\{ re^{i\theta} \in \mathbb{D} : 1 - \frac{|I|}{2\pi} < r < 1 \text{ and } e^{i\theta} \in I \right\}.$$

As usual, $|I|$ denotes the length of I and $dA(z)$ the normalized Lebesgue measure on $\mathbb{D}$. The corresponding $BMOA$ norm is

$$\|f\|_{BMOA} := |f(0)| + \sup_{I \subset \partial\mathbb{D}} \left(\frac{1}{|I|} \int_{R(I)} |f'(z)|^2 (1-|z|^2) dA(z) \right)^{1/2}.$$

Trivially, each polynomial belongs to $BMOA$. The closure of all polynomials in $BMOA$ is denoted by $VMOA$. Alternatively, $VMOA$ is the subspace of $BMOA$ formed by those $f \in BMOA$ such that

$$\lim_{|I| \to 0} \frac{1}{|I|} \int_{R(I)} |f'(z)|^2 (1-|z|^2) dA(z) = 0.$$

Particular and quite interesting examples of members of $VMOA$ are provided by functions in the Dirichlet space $\mathcal{D}$, which is the space of those functions $f \in \mathcal{H}(\mathbb{D})$ such that $f' \in L^2(\mathbb{D}, dA)$. In fact, for every $f \in \mathcal{D}$,

$$\frac{1}{|I|} \int_{R(I)} |f'(z)|^2 (1-|z|^2) dA(z) \leq 2 \int_{R(I)} |f'(z)|^2 dA(z) \to 0 \text{ as } |I| \to 0. \tag{2.1}$$

For more information on these Banach spaces, we refer the reader to the excellent monographs or [11] or [22].

In our first result, we confirm the existence of a maximal closed linear subspace of $BMOA$ on which a semigroup (φ_t) generates a semigroup of operators. In this context, we recall that any analytic self map φ of the disk induces a bounded composition operator $C_\varphi(f) = f \circ \varphi$ on $BMOA$ and there is a constant $C > 0$, not depending on φ, such that

$$\|C_\varphi\|_{BMOA} \le C\left(1 + \log\frac{1}{1 - |\varphi(0)|}\right). \tag{2.2}$$

Moreover C_φ is bounded on $VMOA$ if and only if $\varphi \in VMOA$, see [2] for details.

Proposition 2.1. *Let (φ_t) be a semigroup of analytic functions. Then there exists a closed subspace Y of $BMOA$ such that (φ_t) generates a semigroup of operators on Y and such that any other subspace of $BMOA$ with this property is contained in Y.*

Proof. Consider the linear subspace of $BMOA$ defined by

$$Y := \left\{ f \in BMOA : \lim_{t\to 0^+} \|f \circ \varphi_t - f\|_{BMOA} = 0 \right\}.$$

Notice that $\sup_{t\in[0,1]} |\varphi_t(0)| = M < 1$. Hence from (2.2),

$$\sup_{t\in[0,1]} \|C_t\|_{BMOA} \le C\left(1 - \log(1 - M)\right) < +\infty.$$

This and the triangle inequality for norms shows that Y is a closed subspace of $BMOA$. Thus in order to prove that (φ_t) generates a semigroup of operators in Y, it remains to check that if $f \in Y$, then $C_s(f) \in Y$ for all $s \ge 0$. To see this let $s, t \ge 0$, then

$$\|C_s(f) \circ \varphi_t - C_s(f)\|_{BMOA} \le C\left(1 + \log\frac{1}{1 - |\varphi_s(0)|}\right) \|f \circ \varphi_t - f\|_{BMOA} \to 0$$

as $t \to 0^+$.

Finally, if W is a subspace of $BMOA$ such that (φ_t) generates a semigroup of operators on W, then for any $f \in W$ we have in particular

$$\lim_{t\to 0^+} \|f \circ \varphi_t - f\|_{BMOA} = 0,$$

thus $f \in Y$ and we conclude $W \subset Y$. □

In what follows, this maximal subspace Y will be denoted as $[\varphi_t, BMOA]$. It is easy to see that if Z is any closed subspace of $[\varphi_t, BMOA]$ which is invariant under (C_t) (i.e. $C_t(Z) \subset Z$ for every $t \ge 0$), then (φ_t) generates a semigroups of operators on Z.

This maximal subspace can be also described directly in terms of the infinitesimal generator.

Theorem 2.2. *Let G be the infinitesimal generator of (φ_t). Then,*

$$[\varphi_t, BMOA] = \overline{\{f \in BMOA : Gf' \in BMOA\}}.$$

Proof. We may assume that (φ_t) is not trivial. Denote by Γ the infinitesimal generator of the operator semigroup (C_t) acting on the Banach space $[\varphi_t, BMOA]$, and by $\mathcal{D}(\Gamma)$ its domain. We will show that if $f \in \mathcal{D}(\Gamma)$, then $Gf' \in BMOA$. Indeed if $f \in \mathcal{D}(\Gamma)$, then $\Gamma(f) \in BMOA$ and

$$\lim_{t\to 0^+} \left\| \frac{1}{t}(C_t(f) - f) - \Gamma(f) \right\|_{BMOA} = 0.$$

Since convergence in the $BMOA$ norm implies uniform convergence on compact subsets of $\mathbb{D}$ and therefore in particular pointwise convergence, for each $z \in \mathbb{D}$ we have

$$\begin{aligned}\Gamma(f)(z) &= \lim_{t\to 0^+} \frac{f(\varphi_t(z)) - f(z)}{t} = \lim_{t\to 0^+} \frac{f(\varphi_t(z)) - f(\varphi_0(z))}{t} \\ &= \left.\frac{\partial f \circ \varphi_t(z)}{\partial t}\right|_{t=0} = f'(\varphi_0(z)) \left.\frac{\partial \varphi_t(z)}{\partial t}\right|_{t=0} = f'(z)G(z),\end{aligned}$$

that is, $Gf' = \Gamma(f) \in BMOA$, and thus $\mathcal{D}(\Gamma) \subset \{f \in BMOA : Gf' \in BMOA\}$. Taking closures and bearing in mind the fact from the general theory of operator semigroups that $\mathcal{D}(\Gamma)$ is dense in $[\varphi_t, BMOA]$ we conclude

$$[\varphi_t, BMOA] \subseteq \overline{\{f \in BMOA : Gf' \in BMOA\}}.$$

Conversely, let $f \in BMOA$ such that $m := Gf' \in BMOA$. First of all, we assert that

$$(f \circ \varphi_t)'(z) - f'(z) = \int_0^t (m \circ \varphi_s)'(z) ds; \text{ for } t \geq 0,\ z \in \mathbb{D}.$$

Indeed,

$$\begin{aligned}G(z)\left((f \circ \varphi_t)'(z) - f'(z)\right) &= f'(\varphi_t(z))G(z)\varphi_t'(z) - m(z) \\ &= f'(\varphi_t(z))\frac{\partial \varphi_t(z)}{\partial t} - m(z) \\ &= m(\varphi_t(z)) - m(z) = \int_0^t \frac{\partial (m \circ \varphi_s)(z)}{\partial s} ds \\ &= \int_0^t G(z) m'(\varphi_s(z))\varphi_s'(z) ds \\ &= G(z) \int_0^t (m \circ \varphi_s)'(z) ds.\end{aligned}$$

Since G is not identically null this verifies our assertion. Next let I be an interval in $\partial\mathbb{D}$ and $R(I)$ the corresponding Carleson rectangle. For $0 \le t \le 1$ we have

$$\begin{aligned}
&\int_{R(I)} \left|(f\circ\varphi_t)'(z) - f'(z)\right|^2 (1-|z|^2)dA(z)\\
&= \int_{R(I)} \left|\int_0^t (m\circ\varphi_s)'(z)ds\right|^2 (1-|z|^2)dA(z)\\
&\le \int_{R(I)} t\left(\int_0^1 \left|(m\circ\varphi_s)'(z)\right|^2 ds\right)(1-|z|^2)dA(z)
\end{aligned}$$

where we have applied Cauchy-Schwarz in the inside integral. Dividing by $|I|$, taking sup and interchanging the integrals we have

$$\begin{aligned}
&\sup_{I\subseteq\partial\mathbb{D}}\left(\frac{1}{|I|}\int_{R(I)} \left|(f\circ\varphi_t)'(z) - f'(z)\right|^2 (1-|z|^2)dA(z)\right)^{\frac12}\\
&\le \sup_{I\subseteq\partial\mathbb{D}}\left(\frac{1}{|I|}\int_{R(I)} t\left(\int_0^1 \left|(m\circ\varphi_s)'(z)\right|^2 ds\right)(1-|z|^2)dA(z)\right)^{\frac12}\\
&\le \sup_{I\subseteq\partial\mathbb{D}}\left(t\int_0^1\left(\frac{1}{|I|}\int_{R(I)} \left|(m\circ\varphi_s)'(z)\right|^2 (1-|z|^2)dA(z)\right)ds\right)^{\frac12}\\
&\le \left(t\int_0^1 \|m\circ\varphi_s\|^2_{BMOA}\,ds\right)^{\frac12}\\
&\le \sqrt{t}\sup_{s\in[0,1]}\|m\circ\varphi_s\|_{BMOA}\\
&\le \sqrt{t}C\|m\|_{BMOA}\sup_{s\in[0,1]}(1-\log(1-|\varphi_s(0)|))\\
&\le C'\sqrt{t},
\end{aligned}$$

where $C' > 0$ is a certain constant not depending on t. Hence,

$$\|C_t f - f\|_{BMOA} \le |f(\varphi_t(0)) - f(0)| + C'\sqrt{t}.$$

Since $\lim_{t\to 0^+}\varphi_t(0) = 0$ we find that $\lim_{t\to 0^+}\|C_t f - f\|_{BMOA} = 0$, hence $f \in [\varphi_t, BMOA]$. We have shown $\{f \in BMOA : Gf' \in BMOA\} \subset [\varphi_t, BMOA]$, and the desired inclusion follows by taking closures. □

If (φ_t) is a semigroup of analytic functions, then every composition operator $C_t(f) = f\circ\varphi_t$ is bounded on $VMOA$. This is because each φ_t belongs to the Dirichlet space $\mathcal{D}$ (recall that φ_t is univalent) and therefore also in $VMOA$. Thus the composition semigroup (C_t) consists of bounded operators on $VMOA$.

Moreover (C_t) is strongly continuous on $VMOA$ for every semigroup (φ_t). This was stated in [18] without proof, and is contained as a special case among the $Q_{p,0}$ spaces in [20, Theorem 4.1]. A short proof goes as follows. Strong continuity requires that $\lim_{t\to 0} \|f\circ\varphi_t - f\|_{BMOA} = 0$ for each $f \in VMOA$. For a polynomial P we can write

$$\begin{aligned}
&\|f\circ\varphi_t - f\|_{BMOA} \\
&\leq \|f\circ\varphi_t - P\circ\varphi_t\|_{BMOA} + \|P\circ\varphi_t - P\|_{BMOA} + \|P - f\|_{BMOA} \\
&\leq (\|C_t\|_{VMOA} + 1)\|P - f\|_{BMOA} + \|P\circ\varphi_t - P\|_{BMOA}.
\end{aligned}$$

Since $VMOA$ contains the polynomials as a dense set and since

$$\sup_{0\leq t<1} \|C_t\|_{VMOA} < \infty,$$

it suffices to show $\lim_{t\to 0} \|P\circ\varphi_t - P\|_{BMOA} = 0$ for each polynomial. This now follows from the inequality $\|g\|_{BMOA} \leq \|g\|_{\mathcal{D}}$ between the $VMOA$-norm and the Dirichlet space norm which is valid for all $g \in \mathcal{D}$, see (2.1), along with the fact that every semigroup generates a semigroup of operators on the Dirichlet space [17, Theorem 1].

We proceed however to provide an alternative direct proof of this result which does not use [17, Theorem 1] and which is based on the $VMOA{-}H^1$ duality. Recall that this duality is realized by the pairing,

$$\langle f, g\rangle := \lim_{r\to 1} \int_0^{2\pi} f(re^{i\theta})\overline{g(re^{i\theta})}\,\frac{d\theta}{2\pi}, \quad f \in VMOA,\ g \in H^1.$$

If we restrict the choice of f and g, for example if both are chosen to lie in H^2, then the pairing can be expressed by the Littlewood-Paley identity (see [22, 8.1.9]):

$$\langle f, g\rangle = f(0)\overline{g(0)} + 2\int_{\mathbb{D}} f'(z)\overline{g'(z)} \log\frac{1}{|z|} dA(z). \tag{2.3}$$

Now, we present another formulation of this dual pair involving functions in spaces which will be more convenient for our purposes.

Lemma 2.3. *If $f \in \mathcal{D}$ and $g \in H^1$, then*

$$\langle f, g\rangle = f(0)\overline{g(0)} + 2\int_{\mathbb{D}} f'(z)\overline{g'(z)} \log\frac{1}{|z|} dA(z).$$

Proof. Select a sequence (g_n) in H^2 converging to g in H^1. Using the Littewood-Paley identity, we have

$$\langle f, g_n\rangle = f(0)\overline{g_n(0)} + 2\int_{\mathbb{D}} f'(z)\overline{g_n'(z)} \log\frac{1}{|z|} dA(z)$$

and $\langle f, g\rangle = \lim_{n\to\infty}\langle f, g_n\rangle$.

Now,

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(z)(g_n'(z) - g'(z))| \log \frac{1}{|z|} dA(z) \\
&\leq \int_{|z|>1/2} |f'(z)(g_n'(z) - g'(z))| \log \frac{1}{|z|} dA(z) \\
&\quad + \int_{|z|\leq 1/2} |f'(z)(g_n'(z) - g'(z))| \log \frac{1}{|z|} dA(z) \\
&\leq C_1 \left(\int_{\mathbb{D}} |f'(z)|^2 \, dA(z) \right)^{\frac{1}{2}} \left(\int_{\mathbb{D}} |g_n'(z) - g'(z)|^2 \, (1-|z|)^2 dA(z) \right)^{\frac{1}{2}} \\
&\quad + \int_{|z|\leq 1/2} |f'(z)(g_n'(z) - g'(z))| \log \frac{1}{|z|} dA(z)
\end{aligned}$$

For any function $h = \sum_{n=0}^{\infty} a_n z^n \in H^1$, we obtain from Hardy's inequality (see [10, Theorem 6.2]) that

$$\int_{\mathbb{D}} |h'(z)|^2 (1-|z|)^2 dA(z) \leq C_2 \sum_{n=0}^{\infty} \frac{|a_n|^2}{n} \leq C_2 \|h\|_{H^1}^2. \tag{2.4}$$

Hence,

$$\int_{\mathbb{D}} |g_n'(z) - g'(z)|^2 \, (1-|z|)^2 dA(z) \leq C_2 \|g_n - g\|_{H^1}^2.$$

Finally, applying the inclusion $f'(z) \log \frac{1}{|z|} \in L^1(dA)$ and Lebesgue's dominated convergence theorem, one also has

$$\lim_{n\to\infty} \int_{|z|\leq 1/2} |f'(z)(g_n'(z) - g'(z))| \log \frac{1}{|z|} dA(z) = 0. \qquad \square$$

Theorem 2.4. *Every semigroup (φ_t) generates a semigroup of operators on $VMOA$.*

Proof. From the general theory of operator semigroups, a semigroup which is weakly continuous on a Banach space is in fact strongly continuous [19, p. 233]). Thus it suffices to prove that for each $f \in VMOA$ we have

$$w - \lim_{t\to 0^+} C_t(f) = f,$$

where $w-$ denotes the weak limit. In other words for each fixed $f \in VMOA$ we want to prove

$$\lim_{t\to 0^+} \langle C_t(f), g\rangle = \langle f, g\rangle$$

for every $g \in H^1$. Arguing as before about the density of polynomials in $VMOA$ and the fact that $\sup_{0\leq t<1} \|C_t\|_{VMOA} < \infty$ we see that it suffices to prove this for $f = P$ a polynomial.

Now, using again the Area Theorem (φ_t is univalent), we find

$$\int_{\mathbb{D}} |P'(\varphi_t(z))\varphi_t'(z)|^2 \, dA(z) \leq \int_{\mathbb{D}} \|P'\|_\infty^2 \, |\varphi_t'(z)|^2 dA(z) \leq \|P'\|_\infty^2,$$

so we can apply Lemma 2.3. Therefore, if P a polynomial and $g \in H^1$ we deduce

$$\langle P \circ \varphi_t - P, g \rangle = (P(\varphi_t(0)) - P(0))\overline{g(0)} + 2\int_{\mathbb{D}} (P \circ \varphi_t - P)'(z)\overline{g'(z)} \log \frac{1}{|z|} |dA(z).$$

For each $\delta > 0$, we split the integral

$$\begin{aligned} &\int_{\mathbb{D}} |(P \circ \varphi_t - P)'(z)| \, |g'(z) \, | \log \frac{1}{|z|} dA(z) \\ &\leq \int_{|z|>\delta} |(P \circ \varphi_t) - P)'(z)||g'(z) \, | \log \frac{1}{|z|} dA(z) \\ &\quad + \int_{|z|\leq\delta} |(P \circ \varphi_t - P)'(z)||g'(z) \, | \log \frac{1}{|z|} dA(z) = (1) + (2). \end{aligned}$$

To estimate the first integral, we use the estimate $\log \frac{1}{|z|} \approx 1 - |z|$ and apply Cauchy-Schwarz to obtain

$$\begin{aligned} (1) &\leq C\|P'\|_\infty \int_{|z|>\delta} (|\varphi_t'(z)| + 1)|g'(z)|(1 - |z|) dA(z) \\ &\leq C\|P'\|_\infty ((\int_{\mathbb{D}} |\varphi_t'(z)|^2 \, dA(z))^{\frac{1}{2}} + 1)(\int_{|z|>\delta} |g'(z)|^2 (1 - |z|)^2 dA(z))^{\frac{1}{2}} \\ &\leq 2C\|P'\|_\infty (\int_{|z|>\delta} |g'(z)|^2 (1 - |z|)^2 dA(z))^{\frac{1}{2}}. \end{aligned}$$

Now using (2.4) one has that $g'(z)(1-|z|) \in L^2(dA)$, which shows that given $\varepsilon > 0$ and, for all $t > 0$, there exists $0 < \delta_0 < 1$ such that

$$\int_{|z|>\delta_0} |(P \circ \varphi_t) - P)'(z)| \; |g'(z) \, | \log \frac{1}{|z|} |dA(z) < \varepsilon \, . \tag{2.5}$$

At the same time, for every $z \in \mathbb{D}$,

$$\lim_{t\to 0} (P \circ \varphi_t - P)'(z) = 0.$$

Therefore, using the inclusion $g'(z) \log \frac{1}{|z|} \in L^1(dA)$ (note that $L^2(dA) \subset L^1(dA)$ and again $\log \frac{1}{|z|} \approx 1 - |z|$) and the Lebesgue dominated convergence theorem, one concludes that

$$\lim_{t\to 0} \int_{|z|\leq\delta_0} |(P(\varphi_t) - P)'(z)||g'(z) \, | \log \frac{1}{|z|} dA(z) = 0. \tag{2.6}$$

□

3. VMOA and Maximal Subspaces

This section is devoted to analyzing those semigroups of analytic functions (φ_t) such that $VMOA = [\varphi_t, BMOA]$. Since $VMOA$ is always contained in the subspace $[\varphi_t, BMOA]$ we see that $VMOA = [\varphi_t, BMOA]$ is equivalent to the following statement: if $f \in BMOA$, then

$$f \in VMOA \quad \text{if and only if} \quad \lim_{t \to 0^+} \|f \circ \varphi_t - f\|_{BMOA} = 0.$$

It is easy to see that the inclusion $VMOA \subset [\varphi_t, BMOA]$ can be proper. The easiest example of this type is the semigroup

$$\varphi_t(z) = e^{-t}z + 1 - e^{-t}, \quad t \geq 0, z \in \mathbb{D}.$$

For this semigroup the function $f(z) = \log(1-z) \in BMOA \setminus VMOA$ satisfies

$$\|f \circ \varphi_t - f\|_{BMOA} = \left\|\log(e^{-t}(1-z)) - \log(1-z)\right\|_{BMOA} = \log e^{-t} \longrightarrow 0,$$

thus $f \in [\varphi_t, BMOA]$. In fact it is easy to construct general examples of semigroups with this behavior. For instance take any starlike univalent function $h : \mathbb{D} \to \mathbb{C}$ with $h(0) = 0$ and $h \in BMOA \setminus VMOA$ and define $\varphi_t(z) = h^{-1}(e^{-t}h(z))$. Then

$$\|h \circ \varphi_t - h\|_{BMOA} = |e^{-t} - 1| \|h\|_{BMOA} \longrightarrow 0,$$

so that $h \in [\varphi_t, BMOA]$ while $h \notin VMOA$.

The following theorem gives a sufficient condition on the infinitesimal generator which assures that $VMOA = [\varphi_t, BMOA]$.

Theorem 3.1. *Let (φ_t) be a semigroup with infinitesimal generator G. Assume that for some $0 < \alpha < 1$,*

$$\frac{(1-|z|)^\alpha}{G(z)} = O(1) \quad \text{as } |z| \to 1. \tag{3.1}$$

Then $VMOA = [\varphi_t, BMOA]$.

Proof. Since $[\varphi_t, BMOA] = \overline{\{f \in BMOA : Gf' \in BMOA\}}$, it suffices to show

$$\{f \in BMOA : Gf' \in BMOA\} \subset VMOA.$$

Let $g \in BMOA$ with $Gg' \in BMOA$. First we choose indices p, p' such that $1/p + 1/p' = 1$, $2 < p < \infty$, and such that $\alpha < \frac{1}{p'} < \alpha + \frac{1}{2}$. Hence $\alpha = \frac{1}{p'} - \varepsilon$ with $0 < \varepsilon < 1/2$. We use the usual notation

$$M_p(f, r) = \left(\int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi}\right)^{1/p}$$

and we have, taking into account that $BMOA \subset H^p$, for $0 < \delta \leq r < 1$,

$$M_p(g', r) = M_p(Gg'\frac{1}{G}, r) \leq M_p(Gg', r)M_\infty(\frac{1}{G}, r) \leq \frac{C_p}{(1-r)^{1/p'-\varepsilon}},$$

where C_p is a constant depending only on p.

In order to show that $g \in VMOA$ we will use the characterization of $VMOA$ in terms of Carleson measures (see [22, 8.2.5 and 8.4.2]); it suffices to prove

$$\lim_{|z|\to 1} \int_{\mathbb{D}} |g'(w)|^2 \frac{(1-|w|^2)(1-|z|^2)}{|1-z\overline{w}|^2} dA(w) = 0.$$

Now let $q = p/2 > 1$ and apply Hölder's inequality for the pair of indices q, q', that is, $\frac{2}{p} + \frac{1}{q'} = 1$ to obtain

$$\begin{aligned}
\int_{\mathbb{D}} |g'(w)|^2 \frac{(1-|w|^2)(1-|z|^2)}{|1-z\overline{w}|^2} dA(w) &= \int_0^1 \int_0^{2\pi} |g'(re^{i\theta})|^2 \frac{(1-r^2)(1-|z|^2)}{|1-zre^{-i\theta}|^2} d\theta r dr \\
&\le \int_0^1 M_p^2(g',r) \left(\int_0^{2\pi} \frac{(1-r^2)^{q'}(1-|z|^2)^{q'}}{|1-zre^{-i\theta}|^{2q'}} d\theta \right)^{\frac{1}{q'}} dr \\
&\le C(1-|z|) \int_0^1 M_p^2(g',r) \frac{(1-r)}{(1-|z|r)^{2-\frac{1}{q'}}} dr = C(1-|z|)Q(|z|),
\end{aligned}$$

where the last inequality follows from the standard estimate ($c > 0$)

$$\int_0^{2\pi} \frac{d\theta}{|1-we^{-i\theta}|^{1+c}} \approx \frac{1}{(1-|w|^2)^c} \qquad \text{as } |w| \to 1$$

(see, for example, [8, Exercise 2.1.4]) and $Q(|z|)$ denotes the last integral. We now have

$$\begin{aligned}
Q(|z|) &= \int_0^\delta M_p^2(g',r) \frac{(1-r)}{(1-|z|r)^{2-\frac{1}{q'}}} dr + \int_\delta^1 M_p^2(g',r) \frac{(1-r)}{(1-|z|r)^{2-\frac{1}{q'}}} dr \\
&\le M_p^2(g',\delta) \int_0^\delta \frac{1}{(1-|z|r)^{\frac{1}{q}}} dr + C_p^2 \int_\delta^1 \frac{(1-r)^{1-2/p'+2\varepsilon}}{(1-|z|r)^{2-\frac{1}{q'}}} dr \\
&\le C_1 \int_0^1 \frac{1}{(1-r)^{\frac{1}{q}}} dr + C_2 \int_0^1 \frac{(1-r)^{1-2/p'+2\varepsilon}}{(1-|z|r)^{2-\frac{1}{q'}}} dr \\
&= C_1' + C_2 \int_0^1 \frac{(1-r)^{1-2/p'+2\varepsilon}}{(1-|z|r)^{2-\frac{1}{q'}}} dr \\
&\le C_1' + C_2'(1-|z|)^{-1+2\varepsilon}
\end{aligned}$$

where the last integral was calculated by integration by parts as in [16, Lemma 6]. For the sake of clearness, we recall that such an estimate is exactly

$$\int_0^1 \frac{(1-r)^\gamma}{(1-\rho r)^m} dr \le C(1-\rho)^{1+\gamma-m}, \qquad \rho \in (0,1),$$

where $\gamma = 1 - \frac{2}{p'} + 2\varepsilon > -1$, $m = 2 - \frac{1}{q'} > 1 + \gamma$, and $\rho = r|z|$.

Putting all these together we find

$$\begin{aligned}\int_{\mathbb{D}} |g'(w)|^2 \frac{(1-|w|^2)(1-|z|^2)}{|1-z\overline{w}|^2}\, dA(w) &\le C(1-|z|)Q(|z|)\\ &\le C(1-|z|)(C_1' + C_2'(1-|z|)^{-1+2\varepsilon})\\ &\le C'' \max\{(1-|z|), (1-|z|)^{2\varepsilon}\},\end{aligned}$$

and the proof is complete. □

As an immediate corollary we have

Corollary 3.2. *Suppose* (φ_t) *is a semigroup whose infinitesimal generator* G *satisfies condition (*3.1*) of Theorem* 3.1*. Then for a function* $f \in BMOA$ *the following are equivalent*

1. $f \in VMOA$.
2. $\lim_{t\to 0^+} \|f\circ\varphi_t - f\|_{BMOA} = 0$.

Clearly the semigroups $\varphi_t(z) = e^{-t}z$ and $\varphi_t(z) = e^{it}z$ satisfy the condition (3.1) since, in both cases, the infinitesimal generator is $G(z) = cz$ for a certain nonzero constant c. Thus Theorem 3.1 gives an alternative proof (with entirely different techniques, see also [22]) of Sarason's result.

But there is a plethora of different semigroups (φ_t) for which $VMOA = [\varphi_t, BMOA]$. A specific class of examples of this type are given by the semigroups associated with the generators

$$G(z) = -z(1-z)^{\alpha}, \quad 0<\alpha<1.$$

To appreciate the breadth of the theorem recall that infinitesimal generators of semigroups with Denjoy-Wolff point $b=0$ have the form $G(z) = -zP(z)$ where $\operatorname{Re} P(z) \ge 0$. By Schwarz's lemma applied to $1/P$ which also has nonnegative real part, $P(z)$ satisfies the growth condition

$$\left|\frac{1}{P(z)}\right| \le C_2 \frac{1+|z|}{1-|z|} \quad \text{as } |z|\to 1.$$

Thus the most general infinitesimal generator of semigroups with inner Denjoy-Wolff point, fulfills the condition $\frac{1-|z|}{G(z)} = O(1)$.

Remark. Clearly, our O(1) condition is intimately related to the number and location of the zeros (in the angular sense) of the infinitesimal generator on the boundary of the unit disk. This topic has been partly analyzed in [6]. Another sufficient condition of the same nature which implies the conclusion of the theorem is

$$\int_{|z|\ge\delta} \frac{dA(z)}{|G(z)|^p} < \infty,$$

for some $0<\delta<1$ and $p>2$. Indeed for $\delta<r<1$ we have

$$(1-r)M_p^p(1/G, r) \le \int_r^1 M_p^p(1/G, s)ds,$$

so by the finiteness of the integral we have

$$M_p(1/G, r) = o\left(\frac{1}{(1-r)^{\frac{1}{p}}}\right).$$

Now, using the Hardy-Littlewood estimates (see [10, 5.9]) one has

$$M_\infty(1/G, r) \leq \frac{CM_p(1/G, r)}{(1-r)^{\frac{1}{p}}} = o\left(\frac{1}{(1-r)^{\frac{2}{p}}}\right),$$

and this little-oh condition implies our big-Oh condition.

We now present a necessary condition for semigroups with inner Denjoy-Wolff point for $VMOA = [\varphi_t, BMOA]$ to hold. Observe that this necessary condition is quite close to the sufficient condition of Theorem 3.1.

Theorem 3.3. *Let (φ_t) be a semigroup with infinitesimal generator G and Denjoy-Wolff point $b \in \mathbb{D}$. If $VMOA = [\varphi_t, BMOA]$, then*

$$\lim_{|z|\to 1} \frac{1-|z|}{G(z)} = 0.$$

Proof. Without loss of generality, we may assume that $b = 0$. The infinitesimal generator then is

$$G(z) = -zP(z),$$

where P is analytic with $\operatorname{Re} P \geq 0$. If P is constant, the result is clear. Otherwise consider the function

$$m(z) = \int_0^z \frac{u}{G(u)}\, du = -\int_0^z \frac{1}{P(u)}\, du.$$

Since $\operatorname{Re}(1/P) \geq 0$ we have m is univalent (see [13, Proposition 1.10]) so, according to [11, Chap. IV, Exercise 25], we see that $m \in BMOA$ if and only if m belongs to the Bloch space. Moreover, using [7, Chap. 17, Proposition 1.5], we deduce that m belongs to the Bloch space and thus $m \in BMOA$.

Now observe that

$$\begin{aligned}(m \circ \varphi_t)'(z) - m'(z) &= \frac{\varphi_t(z)\varphi_t'(z)}{G(\varphi_t(z))} - \frac{z}{G(z)} = \frac{\varphi_t(z) - z}{G(z)} \\ &= \frac{1}{G(z)} \int_0^t \frac{\partial \varphi_s(z)}{\partial s}\, ds = \int_0^t \varphi_s'(z)\, ds,\end{aligned}$$

where we have used (1.1) twice. Hence

$$|(m \circ \varphi_t)'(z) - m'(z)|^2 = \left|\int_0^t \varphi_s'(z)\, ds\right|^2 \leq t \int_0^t |\varphi_s'(z)|^2\, ds.$$

Now let $I \subset \partial\mathbb{D}$ be an interval and $R(I)$ the corresponding Carleson rectangle. We have

$$\begin{aligned}
\frac{1}{|I|}\int_{R(I)} &|(m\circ\varphi_t)'(z)-m'(z)|^2(1-|z|^2)\,dA(z)\\
&\le \frac{1}{|I|}\int_{R(I)}\left(t\int_0^t|\varphi_s'(z)|^2\,ds\right)(1-|z|^2)\,dA(z)\\
&= t\int_0^t\frac{1}{|I|}\int_{R(I)}|\varphi_s'(z)|^2(1-|z|^2)\,dA(z)ds\\
&\le Ct\int_0^t\int_{R(I)}|\varphi_s'(z)|^2\,dA(z)\,ds\\
&\le Ct\int_0^t\int_{\mathbb{D}}|\varphi_s'(z)|^2\,dA(z)\,ds\\
&= Ct\int_0^t[\mathrm{Area}(\varphi_s(\mathbb{D}))]^2\,ds \le C_1t^2.
\end{aligned}$$

Now the norm $\|m\circ\varphi_t - m\|_{BMOA}$ equals

$$\begin{aligned}
&|m(\varphi_t(0))-m(0)| + \sup_{I\subset\partial\mathbb{D}}\left(\frac{1}{|I|}\int_{R(I)}|(m\circ\varphi_t)'(z)-m'(z)|^2(1-|z|^2)\,dA(z)\right)^{1/2}\\
&\le |m(\varphi_t(0))-m(0)|+C_2t,
\end{aligned}$$

so $\lim_{t\to0}\|m\circ\varphi_t-m\|_{BMOA}=0$. Thus $m\in[\varphi_t, BMOA]$ and by the hypothesis $m\in VMOA$. The following standard argument for functions in $VMOA$ completes the proof. For $a\in\mathbb{D}$ write $\phi_a(z)=\frac{a-z}{1-\bar{a}z}$, then

$$\begin{aligned}
(1-|a|^2)^2|m'(a)|^2 &= |(m\circ\phi_a)'(0)|^2 \le \|m\circ\phi_a\|_{H^2}^2\\
&\le C\int_{\mathbb{D}}|(m\circ\phi_a(z))'|^2(1-|z|^2)\,dA(z)
\end{aligned}$$

(by the change of variables $w=\phi_a(z)$)

$$\begin{aligned}
&= C\int_{\mathbb{D}}|m'(w)|^2(1-|\phi_a(w)|^2)\,dA(w)\\
&= C\int_{\mathbb{D}}|m'(w)|^2\frac{(1-|a|^2)(1-|w|^2)}{|1-\bar{a}w|^2}\,dA(w)
\end{aligned}$$

and this last integral tends to 0 as $|a|\to1$ because $m\in VMOA$. It follows that

$$\lim_{|a|\to1}\frac{1-|a|}{G(a)}=\lim_{|a|\to1}\frac{(1-|a|)m'(a)}{a}=0. \qquad \square$$

We end this paper by characterizing those semigroups (φ_t) of linear fractional maps such that $VMOA=[\varphi_t, BMOA]$. Roughly speaking, we show that, when dealing with semigroups of linear fractional maps, those found by Sarason are the unique ones for which his theorem is true. For a detailed analysis of these types of semigroups in one and several variables we refer the reader to [3].

Proposition 3.4. *Let* (φ_t) *be a semigroup such that each iterate is a linear fractional map. Then,* $VMOA = [\varphi_t, BMOA]$ *if and only if* (φ_t) *has a fixed point in the unit disk but is without fixed points in the boundary of the unit disk.*

Proof. We freely use some results from [3]. In particular, we use that the infinitesimal generator of a semigroup of linear fractional maps is a polynomial of degree two.

Assume that $VMOA = [\varphi_t, BMOA]$. First of all, we are going to prove that the Denjoy-Wolff point of the semigroup (φ_t) must be in the unit disk. Let σ be the Koenigs or univalent map of the semigroup (see [17]). It is known that σ satisfies

$$\sigma \circ \varphi_t = \sigma + t, \text{ for all } t \geq 0.$$

If the semigroup is hyperbolic, using [5, Theorem 2.1], [11, page 283] and [13, page 78], we conclude that $\sigma \in BMOA \setminus VMOA$. Moreover, we observe that

$$\|\sigma \circ \varphi_t - \sigma\|_{BMOA} = t \stackrel{t\to 0}{\longrightarrow} 0,$$

and, therefore, $\sigma \in [\varphi_t, BMOA]$. Hence, the equality we are dealing with is impossible for this type of semigroup (we want to mention that this argument is indeed completely general, not only valid in the framework of semigroups of linear fractional maps).

If the semigroup is parabolic, then σ is the Riemann map of a half-plane. In this case, taking $c \in \mathbb{C}\setminus\sigma(\mathbb{D})$, we see that $f(z) = \mathrm{Log}(\sigma(z) - c) \in BMOA\setminus VMOA$ and, arguing as in the hyperbolic case, we obtain $f \in [\varphi_t, BMOA]$. Therefore, for this type of semigroup the equality is also impossible and we conclude that the semigroup is necessarily elliptic (it has a fixed point in $\mathbb{D}$).

Therefore, we may assume in our proof that the semigroup has Denjoy-Wolff point in $\mathbb{D}$. Now, since $VMOA = [\varphi_t, BMOA]$, using Theorem 3.3, we have

$$\lim_{|z|\to 1} \frac{1-|z|}{G(z)} = 0.$$

If the other fixed point is in the boundary, then the infinitesimal generator has the form

$$G(z) = \lambda(z-a)(z-b), \text{ with } a \in \partial\mathbb{D}, b \in \mathbb{D} \text{ and } \lambda \neq 0.$$

Taking limits, we obtain a contradiction.

Likewise, assume now that (φ_t) has no fixed point in the boundary of $\mathbb{D}$. In this case, the infinitesimal generator fits the following scheme:

$$\begin{aligned} G(z) &= \lambda(z-a)(z-b), \text{ with } a \in \mathbb{C}\setminus\overline{\mathbb{D}}, b \in \mathbb{D} \text{ and } \lambda \neq 0 \\ \text{or } G(z) &= \lambda(z-b), \text{ with } b \in \mathbb{D} \text{ and } \lambda \neq 0. \end{aligned}$$

Finally, apply Theorem 3.1 □

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