Chapter VII

Sequences and Series of Analytic Functions

Introduction

Some of the most significant properties of analytic functions only come to light because of the special means available for representing these functions. We have already experienced one occurrence of this phenomenon in the Cauchy integral formula and the many conclusions we were able to draw with that formula in hand. A second important method of representing analytic functions — as sums of certain kinds of infinite series — constitutes the subject matter of the present chapter. In particular, the Taylor series expansion of a function in a disk where it is analytic and the Laurent series development of a function that is analytic in an annulus will be discussed in detail. The former provides the key to elucidating many of the delicate local features of analytic functions; the latter plays a decisive role in the classification of isolated singularities of such functions and in our treatment of the residue theorem.

We begin the chapter with a review of some basic facts and terminology relating to the convergence of sequences and series of functions.

1 Sequences of Functions

1.1 Uniform Convergence

Suppose that $\langle f_n \rangle$ is a sequence of complex-valued functions, each of whose domain-sets contains a subset A of the complex plane. There is a perfectly obvious and natural way to speak of this sequence being convergent in A and having the function $f: A \to \mathbb{C}$ as its limit when $n \to \infty$: it can simply be demanded that for each point z of A the sequence of values $\langle f_n(z) \rangle$, which is just an ordinary sequence of complex numbers, be convergent and have the value $f(z)$ for its limit; i.e., we can insist that

$$f(z) = \lim_{n\to\infty} f_n(z)$$

hold at every z in A. When this happens we say that the sequence $\langle f_n\rangle$ *converges pointwise in* A to the limit function f and express the fact symbolically by writing $f_n \to f$ in A or $f = \lim_{n\to\infty} f_n$ in A. (N.B. In the absence of any pronouncement to the contrary, it will be our convention when talking of pointwise convergence in A to regard A as the complete domain-set of the limit function.) Although it provides an appropriate starting point for a discussion of the convergence of function sequences, pointwise convergence has a number of serious drawbacks that render it less than satisfactory for the ends we have in mind. Consider, for example, the sequence of functions $f_n: [0,1] \to \mathbb{C}$ given by $f_n(t) = t^n$. Clearly $\langle f_n\rangle$ converges pointwise in $[0,1]$ to the function f defined by $f(t) = 0$ when $0 \le t < 1$ and $f(1) = 1$. Thus, whereas each of the functions f_n is continuous, the limit function f is not. This example exposes one of the major shortcomings of pointwise convergence: the property of continuity is not necessarily preserved under pointwise passage to a limit.

There is a stronger mode of convergence that is often used in conjunction with sequences of functions, one from which many of the negative features inherent in pointwise convergence are absent. It is called "uniform convergence." To say that a function sequence $\langle f_n\rangle$ converges uniformly to f on a set A means not only that $f_n(z)$ tends to $f(z)$ for each z in A, but also that this convergence takes place, roughly speaking, at the same rate everywhere in A. Formulated precisely the definition states: a sequence $\langle f_n\rangle$ of functions defined on a plane set A *converges uniformly on* A to the limit function f if corresponding to each $\epsilon > 0$ there is an index $N = N(\epsilon)$ such that $|f_n(z) - f(z)| < \epsilon$ holds for every z in A once $n \ge N$. When n is large, therefore, $f_n(z)$ is required to be "uniformly close" to $f(z)$ throughout A. We emphasize that the uniform convergence of a sequence $\langle f_n\rangle$ on A carries with it the pointwise convergence of this sequence in A. The next theorem describes two of the principal virtues of uniform convergence as they touch on matters of interest in this book.

Theorem 1.1. *Suppose that each function in a sequence $\langle f_n\rangle$ is continuous on a set A and that this sequence converges uniformly on A to the limit function f. Then f is also continuous on A. Furthermore,*

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz$$

for every piecewise smooth path γ in A.

Proof. We fix a point z_0 of A and verify that $f: A \to \mathbb{C}$ is continuous at

z_0. Let $\epsilon > 0$ be given. We must find a $\delta > 0$ with the property that

$$|f(z) - f(z_0)| < \epsilon$$

whenever z belongs to A and $|z - z_0| < \delta$. We observe using the triangle inequality that for any index n and any point z of A

$$\begin{aligned} |f(z) - f(z_0)| &\leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| \\ &\quad + |f_n(z_0) - f(z_0)| \, . \end{aligned} \tag{7.1}$$

We now exploit the uniform convergence of $\langle f_n \rangle$ on A to select and fix an index n for which it is true that

$$|f_n(z) - f(z)| < \frac{\epsilon}{3}$$

for every z in A. Next, since the function f_n is by hypothesis continuous on A, it is possible to choose $\delta > 0$ with the property that

$$|f_n(z) - f_n(z_0)| < \frac{\epsilon}{3}$$

whenever z lies in A and $|z - z_0| < \delta$. Referring to (7.1) we conclude that

$$|f(z) - f(z_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

for every z belonging to A and satisfying $|z - z_0| < \delta$. This establishes the continuity of f at z_0, an arbitrary point of A — hence, its continuity on A.

Let γ be a piecewise smooth path in A. Given $\epsilon > 0$, we can again take advantage of the uniform convergence of $\langle f_n \rangle$ to pick N so that

$$|f_n(z) - f(z)| < \frac{\epsilon}{1 + \ell(\gamma)}$$

holds for every z in A as soon as $n \geq N$. It follows that

$$\left| \int_\gamma f_n(z)\, dz - \int_\gamma f(z)\, dz \right| = \left| \int_\gamma [f_n(z) - f(z)]\, dz \right|$$

$$\leq \int_\gamma |f_n(z) - f(z)|\, |dz| \leq \int_\gamma \frac{\epsilon}{\ell(\gamma) + 1}\, |dz| = \frac{\epsilon \ell(\gamma)}{1 + \ell(\gamma)} < \epsilon$$

whenever $n \geq N$. This proves that $\int_\gamma f_n(z)dz \to \int_\gamma f(z)dz$, the second assertion of the theorem. ∎

How is it possible to detect whether a sequence of functions converges uniformly on a set? In general, demonstrating uniform convergence can be a tricky business. There is, however, a criterion for the uniform convergence of function sequences that is analogous to the Cauchy criterion for the convergence of complex numerical sequences. A sequence of functions $\langle f_n \rangle$

is called a *uniform Cauchy sequence on a set A* if corresponding to each $\epsilon > 0$ there exists an index $N = N(\epsilon)$ such that $|f_m(z) - f_n(z)| < \epsilon$ is satisfied for all z in A whenever $m > n \geq N$. If $\langle f_n \rangle$ converges uniformly on A to the limit function f, then the inequaltiy

$$|f_m(z) - f_n(z)| \leq |f_m(z) - f(z)| + |f_n(z) - f(z)|$$

makes it clear that $\langle f_n \rangle$ is a uniform Cauchy sequence on A. It is the other direction in the following proposition that is frequently helpful in substantiating the uniform convergence of a sequence of functions.

Theorem 1.2. (Cauchy Criterion For Uniform Convergence) *Suppose that each function in a sequence $\langle f_n \rangle$ is defined on a set A. The sequence converges uniformly on A if and only if it is a uniform Cauchy sequence on A.*

Proof. As noted, only the sufficiency is a question mark. We assume, therefore, that $\langle f_n \rangle$ is a uniform Cauchy sequence on A and establish the existence of a function $f: A \to \mathbb{C}$ to which $\langle f_n \rangle$ converges uniformly. It is clear from the definition of a uniform Cauchy sequence that for each fixed z in A the sequence of values $\langle f_n(z) \rangle$ is a Cauchy sequence of complex numbers. Theorem II.4.2 thus certifies the existence of the pointwise limit of $\langle f_n \rangle$ in A; i.e., a function $f: A \to \mathbb{C}$ is properly defined by the rule of correspondence $f(z) = \lim_{n\to\infty} f_n(z)$. It remains to check that the convergence of $\langle f_n \rangle$ to f is actually uniform on A. Let $\epsilon > 0$ be given. We wish to exhibit an index N with the property that

$$|f_n(z) - f(z)| < \epsilon \tag{7.2}$$

for every z in A, provided only that $n \geq N$. The uniform Cauchy condition enables us to choose N so that

$$|f_n(z) - f_m(z)| < \frac{\epsilon}{2} \tag{7.3}$$

is satisfied for all z in A whenever $m > n \geq N$. If we now fix $n \geq N$ and let $m \to \infty$ in (7.3), we find that

$$|f_n(z) - f(z)| \leq \frac{\epsilon}{2} < \epsilon$$

for every z in A. As $n \geq N$ was arbitrary, we have verified (7.2) and so demonstrated that $f_n \to f$ uniformly on the set A. ∎

1.2 Normal Convergence

Our primary concern within the realm of function sequences is the convergence of sequences of analytic functions. We now indicate the framework in which the convergence of such sequences is usually discussed. Suppose that each function in a sequence $\langle f_n \rangle$ is defined — but not necessarily analytic — in an open subset U of $\mathbb{C}$. There is notion of convergence for $\langle f_n \rangle$ that lies between pointwise convergence in U and uniform convergence on U. We refer to it as "normal convergence" in U: *$\langle f_n \rangle$ converges normally in U to the limit function f* if $\langle f_n \rangle$ is pointwise convergent to f in U and if, in addition, the convergence is uniform on each compact set in U. The literature contains an assortment of other names for this type of convergence, two of the most popular being "locally uniform convergence in U" and "uniform convergence on compacta in U." We favor "normal convergence" for its conciseness. To test $\langle f_n \rangle$ for normal convergence in U it is not really necessary to check uniform convergence on every compact set in U — checking it on the closed disks in U is enough, as the next result points out.

Lemma 1.3. *Let $\langle f_n \rangle$ be a sequence of functions defined in an open set U. The sequence converges normally in U if and only if it converges uniformly on each closed disk that is contained in U.*

Proof. The "only if" implication is obvious. We must prove the converse. Suppose then that $\langle f_n \rangle$ converges uniformly on each closed disk in U. This, of course, entails the pointwise convergence of the sequence in U. We denote its limit by f. Let K be an arbitrary non-empty compact set in U. We claim that $f_n \to f$ uniformly on K. To see this, we first invoke Lemma II.4.4 and fix $r > 0$ with the following property: for each z in K the closed disk $\overline{\Delta}(z, r)$ is contained in U. We now pick a point z_1 of K. Either K is contained in $\Delta_1 = \overline{\Delta}(z_1, r)$, or we can select a point z_2 from $K \sim \Delta_1$. In the latter case, set $\Delta_2 = \overline{\Delta}(z_2, r)$. Then either K is contained in $\Delta_1 \cup \Delta_2$, or we can choose a point z_3 from the set $K \sim (\Delta_1 \cup \Delta_2)$ — and so forth. After a finite number of steps — call that number p — we must arrive at a collection $\Delta_1, \Delta_2, \ldots, \Delta_p$ of closed disks in U whose union covers K. (The alternative would be the construction of a sequence $\langle z_n \rangle$ in K having the feature that $|z_n - z_m| \geq r$ when $n \neq m$. Such a sequence could hardly have an accumulation point in K — or anywhere else in $\mathbb{C}$, for that matter — contrary to the definition of compactness.) Because $f_n \to f$ uniformly on each of the disks $\Delta_1, \Delta_2, \ldots, \Delta_p$, it is easy to see that convergence is uniform on $\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_p$ — hence, on K. It follows that $f_n \to f$ normally in U. ∎

With a little extra effort, we could improve Lemma 1.3 somewhat: *$\langle f_n \rangle$ converges normally in U if and only if each point z of U is the center of some closed disk on which this sequence converges uniformly.* We shall make no use of this fact and, for that reason, do not include its proof. For reference purposes we record the counterparts of Theorems 1.1 and 1.2 in the setting of normal convergence. The straightforward proofs are left as exercises.

Theorem 1.4. *Suppose that each function in a sequence $\langle f_n \rangle$ is continuous in an open set U and that the sequence converges normally in U to the limit function f. Then f is continuous in U. Furthermore,*

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_\gamma f_n(z)\,dz$$

for every piecewise smooth path γ in U.

Theorem 1.5. *A sequence $\langle f_n \rangle$ of functions defined in an open set U converges normally in U if and only if it is a uniform Cauchy sequence on each closed disk that is contained in U.*

2 Infinite Series

2.1 Complex Series

We are about ready to take up the topic of paramount interest in this chapter, convergent infinite series of functions. Before embarking on a treatment of such series, however, it will be wise to recall some elementary facts about series of complex numbers.

Presented with a sequence $\langle z_n \rangle$ of complex numbers, we can use it to generate a sequence $\langle s_n \rangle$, the *sequence of partial sums associated with* $\langle z_n \rangle$, by defining $s_1 = z_1$, $s_2 = z_1 + z_2$, $s_3 = z_1 + z_2 + z_3, \cdots$. Thus

$$s_n = z_1 + z_2 + \cdots + z_n \ .$$

For the sequence $\langle s_n \rangle$ there are, naturally, two possibilities. First, it may happen that $\langle s_n \rangle$ is convergent, say with the limit s. It is customary to express this fact by writing $s = \sum_{n=1}^\infty z_n$ and, in these circumstances, to speak of $\sum_{n=1}^\infty z_n$ as a *convergent infinite series with sum* s. Alternatively, $\langle s_n \rangle$ may fail to have a limit in the complex plane, in which case we declare the infinite series $\sum_{n=1}^\infty z_n$ to be *divergent*. The term z_n from the original sequence is also referred to as the n^{th} *term* of the series $\sum_{n=1}^\infty z_n$.

Although it can sometimes be a far from trivial task to decide whether a series $\sum_{n=1}^n z_n$ converges or diverges, we point out one condition that must be met if it is to have the slightest chance of converging: *the n^{th} term of a convergent infinite series $\sum_{n=1}^\infty z_n$ must tend to zero as $n \to \infty$.* The reason for this is that when $n \geq 2$ we can rewrite z_n as $s_n - s_{n-1}$, with the result that $z_n \to s - s = 0$ as $n \to \infty$ in case $s = \sum_{n=1}^\infty z_n$ exists. When trying to determine if the partial sum sequence $\langle s_n \rangle$ associated with a sequence $\langle z_n \rangle$ has a limit, one always has recourse to the Cauchy criterion. Since $s_m - s_{n-1} = \sum_{k=n}^m z_k$ for $m \geq n$, we can formulate the "Cauchy criterion for infinite series" as follows: *a series of complex numbers $\sum_{n=1}^\infty z_n$ converges* *if and only if corresponding to each $\epsilon > 0$ there is an index $N = N(\epsilon)$ such that $|\sum_{k=n}^{m} z_k| < \epsilon$ holds whenever $m \geq n \geq N$.*

In attempting to ascertain whether a complex series $\sum_{n=1}^{\infty} z_n$ converges or diverges, it is often instructive to consider along with the given series the corresponding series of moduli, $\sum_{n=1}^{\infty} |z_n|$. If the latter series converges, the original series $\sum_{n=1}^{\infty} z_n$ is said to be ***absolutely convergent***. The inequality $|\sum_{k=n}^{m} z_k| \leq \sum_{k=n}^{m} |z_k|$ and the Cauchy criterion for infinite series insure that an absolutely convergent series does, in fact, converge. Furthermore, it is then an easy matter to check that

$$\left|\sum_{n=1}^{\infty} z_n\right| \leq \sum_{n=1}^{\infty} |z_n| \, .$$

The advantage gained by passing to the absolute value series $\sum_{n=1}^{\infty} |z_n|$ is that, being a series of non-negative real numbers, this series is subject to the standard convergence tests studied in calculus — in particular, to the comparison, root, ratio, and integral tests.

An important characteristic of an absolutely convergent series $\sum_{n=1}^{\infty} z_n$ is that the terms of such a series can be permuted in an arbitrary fashion without influencing either the fact of convergence or the value of the sum. We mean by this statement that

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} z_{\sigma(n)}$$

for every one-to-one mapping σ of the set of positive integers onto itself (Exercise 5.17). This rearrangement invariance (it is a property definitely not enjoyed by a series that converges but fails to converge absolutely, a so-called ***conditionally convergent series***) can sometimes be instrumental in actually computing the sum of an absolutely convergent series.

If each of two series $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ is convergent (respectively, absolutely convergent) and if c is a complex number, then the series $\sum_{n=1}^{\infty} cz_n$ and $\sum_{n=1}^{\infty}(z_n + w_n)$ also converge (resp., converge absolutely). Moreover,

$$\sum_{n=1}^{\infty} cz_n = c\sum_{n=1}^{\infty} z_n \quad , \quad \sum_{n=1}^{\infty}(z_n + w_n) = \sum_{n=1}^{\infty} z_n + \sum_{n=1}^{\infty} w_n \, .$$

Notice especially that, if $z_n = x_n + iy_n$, the complex series $\sum_{n=1}^{\infty} z_n$ converges (resp., converges absolutely) if and only if both real series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converge (resp., converge absolutely) — in which case, of course,

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} x_n + i\sum_{n=1}^{\infty} y_n \, .$$

We shall frequently be forced to deal with infinite series of a more general type than those we've considered up to this point, series such as $\sum_{n=0}^{\infty} z_n$ or $\sum_{n=-3}^{\infty} z_n$ or even doubly infinite series $\sum_{n=-\infty}^{\infty} z_n$. Examples like the first two cause no problems: one simply makes the obvious adjustment in the definition of the partial sum s_n — in the first case use $s_n = z_0 + z_1 + \cdots + z_n$, in the second $s_n = z_{-3} + z_{-2} + \cdots + z_n$ — and proceeds as earlier. The situation with a series of the form $\sum_{n=-\infty}^{\infty} z_n$ is more complicated, but only mildly so. For each pair of positive integers m and n we form the partial sum $s_{m,n} = z_{-m} + z_{-m+1} + \cdots + z_{n-1} + z_n$. The series $\sum_{n=-\infty}^{\infty} z_n$ is defined to be convergent and to have the complex number s as its sum — again indicated by writing $s = \sum_{n=-\infty}^{\infty} z_n$ — if $s_{m,n} \to s$ as m and n tend independently to ∞. (The technical definition of the last condition reads: corresponding to each $\epsilon > 0$ there is a positive integer $N = N(\epsilon)$ such that $|s_{m,n} - s| < \epsilon$ is true whenever both $n \geq N$ and $m \geq N$.) When no complex number s with this property exists, $\sum_{n=-\infty}^{\infty} z_n$ is pronounced divergent. Often the most efficient method of handling a doubly infinite series $\sum_{n=-\infty}^{\infty} z_n$ is to deal separately with the series $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=-\infty}^{-1} z_n = \sum_{n=1}^{\infty} z_{-n}$. This approach is the subject of a short lemma.

Lemma 2.1. *A doubly infinite series of complex numbers $\sum_{n=-\infty}^{\infty} z_n$ converges if and only if both of the series $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_{-n}$ converge, in which event*

$$\sum_{n=-\infty}^{\infty} z_n = \sum_{n=1}^{\infty} z_{-n} + \sum_{n=0}^{\infty} z_n \; . \tag{7.4}$$

Proof. For $m \geq 1$ and $n \geq 1$, we write

$$s_{-m} = z_{-m} + z_{-m+1} + \cdots + z_{-1} \quad , \quad s_n = z_0 + z_1 + \cdots + z_n \; ,$$

$$s_{m,n} = z_{-m} + z_{-m+1} + \cdots + z_{n-1} + z_n \; .$$

Thus $s_{m,n} = s_{-m} + s_n$. Suppose first that $s^- = \sum_{n=1}^{\infty} z_{-n}$ and $s^+ = \sum_{n=0}^{\infty} z_n$ both exist. Then clearly $s_{m,n} = s_{-m} + s_n \to s^- + s^+$ as $m \to \infty$ and $n \to \infty$. Accordingly, $\sum_{n=-\infty}^{\infty} z_n$ converges and (7.4) holds. For the converse, assume that $\sum_{n=-\infty}^{\infty} z_n$ converges and has the complex number s for its sum. We employ the Cauchy criterion for series to establish the convergence of $\sum_{n=0}^{\infty} z_n$. Let $\epsilon > 0$ be given. We want to produce an index N with the property that $|\sum_{k=n}^{m} z_k| < \epsilon$, provided $m \geq n \geq N$. Since $s_{p,q} \to s$ as p and q tend to ∞, we are able to fix a positive integer M such that $|s_{p,q} - s| < \epsilon/2$ holds whenever both $p \geq M$ and $q \geq M$. In particular, $|s_{M,q} - s| < \epsilon/2$ for $q \geq M$. Set $N = M + 1$. When $m \geq n \geq N$, we see that

$$\left| \sum_{k=n}^{m} z_k \right| = |s_{M,m} - s_{M,n-1}| \leq |s_{M,m} - s| + |s - s_{M,n-1}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \; .$$

The Cauchy criterion for series informs us that $\sum_{n=0}^{\infty} z_n$ is convergent. A similar argument demonstrates the convergence of $\sum_{n=1}^{\infty} z_{-n}$. By the first part of this proof formula (7.4) is then valid. ∎

The notion of absolute convergence carries over to doubly infinite series, with the expected results: if $\sum_{n=-\infty}^{\infty} |z_n|$ converges, then $\sum_{n=-\infty}^{\infty} z_n$ converges; moreover, the terms of the latter series can be arbitrarily rearranged without affecting either the convergence or the value of the sum. Lemma 2.1 implies that $\sum_{n=-\infty}^{\infty} z_n$ is absolutely convergent precisely when both $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=1}^{\infty} z_{-n}$ have that feature. The algebraic properties of convergent infinite series are, of course, inherited by convergent series of the doubly infinite variety.

Before moving on to series of functions, we pause to look at a few examples involving complex numerical series.

EXAMPLE 2.1. Discuss the convergence of the geometric series $\sum_{n=0}^{\infty} z^n$.

The geometric series is one series for which it is possible to express the partial sum $s_n = 1 + z + \cdots + z^n$ in an explicit and convenient form. If $z = 1$, then $s_n = n+1$; if $z \neq 1$, then the formula for the sum of a geometric progression gives

$$s_n = \frac{1 - z^{n+1}}{1 - z} = \frac{1}{1-z} - \frac{z^{n+1}}{1-z} .$$

When $z = 1$, $\langle s_n \rangle$ is unbounded and so has no limit. The geometric series thus diverges for $z = 1$. Assuming that $z \neq 1$, we recall from Section II.1.7 that $\lim_{n\to\infty} z^n = 0$ when $|z| < 1$ and that this limit fails to exist when $|z| \geq 1$. In the former case the geometric series is convergent, and its sum is found to be $\lim_{n\to\infty} s_n = (1-z)^{-1}$; in the latter case the series diverges. To summarize:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

when $|z| < 1$ and $\sum_{n=0}^{\infty} z^n$ diverges for all other complex numbers z. The convergence of $\sum_{n=0}^{\infty} z^n$ is plainly absolute when $|z| < 1$.

EXAMPLE 2.2. Test each of the series $\sum_{n=1}^{\infty} n(1+i)^n(2i)^{-n}$ and $\sum_{n=1}^{\infty} n^2 2^n (1+i)^{-n}$ for convergence.

For the first series we apply the root test to the associated series of moduli; i.e., we compute

$$L = \lim_{n\to\infty} \sqrt[n]{\frac{n|1+i|^n}{|2i|^n}} = \frac{|1+i|}{2} \lim_{n\to\infty} \sqrt[n]{n} = \frac{|1+i|}{2} = \frac{\sqrt{2}}{2} .$$

Since $L < 1$, the series $\sum_{n=1}^{\infty} n(1+i)^n(2i)^{-n}$ converges absolutely. We test the second series for convergence by using the ratio test on the absolute value series. This time we calculate

$$L = \lim_{n\to\infty} \frac{\dfrac{(n+1)^2 2^{n+1}}{|1+i|^{n+1}}}{\dfrac{n^2 2^n}{|1+i|^n}} = \frac{2}{|1+i|} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = \frac{2}{|1+i|} = \sqrt{2}\,.$$

Because $L > 1$, the series $\sum_{n=1}^{\infty} n^2 2^n (1+i)^{-n}$ definitely does not converge absolutely. In fact, more can be said. If, in applying the root test or ratio test to the absolute value series $\sum_{n=1}^{\infty} |z_n|$ corresponding to a complex series $\sum_{n=1}^{\infty} z_n$, it turns out that $L = \lim_{n\to\infty} \sqrt[n]{|z_n|} > 1$ or that $L = \lim_{n\to\infty} |z_{n+1}|/|z_n| > 1$, then it is necessarily true that $z_n \not\to 0$ as $n \to \infty$ — hence, that $\sum_{n=1}^{\infty} z_n$ diverges. In particular, we conclude that the series $\sum_{n=1}^{\infty} n^2 2^n (1+i)^{-n}$ diverges.

EXAMPLE 2.3. Test the series $\sum_{n=1}^{\infty} n^{-1} i^n$ for convergence.

The absolute value series associated with this series is the harmonic series $\sum_{n=1}^{\infty} n^{-1}$. It diverges, as an application of the integral test demonstrates. Consequently, $\sum_{n=1}^{\infty} n^{-1} i^n$ is not absolutely convergent. If we now write $n^{-1} i^n$ in the form $x_n + i y_n$, we find that $x_n = 0$ for odd n and $x_{2n} = (-1)^n/(2n)$ for $n = 1, 2, 3, \cdots$, while $y_n = 0$ for even n and $y_{2n-1} = (-1)^{n-1}/(2n-1)$ for $n = 1, 2, 3, \cdots$. In other words, $\sum_{n=1}^{\infty} x_n$ reduces to $\sum_{n=1}^{\infty} (-1)^n/(2n)$ and $\sum_{n=1}^{\infty} y_n$ to $\sum_{n=1}^{\infty} (-1)^{n-1}/(2n-1)$. That each of these real series converges follows from the alternating series test. We deduce that $\sum_{n=1}^{\infty} n^{-1} i^n$ is conditionally convergent.

EXAMPLE 2.4. Discuss the convergence of $\sum_{n=-\infty}^{\infty} 2^{-|n|} z^n$.

According to Lemma 2.1 we are allowed to split the given series into the parts corresponding to $n \ge 0$ and $n < 0$, respectively, and to consider each of these subseries on its own. For the portion whose terms go with non-negative indices, Example 2.2 shows that

$$\sum_{n=0}^{\infty} 2^{-|n|} z^n = \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \frac{1}{1-(z/2)} = \frac{2}{2-z}$$

when $|z/2| < 1$ — i.e., when $|z| < 2$ — and that this series diverges for all remaining z. For the negatively indexed half of the series we obtain

$$\sum_{n=-\infty}^{-1} 2^{-|n|} z^n = \sum_{n=1}^{\infty} 2^{-|-n|} z^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{2z}\right)^n = \frac{1}{2z} \sum_{n=1}^{\infty} \left(\frac{1}{2z}\right)^{n-1}$$

$$= \frac{1}{2z} \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n = \frac{1}{2z}\, \frac{1}{1-(1/2z)} = \frac{1}{2z-1}$$

when $|1/(2z)| < 1$ — i.e., when $|z| > 1/2$ — but we observe divergence otherwise. On the strength of Lemma 2.1 we can say that

$$\sum_{n=-\infty}^{\infty} 2^{-|n|} z^n = \frac{2}{2-z} + \frac{1}{2z-1} = \frac{3z}{(2-z)(2z-1)}$$

when $1/2 < |z| < 2$, whereas this series is divergent for every other z.

2.2 Series of Functions

We return attention once more to a sequence $\langle f_n \rangle$ of complex-valued functions, each of which we assume to be defined in an open subset U of the complex plane. It is natural to mimic the process carried out for complex numerical series and to associate with $\langle f_n \rangle$ its sequence of partial sums s_n:

$$s_n = f_1 + f_2 + \cdots + f_n \ .$$

If the sequence of functions $\langle s_n \rangle$ converges pointwise in the set U to the limit function f, then we write $f = \sum_{n=1}^{\infty} f_n$ and say that the infinite series $\sum_{n=1}^{\infty} f_n$ is *pointwise convergent in U with sum f*. (Another way to phrase this definition is to state that for each z in U the series of complex numbers $\sum_{n=1}^{\infty} f_n(z)$ is convergent and has sum $f(z)$.) If $\langle s_n \rangle$ converges uniformly on a subset A of U, we refer to $\sum_{n=1}^{\infty} f_n$ as *uniformly convergent on A*. Finally, if $\langle s_n \rangle$ converges uniformly on each compact set in U, $\sum_{n=1}^{\infty} f_n$ is termed *normally convergent in U*. Owing to Lemma 1.3, in order to certify that a series $\sum_{n=1}^{\infty} f_n$ is normally convergent in U we need only check that it is uniformly convergent on each closed disk in U. We speak of $\sum_{n=1}^{\infty} f_n$ as *absolutely convergent in U* if the absolute value series $\sum_{n=1}^{\infty} |f_n|$ is pointwise convergent there. When this is so, the series $\sum_{n=1}^{\infty} f_n$ is itself pointwise convergent in U, and its sum remains unchanged under an arbitrary reordering of its terms.

A necessary condition for a series of functions $\sum_{n=1}^{\infty} f_n$ to converge uniformly on a set A is that $f_n \to 0$ uniformly on that set. This is evident from the fact that $f_n = s_n - s_{n-1}$ when $n \geq 2$. Theorem 1.2 leads directly to the counterpart in the context of uniform convergence of function series to the Cauchy convergence criterion for series of complex numbers: *the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on A if and only if corresponding to each $\epsilon > 0$ there is an index $N = N(\epsilon)$ such that $|\sum_{k=n}^{m} f_k(z)| < \epsilon$ is satisfied for every z in A whenever $m \geq n \geq N$*. In practice this criterion is usually implemented via comparison with an appropriate numerical series, as detailed in the next theorem.

Theorem 2.2. **(Weierstrass M-test)** *Suppose that each term in a function series $\sum_{n=1}^{\infty} f_n$ is defined on a set A. If there exists a sequence $\langle M_n \rangle$ of real numbers such that the estimate $|f_n(z)| \leq M_n$ holds for every z in A and such that the series $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges absolutely and uniformly on A.*

Proof. That $\sum_{n=1}^{\infty}|f_n(z)|$ converges for each z in A follows immediately from the comparison test for real series. To verify that the convergence of $\sum_{n=1}^{\infty} f_n$ is uniform on A we invoke the Cauchy criterion cited above. Given $\epsilon > 0$, we first appeal to the Cauchy criterion for numerical series to select an index N with the property that $\sum_{k=n}^{m} M_k < \epsilon$ whenever $m \geq n \geq N$. This is possible because $\sum_{n=1}^{\infty} M_n$ is, by assumption, convergent. It follows that for every z in A

$$\left|\sum_{k=n}^{m} f_k(z)\right| \leq \sum_{k=n}^{m} |f_k(z)| \leq \sum_{k=n}^{m} M_k < \epsilon ,$$

as soon as $m \geq n \geq N$. By the Cauchy criterion for uniform convergence $\sum_{n=1}^{\infty} f_n$ is seen to converge uniformly on A. ∎

We illustrate the application of the last theorem with two examples.

EXAMPLE 2.5. Verify that the geometric series $\sum_{n=0}^{\infty} z^n$ converges uniformly on $A_r = \overline{\Delta}(0,r)$ when $0 < r < 1$. Conclude that this series is normally convergent in the open disk $\Delta = \Delta(0,1)$.

We are dealing here with the series of functions $\sum_{n=0}^{\infty} f_n$ in Δ, where $f_n(z) = z^n$. Since $|f_n(z)| = |z^n| = |z|^n \leq r^n$ for every z in A_r and since $\sum_{n=0}^{\infty} r^n$ converges when $0 < r < 1$, we can appeal to the Weierstrass M-test with $M_n = r^n$ to establish the uniform convergence of the geometric series on A_r for each fixed r in $(0,1)$. Because any compact subset K of Δ lies in A_r for r suitably close to 1, we conclude that $\sum_{n=0}^{\infty} z^n$ converges uniformly on each such K — and, as a result, normally in Δ. Note, however, that $\sum_{n=0}^{\infty} z^n$ does not converge uniformly on Δ. If it did, it would follow that $f_n \to 0$ uniformly there. This is simply not the case. For instance, given any positive integer n, one can easily exhibit points z in Δ for which $|z^n| \geq 1/2$; e.g., one can take $z = \exp(-n^{-1}\,\mathrm{Log}\,2)$.

EXAMPLE 2.6. Show that the series $\sum_{n=1}^{\infty} n^{-z}$ converges absolutely and uniformly on the set $A_\sigma = \{z : \mathrm{Re}\, z \geq \sigma\}$ for $\sigma > 1$. Deduce that this series is normally convergent in the open half-plane $U = \{z : \mathrm{Re}\, z > 1\}$.

This example is concerned with $\sum_{n=1}^{\infty} f_n$, where $f_n(z) = n^{-z}$. For $z = x + iy$ in A_σ we have

$$|f_n(z)| = |n^{-z}| = \left|e^{-z\,\mathrm{Log}\,n}\right| = e^{-x\,\mathrm{Log}\,n} \leq e^{-\sigma\,\mathrm{Log}\,n} = n^{-\sigma} .$$

The integral test shows that $\sum_{n=1}^{\infty} n^{-\sigma}$ converges when $\sigma > 1$. Using $M_n = n^{-\sigma}$ in the Weierstrass M-test we infer the absolute and uniform convergence of $\sum_{n=1}^{\infty} n^{-z}$ on A_σ, provided $\sigma > 1$. As each compact subset of U is contained in A_σ for some $\sigma > 1$, the normal convergence of the series in U then becomes evident.

Doubly infinite series of functions will also be commonplace throughout the rest of this book. The definitions of pointwise, uniform, and normal convergence for a series $\sum_{n=-\infty}^{\infty} f_n$ merely demand the respective type of convergence for the partial sums $s_{m,n} = f_{-m} + f_{-m+1} + \cdots + f_{n-1} + f_n$ as m and n tend independently to ∞. To test such a series for uniform convergence we can take a hint from Lemma 2.1 and break the series into its "positive" and "negative" parts, meaning $\sum_{n=0}^{\infty} f_n$ and $\sum_{n=1}^{\infty} f_{-n}$, for separate treatment. This method is justified by the analogue of Lemma 2.1 in the framework of uniform convergence of function series. Its proof, which essentially retraces the steps in the proof of Lemma 2.1, is relegated to the exercises (Exercise 5.28).

Lemma 2.3. *A doubly infinite series of functions $\sum_{n=-\infty}^{\infty} f_n$ converges uniformly on a set A if and only if both of the series $\sum_{n=0}^{\infty} f_n$ and $\sum_{n=1}^{\infty} f_{-n}$ converge uniformly on A, in which event it is true that*

$$\sum_{n=-\infty}^{\infty} f_n = \sum_{n=1}^{\infty} f_{-n} + \sum_{n=0}^{\infty} f_n$$

in this set.

We close this section by establishing the series version of Theorem 1.4.

Theorem 2.4. *Suppose that each term in a function series $\sum_{n=1}^{\infty} f_n$ is continuous in an open set U and that the series converges normally in U, with the function f as its sum. Then f is continuous in U. Furthermore,*

$$\int_\gamma f(z)\,dz = \sum_{n=1}^{\infty} \int_\gamma f_n(z)\,dz \tag{7.5}$$

for every piecewise smooth path γ in U.

Proof. If $s_n = f_1 + f_2 + \cdots + f_n$, then s_n is continuous in U and $s_n \to f$ normally in this open set. Theorem 1.4 informs us that f is a continuous function in U and that

$$\int_\gamma f(z)\,dz = \lim_{N\to\infty} \int_\gamma s_N(z)\,dz = \lim_{N\to\infty} \sum_{n=1}^{N} \int_\gamma f_n(z)\,dz = \sum_{n=1}^{\infty} \int_\gamma f_n(z)\,dz$$

for every piecewise smooth path γ in U. ∎

We remark that it is only the uniform convergence of $\sum_{n=1}^{\infty} f_n$ on the specific compact set $|\gamma|$ that was required for the proof of (7.5). Theorem 2.4 has an obvious counterpart for doubly infinite series. We leave its formulation and proof to the reader.

3 Sequences and Series of Analytic Functions

3.1 General Results

The discussions in the preceding sections lay the groundwork for two theorems of the utmost importance in the theory of analytic functions.

Theorem 3.1. *Suppose that each function in a sequence $\langle f_n \rangle$ is analytic in an open set U and that the sequence converges normally in U to the limit function f. Then f is analytic in U. Moreover, $f_n^{(k)} \to f^{(k)}$ normally in U for each positive integer k.*

Proof. Theorem 1.4 certifies the continuity of the limit f in U. The same theorem, in combination with Lemma V.1.1, tells us that

$$\int_{\partial R} f(z)\,dz = \lim_{n\to\infty} \int_{\partial R} f_n(z)\,dz = \lim_{n\to\infty} 0 = 0$$

for every closed rectangle R in U. Morera's theorem then bears witness to the analyticity of f in that open set.

We next show that $f_n' \to f'$ normally in U. By Lemma 1.3 it is enough to check that $f_n' \to f'$ uniformly on each closed disk in U. Fix such a disk Δ, say $\Delta = \overline{\Delta}(z_0, r)$. Given $\epsilon > 0$ we wish to determine an index N with the property that $|f_n'(z) - f'(z)| < \epsilon$ for every z in Δ and every $n \geq N$. We begin by fixing $s > r$ such that the disk $\overline{\Delta}(z_0, s)$ is still contained in U. Let z belong to Δ. We appeal to Theorem V.3.4 in making the estimate

$$|f_n'(z) - f'(z)| = \left| \frac{1}{2\pi i} \int_{|\zeta - z_0| = s} \frac{f_n(\zeta)\,d\zeta}{(\zeta - z)^2} - \frac{1}{2\pi i} \int_{|\zeta - z_0| = s} \frac{f(\zeta)\,d\zeta}{(\zeta - z)^2} \right|$$

$$= \frac{1}{2\pi} \left| \int_{|\zeta - z_0| = s} \frac{[f_n(\zeta) - f(\zeta)]\,d\zeta}{(\zeta - z)^2} \right| \leq \frac{1}{2\pi} \int_{|\zeta - z_0| = s} \frac{|f_n(\zeta) - f(\zeta)|\,|d\zeta|}{|\zeta - z|^2}$$

$$\leq \frac{s}{(s-r)^2} \max\left\{ |f_n(\zeta) - f(\zeta)| : \zeta \in K \right\} ,$$

where $K = K(z_0, s)$. (N.B. $|\zeta - z| \geq s - r$ when z belongs to Δ and ζ lies on the circle K.) By hypothesis, $f_n \to f$ uniformly on K. This fact enables us to select N so that, once $n \geq N$, we can count on the inequality

$$|f_n(\zeta) - f(\zeta)| < \frac{(s-r)^2 \epsilon}{s}$$

being in force for every ζ on K. In particular, it is applicable at any point ζ of K where the quantity $|f_n(\zeta) - f(\zeta)|$ is maximized. If $n \geq N$, therefore, the previous estimate insures that $|f_n'(z) - f'(z)| < \epsilon$ holds for every z in Δ, as desired. Accordingly, $f_n' \to f'$ uniformly on each closed disk in U

— hence, normally in U. Applying this fact to the sequence of derivatives $\langle f_n' \rangle$, we find that $f_n'' = (f_n')' \to (f')' = f''$ normally in U — and so forth for higher derivatives. ∎

The conclusion in Theorem 3.1 that the limit function f is analytic in U would not follow, in general, if the convergence assumption in the theorem were weakened from normal convergence in U to pointwise convergence there, although it is not an easy matter to write down an explicit counterexample. For more information on this topic we refer the reader to the article "Pointwise limits of analytic functions" by K.R. Davidson in *The American Mathematical Monthly,* Vol. 90, No. 6, 1983.

The series companion to Theorem 3.1 is:

Theorem 3.2. *Suppose that each term in a function series $\sum_{n=1}^{\infty} f_n$ is analytic in an open set U and that the series converges normally in U, with the function f as its sum. Then f is analytic in U. Moreover, $f^{(k)} = \sum_{n=1}^{\infty} f_n^{(k)}$ in U for each positive integer k. The convergence of these derived series is also normal in U.*

Proof. The partial sum $s_n = f_1 + f_2 + \cdots + f_n$ is analytic in U, and $s_n \to f$ normally in U. On the basis of Theorem 3.1 we can assert that f is analytic in U and that $s_n^{(k)} \to f^{(k)}$ normally in this set for each positive integer k. As a result, we discover that

$$f^{(k)} = \lim_{N\to\infty} s_N^{(k)} = \lim_{N\to\infty} \sum_{n=1}^{N} f_n^{(k)} = \sum_{n=1}^{\infty} f_n^{(k)}$$

in U and learn, too, that the convergence of the series of k^{th}-derivatives is normal there. ∎

Theorem 3.2 can also be stated for a doubly infinite series of analytic functions. As all the necessary changes are obvious and purely cosmetic in nature, we omit the details. Theorems 3.1 and 3.2 equip us with a powerful new set of tools for constructing analytic functions. The extent of their power will not be fully appreciated until Chapters IX and X, where these theorems will be put to effective use in establishing basic results on the existence of analytic functions subject to various kinds of constraints. For the time being we must be content with a couple of examples that give the flavor of those later constructions.

EXAMPLE 3.1. Show that the formula $f(z) = \sum_{n=1}^{\infty} n^{-z}$ defines an analytic function in the open set $U = \{z : \operatorname{Re} z > 1\}$. Find a series expansion for f' that is valid in U.

It was observed in Example 2.6 that the series $\sum_{n=1}^{\infty} n^{-z}$ converges normally in U. Because $f_n(z) = n^{-z}$ describes an entire function, Theorem

3.2 implies that f is analytic in U and that

$$f'(z) = \sum_{n=1}^{\infty} f_n'(z) = -\sum_{n=1}^{\infty} n^{-z} \operatorname{Log} n = -\sum_{n=2}^{\infty} n^{-z} \operatorname{Log} n$$

for each z in U. More generally, for any positive integer k we obtain a series expansion of $f^{(k)}$ in U:

$$f^{(k)}(z) = (-1)^k \sum_{n=2}^{\infty} n^{-z} (\operatorname{Log} n)^k .$$

The function f in Example 3.1 has a multitude of interesting properties. For starters, it is known that f can be extended beyond the confines of U to a function that is analytic in the set $\mathbb{C} \sim \{1\}$. Since the series $\sum_{n=1}^{\infty} n^{-z}$ diverges when $\operatorname{Re} z < 1$, it is not at all apparent from the definition of f that such an extension would be possible, to say nothing of how it might be achieved. We give no details of the extension process. The only point we wish to make here is that through this extension one can construct from our example f a function that ranks among the most fascinating in all of mathematics. It is called the *Riemann zeta-function.* (In his studies of this function Riemann used the Greek letter ζ to designate it.) With the zeta-function is associated one of the truly monumental unsolved problems of mathematics, to prove or disprove the veracity of the "Riemann hypothesis." Riemann was able to show that $\zeta(-2k) = 0$ for every positive integer k. (These zeros are now referred to as the "trivial zeros" of the zeta-function.) Every other zero of the zeta-function that Riemann managed to identify had real part 1/2. This led to the *Riemann hypothesis:* every non-trivial zero of the Riemann zeta-function has real part 1/2. Whether this statement is true or false remains an unanswered question despite sustained efforts over the past century by some of the world's finest mathematicians to settle it. The Riemann hypothesis continues to be a great stimulus to mathematical research — especially in the field of analytic number theory, where progress on this problem concerned with a specific, granted somewhat exotic, analytic function translates in the long run into better knowledge of something as down to earth as the distribution of primes among the natural numbers. From simple examples mighty theories sometimes spring!

EXAMPLE 3.2. Verify that the formula $f(z) = \sum_{n=-\infty}^{\infty} (z-n)^{-2}$ defines an analytic function in the open set $U = \mathbb{C} \sim \{0, \pm 1, \pm 2, \cdots\}$.

For each integer n the function f_n given by $f_n(z) = (z-n)^{-2}$ is analytic in U. If we can demonstrate that the series $\sum_{n=-\infty}^{\infty} f_n$ converges normally in U, then the analyticity of f in U will follow from the analogue of Theorem 3.2 for doubly infinite series. In view of Lemma 2.3 it is enough to check that the series $\sum_{n=0}^{\infty} f_n$ and $\sum_{n=0}^{\infty} f_{-n}$ both converge normally in U. Let p be a positive integer. We shall prove that the truncated series $\sum_{n=p}^{\infty} f_n$ and $\sum_{n=p}^{\infty} f_{-n}$ converge uniformly on the set $A_p = \overline{\Delta}(0, p/2)$. As any given compact subset K of U lies in A_p for p suitably large, we conclude that the full series $\sum_{n=0}^{\infty} f_n$ and $\sum_{n=1}^{\infty} f_{-n}$ converge uniformly on K — the extra terms $\sum_{n=0}^{p-1} f_n$ and $\sum_{n=1}^{p-1} f_{-n}$ clearly do not affect the convergence — and, thus, normally in U. Assuming that z belongs to A_p and that $|n| \geq p$, we remark that

$$|z-n| \geq |n| - |z| \geq |n| - \frac{p}{2} \geq |n| - \frac{|n|}{2} = \frac{|n|}{2} .$$

This gives rise to the estimate

$$|f_n(z)| = \frac{1}{|z-n|^2} \leq \frac{4}{n^2}$$

for all z in A_p, provided $|n| \geq p$. Because $\sum_{n=p}^{\infty}(4/n^2)$ converges, the Weierstrass M-test stamps $\sum_{n=p}^{\infty} f_n$ and $\sum_{n=p}^{\infty} f_{-n}$ as absolutely and uniformly convergent on A_p. We can now safely proclaim the function f analytic in U, where according to Theorem 3.2 its derivative is given by

$$f'(z) = -2 \sum_{n=-\infty}^{\infty} (z-n)^{-3} .$$

Notice that the series defining f is absolutely convergent in U, which means that we are free to rearrange its terms without changing the sum. For instance, we can place the terms $(z-n)^{-2}$ and $(z+n)^{-2}$ for $n \geq 1$ side by side and then combine them to arrive at a different representation for f; namely,

$$f(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} \left\{ \frac{1}{(z-n)^2} + \frac{1}{(z+n)^2} \right\} = \frac{1}{z^2} + 2 \sum_{n=1}^{\infty} \frac{z^2+n^2}{(z^2-n^2)^2} .$$

Observe, also, that the function f is periodic, with period 1: $f(z+1) = f(z)$ for every z in U. This fact is confirmed by the calculation

$$f(z+1) = \sum_{n=-\infty}^{\infty} \frac{1}{[(z+1)-n]^2} = \sum_{n=-\infty}^{\infty} \frac{1}{[z-(n-1)]^2}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = f(z) .$$

3.2 Limit Superior of a Sequence

To insure a smooth treatment of Taylor and Laurent series we must say a few words about the so-called "limit superior" of a sequence $\langle r_n \rangle$ of non-negative real numbers. (For additional discussion of this concept consult Appendix A.) Suppose first that the sequence $\langle r_n \rangle$ is bounded. Its set of accumulation points is then easily seen to be non-empty, closed, and bounded. As a non-empty compact set of real numbers, the set of accumulation points of $\langle r_n \rangle$ has a largest element. That element is called the *limit superior of* $\langle r_n \rangle$. We denote it by $\limsup_{n\to\infty} r_n$. If the given sequence $\langle r_n \rangle$ is unbounded, on the other hand, we express that fact by writing $\limsup_{n\to\infty} r_n = \infty$. In the bounded case the limit superior of $\langle r_n \rangle$ is the unique real number r that meets the following conditions: for each $\epsilon > 0$ it is the case that (i) $r_n \geq r + \epsilon$ holds for at most finitely many values of n and (ii) there are infinitely many indices n for which $r_n > r - \epsilon$. The first requirement makes certain that $\langle r_n \rangle$ has no accumulation points larger than r; the two conditions together imply that r is itself an accumulation point of $\langle r_n \rangle$. Of course, if $\langle r_n \rangle$ happens to be a convergent sequence, then we note that $\limsup_{n\to\infty} r_n = \lim_{n\to\infty} r_n$. Some examples:

$$\limsup_{n\to\infty} [1 + (-1)^n] = 2 ,$$

since 0 and 2 are the only accumulation points of the sequence 0,2,0,2,$\cdots$;

$$\limsup_{n\to\infty} [n + (-1)^n n] = \infty ,$$

because the sequence 0,4,0,8,$\cdots$ is unbounded;

$$\limsup_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} \sqrt[n]{n} = 1 ;$$

$$\limsup_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e .$$

3.3 Taylor Series

Suppose that z_0 is a point of the complex plane. We refer to a function series of the type

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots ,$$

where $a_0, a_1, a_2, \cdots$ is a sequence of complex numbers, as a *Taylor series* (or, alternatively, as a *power series*) *centered at* z_0. (The Taylor in the name is Brook Taylor (1685-1731), who along with Colin Maclauren (1698-1746) pioneered the study of these series.) The numbers a_n in this polynomial-like expression are known as its *coefficients.* With any such Taylor series we associate an extended real number ρ by the rule

$$\rho = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1} . \tag{7.6}$$

Here we observe the conventions $1/0 = \infty$ and $1/\infty = 0$. The quantity ρ is known as the *radius of convergence* of the given series. When $\rho > 0$ the open disk $\Delta(z_0, \rho)$ is called its *disk of convergence.* (For $\rho = \infty$ this "disk" is actually the whole complex plane.) An explanation for the terminology is supplied by the following theorem.

Theorem 3.3. *Suppose that ρ is the radius of convergence of a Taylor series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ centered at z_0. The series diverges for any z satisfying $|z - z_0| > \rho$. If $\rho > 0$, the series converges absolutely and normally in the disk $\Delta = \Delta(z_0, \rho)$, so the function f defined by $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is analytic in Δ. The coefficient a_n is then related to f through the formula*

$$a_n = \frac{f^{(n)}(z_0)}{n!}\,. \tag{7.7}$$

Proof. Assume first that z satisfies $|z - z_0| = r > \rho$. Thus $r^{-1} < \rho^{-1}$. It follows from (7.6) and the definition of $\limsup_{n\to\infty} \sqrt[n]{|a_n|}$ that the statement $\sqrt[n]{|a_n|} > r^{-1}$ — hence, $|a_n| > r^{-n}$ — must be true for infinitely many values of n. Now

$$|a_n(z - z_0)^n| = |a_n||z - z_0|^n > r^{-n}r^n = 1$$

for any such n. This implies that $a_n(z - z_0)^n \not\to 0$ as $n \to \infty$, a fact which marks the series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ as divergent.

We assume next that $\rho > 0$ and write $\Delta = \Delta(z_0, \rho)$. In order to establish that $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges absolutely and normally in Δ it suffices to demonstrate that this series converges absolutely and uniformly on the closed disk $\Delta_r = \overline{\Delta}(z_0, r)$ for each r in the interval $(0, \rho)$. (Any compact set K in Δ can be enclosed in Δ_r by taking r suitably close to ρ.) We fix r in $(0, \rho)$ and then fix s satisfying $r < s < \rho$. Because $\rho^{-1} < s^{-1}$, (7.6) implies that $\sqrt[n]{|a_n|} < s^{-1}$ once n is sufficiently large. Assume this to be so for all n larger than N. Setting $c = \max\{1, |a_0|, |a_1|s, \ldots, |a_N|s^N\}$, we observe that $|a_n| \le cs^{-n}$ then holds for every n. This gives rise to the estimate $|a_n(z - z_0)^n| \le c(r/s)^n = M_n$ for any z belonging to the disk Δ_r. As $r/s < 1$, $\sum_{n=1}^{\infty} M_n$ converges. The Weierstrass M-test thus guarantees that $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is absolutely and uniformly convergent on Δ_r, as we had set cut to prove.

If $\rho > 0$, Theorem 3.2 assures us that the rule of correspondence $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ defines a function which is analytic in the disk Δ. Furthermore, the theorem tells us how to compute the derivatives of f:

$$f'(z) = \sum_{n=1}^{\infty} na_n(z - z_0)^{n-1} = a_1 + 2a_2(z - z_0) + \cdots\,,$$

$$f''(z) = \sum_{n=2}^{\infty} n(n-1)a_n(z - z_0)^{n-2} = 2a_2 + 6a_3(z - z_0) + \cdots\,,$$

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-z_0)^{n-k}$$

$$= k!a_k + (k+1)!a_{k+1}(z-z_0) + \cdots .$$

Inserting $z = z_0$ into the formula for $f^{(k)}$ for any non-negative integer k leads to $f^{(k)}(z_0) = k!a_k$, which corroborates (7.7), the final assertion of the theorem. ∎

We emphasize that, when $0 < \rho < \infty$, Theorem 3.3 makes no statement whatsoever about the convergence or divergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ for z satisfying $|z - z_0| = \rho$. Depending on the coefficients, convergence may occur for all, for some, or for no such z. The use of formula (7.6) is by no means the exclusive way (or, for that matter, invariably the preferred way) to determine the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. For instance, if $a_n \neq 0$ for all n — or, more generally, for all sufficiently large n — and if it happens that either $\lim_{n\to\infty}|a_n|/|a_{n+1}|$ exists in the strict sense or $\lim_{n\to\infty}|a_n|/|a_{n+1}| = \infty$, then this limit is equal to ρ (Exercise 5.38). In some situations — see Example 3.4 — a quite effective method of finding ρ is simply to apply the root test or ratio test to $\sum_{n=0}^{\infty} |a_n||z-z_0|^n$ on a pointwise basis.

EXAMPLE 3.3. Determine the radii of convergence of the Taylor series $\sum_{n=1}^{\infty} n^{-1}i^n z^n$ and $\sum_{n=0}^{\infty} \left[(n!)^2/(2n)!\right](z+i)^n$.

For the first series we calculate

$$\frac{1}{\rho} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \frac{1}{\sqrt[n]{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 ,$$

which gives $\rho = 1$ as its radius of convergence. Due to the presence of the factorials in the latter series it becomes more convenient to evaluate

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \lim_{n\to\infty} \frac{(n!)^2(2n+2)!}{(2n)![(n+1)!]^2} = \lim_{n\to\infty} \frac{4n+2}{n+1} = 4$$

and to conclude that $\rho = 4$ for this series. Its disk of convergence is $\Delta(-i, 4)$.

EXAMPLE 3.4. Discuss the convergence of the series $\sum_{n=0}^{\infty} 2^n z^{n^2}$.

The given series $\sum_{n=0}^{\infty} 2^n z^{n^2} = 1 + 2z + 4z^4 + 8z^9 + \cdots$ is a Taylor series centered at the origin in which the coefficient a_n is zero for any non-square value of n. Rather than using (7.6) we test convergence by resorting to the root test, applied pointwise to the absolute value series $\sum_{n=0}^{\infty} 2^n |z|^{n^2}$. From the computation

$$\lim_{n\to\infty} \sqrt[n]{2^n|z|^{n^2}} = \lim_{n\to\infty} 2|z|^n = \begin{cases} 0 & \text{if } |z| < 1 , \\ 2 & \text{if } |z| = 1 , \\ \infty & \text{if } |z| > 1 , \end{cases}$$

we learn that the series $\sum_{n=0}^{\infty} 2^n z^{n^2}$ converges absolutely when $|z| < 1$ and diverges otherwise. This state of affairs is only compatible with $\rho = 1$ being its radius of convergence. The approach we've opted to take here has the extra advantage of making clear in this example the status of convergence at all points on the circle bounding the disk of convergence, not always an easy thing to do.

The next theorem declares that the collection of functions which are representable in an open disk $\Delta = \Delta(z_0, r)$ as sums of convergent Taylor series centered at z_0 embraces all of the functions that are analytic in Δ. This important fact opens the door to a systematic examination of the local structural properties of analytic functions, a line of inquiry we shall pursue in the succeeding chapters of this book.

Theorem 3.4. *Suppose that a function f is analytic in an open set U, that z_0 is a point of U, and that the open disk $\Delta = \Delta(z_0, r)$ is contained in U. Then f can be represented in Δ as the sum of a Taylor series centered at z_0. This expansion is uniquely determined by f: if $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ in Δ, then the coefficient a_n is given by $a_n = f^{(n)}(z_0)/n!$.*

Proof. Define a sequence $\langle a_n \rangle_{n=0}^{\infty}$ by $a_n = f^{(n)}(z_0)/n!$. It is our contention that $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ for every z in Δ. Fix such a point z, and fix along with it a number s satisfying $|z - z_0| < s < r$. For any ζ on the circle $K = K(z_0, s)$ it is true that $|(z-z_0)/(\zeta - z_0)| = |z - z_0|/s < 1$. On the basis of Example 2.1 the following expansion of $f(\zeta)/(\zeta - z)$ is permitted:

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{\zeta - z_0} \, \frac{1}{1 - \left(\dfrac{z - z_0}{\zeta - z_0}\right)} = \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n ;$$

i.e., for ζ on K we can write

$$\frac{f(\zeta)}{\zeta - z} = \sum_{n=0}^{\infty} \frac{f(\zeta)(z-z_0)^n}{(\zeta - z_0)^{n+1}} . \tag{7.8}$$

Viewed as a series of functions of the variable ζ the series in (7.8) converges uniformly on K. (This assertion is supported by the Weierstrass M-test — take $M_n = ct^n/s$, where $t = |z - z_0|/s < 1$ and $c = \max\{|f(\zeta)| : \zeta \in K\}$.) Appealing to the Cauchy integral formula, then to Theorem 2.4 for authorization to interchange summation and integration, and lastly to Cauchy's integral formula for derivatives, we find that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|\zeta - z_0| = s} \frac{f(\zeta)\, d\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i} \int_{|\zeta - z_0| = s} \left[\sum_{n=0}^{\infty} \frac{f(\zeta)(z-z_0)^n}{(\zeta - z_0)^{n+1}} \right] d\zeta \end{aligned}$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{|\zeta - z_0| = s} \frac{f(\zeta)(z - z_0)^n \, d\zeta}{(\zeta - z_0)^{n+1}}$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{|\zeta - z_0| = s} \frac{f(\zeta) \, d\zeta}{(\zeta - z_0)^{n+1}} \right] (z - z_0)^n$$

$$= \sum_{n=0}^{\infty} \left[\frac{f^{(n)}(z_0)}{n!} \right] (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n \ ,$$

the result desired.

The existence of at least one Taylor series development for f in Δ has now been substantiated. If $f(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$ were a potentially different Taylor series representation of f in Δ, then the radius of convergence ρ of $\sum_{n=0}^{\infty} b_n(z - z_0)^n$ could clearly be no smaller than r, and the function $g(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$, known by Theorem 3.3 to be analytic in $\Delta(z_0, \rho)$, would agree with f in Δ. According to the last statement in Theorem 3.3 this would mean that

$$b_n = \frac{g^{(n)}(z_0)}{n!} = \frac{f^{(n)}(z_0)}{n!} = a_n \ ,$$

so the second expansion would not really be different after all. The uniqueness assertion in the present theorem follows. ∎

Given that a function f is analytic in an open set U, we are now at liberty to expand f in a Taylor series about an arbitrary point z_0 of U, say $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. We are assured by Theorem 3.4 that the radius of convergence ρ of this Taylor series is not less than d, the radius of the largest open disk centered at z_0 that is contained in U. It may, in fact, happen that $\rho > d$. (See Figure 1.) Assuming this to be the situation, we know that $g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ defines a function which is analytic in the disk $\Delta(z_0, \rho)$. *We wish to maintain that $f(z) = g(z)$, however, only for those z belonging to $\Delta(z_0, d)$.* For z outside $\Delta(z_0, d)$, as in Figure 1, it is frequently the case that $f(z) \neq g(z)$. Example 3.11 will provide a concrete illustration of this phenomenon. We remark that the condition $\rho > d$ sometimes opens the possibility of extending the function f, originally assumed to be analytic just in U, to a function that is analytic in an open set larger than U. For instance, we shall learn in the next chapter that, if $\rho > d$ and if the set $U \cap \Delta(z_0, \rho)$ is connected, then the rule of correspondence

$$h(z) = \begin{cases} f(z) & \text{if } z \in U \ , \\ g(z) & \text{if } z \in \Delta(z_0, \rho) \ , \end{cases}$$

describes a function h that is analytic in $V = U \cup \Delta(z_0, \rho)$, an open set which properly includes U. This method of extending an analytic function is the basic step in a process called "analytic continuation," a topic that will be taken up in Chapter X.

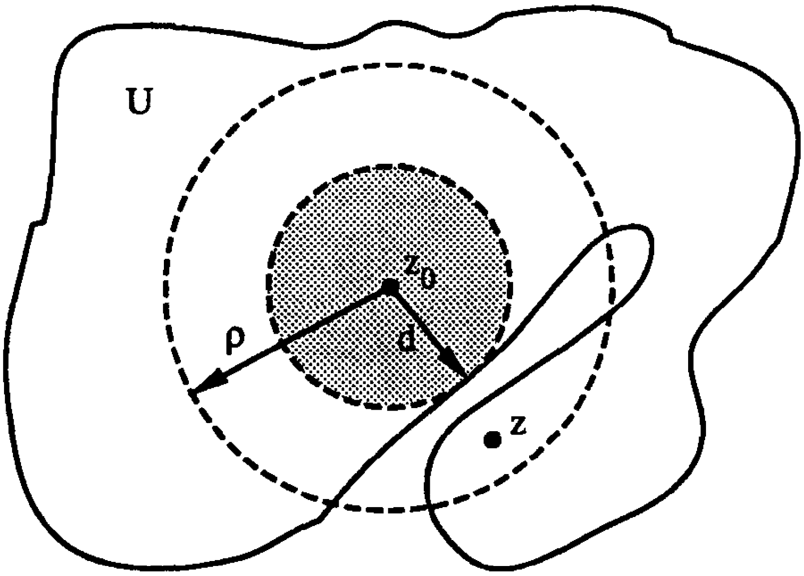

Figure 1.

We look now at a string of examples.

EXAMPLE 3.5. Find the Taylor series expansion of $f(z) = e^z$ that is centered at the origin.

Since $f^{(n)}(z) = e^z$ for every non-negative integer n, we obtain $f^{(n)}(0) = 1$ for all n. This produces the Taylor series representation

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

for any z in the complex plane.

EXAMPLE 3.6. Determine the power series expansions of $f(z) = \sin z$ and $g(z) = \cos z$ about the origin.

Here we have $f^{(n)}(0) = 0$ for even n and $f^{(2n+1)}(0) = (-1)^n$ for $n = 0, 1, 2, \cdots$. The resulting Taylor series expansion,

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots ,$$

is valid throughout the complex plane. To arrive at the expansion of g we need only differentiate the series representing f term by term,

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots .$$

EXAMPLE 3.7. Expand $f(z) = \operatorname{Log} z$ in a Taylor series about the point $z_0 = 1$.

The function f is analytic in $U = \mathbb{C} \sim (-\infty, 0]$, where its derivatives are given by $f^{(n)}(z) = (-1)^{n-1}(n-1)!\, z^{-n}$ for $n \geq 1$. Hence $f(1) = 0$ and $f^{(n)}(1) = (-1)^{n-1}(n-1)!$ for $n \geq 1$, which yields the expansion

$$\begin{aligned} \operatorname{Log} z &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(z-1)^n}{n} \\ &= (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \cdots . \end{aligned}$$

This representation is applicable in $\Delta(1,1)$, the largest open disk centered at 1 that is contained in U.

EXAMPLE 3.8. Expand $f(z) = 2z(z^2-1)^{-1}$ in a power series centered at $z_0 = i$.

In doing this example one could, of course, attempt to find $f^{(n)}(z)$ and evaluate $f^{(n)}(i)/n!$ straight away. Taking just a couple of derivatives should quickly disabuse the reader of the wisdom of that approach. A slicker method uses a partial fractions decomposition of f and puts the geometric series to work:

$$\begin{aligned} \frac{2z}{z^2-1} &= \frac{1}{z-1} + \frac{1}{z+1} = \frac{1}{(i-1)+(z-i)} + \frac{1}{(i+1)+(z-i)} \\ &= \frac{1}{i-1}\,\frac{1}{1-\left(-\dfrac{z-i}{i-1}\right)} + \frac{1}{i+1}\,\frac{1}{1-\left(-\dfrac{z-i}{i+1}\right)} \\ &= \frac{1}{i-1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{i-1}\right)^n + \frac{1}{i+1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{i+1}\right)^n \\ &= \sum_{n=0}^{\infty}(-1)^n\left[(i-1)^{-n-1} + (i+1)^{-n-1}\right](z-i)^n . \end{aligned}$$

We have already accomplished what we intended to in this example : we have determined the Taylor expansion of f about i. With a small amount of extra effort, however, we can express its coefficients in a more readily computed form. First, by rationalizing denominators we see that

$$\begin{aligned} (i-1)^{-n-1} + (i+1)^{-n-1} &= \frac{(-i-1)^{n+1} + (-i+1)^{n+1}}{2^{n+1}} \\ &= \frac{(-1)^{n+1}\left[(i+1)^{n+1} + (i-1)^{n+1}\right]}{2^{n+1}} . \end{aligned}$$

Next, for any positive integer k we have

$$(i+1)^k + (i-1)^k = 2^{k/2}e^{k\pi i/4} + 2^{k/2}e^{3k\pi i/4}$$

$$= 2^{k/2}e^{k\pi i/2}\left(e^{k\pi i/4} + e^{-k\pi i/4}\right) = 2^{(k+2)/2}i^k\cos(k\pi/4)\ .$$

These observations lead to the revised expansion

$$\frac{2z}{z^2-1} = \sum_{n=0}^{\infty} 2^{-(n-1)/2}i^{n+3}\cos[(n+1)\pi/4](z-i)^n$$

$$= -i - \frac{i(z-i)^2}{2} + \frac{(z-i)^3}{2} + \cdots\ .$$

It is valid for z in $\Delta(i,\sqrt{2})$, the largest open disk with center at i that lies in the set $U = \mathbb{C} \sim \{\pm 1\}$ where f is analytic.

EXAMPLE 3.9. Find the Taylor series expansion of $f(z) = e^{z^2}$ about the origin. Use this to determine $f^{(n)}(0)$ for every non-negative integer n.

Again in this example direct computation of $f^{(n)}(z)$ would be an unpleasant task at best. Instead, we start with the power series expansion

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$$

derived in Example 3.5 and substitute z^2 for z. This procedure results in

$$e^{z^2} = \sum_{n=0}^{\infty}\frac{z^{2n}}{n!} = 1 + z^2 + \frac{z^4}{2!} + \cdots$$

for every z in $\mathbb{C}$. Since the series on the right definitely is a Taylor series centered at the origin, the uniqueness statement in Theorem 3.4 certifies it as the expansion of f we were after. Formula (7.7) then makes it clear that $f^{(n)}(0) = 0$ when n is odd, while $f^{(2n)}(0) = (2n)!/n!$ for $n = 0, 1, 2, \cdots$.

EXAMPLE 3.10. Determine the Taylor series expansion of $f(z) = \cos^2 z$ that is centered at the origin.

We take advantage of the identity $\cos^2 z = [1 + \cos(2z)]/2$ and of the power series representation for $g(z) = \cos z$ described in Example 3.6 to calculate

$$\cos^2 z = \frac{1+\cos(2z)}{2} = \frac{1}{2} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2z)^{2n}}{(2n)!}$$

$$= 1 + \sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n-1}z^{2n}}{(2n)!} = 1 - z^2 + \frac{z^4}{3} - \cdots\ ,$$

an expansion valid for all complex z.

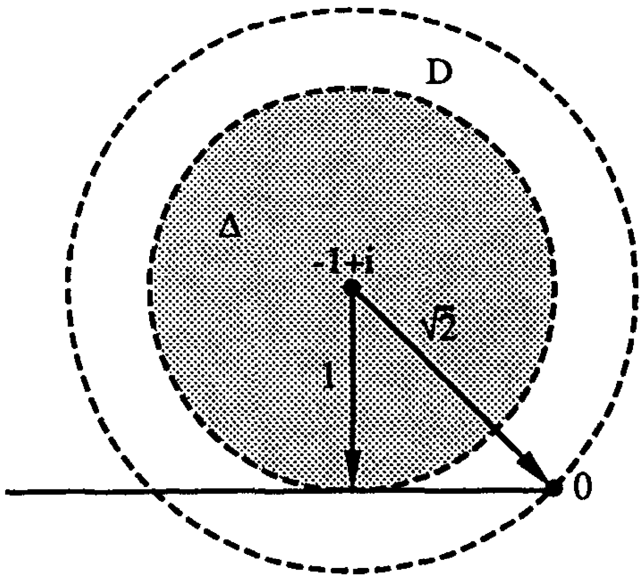

Figure 2.

EXAMPLE 3.11. Discuss the Taylor series expansion of the function $f(z) = \operatorname{Log} z$ about the point $z_0 = -1 + i$.

As observed in Example 3.7 this function is analytic in the set $U = \mathbb{C} \sim (-\infty, 0]$, where $f^{(n)}(z) = (-1)^{n-1}(n-1)!\, z^{-n}$ for $n \geq 1$. It follows that

$$\operatorname{Log} z = \operatorname{Log}\sqrt{2} + \frac{3\pi i}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(z+1-i)^n}{n(-1+i)^n}$$

for z in $\Delta = \Delta(-1+i, 1)$, the largest open disk centered at $z_0 = -1 + i$ that is contained in U. On the other hand, the radius of convergence ρ of this Taylor series is quickly computed: $\rho = \sqrt{2}$. By Theorem 3.3

$$g(z) = \operatorname{Log}\sqrt{2} + \frac{3\pi i}{4} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(z+1-i)^n}{n(-1+i)^n}$$

actually defines a function that is analytic in the disk in $D = \Delta(-1+i, \sqrt{2})$. (See Figure 2.) What function could it be? To answer this question we consider the function L given by

$$L(z) = \begin{cases} \operatorname{Log} z & \text{if } z \in D \text{ and } \operatorname{Im} z \geq 0 \,, \\ \operatorname{Log} z + 2\pi i & \text{if } z \in D \text{ and } \operatorname{Im} z < 0 \,. \end{cases}$$

Then L is continuous and satisfies $e^{L(z)} = z$ for z in D — i.e., L is a branch of the logarithm in D. In particular, L is an analytic function. As such, L admits a Taylor series expansion about the point $-1+i$, an expansion which is definitely valid throughout the disk D. Moreover, since $L(z) = f(z)$ for every z in Δ, we see that $L^{(n)}(-1+i) = f^{(n)}(-1+i)$ for $n = 0, 1, 2, \cdots$, which means that the functions f and L generate exactly the same Taylor series at $-1+i$. It follows that $g = L$. We note especially that, for z in D with $\operatorname{Im} z < 0$, $g(z) = \operatorname{Log} z + 2\pi i \neq f(z)$. In the present example we see exhibited in a concrete situation the behavior alluded to in the comments following Theorem 3.4.

3.4 Laurent Series

Let z_0 again be a point in the complex plane. By a *Laurent series centered at* z_0 is meant a doubly infinite function series of the form

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

$$= \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots ,$$

where for $n = 0, \pm 1, \pm 2, \cdots$ the coefficient a_n is a complex constant. Included among these series, which are named in honor of Pierre Alphonse Laurent (1813-1854), are all the Taylor series centered at z_0, the latter being the Laurent series in which $a_n = 0$ for every negative integer n. In line with our policy for Taylor series we assign to any such Laurent series two non-negative extended real numbers ρ_0 and ρ_I, its *outer* and *inner radii of convergence,* via the formulas

(7.9) $$\rho_0 = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)^{-1} , \quad \rho_I = \limsup_{n\to\infty} \sqrt[n]{|a_{-n}|} .$$

When $\rho_I < \rho_0$, $D = \{z : \rho_I < |z - z_0| < \rho_0\}$ is called the *ring* (or *annulus*) *of convergence* of the given series. In the extreme case where $\rho_I = 0$ and $\rho_0 = \infty$ this set is just the punctured plane $\mathbb{C} \sim \{z_0\}$. Figure 3 indicates the other possibilities for D. The terminology is suggested by a Laurent series companion to Theorem 3.3.

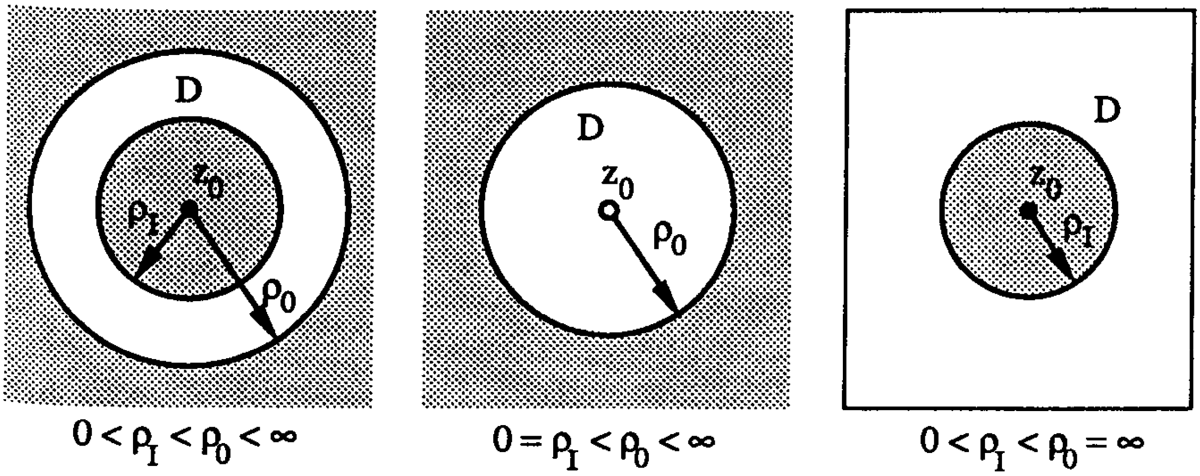

Figure 3.

Theorem 3.5. *Suppose that ρ_O and ρ_I are the outer and inner radii of convergence of a Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ centered at z_0. The series diverges for any z satisfying $|z-z_0|>\rho_O$ or $|z-z_0|<\rho_I$. If $\rho_O>0$, the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely and normally in the disk $D_O=\Delta(z_0,\rho_O)$, so $f_O(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ defines a function f_O that is analytic in D_O. If $\rho_I<\infty$, the series $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges absolutely and normally in the open set $D_I=\{z:|z-z_0|>\rho_I\}$, so $f_I(z)=\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ defines a function f_I that is analytic in D_I. If $\rho_I<\rho_O$, the full Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ converges absolutely and normally in the set $D=\{z:\rho_I<|z-z_0|<\rho_O\}$, so the function f defined by $f(z)=\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n=f_I(z)+f_O(z)$ is analytic in D. The coefficient a_n is then related to f through the formula*

$$a_n=\frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f(z)\,dz}{(z-z_0)^{n+1}} \tag{7.10}$$

for any number r in the interval (ρ_I,ρ_O).

Proof. First, the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is nothing but a Taylor series with radius of convergence ρ_O. It thus diverges for any z satisfying $|z-z_0|>\rho_O$. Under the condition that $\rho_O>0$ it converges absolutely and normally in $D_O=\Delta(z_0,\rho_O)$, and the function f_O given by $f_O(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is analytic in D_O (Theorem 3.3). We next consider $\sum_{n=1}^{\infty} a_{-n}\zeta^n$, a Taylor series in the variable ζ whose radius of convergence is clearly $1/\rho_I$. This series diverges for any ζ with $|\zeta|>1/\rho_I$. Also, assuming that $\rho_I<\infty$, it converges absolutely and normally in the disk $\Delta=\Delta(0,1/\rho_I)$. If we make the substitution $\zeta=(z-z_0)^{-1}$ in $\sum_{n=1}^{\infty} a_{-n}\zeta^n$, we obtain the series $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$. This process plainly results in a divergent series when $|z-z_0|<\rho_I$ and an absolutely convergent one when $|z-z_0|>\rho_I$. Suppose now that $\rho_I<\infty$. The function $g(z)=(z-z_0)^{-1}$ is continuous and maps the set $D_I=\{z:|z-z_0|>\rho_I\}$ into Δ. In particular, g carries any compact set in D_I to a compact set in Δ. Given a compact set K in D_I and given $\epsilon>0$, we can exploit the uniform convergence of $\sum_{n=1}^{\infty} a_{-n}\zeta^n$ on the compact set $g(K)$ to choose an index N such that the inequality $\left|\sum_{k=n}^{m} a_{-k}\zeta^k\right|<\epsilon$ is satisfied for every ζ in $g(K)$, provided $m\ge n\ge N$. It follows immediately that $\left|\sum_{k=n}^{m} a_{-k}(z-z_0)^{-k}\right|<\epsilon$ holds for every z in K whenever $m\ge n\ge N$. The Cauchy criterion for uniform convergence then attests to the uniform convergence of $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ on K. We conclude that this series is normally convergent in D_I. By Theorem 3.2 the function f_I with rule of correspondence $f_I(z)=\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ is analytic in D_I.

We turn to the doubly infinite series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ and draw some conclusions about it. In light of Lemma 2.1 this series diverges when z satisfies either $|z-z_0|>\rho_O$ or $|z-z_0|<\rho_I$. If $\rho_I<\rho_O$ — which implies that $\rho_O>0$ and $\rho_I<\infty$ — both the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and its mate $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converge absolutely and normally in the set $D = D_0 \cap D_I = \{z : \rho_I < |z-z_0| < \rho_0\}$, which fact insures the absolute and normal convergence of $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ in D. (Lemma 2.3 comes into play here.) The function f defined by $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is analytic in D. Indeed, $f(z) = f_I(z) + f_0(z)$ for z in D.

Finally, let r satisfy $\rho_I < r < \rho_0$. If k is an arbitrary integer, we can write

$$\frac{f(z)}{(z-z_0)^{k+1}} = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^{n-k-1}$$

for every z belonging to D, a set in which the convergence of the series on the right is normal. (See Exercise 5.21.) Invoking the analogue of (7.5) for doubly infinite series to justify an interchange of summation and integration, we discover that

$$\frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)\,dz}{(z-z_0)^{k+1}} = \frac{1}{2\pi i}\int_{|z-z_0|=r} \left\{\sum_{n=-\infty}^{\infty} a_n(z-z_0)^{n-k-1}\right\} dz$$

$$= \sum_{n=-\infty}^{\infty} \frac{a_n}{2\pi i}\int_{|z-z_0|=r} (z-z_0)^{n-k-1}\,dz = a_k \ ,$$

because

$$\int_{|z-z_0|=r} (z-z_0)^{n-k-1}\,dz = \begin{cases} 2\pi i & \text{if } n = k \ , \\ 0 & \text{if } n \neq k \ . \end{cases}$$

This verifies (7.10). ∎

Theorem 3.4, too, has its dual in the setting of Laurent series. Theorem 3.6 will provide us with a jumping-off point for our discussion of isolated singularities of analytic functions, the topic of the next chapter.

Theorem 3.6. *Suppose that a function f is analytic in an annulus $D = \{z : a < |z-z_0| < b\}$, where $0 \le a < b \le \infty$. Then f can be represented in D as the sum of a Laurent series centered at z_0. This expansion is uniquely determined by f: if $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ in D, then the coefficient a_n is given by*

$$a_n = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

for any number r satisfying $a < r < b$.

Proof. For each integer n we define a complex number a_n in the manner prescribed by the statement of the theorem: we select r in the interval (a, b) and set $a_n = (2\pi i)^{-1}\int_{|z-z_0|=r}(z-z_0)^{-n-1}f(z)dz$. The specific choice of r is of no consequence, for the number a_n is independent of that choice. Indeed, since the paths γ and β defined on $[0, 2\pi]$ by $\gamma(t) = z_0 + re^{it}$ and $\beta(t) = z_0 + se^{it}$ (Figure 4) are homologous in D whenever $a < r < s < b$

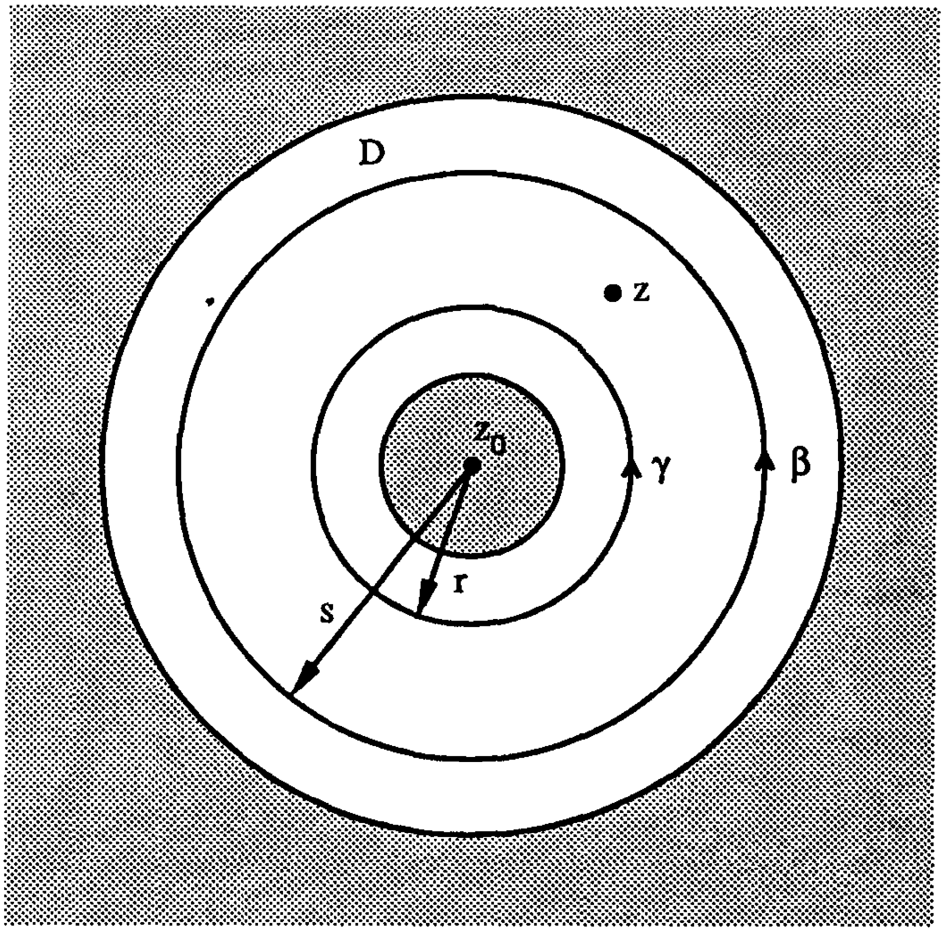

Figure 4.

and since the function g defined by $g(z) = (z - z_0)^{-n-1} f(z)$ is analytic in D, Corollary V.5.2 gives

$$\text{(7.11)} \quad a_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)\,dz}{(z-z_0)^{n+1}} = \frac{1}{2\pi i} \int_{|z-z_0|=s} \frac{f(z)\,dz}{(z-z_0)^{n+1}} \,.$$

With this determination of coefficients a_n we make the assertion that $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ for every z in D.

Fix z in D. Fix, also, numbers r and s obeying $a < r < |z - z_0| < s < b$. Let paths γ and β be defined as they were above. We apply the Cauchy integral formula in D to the function f and the cycle $\sigma = (\beta, -\gamma)$, which is obviously homologous to zero in D. Because $n(\sigma, z) = 1$, this step yields

$$f(z) = \frac{1}{2\pi i} \int_\sigma \frac{f(\zeta)\,d\zeta}{\zeta - z} \,,$$

which in expanded form becomes

$$\text{(7.12)} \qquad f(z) = \frac{1}{2\pi i} \int_{|\zeta-z_0|=s} \frac{f(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{|\zeta-z_0|=r} \frac{f(\zeta)\,d\zeta}{\zeta - z} \,.$$

The argument now proceeds along lines established in the proof of Theorem 3.4. For ζ belonging to the circle $K = K(z_0, s)$ we capitalize on the fact that $|(z-z_0)/(\zeta-z_0)| = |z-z_0|/s < 1$ to write

$$\frac{f(\zeta)}{\zeta-z} = \frac{f(\zeta)}{\zeta-z_0}\,\frac{1}{1-\left(\dfrac{z-z_0}{\zeta-z_0}\right)} = \sum_{n=0}^{\infty}\frac{f(\zeta)(z-z_0)^n}{(\zeta-z_0)^{n+1}}\ .$$

We check, as we did in the proof of Theorem 3.4, that convergence is uniform on K and then integrate the series termwise to get

$$\frac{1}{2\pi i}\int_{|\zeta-z_0|=s}\frac{f(\zeta)\,d\zeta}{\zeta-z} = \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{|\zeta-z_0|=s}\frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}}\right](z-z_0)^n\ .$$

In accordance with (7.11) this reduces to

$$\text{(7.13)}\qquad \frac{1}{2\pi i}\int_{|\zeta-z_0|=s}\frac{f(\zeta)\,d\zeta}{\zeta-z} = \sum_{n=0}^{\infty}a_n(z-z_0)^n\ .$$

In a like manner, since $|(\zeta-z_0)/(z-z_0)| = r/|z-z_0| < 1$ for ζ lying on $K^* = K(z_0,r)$, we obtain for such ζ the expansion

$$\frac{f(\zeta)}{\zeta-z} = -\frac{f(\zeta)}{z-z_0}\,\frac{1}{1-\left(\dfrac{\zeta-z_0}{z-z_0}\right)} = -\sum_{n=0}^{\infty}\frac{f(\zeta)(\zeta-z_0)^n}{(z-z_0)^{n+1}}$$

$$= -\sum_{n=1}^{\infty}\frac{f(\zeta)(\zeta-z_0)^{n-1}}{(z-z_0)^n}\ .$$

Convergence here is uniform on K^*. In this instance, integration leads to

$$\frac{1}{2\pi i}\int_{|\zeta-z_0|=r}\frac{f(\zeta)\,d\zeta}{\zeta-z} = -\sum_{n=1}^{\infty}\left[\frac{1}{2\pi i}\int_{|\zeta-z_0|=r}\frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{-n+1}}\right](z-z_0)^{-n}$$

and so by way of (7.11) to

$$\text{(7.14)}\qquad \frac{1}{2\pi i}\int_{|\zeta-z_0|=r}\frac{f(\zeta)\,d\zeta}{\zeta-z} = -\sum_{n=1}^{\infty}a_{-n}(z-z_0)^{-n}\ .$$

The effect of (7.13), (7.14), and (7.12) is to confirm that both of the series $\sum_{n=0}^{\infty}a_n(z-z_0)^n$ and $\sum_{n=1}^{\infty}a_{-n}(z-z_0)^{-n}$ are convergent and that

$$f(z) = \sum_{n=1}^{\infty}a_{-n}(z-z_0)^{-n} + \sum_{n=0}^{\infty}a_n(z-z_0)^n\ .$$

Owing to Lemma 2.1 we can conclude that $f(z) = \sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$ — this for arbitrary z in D.

Suppose that $f(z) = \sum_{n=-\infty}^{\infty} b_n(z-z_0)^n$ were a competing Laurent series expansion of f in D. The convergence of this series in D would insure that its radii of convergence ρ_0 and ρ_I satisfy $\rho_I \le a$ and $\rho_0 \ge b$. If r belongs to the interval (a,b), then Theorem 3.5 and (7.11) would imply that

$$b_n = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)\,dz}{(z-z_0)^{n+1}} = a_n\ ,$$

so the presumed alternative expansion would coincide with our original one. The uniqueness assertion in the theorem is the consequence. ∎

Although the formula $a_n = (2\pi i)^{-1}\int_{|z-z_0|=r}(z-z_0)^{-n-1}f(z)dz$ for the n^{th} Laurent coefficient of a function is of theoretical value, it is safe to say that this formula is all but useless when it ultimately gets down to producing the Laurent series representations of functions in concrete situations. As a matter of fact the integral involved here is, more often than not, sufficiently horrendous in its own right to dampen any enthusiasm for using the formula. Several of the more practical techniques employed to determine Laurent series expansions are indicated in the succeeding examples.

EXAMPLE 3.12. Expand the function $f(z) = e^{1/z}$ in a Laurent series in the ring $D = \{z : 0 < |z| < \infty\}$.

From Example 3.5 we recall that the Taylor series development

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$$

is valid for all complex numbers z. Assuming that $0 < |z| < \infty$, therefore, we are permitted to substitute $1/z$ for z in this series to obtain the expansion

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots\ .$$

When expressed in standard Laurent series form this turns into

$$e^{1/z} = \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!}\ .$$

EXAMPLE 3.13. Determine the Laurent series representation of $f(z) = (z-1)^{-3}\sin \pi z$ in the ring $D = \{z : 0 < |z-1| < \infty\}$.

We start by finding the Taylor series expansion of $g(z) = \sin \pi z$ about $z_0 = 1$. Computing derivatives we learn that $g^{(n)}(1) = 0$ for even n and $g^{(2n+1)}(1) = (-1)^{n+1}\pi^{2n+1}$ for $n = 0, 1, 2, \cdots$. We conclude that

$$\sin \pi z = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\pi^{2n+1}(z-1)^{2n+1}}{(2n+1)!}$$

$$= -\pi(z-1) + \frac{\pi^3(z-1)^3}{3!} - \frac{\pi^5(z-1)^5}{5!} + \cdots ,$$

an expansion which applies everywhere in $\mathbb{C}$. Division by $(z-1)^3$ gives

$$\frac{\sin \pi z}{(z-1)^3} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\pi^{2n+1}(z-1)^{2n-2}}{(2n+1)!}$$

$$= -\frac{\pi}{(z-1)^2} + \frac{\pi^3}{3} - \frac{\pi^5(z-1)^2}{5!} + \cdots ,$$

as the Laurent series that represents f in D.

EXAMPLE 3.14. Develop the functions $f(z) = z^{-1}$ and $g(z) = z^{-2}$ in Laurent series in the annulus $D = \{z : 1 < |z-i| < \infty\}$.

The secret here is to make shrewd use of the geometric series. The initial temptation might be to write

$$\frac{1}{z} = \frac{1}{i + (z-i)} = \frac{1}{i}\,\frac{1}{1 - i(z-i)} .$$

The last expression does appear primed for application of the geometric series. Unfortunately, however, $|i(z-i)| > 1$ when z belongs to D. A little algebraic reorganization is sufficient to remedy the situation; namely, we manipulate the above expression into

$$\frac{1}{z} = \frac{1}{z-i}\,\frac{1}{1 - \dfrac{1}{i(z-i)}} .$$

For z in D the geometric series can then be utilized to arrive at

$$\frac{1}{z} = \frac{1}{z-i}\sum_{n=0}^{\infty} \frac{1}{i^n(z-i)^n} = \frac{1}{z-i}\sum_{n=-\infty}^{0} i^n(z-i)^n$$

$$= \sum_{n=-\infty}^{0} i^n(z-i)^{n-1} = \sum_{n=-\infty}^{-1} i^{n+1}(z-i)^n ;$$

i.e., the desired Laurent expansion of f in D takes the form

$$\frac{1}{z} = \sum_{n=-\infty}^{-1} i^{n+1}(z-i)^n = \frac{1}{z-i} - \frac{i}{(z-i)^2} - \frac{1}{(z-i)^3} + \cdots .$$

Concerning the function g, we note that $g(z) = -f'(z)$. To get the Laurent series for f' in D we have only to differentiate termwise the Laurent series obtained for f there. This produces the representation

$$-\frac{1}{z^2} = \sum_{n=-\infty}^{-1} n i^{n+1}(z-i)^{n-1} = \sum_{n=-\infty}^{-2} (n+1) i^{n+2}(z-i)^n$$

$$= -\sum_{n=-\infty}^{-2} (n+1) i^n (z-i)^n$$

for z in D. Accordingly,

$$\frac{1}{z^2} = \sum_{n=-\infty}^{-2} (n+1) i^n (z-i)^n = \frac{1}{(z-i)^2} - \frac{2i}{(z-i)^3} - \frac{3}{(z-i)^4} + \cdots$$

delivers the Laurent expansion of g in D.

EXAMPLE 3.15. Find the Laurent series expansion of the function $f(z) = (2 - 3z + z^2)^{-1}$ in each of the annuli $D = \{z : 1 < |z| < 2\}$ and $D^* = \{z : \sqrt{2} < |z+i| < \sqrt{5}\}$ (Figure 5).

We first express f in its partial fraction decomposition,

$$f(z) = \frac{1}{2 - 3z + z^2} = \frac{1}{1-z} - \frac{1}{2-z} .$$

For z lying in D we appeal to the geometric series in computing

$$\frac{1}{1-z} = -\frac{1}{z}\,\frac{1}{1-(1/z)} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} = -\sum_{n=-\infty}^{-1} z^n$$

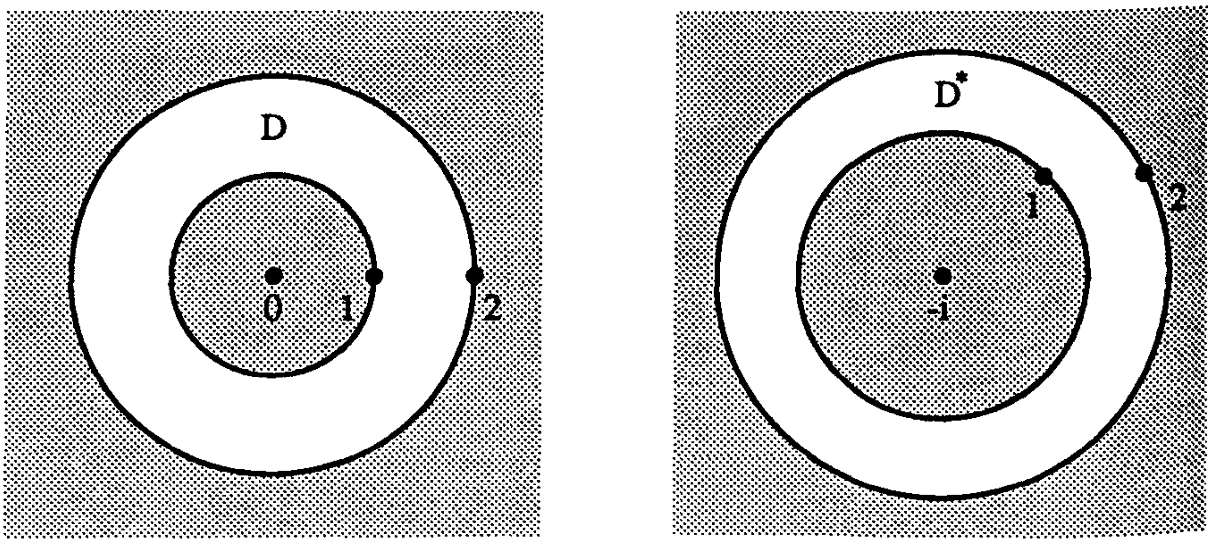

Figure 5.

and

$$\frac{1}{2-z}=\frac{1}{2}\,\frac{1}{1-(z/2)}=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n=\sum_{n=0}^{\infty}2^{-n-1}z^n .$$

Since $|1/z|<1$ and $|z/2|<1$ when z belongs to D, the use of the geometric series is legitimate in each instance. Consequently,

$$\frac{1}{2-3z+z^2}=-\sum_{n=-\infty}^{-1}z^n-\sum_{n=0}^{\infty}2^{-n-1}z^n$$

$$=\cdots-\frac{1}{z^2}-\frac{1}{z}-\frac{1}{2}-\frac{z}{4}-\frac{z^2}{8}-\cdots$$

for every point z of D.

When z is in D^* the corresponding calculations run as follows:

$$\frac{1}{1-z}=\frac{1}{(1+i)-(z+i)}=-\frac{1}{z+i}\,\frac{1}{1-[(1+i)/(z+i)]}$$

$$=-\frac{1}{z+i}\sum_{n=0}^{\infty}\left(\frac{1+i}{z+i}\right)^n=-\frac{1}{z+i}\sum_{n=-\infty}^{0}\left(\frac{z+i}{1+i}\right)^n$$

$$=-\sum_{n=-\infty}^{0}(1+i)^{-n}(z+i)^{n-1}=-\sum_{n=-\infty}^{-1}(1+i)^{-n-1}(z+i)^n ,$$

$$\frac{1}{2-z}=\frac{1}{(2+i)-(z+i)}=\frac{1}{2+i}\,\frac{1}{1-[(z+i)/(2+i)]}$$

$$=\frac{1}{2+i}\sum_{n=0}^{\infty}\left(\frac{z+i}{2+i}\right)^n=\sum_{n=0}^{\infty}(2+i)^{-n-1}(z+i)^n .$$

Here the fact that $|(1+i)/(z+i)|=\sqrt{2}/|z+i|<1$ and $|(z+i)/(2+i)|=|z+i|/\sqrt{5}<1$ for z belonging to D^* justifies the applications made of the geometric series. The Laurent series development of f in D^* thus reads

$$\frac{1}{2-3z+z^2}=-\sum_{n=-\infty}^{-1}(1+i)^{-n-1}(z+i)^n-\sum_{n=0}^{\infty}(2+i)^{-n-1}(z+i)^n$$

$$=\cdots-\frac{1+i}{(z+i)^2}-\frac{1}{z+i}-\frac{1}{2+i}-\frac{(z+i)}{(2+i)^2}-\frac{(z+i)^2}{(2+i)^3}-\cdots .$$

The combined methods of the last two examples — i.e., partial fraction decomposition, use of the geometric series, and termwise differentiation — can, in principle at least, be adapted to determine the Laurent series development of an arbitrary rational function in any ring where it is analytic.

4 Normal Families

4.1 Normal Subfamilies of $C(U)$

In Chapter II we found out that a bounded sequence of complex numbers $\langle z_n \rangle$ always possesses at least one accumulation point, which fact implies that $\langle z_n \rangle$ has convergent subsequences. In this section we consider the analogous situation for sequences of functions. Specifically, given a sequence of functions $\langle f_n \rangle$ each of whose terms is analytic in an open set U, we explore the possibility of extracting from $\langle f_n \rangle$ a subsequence $\langle f_{n_k} \rangle$ that converges normally in U. The principal result of the section, Montel's theorem, asserts that, if the sequence $\langle f_n \rangle$ is "bounded" in a sense soon to be explained, then the existence of such a subsequence is assured. Thus, the situation for sequences of analytic functions exactly parallels that for sequences of complex numbers. The material presented here will resurface just once later in this book — namely, in the proof of the "Riemann Mapping Theorem" (Theorem IX.3.4). Readers prepared to gloss over the single step in that proof where Montel's theorem gets invoked should not feel hesitant about proceeding directly to Chapter VIII.

In order to render less cumbersome the statements of propositions in this section, we adopt some convenient terminology. Recall that the notation $C(U)$ signifies the collection of all complex-valued functions which are continuous in an open subset U of the complex plane. We define a subfamily $\mathcal{F}$ of $C(U)$ to be *normal in U* — *pre-compact in U* is a different name for the same concept — provided each sequence $\langle f_n \rangle$ from $\mathcal{F}$ has at least one subsequence $\langle f_{n_k} \rangle$ that converges normally in U. (N.B. The reader is advised that this interpretation of "normal family" does not totally jibe with the general usage of the term in the literature of complex analysis, where allowances are usually made for sequences that "tend to infinity uniformly on compact sets in U." See, for example, Theorem 4.6. It would be wise for the reader to keep the discrepancy in mind when consulting other references.) We stress: the normality of $\mathcal{F}$ does not require that every function obtained as the limit of a normally convergent sequence from $\mathcal{F}$ be itself a member of $\mathcal{F}$. A normal subfamily of $C(U)$ endowed with this extra property is called a *compact subfamily* of $C(U)$, a definition that accords nicely with the concept of a compact subset of $\mathbb{C}$. Our ultimate objective in this section is the derivation of a reasonably simple criterion for detecting normality in those subfamilies of $C(U)$ whose members are analytic in U. Along the way, however, we shall also obtain a useful characterization of an arbitrary normal subfamily of $C(U)$.

4.2 Equicontinuity

How does one recognize that a given subfamily $\mathcal{F}$ of $C(U)$ is normal in U? An important element in the eventual answer to this question is the idea of "equicontinuity." The family $\mathcal{F}$ is said to be *equicontinuous at a point* z_0 *of* U if corresponding to each $\epsilon > 0$ there exists a number $\delta > 0$ with the property that $|f(z) - f(z_0)| < \epsilon$ holds for every f in $\mathcal{F}$ whenever $|z - z_0| < \delta$. Crucial here is that δ does not depend on the function f (although it will typically depend on the point z_0): the same δ must work for all members of $\mathcal{F}$. When the family $\mathcal{F}$ is equicontinuous at every point of U, we pronounce it *equicontinuous in* U. (In slightly less precise language we sometimes just call $\mathcal{F}$ an *equicontinuous subfamily of* $C(U)$.) For example, the family $\mathcal{F}$ that consists of all functions of the form $f(z) = 2z + c$, where c is an arbitrary complex constant, is an equicontinuous subfamily of the continuous functions on $\mathbb{C}$. In fact, since $|f(z) - f(z_0)| = 2|z - z_0|$ for all f in this family and for all complex numbers z and z_0, the choice $\delta = \epsilon/2$ corresponding to a given $\epsilon > 0$ meets the stated requirement for any z_0. This is more than the definition of equicontinuity for $\mathcal{F}$ in $\mathbb{C}$ strictly demands — δ would, in general, be permitted to vary from one point z_0 to another.

For present purposes the most significant feature of equicontinuity is that it bridges the gap between pointwise convergence and normal convergence.

Theorem 4.1. *Let $\langle f_n \rangle$ be a sequence from an equicontinuous subfamily $\mathcal{F}$ of $C(U)$. Suppose that this sequence converges pointwise in U. Then it converges normally in U.*

Proof. Denote by f the pointwise limit of $\langle f_n \rangle$ in U. Let K be an arbitrary compact set in U. We must demonstrate that $f_n \to f$ uniformly on K. On the basis of Theorem 1.2, we need only establish that $\langle f_n \rangle$ is a uniform Cauchy sequence on K. We do this indirectly; i.e., we assume the contrary and argue to a contradiction. Under the assumption that $\langle f_n \rangle$ fails to be a uniform Cauchy sequence on K, there must be a number $\epsilon > 0$ (we choose one and keep it fixed for the duration of the proof) about which the following statement is true: there is no integer N with the property that the inequality $|f_m(z) - f_n(z)| < \epsilon$ is valid for every z in K and every pair of indices m and n satisfying $m > n \geq N$. In particular, for any positive integer k the choice $N = k$ does not do the job, meaning that there have to be indices n_k and m_k with $m_k > n_k \geq k$ for which the inequality $|f_{m_k}(z) - f_{n_k}(z)| < \epsilon$ breaks down somewhere in K. Accordingly, corresponding to each positive integer k we can assert the existence of integers n_k and m_k with $m_k > n_k \geq k$ and a point z_k of K such that

$$|f_{m_k}(z_k) - f_{n_k}(z_k)| \geq \epsilon\,. \tag{7.15}$$

This reasoning gives rise to a sequence $\langle z_k \rangle$ in K. As K is compact, $\langle z_k \rangle$ has at least one accumulaton point in K — say that z_0 is such a point. Utilizing the equicontinuity of the family $\mathcal{F}$ at z_0, we select $\delta > 0$ in a manner which insures that, for any z obeying the condition $|z - z_0| < \delta$, the inequality

$$|f_n(z) - f_n(z_0)| < \frac{\epsilon}{3} \tag{7.16}$$

holds for every n. By construction $n_k \to \infty$ and $m_k \to \infty$ as $k \to \infty$, from which it follows that

$$|f_{m_k}(z_0) - f_{n_k}(z_0)| \to |f(z_0) - f(z_0)| = 0$$

as $k \to \infty$. This allows us to fix an index k_0 with the property that

$$|f_{m_k}(z_0) - f_{n_k}(z_0)| < \frac{\epsilon}{3} \tag{7.17}$$

once $k \geq k_0$. The fact that $\langle z_k \rangle$ accumulates at z_0 permits us to pick an index $k \geq k_0$ for which $|z_k - z_0| < \delta$. The triangle inequality, in combination with (7.15), (7.16), and (7.17), leads for this selection of k to

$$\begin{aligned}
\epsilon &\leq |f_{m_k}(z_k) - f_{n_k}(z_k)| \\
&\leq |f_{m_k}(z_k) - f_{m_k}(z_0)| + |f_{m_k}(z_0) - f_{n_k}(z_0)| + |f_{n_k}(z_0) - f_{n_k}(z_k)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \ ,
\end{aligned}$$

the contradiction we were seeking. Our conclusion: $\langle f_n \rangle$ is, in the end, a uniform Cauchy sequence on K — hence, $f_n \to f$ uniformly on K. ∎

In actuality it is a mildly stronger version of Theorem 4.1 that we require for the application toward which we are heading. Let U be a nonempty open set in $\mathbb{C}$. We say that a subset S of U is *dense in U* if U is contained in the closure of S or, to put it differently, if $S \cap \Delta(z, r) \neq \phi$ for every z in U and every $r > 0$. An example of such a set (the only example we shall really need) is $S_0 = \{z \in U : \operatorname{Re} z \text{ and } \operatorname{Im} z \text{ are rational}\}$. This particular dense subset of U has another valuable property: its elements can be arranged as the terms of a sequence! To see this, first recall that the set of all rational numbers can be listed in a sequence. Figure 6 suggests a way in which to create such a listing, albeit one with duplicatons. By passing to a subsequence one can exhibit the rationals as the terms of a sequence $\langle q_n \rangle$ in which $q_n \neq q_m$ for $n \neq m$. If we now construct a sequence $\langle w_n \rangle$ by continuing the pattern established in

$$w_1 = q_1 + iq_1 \ ,$$

$$w_2 = q_1 + iq_2 \quad , \quad w_3 = q_2 + iq_1 \ ,$$

$$w_4 = q_1 + iq_3 \quad , \quad w_5 = q_2 + iq_2 \quad , \quad w_6 = q_3 + iq_1 \ , \cdots$$

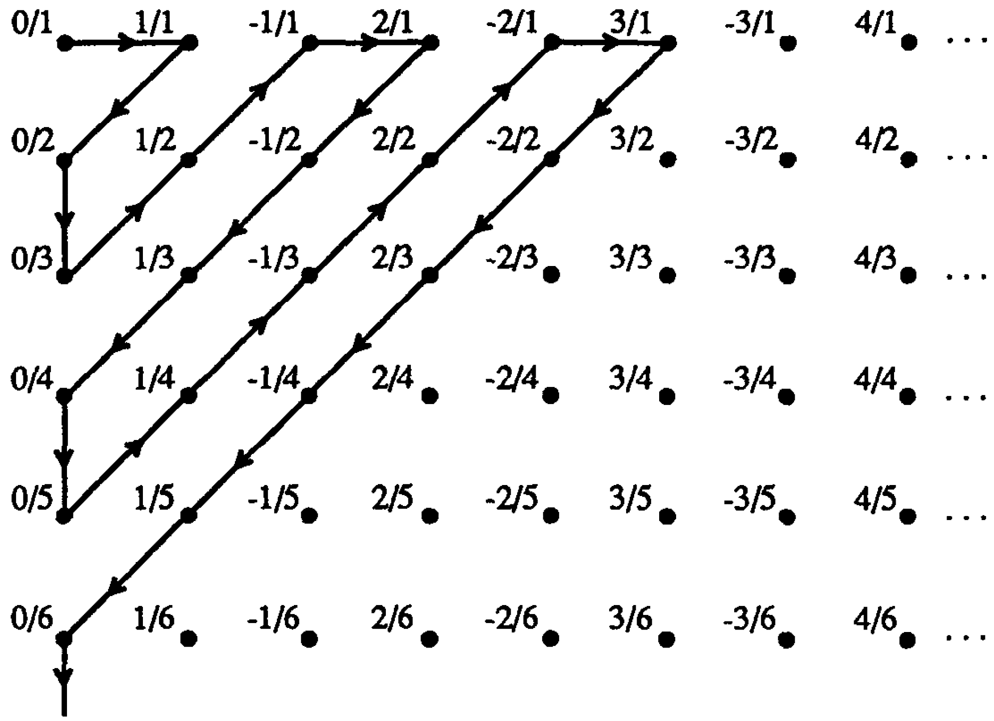

Figure 6.

we obtain a sequential listing of all complex numbers with rational real and imaginary parts. Also, $w_n \neq w_m$ for $n \neq m$. Finally, if we strike from this sequence all terms not belonging to the set U, what remains is a subsequence of $\langle w_n \rangle$ whose terms run through all the elements of S_0.

The refinement of Theorem 4.1 we desire is:

Lemma 4.2. *Let $\langle f_n \rangle$ be a sequence from an equicontinuous subfamily $\mathcal{F}$ of $C(U)$. Suppose that the sequence $\langle f_n(\zeta) \rangle$ is convergent for every ζ belonging to some dense subset S of U. Then $\langle f_n \rangle$ converges normally in U.*

Proof. Owing to Theorem 4.1, it suffices to prove that $\langle f_n \rangle$ converges pointwise in U. To achieve this, we have only to verify that for each element z of U the sequence $\langle f_n(z) \rangle$ is a Cauchy sequence. Fix such a point z, and let $\epsilon > 0$ be given. The challenge is to produce an integer N so that $|f_m(z) - f_n(z)| < \epsilon$ is satisfied whenever $m > n \geq N$. We first make use of the equicontinuity of $\mathcal{F}$ at z to pick a $\delta > 0$ with the property that $|f_n(w) - f_n(z)| < \epsilon/3$ for all indices n, provided only that $|w - z| < \delta$. Next, the fact that the set S is dense in U enables us to choose a point ζ of S satisfying $|\zeta - z| < \delta$. By hypothesis the sequence $\langle f_n(\zeta) \rangle$ is convergent. As a result, this sequence is a Cauchy sequence. There is, consequently, an integer N such that $|f_m(\zeta) - f_n(\zeta)| < \epsilon/3$ is true whenever $m > n \geq N$. We conclude that

$$\begin{aligned} |f_m(z) - f_n(z)| &\leq |f_m(z) - f_m(\zeta)| + |f_m(\zeta) - f_n(\zeta)| + |f_n(\zeta) - f_n(z)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon , \end{aligned}$$

as long as $m > n \geq N$. Thus $\langle f_n(z) \rangle$ is a Cauchy sequence. Since this is the case for every z in U, the pointwise convergence of $\langle f_n \rangle$ in U is established. The quoting of Theorem 4.1 finishes the proof. ∎

4.3 The Arzelà-Ascoli and Montel Theorems

We still lack the final ingredient present in most elementary normal family criteria. It is a fitting notion of boundedness for a subfamily $\mathcal{F}$ of $C(U)$. As far as matters in this book are concerned, there are two relevant concepts of boundedness for such a family. First, we say that $\mathcal{F}$ is *pointwise bounded in U* if for each fixed z in U the set of values $\{f(z) : f \in \mathcal{F}\}$ is a bounded set of complex numbers. Secondly, we speak of the family $\mathcal{F}$ as *locally bounded in U* if its members are uniformly bounded on each compact set in U, which means that there exists for each compact subset K of U a constant $m = m(K)$ with the property that $|f(z)| \leq m$ for every point z in K and every function f in $\mathcal{F}$. Following the precedent set by normal convergence, to verify the local boundedness of $\mathcal{F}$ in U it is enough to check that its members are uniformly bounded on each closed disk in U (or, even less, that each point z of U is the center of some closed disk on which the functions in $\mathcal{F}$ are uniformly bounded). The reason for this is that an arbitrary compact set in U can be covered by a finite number of such disks. (Recall the proof of Lemma 1.3.) If $\mathcal{F}$ is locally bounded in U, then it is plainly pointwise bounded there. The converse is, in general, false.

Our first "normal family theorem" characterizes the normal subfamilies of $C(U)$. The names attached to it are those of Cesare Arzelà (1847-1912) and Giulio Ascoli (1843-1896), who share credit for its discovery. Their theorem and its generalizations have applications in many areas of mathematics.

Theorem 4.3. (Arzelà-Ascoli Theorem) *A subfamily $\mathcal{F}$ of $C(U)$ is normal in U if and only if it is both equicontinuous and pointwise bounded in this open set.*

Proof. Assume initially that $\mathcal{F}$ is both equicontinuous and pointwise bounded in U. Let $\langle f_n \rangle$ be a sequence from $\mathcal{F}$. It is our job to demonstrate the existence of a normally convergent subsequence of $\langle f_n \rangle$. For this we recall that the set $S_0 = \{z \in U : \operatorname{Re} z \text{ and } \operatorname{Im} z \text{ are rational}\}$ is dense in U and that it is possible to list the elements of S_0 in a sequence. Let $\langle z_n \rangle$ be such a listing. We begin a construction by considering the sequence of complex numbers $\langle f_n(z_1) \rangle$. Owing to the pointwise boundedness of $\mathcal{F}$ this sequence is bounded. In light of the Bolzano-Weierstrass theorem it has at least one accumulation point in $\mathbb{C}$. We choose such a point and label it w_1. Then $\langle f_n(z_1) \rangle$ has a subsequence converging to w_1. In other words, it is possible to select a sequence of indices $m_1^{(1)} < m_2^{(1)} < m_3^{(1)} \cdots$ such that

$$\lim_{k\to\infty} f_{m_k^{(1)}}(z_1) = w_1 \ .$$

The superscript in $m_k^{(1)}$ is there to serve notice that this sequence of integers is associated with the point z_1 of S_0. In the interest of preserving some notational sanity we shall abbreviate $f_{m_k^{(1)}}$ to $f_{1,k}$ — and do likewise in similar situations throughout the course of this argument. Now $\langle f_{1,k}(z_2)\rangle_{k=1}^{\infty}$ is another bounded sequence of complex numbers. We can, therefore, single out one of its accumulation points — call it w_2 — and extract a subsequence $m_1^{(2)} < m_2^{(2)} < m_3^{(2)} < \cdots$ from $\langle m_k^{(1)}\rangle$ for which

$$\lim_{k\to\infty} f_{2,k}(z_2) = w_2 \ .$$

Continuing this process inductively we construct corresponding to each positive integer ℓ a complex number w_ℓ and a strictly increasing sequence of positive integers $\langle m_k^{(\ell)}\rangle$ such that

$$\lim_{k\to\infty} f_{\ell,k}(z_\ell) = w_\ell$$

and such that $\langle m_k^{(\ell+1)}\rangle$ is a subsequence of $\left\langle m_k^{(\ell)}\right\rangle$. For $k \geq 1$ we set $n_k = m_k^{(k)}$. By construction $n_1 < n_2 < n_3 < \cdots$. As a result, $\langle f_{n_k}\rangle$ is a legitimate subsequence of $\langle f_n\rangle$. Moreover, for fixed $\ell \geq 1$ the sequence $\langle f_{n_k}\rangle$ is, with the possible exception of its first $\ell - 1$ terms, also a subsequence of $\langle f_{\ell,k}\rangle$. This observation has the consequence that $\lim_{k\to\infty} f_{n_k}(z_\ell) = \lim_{k\to\infty} f_{\ell,k}(z_\ell) = w_\ell$ for each ℓ, which is to say that $\langle f_{n_k}(\zeta)\rangle$ possesses a limit for every point ζ of the set S_0. Lemma 4.2 sees to it that the subsequence $\langle f_{n_k}\rangle$ of $\langle f_n\rangle$ converges normally in U. We have thus established the normality of $\mathcal{F}$ in U.

Turning to the converse, assume that $\mathcal{F}$ is normal in U. Let z_0 be a point of U. If $\mathcal{F}$ fails to be equicontinuous at z_0, then there must exist an $\epsilon > 0$ — we choose such an ϵ and fix it for the rest of the proof — to which there corresponds no $\delta > 0$ fulfilling the condition imposed by the definition of equicontinuity at z_0: the inequality $|f(z) - f(z_0)| < \epsilon$ is supposed to hold for every f in $\mathcal{F}$ and for every z in U satisfying $|z - z_0| < \delta$. In particular, this condition is not met by making the choice $\delta = n^{-1}$, where n is a positive integer. We can therefore select for each n a function f_n in $\mathcal{F}$ and a point z_n of U such that $|z_n - z_0| < n^{-1}$, but such that $|f_n(z_n) - f_n(z_0)| \geq \epsilon$. By hypothesis, the sequence $\langle f_n\rangle$ has a subsequence $\langle f_{n_k}\rangle$ that converges normally in U, say to the limit function f. Then f belongs to $C(U)$ (Theorem 1.4). The continuity of f at z_0 entitles us to choose $\delta > 0$ so that the closed disk $K = \overline{\Delta}(z_0, \delta)$ is contained in U and so that $|f(z) - f(z_0)| < \epsilon/3$ holds whenever z lies in K. Because $f_{n_k} \to f$ uniformly on K and because $z_{n_k} \to z_0$, we can pick an index k with the property that $|f_{n_k}(z) - f(z)| < \epsilon/3$ is satisfied for every z in K and also with the property that z_{n_k} is an element of K. For this choice of k we arrive at

$$\begin{aligned} \epsilon &\le |f_{n_k}(z_{n_k}) - f_{n_k}(z_0)| \\ &\le |f_{n_k}(z_{n_k}) - f(z_{n_k})| + |f(z_{n_k}) - f(z_0)| + |f(z_0) - f_{n_k}(z_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \,. \end{aligned}$$

This contradiction forces us to reject the suggestion that $\mathcal{F}$ is not equicontinuous at the point z_0 and so to conclude that $\mathcal{F}$ must be an equicontinuous subfamily of $C(U)$.

Finally, if $\mathcal{F}$ were not pointwise bounded in U, there would be a point z_0 of U and a sequence $\langle f_n \rangle$ from $\mathcal{F}$ with the property that $|f_n(z_0)| \to \infty$ as $n \to \infty$. Such a sequence could scarcely have a subsequence that converges normally in U, so its existence would conflict with the assumed normality of $\mathcal{F}$ in U. It follows that a normal subfamily of $C(U)$ is of necessity both equicontinuous and pointwise bounded in U. ∎

The Arzelà-Ascoli theorem makes it clear that the property of normality in a subfamily $\mathcal{F}$ of $C(U)$ is a local one: $\mathcal{F}$ is normal in U if and only if corresponding to each point z of U there is an open disk $\Delta = \Delta(z, r)$ contained in U such that the restrictions of the members of $\mathcal{F}$ to Δ constitute a normal subfamily of $C(\Delta)$.

The statement that a normal subfamily of $C(U)$ is pointwise bounded in U can be strengthened. We express the stronger conclusion in the form of a theorem.

Theorem 4.4. *A normal subfamily $\mathcal{F}$ of $C(U)$ is locally bounded in U.*

Proof. Suppose the assertion to be false; i.e., suppose that there is a compact set K in U on which the members of $\mathcal{F}$ are not uniformly bounded. Then for each positive integer n we are at liberty to choose a function f_n in $\mathcal{F}$ and a point z_n of K for which $|f_n(z_n)| \ge n$. (Otherwise n would be a uniform bound for the members of $\mathcal{F}$ on K.) The normality of $\mathcal{F}$ enables us to extract from the sequence $\langle f_n \rangle$ a subsequence $\langle f_{n_k} \rangle$ that converges normally in U. Call its limit f, a function that belongs to $C(U)$. The continuous function $|f|$ attains a maximum value — let m be that value — on K (Corollary II.4.7). Because $f_{n_k} \to f$ uniformly on K there is an index k_0 with the property that $|f_{n_k}(z) - f(z)| < 1$ holds for all z in K once $k \ge k_0$. When $k \ge k_0$, therefore, we can use the triangle inequality to infer that

$$n_k \le |f_{n_k}(z_{n_k})| \le |f_{n_k}(z_{n_k}) - f(z_{n_k})| + |f(z_{n_k})| \le 1 + m \,.$$

This is a contradiction, for $n_k \to \infty$ as $k \to \infty$. We conclude that $\mathcal{F}$ must be locally bounded in U, as claimed. ∎

We return at last to the sphere of analytic functions and derive from the Arzelà-Ascoli theorem the prototype of normal family theorems in complex analysis. It is due to Paul Montel (1876-1975).

Theorem 4.5. (Montel's Theorem) *Let $\mathcal{F}$ be a family of functions that are analytic in an open set U. Suppose that $\mathcal{F}$ is locally bounded in U. Then $\mathcal{F}$ is a normal family in this set.*

Proof. The family $\mathcal{F}$ is obviously pointwise bounded in U. In view of the Arzelà-Ascoli theorem, to prove that $\mathcal{F}$ is a normal subfamily of $C(U)$ we need only to check that $\mathcal{F}$ is equicontinuous in U. We fix a point z_0 in U and establish the equicontinuity of $\mathcal{F}$ at z_0. For this we choose $r > 0$ with the property that the closed disk $K = \overline{\Delta}(z_0, 2r)$ lies in U. By hypothesis there exists a constant $m = m(K) > 0$ such that $|f(\zeta)| \leq m$ holds whenever f belongs to $\mathcal{F}$ and ζ to K. For z lying in the disk $\Delta = \Delta(z_0, r)$ we call upon Cauchy's integral formula to provide an estimate valid for each member f of $\mathcal{F}$:

$$|f(z) - f(z_0)| = \left|\frac{1}{2\pi i}\int_{|\zeta - z_0|=2r}\frac{f(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{|\zeta - z_0|=2r}\frac{f(\zeta)\,d\zeta}{\zeta - z_0}\right|$$

$$= \frac{|z - z_0|}{2\pi}\left|\int_{|\zeta - z_0|=2r}\frac{f(\zeta)\,d\zeta}{(\zeta - z)(\zeta - z_0)}\right|$$

$$\leq \frac{|z - z_0|}{2\pi}\int_{|\zeta - z_0|=2r}\frac{|f(\zeta)||d\zeta|}{|\zeta - z||\zeta - z_0|} \leq \frac{m|z - z_0|}{r}\ .$$

(N.B. $|\zeta - z_0| = 2r$ implies that $|\zeta - z| \geq r$ for z in Δ.) Given $\epsilon > 0$, we now set $\delta = \min\{r, r\epsilon/m\}$. The above estimate implies that the inequality $|f(z) - f(z_0)| < \epsilon$ is valid for every f from the family $\mathcal{F}$, as long as z satisfies $|z - z_0| < \delta$. This confirms the equicontinuity of $\mathcal{F}$ at z_0, an arbitrary point of U. The normality of $\mathcal{F}$ in U follows. ∎

Here is a simple example in which Montel's theorem is used to certify the normality of a specific family of analytic functions.

EXAMPLE 4.1. Let $c > 0$ be a constant, and let $\mathcal{F}$ consist of all the functions f which are analytic in the disk $D = \Delta(0,1)$ and enjoy the property that $\int_0^{2\pi} |f(re^{i\theta})|d\theta \leq c$ for every r in $(0,1)$. Show that $\mathcal{F}$ is a normal family in D.

Given a compact set K in D, we shall exhibit a constant $m = m(K)$ such that $|f(z)| \leq m$ holds for every z in K whenever f belongs to $\mathcal{F}$. We first fix s in $(0,1)$ such that K lies in $\Delta(0, s)$, and we then fix r in $(s, 1)$. For f in $\mathcal{F}$ and z in K we obtain from Cauchy's integral formula and from the fact that $|\zeta - z| \geq r - s$ when $|\zeta| = r$ the estimate

$$|f(z)| = \left|\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(\zeta)\,d\zeta}{\zeta - z}\right| \leq \frac{1}{2\pi}\int_{|\zeta|=r}\frac{|f(\zeta)||d\zeta|}{|\zeta - z|}$$

$$\leq \frac{1}{2\pi(r-s)}\int_{|\zeta|=r}|f(\zeta)||d\zeta| = \frac{r}{2\pi(r-s)}\int_0^{2\pi}|f(re^{i\theta})|\,d\theta \leq \frac{c}{2\pi(r-s)}\,.$$

Thus, $m = c/[2\pi(r-s)]$ is a uniform bound for the members of $\mathcal{F}$ on K. We have just verified that $\mathcal{F}$ is locally bounded in D. Montel's theorem states that $\mathcal{F}$ is a normal family in D.

It is not automatically the case that a pointwise bounded subfamily of $C(U)$ whose members are analytic in U is normal in this set. Montel, however, discovered a number of far-reaching generalizations of Theorem 4.5 that sometimes permit one to infer normality from pointwise boundedness. We state a typical example of these results as an apt finishing touch to this section. No proof is included, for all existing proofs rely on mathematical machinery that we do not wish to introduce in the present text.

Theorem 4.6. *Let $\mathcal{F}$ be a family of functions that are analytic in a domain D. Assume the existence of distinct complex numbers a and b such that the conditions $f(z) \neq a$ and $f(z) \neq b$ are met for every z in D by every f in $\mathcal{F}$. If $\langle f_n\rangle$ is a sequence from $\mathcal{F}$, then either $|f_n(z)| \to \infty$ as $n \to \infty$ for every z in D or $\langle f_n\rangle$ has a subsequence $\langle f_{n_k}\rangle$ that converges normally in D. In particular, if such a family $\mathcal{F}$ is bounded at some point of D, then it is a normal family in this domain.*

5 Exercises for Chapter VII

5.1 Exercises for Section VII.1

5.1. Let $\langle f_n\rangle$ be the sequence of functions defined on the interval $[0,1]$ as follows: $f_n(t) = 4n^2t$ when $0 \leq t \leq (2n)^{-1}$, $f_n(t) = 4n - 4n^2t$ for t satisfying $(2n)^{-1} \leq t \leq n^{-1}$, and $f_n(t) = 0$ if $n^{-1} \leq t \leq 1$. Check that $f_n \to 0$ in $[0,1]$, but that $\int_0^1 f_n(t)dt \not\to 0$. This example exposes another of the disadvantages of pointwise convergence: under this type of convergence the integral of a limit function is not always the limit of the integrals of the functions that converge to it.

5.2. If the functions in a sequence $\langle f_n\rangle$ are all continuous on a set A and if $f_n \to f$ uniformly on A, then $\int_\gamma f(z)|dz| = \lim_{n\to\infty}\int_\gamma f_n(z)|dz|$ for every piecewise smooth path γ in A. Establish this variation on the final conclusion of Theorem 1.1.

5.3. If $\langle f_n\rangle$ is a sequence of functions that converges uniformly on a set A and if $g\colon B \to \mathbb{C}$ is a function whose range $g(B)$ is contained in A, then the sequence of compositions $\langle f_n \circ g \rangle$ is uniformly convergent on B. Prove this.

5.4. Suppose that a function sequence $\langle f_n \rangle$ converges uniformly on a set A and that S is a set which contains $f_n(A)$ for every n. Given that a function g is uniformly continuous on S, show that the sequence $\langle g \circ f_n \rangle$ converges uniformly on A.

5.5. Let $\langle f_n \rangle$ be a function sequence that is uniformly convergent on a set A, and let $g: A \to \mathbb{C}$ be a bounded function. Demonstrate that the sequence of products $\langle g f_n \rangle$ converges uniformly on A.

5.6. Function sequences $\langle f_n \rangle$ and $\langle g_n \rangle$ are known to converge uniformly on a set A. Show that the sequence $\langle f_n + g_n \rangle$ does the same. With the added hypothesis that $\langle f_n \rangle$ and $\langle g_n \rangle$ are uniformly bounded on A (i.e., there is a constant c such that $|f_n(z)| \leq c$ and $|g_n(z)| \leq c$ hold for every index n and every z in A), prove that the product sequence $\langle f_n g_n \rangle$ also converges uniformly on A. Confirm that the latter is true, in particular, when A is a compact set and all the terms in the given sequences $\langle f_n \rangle$ and $\langle g_n \rangle$ are continuous on A.

5.7. Each function in $\langle f_n \rangle$ is continuous on a compact set K and $f_n \to 0$ in K. Under the assumption that $|f_1(z)| \geq |f_2(z)| \geq |f_3(z)| \geq \cdots$ for every z in K, show that $f_n \to 0$ uniformly on K. (*Hint.* Given $\epsilon > 0$, consider the sequence of sets $\langle K_n \rangle$ — $K_n = \{z \in K : |f_n(z)| \geq \epsilon\}$. Prove that there is an index N with the property that $K_n = \phi$ for every $n \geq N$. Do this by assuming the contrary to be true and deriving a contradiction to Cantor's theorem.)

5.8. Let D be a bounded domain in the complex plane. Suppose that every function in a sequence $\langle f_n \rangle$ is continuous on $\overline{D}$ and analytic in D. Given that this sequence converges uniformly on ∂D, prove that it converges uniformly on D.

5.9. Let $\langle f_n \rangle$ be a sequence of functions that are continuous on the closed disk $\overline{\Delta}(z_0, r)$ and analytic in its interior D. Assume that $\langle f_n \rangle$ converges pointwise on the circle ∂D to a continuous function φ. Assume, additionally, that $\int_{|\zeta - z_0| = r} |f_n(\zeta) - \varphi(\zeta)||d\zeta| \to 0$ as $n \to \infty$. (The latter condition will certainly be met if $f_n \to \varphi$ uniformly on ∂D.) Show that $f_n \to f$ normally in D, where f is the function defined in D by the rule of correspondence $f(z) = (2\pi i)^{-1} \int_{|\zeta - z_0| = r} (\zeta - z)^{-1} \varphi(\zeta) d\zeta$.

5.10. Let the terms of a function sequence $\langle f_n \rangle$ be continuous in an open set U. If $\langle f_n \rangle$ converges normally in U to the limit function f and if $\langle z_n \rangle$ is a sequence in U that converges to a point z_0 of U, then $f_n(z_n) \to f(z_0)$ as $n \to \infty$. Corroborate this assertion.

5.11. Each function in a sequence $\langle u_n \rangle$ is harmonic in an open set U and $u_n \to u$ normally in U. Confirm that the limit function u is harmonic in this set.

5.12. Assume that each function in a sequence $\langle u_n \rangle$ is harmonic in a plane domain D and that $0 \le u_1 \le u_2 \le u_3 \le \cdots$ in D. Then for each fixed z in D one of two things happens: either $\lim_{n\to\infty} u_n(z)$ exists (in the strict sense) or else $u_n(z) \to \infty$ as $n \to \infty$. Let $U = \{z \in D\colon \lim_{n\to\infty} u_n(z) \text{ exists}\}$ and $V = \{z \in D\colon u_n(z) \to \infty\}$. Show that either $D = U$ — in which event $\langle u_n \rangle$ is pointwise convergent in D — or $D = V$. (*Hint.* Let z_0 belong to D and let $\Delta = \Delta(z_0, r)$ be an open disk whose closure lies in D. Use Exercise VI.4.41 to establish the existence of a constant $c > 0$, independent of n, with the property that $c^{-1}u_n(z_0) \le u_n(z) \le cu_n(z_0)$ for every index n and every z in Δ. Conclude from this that Δ lies in either U or V.)

5.13. With the aid of Exercises 5.11 and 5.12 prove the following theorem of Harnack: if $\langle u_n \rangle$ is a non-decreasing sequence of functions that are harmonic in a domain D, then either $\langle u_n \rangle$ converges normally in D to a limit function u that is itself harmonic in D or $u_n(z) \to \infty$ as $n \to \infty$ for every z in D. (*Hint.* We may suppose that $u_n \ge 0$ in D for every n — if not, just consider in place of $\langle u_n \rangle$ the sequence $\langle v_n \rangle$, where $v_n = u_n - u_1$. Thus, we may assume that we find ourselves in the context of Exercise 5.12. If $\langle u_n \rangle$ is known to converge pointwise in D and if $\Delta = \overline{\Delta}(z_0, r)$ is an arbitrary closed disk in D, prove that $\langle u_n \rangle$ converges uniformly on Δ. Do this by applying Exercise VI.4.41 to $u_m - u_n$ for $m > n$ in a disk slightly larger than Δ.)

5.2 Exercises for Section VII.2

5.14. Decide whether the following series converge or diverge; in the case of convergence, indicate whether it is absolute: (i) $\sum_{n=1}^{\infty} n^{-1/2} i^n$; (ii) $\sum_{n=2}^{\infty} n^{-1}[\operatorname{Log}(n+iy)]^{-2}$; (iii) $\sum_{n=0}^{\infty}(n!)^2[(2n)!]^{-1}(3+4i)^n$; (iv) $\sum_{n=-\infty}^{\infty} 2^n \sec(ni)$; (v) $\sum_{n=1}^{\infty}(1+n^{-1})^{-n^2}(1+2i)^n e^{ni}$.

5.15. Let $\langle r_n \rangle$ be a sequence of non-negative real numbers, let $\langle s_n \rangle$ be the sequence of partial sums associated with $\langle r_n \rangle$, and let $c \ge 0$. Show that the series $\sum_{n=1}^{\infty} r_n$ is convergent and satisfies $\sum_{n=1}^{\infty} r_n \le c$ if and only if $s_n \le c$ for every n.

5.16. Let $\langle z_n \rangle$ be a sequence of complex numbers whose associated sequence of partial sums $\langle s_n \rangle$ is bounded — but not necessarily convergent — and let $\langle r_n \rangle$ be a non-increasing sequence of real numbers with the property that $\lim_{n\to\infty} r_n = 0$. Use the Cauchy criterion to prove that the series $\sum_{n=1}^{\infty} r_n z_n$ converges. (N.B. The classic example here is to take $z_n = (-1)^{n-1}$ — then $s_n = 1$ for odd n and $s_n = 0$ for even n — and to conclude that $\sum_{n=1}^{\infty}(-1)^{n-1} r_n$ is convergent, the result known as the "alternating series test." (*Hint.* Start by demonstrating that $\sum_{k=n}^{m} r_k z_k =$

$r_m s_m - r_n s_{n-1} + \sum_{k=n}^{m-1}(r_k - r_{k+1})s_k$ when $m > n \geq 2$. Don't forget: $z_k = s_k - s_{k-1}$ for $k \geq 2$.)

5.17. Let $\sum_{n=1}^{\infty} z_n$ be an absolutely convergent series of complex numbers with sum s. Show that $s = \sum_{n\in\mathbb{N}} z_n$ — this notation is intended to suggest that the sum is independent of the actual manner in which its terms are put into sequence — interpreted as follows: corresponding to each $\epsilon > 0$ there is a finite subset F_0 of $\mathbb{N}$ such that $|s - \sum_{n\in F} z_n| < \epsilon$ holds for every finite subset F of $\mathbb{N}$ which contains F_0. Confirm, in particular, that a series $\sum_{n=1}^{\infty} w_n$ which is derived from $\sum_{n=1}^{\infty} z_n$ by permuting its terms — i.e., $w_n = z_{\sigma(n)}$, where $\sigma\colon \mathbb{N} \to \mathbb{N}$ is a one-to-one function with range $\mathbb{N}$ — is also convergent and has sum s. (*Hint.* Given $\epsilon > 0$, one can choose N so that $\sum_{n=N+1}^{\infty} |z_n| < \epsilon$. (Why?) Consider $F_0 = \{1, 2, \ldots, N\}$.)

5.18. Establish the following "grouping principle" for absolutely convergent series: if $\sum_{n=1}^{\infty} z_n$ is an absolutely convergent series of complex numbers with sum s and if $A_1, A_2, A_3, \cdots$ is a (finite or infinite) collection of nonempty, disjoint subsets of $\mathbb{N}$ whose union is $\mathbb{N}$, then $s = \sum_{j=1}^{\infty}(\sum_{n\in A_j} z_n)$.

5.19. Let $\langle z_n \rangle$ and $\langle w_n \rangle$ be complex sequences such that both $\sum_{n=1}^{\infty} |z_n|^2$ and $\sum_{n=1}^{\infty} |w_n|^2$ are convergent. Prove that the series $\sum_{n=1}^{\infty} z_n w_n$ is absolutely convergent and that $|\sum_{n=1}^{\infty} z_n w_n|^2 \leq (\sum_{n=1}^{\infty} |z_n|^2)(\sum_{n=1}^{\infty} |w_n|^2)$. (*Hint.* Recall Exercise I.4.22 and Exercise 5.15.)

5.20. Suppose that a series of functions $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on a set A and that $g\colon B \to \mathbb{C}$ is a function whose range lies in A. Show that the series $\sum_{n=1}^{\infty} f_n \circ g$ converges uniformly on B.

5.21. A function series $\sum_{n=1}^{\infty} f_n$ converges uniformly on a set A. Assuming that $g\colon A \to \mathbb{C}$ is a bounded function, prove that the series $\sum_{n=1}^{\infty} g f_n$ is also uniformly convergent on A.

5.22. Assume that each term in a function series $\sum_{n=1}^{\infty} f_n$ is continuous in an open set U, that the series $\sum_{n=1}^{\infty} |f_n|$ converges pointwise in U, and that the function $f = \sum_{n=1}^{\infty} |f_n|$ is continuous in U. Prove that $\sum_{n=1}^{\infty} f_n$ is normally convergent in U. (*Hint.* Apply Exercise 5.7 on any compact subset K of U to the sequence $\langle g_n \rangle$, where $g_n = f - \sum_{k=1}^{n} |f_k|$.)

5.23. Show that the representation $\sec \pi z = 2\sum_{n=0}^{\infty}(-1)^n e^{(2n+1)\pi i z}$ is valid for every z in the half-plane $H = \{z : \operatorname{Im} z > 0\}$ and that the convergence of this series of functions is absolute and normal there.

5.24. Verify that the series $\sum_{n=0}^{\infty}(1 - z)^n(1 + z)^{-n}$ converges absolutely and uniformly on the set $A_t = \{z : x \geq t, |z| \leq t^{-1}\}$ for every $t > 0$ — hence, absolutely and normally in the half-plane $U = \{z : x > 0\}$ — and find its sum. Show that the series diverges for every other z. Disregard the point $z = -1$, where the terms of the series are undefined.

5.25. A function sequence $\langle f_n \rangle$ converges normally in an open set U to the limit function f. Show that the series $\sum_{n=1}^{\infty}(f_n - f_{n+1})$ converges normally in U, and identify its sum.

5.26. Verify that the series $\sum_{n=1}^{\infty}[z^2 + (2n+1)z + n^2 + n]^{-1}$ converges normally in the open set $U = \mathbb{C} \sim \{-1, -2, -3, \cdots\}$. Determine its sum.

5.27. Show that the series $\sum_{n=1}^{\infty}(z^2 + n^2)^{-1}$ converges absolutely and normally in the set $U = \mathbb{C} \sim \{ni : n = \pm 1, \pm 2, \cdots\}$. If f denotes its sum, verify that $\int_{|z|=r} f(z)dz = 0$ for every non-integral radius $r > 0$ and that

$$\int_{|z-ki|=r} f(z)\,dz = \pi \sum_{|k-r|<n<k+r} n^{-1}$$

for every such r and every positive integer k.

5.28. Prove Lemma 2.3.

5.29. Show that the function series $\sum_{n=-\infty}^{\infty} a^n e^{-|n|z}$, where a is a non-zero constant, converges absolutely and normally in the open half-plane $U = \{z : \operatorname{Re} z > |\operatorname{Log}|a||\}$. Compute its sum.

5.30. Verify that the series $\sum_{n=-\infty}^{\infty}(n^2+1)^n e^{n^2 z}$ converges absolutely and normally in the half-plane $U = \{z : \operatorname{Re} z < 0\}$ and diverges in $\mathbb{C} \sim U$.

5.3 Exercises for Section VII.3

5.31. Assume that each function in a sequence $\langle f_n \rangle$ is continuous in a domain D, where the sequence is known to converge normally to the limit function f. Assume, beyond this, that a primitive F_n for f_n in D has been found and normalized so as to insure the convergence of the sequence $\langle F_n(z_0) \rangle$ for some predetermined point z_0 of D. Show that $\langle F_n \rangle$ converges normally in D and that its limit F is a primitive for f in this domain. (*Hint.* First demonstrate that $\langle F_n \rangle$ converges pointwise in D.)

5.32. Show that the series $\sum_{n=1}^{\infty} \operatorname{Arcsin}(n^{-2}z)$ converges normally in the whole complex plane and that the function $f\colon \mathbb{C} \to \mathbb{C}$ defined by the rule of correspondence $f(z) = \sum_{n=1}^{\infty} \operatorname{Arcsin}(n^{-2}z)$ is analytic in the domain $D = \mathbb{C} \sim \{z : \operatorname{Im} z = 0 \text{ and } |z| \geq 1\}$. Deduce from this that $g(z) = \sum_{n=1}^{\infty}(n^4 - z^2)^{-1/2}$ also defines a function that is analytic in D.

5.33. Confirm that the formula $F(z) = \sum_{n=1}^{\infty} n^{-1} \operatorname{Arctan}(n^{-1}z)$ describes a primitive for the function introduced in Exercise 5.27 in the domain $D = \mathbb{C} \sim \{z : \operatorname{Re} z = 0 \text{ and } |z| \geq 1\}$.

5.34. Demonstrate that the formula $f(z) = \sum_{n=1}^{\infty}[1 - \cos(n^{-1}z)]$ defines an entire function.

5.35. Show that $g(z) = (1/2)z^{-2} + \sum_{n=1}^{\infty}(z^2 - n^2)^{-1}$ defines an analytic function g in the open set $U = \mathbb{C} \sim \{0, \pm 1, \pm 2, \cdots\}$. Show, in addition, that $F(z) = -2zg(z)$ furnishes a primitive in U for the function f presented in

Example 3.2.

5.36. Verify that the function f defined by $f(z) = \sum_{n=0}^{\infty} z^n(1+z^{2n})^{-1}$ is analytic in the open set $U = \{z : |z| \neq 1\}$.

5.37. Suppose that a function f is analytic in the open disk $D = \Delta(0,1)$, that $f(0) = 0$, and that $|f(z)| \leq 1$ for every z in D. Document the fact that $g(z) = \sum_{n=1}^{\infty} f(z^n)$ defines a function g which is analytic in D and which has $|g'(0)| \leq 1$. For which initial functions f will $|g'(0)| = 1$ hold?

5.38. Let $a_0, a_1, a_2, \cdots$ be a sequence of non-zero complex numbers, and let $\lambda = \limsup_{n\to\infty} |a_n|/|a_{n+1}|$. Show that the radius of convergence ρ of the Taylor series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ satisfies $\rho \leq \lambda$ and that equality holds when $\lim_{n\to\infty} |a_n|/|a_{n+1}|$ exists or when $|a_n|/|a_{n+1}| \to \infty$ as $n \to \infty$. Illustrate, by example, that $\rho < \lambda$ is a possibility otherwise.

5.39. Determine the disks of convergence of: (i) $\sum_{n=1}^{\infty} \sqrt[n]{n}\,(z-1)^n$; (ii) $\sum_{n=1}^{\infty} 3^n n^3 (n!)^3 [(3n)!]^{-1} z^{3n}$; (iii) $\sum_{n=1}^{\infty} n^2(z+2)^{2^n}$; (iv) $\sum_{n=1}^{\infty} n!(z-i)^{n!}$; (v) $\sum_{n=2}^{\infty} (\operatorname{Log} n)^n (z+1)^{n^2}$.

5.40. Show that the Taylor series $\sum_{n=1}^{\infty} n^{-1}z^n$ has radius of convergence $\rho = 1$ and that it converges conditionally for any z with $|z| = 1$ save for $z = 1$, at which point it diverges. (*Hint.* Exercise 5.16 is useful in dealing with z for which $|z| = 1$.)

5.41. Demonstrate that the radius of convergence of the Taylor series $\sum_{n=0}^{\infty} (-1)^n (2n+1)^{-1} z^{2n+1}$ is one and that the series is conditionally convergent at any point z of the circle $K(0,1)$ with the exception of i and $-i$, where it is divergent.

5.42. Verify that the Taylor series $\sum_{n=1}^{\infty} n! n^{-n} z^n$ has radius of convergence $\rho = e$ and that it diverges for every z satisfying $|z| = e$.

5.43. The "Fibonacci sequence" $\langle a_n \rangle$ is defined recursively by $a_0 = 0$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Verify that the radius of convergence ρ of the associated Taylor series $\sum_{n=1}^{\infty} a_n z^n$ is positive. Then, through the process of actually finding the sum of this series, determine ρ exactly. (*Hint.* For the first part check that $0 \leq a_n \leq 2^n$ for every n.)

5.44. Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a Taylor series with radius of convergence $\rho > 0$, and let f be its sum in the disk $\Delta = \Delta(z_0, \rho)$. Support the statement that the series $\sum_{n=0}^{\infty} (n+1)^{-1} a_n (z-z_0)^{n+1}$ also converges in Δ, where its sum F furnishes a primitive for f.

5.45. Confirm the observation that a Taylor series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, its corresponding derived series $\sum_{n=0}^{\infty} n a_{n-1}(z-z_0)^{n-1}$, and its associated primitive series $\sum_{n=0}^{\infty} (n+1)^{-1} a_n (z-z_0)^n$ all share the same radius of convergence ρ.

5.46. Find the Taylor series expansion of f about the origin, and identify the largest open disk $\Delta(0,d)$ in which the expansion is valid: (i) $f(z) =$

$(1-z)^{-2}$; (ii) $f(z) = (1+z^2)^{-3}$; (iii) $f(z) = \operatorname{Arctan} z$; (iv) $f(z) = \operatorname{Log}(1+z^2)$; (v) $f(z) = \operatorname{Log}[(1-z)(1+z)^{-1}]$;(vi) $f(z) = (z^2+z-2)^{-2}$; (vii) $f(z) = \sin(3z)\cos(2z)$.

5.47. Obtain the Taylor series development of the function f about the specified point z_0, and determine the largest d for which the expansion is valid in $\Delta(z_0, d)$: (i) $f(z) = z^{-1}, z_0 = i$; (ii) $f(z) = (z^2-z-2)^{-1}, z_0 = 1$; (iii) $f(z) = (z^2+1)^{-1}$, $z_0 = 1$; (iv) $f(z) = \sin z, z_0 = \pi/2$; (v) $f(z) = z\sin z, z_0 = \pi/2$; (vi) $f(z) = \cos^2 z, z_0 = \pi$; (vii) $f(z) = z\cos^2 z, z_0 = \pi$.

5.48. (i) Derive the expansion: $e^z\cos z = \sum_{n=0}^{\infty}(n!)^{-1}2^{n/2}\cos(n\pi/4)z^n$. (ii) Use the fact that $f^{(n)}(0)/n! = (2\pi i)^{-1}\int_{|z|=1} z^{-n-1}f(z)dz$ for any entire function f to arrive at an alternate description of the expansion in part (i) — namely,

$$e^z\cos z = \sum_{n=0}^{\infty}\left\{\sum_{0\le k\le n/2}(-1)^k[(2k)!(n-2k)!]^{-1}\right\}z^n .$$

5.49. Recall that for any non-zero complex number λ and any non-negative integer n the *binomial coefficient* $\binom{\lambda}{n}$ is defined by $\binom{\lambda}{0} = 1$ and $\binom{\lambda}{n} = \lambda(\lambda-1)\cdots(\lambda-n+1)/n!$ for $n \ge 1$. Thus $\binom{\lambda}{1} = \lambda$, $\binom{\lambda}{2} = \lambda(\lambda-1)/2$, etc. Certify that the expansion $(1+z)^\lambda = \sum_{n=0}^{\infty}\binom{\lambda}{n}z^n$ is valid when $|z| < 1$.

5.50. Expand $f(z) = \operatorname{Arcsin} z$ in a Taylor series centered at the origin. In what open disk about the origin does this series represent f? (*Hint.* To get started, consider f'.)

5.51. Let $z_t = e^{it}$, where $-\pi < t < \pi$. Demonstrate that the Taylor series expansion of $f(z) = \sqrt{z}$ centered at z_t takes the form $f(z) = \sum_{n=0}^{\infty}\binom{1/2}{n}z_t^{(1-2n)/2}(z-z_t)^n$, and determine the radius d_t of the largest open disk centered at z_t in which this representation is valid. Show that the above series has radius of convergence $\rho = 1$. For those t with $d_t < 1$ identify the function to which this series sums in the disk $\Delta(z_t, 1)$.

5.52. Find the disk of convergence Δ of the series $\sum_{n=1}^{\infty}(-1)^{n-1}n^2z^n$; compute its sum in this disk.

5.53. Confirm that the function series $\sum_{n=0}^{\infty}(n!)^{-1}\cos(nz)$ converges normally in the complex plane, and find the entire function that is its sum.

5.54. Let f and g be functions that are analytic in an open disk $\Delta = \Delta(z_0, r)$, where their Taylor expansions read $f(z) = \sum_{n=0}^{\infty}a_n(z-z_0)^n$ and $g(z) = \sum_{n=0}^{\infty}b_n(z-z_0)^n$. Verify that Taylor expansion of the product

$h = fg$ about z_0 has the structure $h(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$, in which $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Deduce from this that the radius of convergence of the Taylor series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ is at least the minimum of the radii of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n(z-z_0)^n$.

5.55. Show that $(1-z)^{-1}e^z = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(k!)^{-1}\right] z^n$ when $|z| < 1$.

5.56. Verify that $z^{-1}\operatorname{Log} z = \sum_{n=1}^{\infty}(-1)^{n-1}\left[\sum_{k=1}^{n} k^{-1}\right](z-1)^n$ and that $\operatorname{Log}^2 z = \sum_{n=2}^{\infty}(-1)^n\left[\sum_{k=1}^{n-1} k^{-1}(n-k)^{-1}\right](z-1)^n$ when $|z-1| < 1$.

5.57. If a function f has the Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ in the disk $\Delta(z_0, r)$ and if $f(z_0) \neq 0$, then the coefficients in the Taylor expansion $1/f(z) = \sum_{n=0}^{\infty} b_n(z-z_0)^n$ of its reciprocal are given recursively by $b_0 = a_0^{-1}$ and $b_n = -a_0^{-1}\sum_{k=1}^{n} a_k b_{n-k}$ for $n \geq 1$. Prove this.

5.58. Use Exercises 5.54 and 5.57 to find the terms up to and including the one involving z^5 in the Taylor expansions about the origin of the functions: (i) $f(z) = \sec z$; (ii) $g(z) = \tan z$; (iii) $h(z) = (1+e^z)^{-1}$; (iv) $k(z) = (1-e^z)(1+e^z)^{-1}$; (v) $\ell(z) = [\operatorname{Log}(1+z)]^{-1}$.

5.59. Let a function f be analytic in an open disk Δ centered at the origin, where it thus admits a Taylor series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Demonstrate that f is an even function in Δ if and only if $a_n = 0$ for all odd n; that f is an odd function in Δ if and only if $a_n = 0$ for all even n; that f is symmetric about the real axis in Δ — i.e., $f(\bar{z}) = \overline{f(z)}$ for every z in Δ — if and only if a_n is real for every n.

5.60. If a function f is both analytic and even in a disk $\Delta = \Delta(0, r)$, then the function $g\colon \Delta \to \mathbb{C}$ defined by $g(z) = f(\sqrt{z})$ is an analytic function and $g^{(n)}(0) = [(2n)(2n-1)\cdots(n+1)]^{-1} f^{(2n)}(0)$. Justify these claims. (This exercise redoes, only much more simply, Exercise III.6.38.)

5.61. A non-constant entire function f has the feature that $f(\lambda z) = f(z)$ for every z, where $\lambda \neq 1$ is a constant. Prove that there must be a positive integer m for which $\lambda^m = 1$. Moreover, if m is the smallest such integer, prove that f has the structure $f(z) = g(z^m)$ for some entire function g.

5.62. The function f defined in the disk $\Delta(0, 2\pi)$ by $f(z) = z(e^z - 1)^{-1}$ if $z \neq 0$ and $f(0) = 1$ is an analytic function. (Why is f differentiable at the orgin?) Show that the Taylor expansion of f about the origin can be written in the form $f(z) = 1 - (1/2)z + \sum_{n=1}^{\infty}(-1)^{n-1}[(2n)!]^{-1}B_n z^{2n}$. The numbers $B_1, B_2, B_3, \cdots$ that turn up here — i.e., $B_n = (-1)^{n-1}f^{(2n)}(0)$ — are called the *Bernoulli numbers*. Verify that

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{2n+1}{2k} B_k = \frac{2n-1}{2}$$

for $n = 1, 2, 3, \cdots$ and use this combinatorial identity to compute B_1 through B_7. (*Hint.* For the first part of the problem confirm the observation that the function g given by $g(z) = (1/2)z + f(z)$ is an even function.)

5.63. Derive the expansion

$$z \cot z = 1 - \sum_{n=1}^{\infty} \frac{B_n 4^n z^{2n}}{(2n)!}$$

for z in $\Delta^*(0, \pi)$, where $B_1, B_2, B_3, \cdots$ are the Bernoulli numbers. Use this to conclude that

$$\tan z = \sum_{n=1}^{\infty} \frac{B_n 4^n (4^n - 1) z^{2n-1}}{(2n)!}$$

when $|z| < \pi/2$. (*Hint.* Exploit the identities $z \cot z = iz + 2iz(e^{2iz} - 1)^{-1}$ and $\tan z = \cot z - 2\cot(2z)$.)

5.64. Demonstrate that the Taylor series $\sum_{n=1}^{\infty} z^{n!} = z + z^2 + z^6 + \cdots$ has radius of convergence $\rho = 1$ and that the analytic function g defined in $\Delta = \Delta(0,1)$ by $g(z) = \sum_{n=1}^{\infty} z^{n!}$ has the following behavior: for every rational number θ, $|g(re^{2\pi i\theta})| \to \infty$ as $r \to 1$ from the left.

5.65. Show that $f(z) = \sum_{n=1}^{\infty}(n!)^{-1} z^{n!}$ describes a function that is continuous on the closed disk $\bar{\Delta}(0,1)$, is analytic in the open disk $\Delta = \Delta(0,1)$, and is endowed with the following property: if D is any domain that properly includes Δ, then there exists no analytic function $F: D \to \mathbb{C}$ whose restriction to Δ coincides with f there. (*Hint.* Assume such a function F were to exist. Obtain a contradiction by looking at F'.)

5.66. Let $\sum_{n=1}^{\infty} a_n z^n$ be a Taylor series with non-negative coefficients and with radius of convergence $\rho = 1$. If U is an open set that contains both the disk $\Delta(0,1)$ and the point $z = 1$, demonstrate that there can be no function f which is analytic in U and satisfies $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for every z in $\Delta(0,1)$. (*Hint.* Argue by contradiction. Assume that a function f of this description exists, expand f in a Taylor series about the point $z_0 = 1/2$, and, by regrouping the terms of this series, arrive at the conclusion that the series $\sum_{n=0}^{\infty} a_n(1+r)^n$ converges for some $r > 0$. Why is this unacceptable?)

5.67. Let $\sum_{n=0}^{\infty} a_n$ be a convergent series of complex numbers, and let s be its sum. Making note that the radius of convergence of the Taylor series $\sum_{n=0}^{\infty} a_n z^n$ is at least one, prove that the analytic function f defined in $\Delta(0,1)$ by $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfies $\lim_{r\to 1^-} f(r) = s$. (N.B. This result is called "Abel's Theorem" after its discoverer, Henrik Abel (1802-1829). *Hint.* Let $s_n = \sum_{k=0}^{n} a_k$ for $n = 0, 1, 2, \cdots$ and set $s_{-1} = 0$. Use the fact that $a_n = s_n - s_{n-1}$ to get the formula $f(z) = (1-z)\sum_{n=0}^{\infty} s_n z^n$ for f in $\Delta(0,1)$. Then observe that $s = (1-z)\sum_{n=0}^{\infty} s z^n$. This information enables one to get a handle on $|f(r) - s|$ when $0 \le r < 1$.)

5.68. Compute the sum of the series $\sum_{n=1}^{\infty} n^{-1} e^{in\theta}$ for θ in the interval $(0, 2\pi)$. Use it to find $\sum_{n=1}^{\infty} n^{-1}\cos(n\theta)$ and $\sum_{n=1}^{\infty} n^{-1}\sin(n\theta)$ for such θ.

(*Hint.* For fixed θ consider the function $f(z) = -\operatorname{Log}(1 - e^{i\theta}z)$ in the disk $\Delta(0,1)$. Recall Exercise 5.40.)

5.69. Calculate the sum of the series $\sum_{n=0}^{\infty}(-1)^n(2n+1)^{-1}e^{(2n+1)i\theta}$ for θ that is not an odd multiple of $\pi/2$. With the aid of the result sum the series $\sum_{n=0}^{\infty}(-1)^n(2n+1)^{-1}\cos[(2n+1)\theta]$ and $\sum_{n=0}^{\infty}(-1)^n(2n+1)^{-1}\sin[(2n+1)\theta]$ for such θ. (*Hint.* Consider the function $f(z) = \operatorname{Arctan}(e^{i\theta}z)$.)

5.70. A function f is analytic in the disk $\Delta = \Delta(0,1)$, where it has the Taylor series representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If $0 < r < 1$, verify that $\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$. Conclude, in particular, that the latter series is convergent. (*Hint.* $|f(z)|^2 = f(z)\overline{f(z)} = \sum_{n=0}^{\infty} \bar{a}_n \bar{z}^n f(z)$ in Δ, and the convergence of this series is normal there.)

5.71. If a function f is continuous on the closed disk $\bar{\Delta}(0,1)$ and analytic in the open disk $\Delta(0,1)$, and if it has the Taylor series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ about the origin, demonstrate (i) that the series $\sum_{n=0}^{\infty} |a_n|^2$ is convergent and (ii) that $\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = 2\pi \sum_{n=0}^{\infty} |a_n|^2$. (*Hint.* First show that $\sum_{n=0}^{N} |a_n|^2 \le (2\pi)^{-1} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$ by considering $\sum_{n=0}^{N} |a_n|^2 r^{2n}$ for r in $(0,1)$ and proving that $\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \to \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$ as $r \to 1^-$.)

5.72. Let $\Delta = \Delta(0,1)$ and let $g: \Delta \to \mathbb{C}$ be an analytic function satisfying $g(0) = 0$ and $|g'(0)| = 1$ whose derivative is a bounded function in Δ. Show that $|w| \ge (4m)^{-1}$ for every point w of $\mathbb{C} \sim g(\Delta)$, where $m = \sup\{|g'(z)| : z \in \Delta\}$; i.e., show that the range of g contains the disk $\Delta(0,(4m)^{-1})$. (*Hint.* Fix w belonging to $\mathbb{C} \sim g(\Delta)$. Then $w \neq 0$. The function h defined by $h(z) = 1 - w^{-1}g(z)$ is analytic and zero-free in Δ. Let f denote the branch of the square root of h in Δ for which $f(0) = 1$, and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Taylor expansion of this function about the origin. Make use of the knowledge, a corollary of Exercise 5.70, that $2\pi(1 + |a_1|^2 r^2) \le \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta$ whenever $0 < r < 1$. Notice, too, that $|g(z)| \le m$ for any z in Δ. Why?)

5.73. Under the assumptions that a function f is analytic in the open disk $\Delta = \Delta(0,1)$, that f' is bounded in Δ, and that $|f'(0)| = 1$, prove that the set $f(\Delta)$ must contain some open disk of radius $1/16$. (N.B. The disk in question need not have center $f(0)$. The result stated here is a watered-down version of a famous theorem of André Bloch (1893-1948). The argument we outline is ascribed to Edmund Landau (1877-1938). *Hint.* Define $\varphi: (0,1] \to \mathbb{R}$ by $\varphi(s) = \max\{|f'(z)| : |z| \le 1 - s\}$ and set $r = \min\{s : s\varphi(s) = 1\}$. Since φ is a continuous function with $\varphi(1) = 1$ and since $s\varphi(s) \to 0$ as $s \to 0$ — φ is a bounded function — r is well-defined. Then $0 < r \le 1$, $r\varphi(r) = 1$ by continuity, and $s\varphi(s) < 1$ when $0 < s < r$. The maximum principle allows us to choose a point z_0 with $|z_0| = 1 - r$ for which $|f'(z_0)| = \varphi(r) = r^{-1}$. Apply Exercise 5.72 in Δ to the function

g defined in $\Delta(0,2)$ by $g(z) = 2\left[f(z_0 + 2^{-1}rz) - f(z_0)\right]$.)

5.74. Let $D = \Delta(0,r)$ and let f be a function that is continuous on $\overline{D}$ and analytic in D. Confirm the existence of a sequence $\langle p_n \rangle$ whose terms are polynomial functions of z such that $p_n \to f$ uniformly on $\overline{D}$. (N.B. This is a special case of a celebrated approximation theorem of S.N. Mergelyan: if K is a compact set in the complex plane and if $\mathbb{C} \sim K$ is a domain, then every function that is continuous on K and analytic in the interior of K is the uniform limit on K of some sequence of polynomials in z.)

5.75. Let $f(z) = z^{-2}\sin z$. Demonstrate that there is no sequence of polynomial functions of z that converges to f uniformly on the circle $K(0,1)$. Can the same statement be made about $f(z) = z^{-1}\sin z$? What about $f(z) = z^{-3}\sin z$?

5.76. Let D be a bounded domain in the complex plane such that $\partial\overline{D} = \partial D$. (This condition would not allow D to be a punctured disk, for example.) Assume that D is blessed with the following property: for every function f that is continuous on $\overline{D}$ and analytic in D there exists a sequence $\langle p_n \rangle$ of polynomials in z such that $p_n \to f$ uniformly on $\overline{D}$. Prove that D has to be simply connected. Find an example of a bounded simply connected domain D that satisfies $\partial\overline{D} = \partial D$, yet fails to have the approximation property just described.

5.77. For which complex numbers λ does the Laurent series $\sum_{n=-\infty}^{\infty} \lambda^{|n|} z^n$ have a non-empty ring of convergence D? For such λ identify the function represented by this series in D.

5.78. Determine the annulus of convergence of the Laurent series $\sum_{n=-\infty}^{\infty} n(-1)^n 2^{-|n|} z^n$, and compute its sum there.

5.79. Determine the Laurent series expansion of the function f in the specified ring D: (i) $f(z) = (z^2+1)^{-1}$, $D = \{z : 1 < |z-2i| < 3\}$; (ii) $f(z) = z^{-4}(e^{z^2} - 1)$, $D = \Delta^*(0,\infty)$; (iii) $f(z) = (z-1)^{-2}\operatorname{Log} z$, $D = \Delta^*(1,1)$; (iv) $f(z) = (z^2 - 4z)^{-1}$, $D = \{z : 3 < |z-3i| < 5\}$; (v) $f(z) = z(z-2)^{-4}\cos \pi z$, $D = \Delta^*(2,\infty)$; (vi) $f(z) = (z-1)\sin(z^{-1})$, $D = \Delta^*(0,\infty)$; (vii) $f(z) = z^6\cos^2(z^{-2})$, $D = \Delta^*(0,\infty)$.

5.80. Obtain the Laurent expansions of the functions $f(z) = (z^2 - z)^{-1}$ and $g(z) = (1-2z)(z^2-z)^{-2}$ in the annulus D: (i) $D = \Delta^*(0,1)$; (ii) $D = \Delta^*(1,1)$; (iii) $D = \{z : |z| > 1\}$; (iv) $D = \{z : |z-1| > 1\}$; (v) $D = \{z : 1 < |z+1| < 2\}$; (vi) $D = \{z : 1 < |z+i| < \sqrt{2}\}$.

5.81. Functions f and g are analytic in $D = \{z : a < |z - z_0| < b\}$, with $0 \le a < b \le \infty$, where they exhibit the Laurent expansions $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ and $g(z) = \sum_{n=-\infty}^{\infty} b_n(z-z_0)^n$. Demonstrate that the Laurent expansion of the product $h = fg$ in D takes the form $h(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$, in which $c_n = \sum_{k=-\infty}^{\infty} a_k b_{n-k}$. Show, additionally, that the series which gives c_n is absolutely convergent. (*Hint.* Observe that

$h(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n g(z)$ in D, the convergence being uniform on $K(z_0, r)$ for r in (a, b).)

5.82. Obtain the expansions

$$\exp(z + z^{-1}) = \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=\max\{0,n\}}^{\infty} \frac{1}{k!(k-n)!} \right\} z^n$$

and

$$\exp(z - z^{-1}) = \sum_{n=-\infty}^{\infty} (-1)^n \left\{ \sum_{k=\max\{0,n\}}^{\infty} \frac{(-1)^k}{k!(k-n)!} \right\} z^n$$

for z satisfying $0 < |z| < \infty$.

5.83. A function f that is analytic and free of zeros in the domain $D = \mathbb{C} \sim \{0\}$ can be written in the form $f(z) = z^p g(z) h(z^{-1})$, where p is an integer and where both g and h are zero-free entire functions. Justify this statement. (*Hint.* Set $p = (2\pi i)^{-1} \int_{|z|=1} [f'(z)/f(z)]dz$; i.e., $p = n(\beta, 0)$, where $\beta(t) = f(e^{it})$ for $0 \le t \le 2\pi$. Show that in D there exists a branch of $\log k(z)$, where $k: D \to \mathbb{C}$ is the function given by $k(z) = z^{-p} f(z)$. For this use Theorem V.4.1.)

5.84. A function series of the type $\sum_{n=1}^{\infty} a_n n^{-z}$, where $\langle a_n \rangle$ is a sequence of complex constants, is called a *Dirichlet series.* (The series defining the Riemann zeta-function when $\operatorname{Re} z > 1$ is one such series.) Assuming that $\langle s_n \rangle$, the sequence of partial sums associated with $\langle a_n \rangle$, is bounded — say $|s_n| \le c$ for all n — show that $\sum_{n=1}^{\infty} a_n n^{-z}$ converges uniformly on the set $A_\theta = \{z : |\operatorname{Arg} z| < \theta, \operatorname{Re} z \ge (\pi/2) - \theta\}$ for any θ in the interval $(0, \pi/2)$ — hence, normally in the half-plane $D = \{z : \operatorname{Re} z > 0\}$. (*Hint.* Check that $\sum_{k=n}^{m} a_k k^{-z} = s_m m^{-z} - s_{n-1} n^{-z} + \sum_{k=n}^{m-1} s_k [k^{-z} - (k+1)^{-z}]$ when $m > n \ge 2$. Make use of this information, along with the identity $z \int_k^{k+1} t^{-z-1} dt = k^{-z} - (k+1)^{-z}$, to derive the estimate $|\sum_{k=n}^{m} a_k k^{-z}| \le 2c(1 + x^{-1}|z|) n^{-x}$ for $z = x + iy$ in D and for $m \ge n \ge 2$. Apply the Cauchy criterion in A_θ.)

5.85. Given that a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-z}$ converges at the point $z_0 = x_0 + iy_0$, certify that it converges normally in the half-plane $\{z : \operatorname{Re} z > x_0\}$ and that the convergence is absolute when $\operatorname{Re} z > 1 + x_0$. Infer the existence of an extended real number σ_0, known as the ***abscissa of convergence*** of $\sum_{n=1}^{\infty} a_n n^{-z}$, such that this series diverges for any z with $\operatorname{Re} z < \sigma_0$ and converges normally in the half-plane $D = \{z : \operatorname{Re} z > \sigma_0\}$, the convergence being absolute if $\operatorname{Re} z > 1 + \sigma_0$. Conclude that $f(z) = \sum_{n=1}^{\infty} a_n n^{-z}$ defines an analytic function in D, provided $\sigma_0 < \infty$. (*Hint.* Apply Exercise 5.84 to the series $\sum_{n=1}^{\infty} (a_n n^{-z_0}) n^{-\zeta}$ and then substitute $\zeta = z - z_0$.)

5.86. Verify that the formula $g(z) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-z}$ defines an analytic function in the half-plane $D = \{z : \operatorname{Re} z > 0\}$ and that $g(z) = (1 - 2^{1-z}) f(z)$

when $\operatorname{Re} z > 1$, where f is the function in Example 3.1. Deduce from this fact that f admits an analytic extension to the set $\{z : \operatorname{Re} z > 0, z \neq 1\}$.

5.4 Exercises for Section VII.4

5.87. Let $\langle f_n \rangle$ be a sequence from a normal subfamily $\mathcal{F}$ of $C(U)$. Under the assumption that $\langle f_n \rangle$ is not itself normally convergent in U, prove that $\langle f_n \rangle$ must have subsequences $\langle f_{n_k} \rangle$ and $\langle f_{m_k} \rangle$ which converge normally in U to different limit functions.

5.88. Let $\mathcal{F}$ be a normal subfamily of $C(U)$ each of whose members is analytic in U. Demonstrate that the family $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is also normal in U.

5.89. Let $\mathcal{F}$ be a normal subfamily of $C(D)$, where D is a domain in the complex plane. Suppose that $\widetilde{\mathcal{F}}$ is a family of functions which are analytic in D and that $\widetilde{\mathcal{F}}$ has these two properties: (i) F' belongs to $\mathcal{F}$ whenever F is a member of $\widetilde{\mathcal{F}}$ and (ii) for some point z_0 of D the set $\{F(z_0) : F \in \widetilde{\mathcal{F}}\}$ is a bounded set of complex numbers. Prove that $\widetilde{\mathcal{F}}$ is a normal family in D.

5.90. Let $c > 0$ and $\lambda \geq 0$ be constants. Confirm that the family $\mathcal{F}$ of functions f which (i) are analytic in the disk $D = \Delta(0,1)$, (ii) satisfy $f(0) = 0$, and (iii) admit the derivative estimate $|f'(z)| \leq c(1 - |z|)^{-\lambda}$ for every z in D is a normal family in that disk.

5.91. Let $c > 0$ and let $\mathcal{F}$ be the family of all analytic functions f in the disk $D = \Delta(0,1)$ whose Taylor coefficients $a_n = f^{(n)}(0)/n!$ obey the condition $\sum_{n=0}^{\infty} |a_n|^2 \leq c$. Prove that $\mathcal{F}$ is a normal family in D. (*Hint.* Derive the estimate $|f(z)| \leq c^{1/2}(1-r^2)^{-1/2}$ for f in $\mathcal{F}$ and z in the closed disk $\overline{\Delta}(0,r)$, where $0 < r < 1$. Exercise 5.19 might be of some use.)

5.92. Let $c > 0$ be a constant. If $\mathcal{F}$ is the family of all analytic functions f in the disk $D = \Delta(0,1)$ for which $\iint_{\overline{\Delta}(0,r)} |f(z)|^2 dx dy \leq c$ when $0 < r < 1$, then $\mathcal{F}$ is a normal family in D. Prove this. (*Hint:* One solution begins by verifying that $\iint_{\overline{\Delta}(0,r)} |f(z)|^2 dx dy = \pi \sum_{n=0}^{\infty} (n+1)^{-1} |a_n|^2 r^{2n+2}$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is the Taylor expansion of f about the origin.)

5.93. Let D be a plane domain, and let $\mathcal{G}$ be a pointwise bounded family of analytic functions $g : D \to \mathbb{C}$. Assume the existence of a constant $c \geq 1$ about which the following is true: for each member g of $\mathcal{G}$ the set $\mathbb{C} \sim g(D)$ contains a pair of points a_g and b_g satisfying $|a_g| \leq c, |b_g| \leq c$, and $|a_g - b_g| \geq c^{-1}$. Use Theorem 4.6 to prove that $\mathcal{G}$ is a normal family in D. (*Hint.* Consider the family $\mathcal{F}$ whose member functions f have the form $f(z) = [g(z) - a_g]/[b_g - a_g]$, where g belongs to $\mathcal{G}$.)

5.94. Let $c \geq 1$ be a constant. With the help of Theorem 4.6 confirm the following fact: corresponding to each pair of distinct complex numbers a and b there exists a radius $r > 0$ such that any analytic function $g: \Delta(0, r) \to \mathbb{C}$ for which $|g(0)| \leq c$ and $|g'(0)| \geq c^{-1}$ must have at least one of the points a or b in its range. (*Hint.* Suppose that no such r existed. Then for each positive integer n there would have to be an analytic function $g_n: \Delta(0, n) \to \mathbb{C}$ satisfying $|g_n(0)| \leq c$ and $|g_n'(0)| \geq c^{-1}$, but having its range in $\mathbb{C} \sim \{a, b\}$. Consider the sequence $\langle f_n \rangle$ in the disk $D = \Delta(0, 1)$, where $f_n(z) = g_n(nz)$.)

5.95. From Exercise 5.94 deduce the following theorem of Picard: if f is a non-constant entire function, then $\mathbb{C} \sim f(\mathbb{C})$ contains at most one point. (*Hint.* Suppose that $\mathbb{C} \sim f(\mathbb{C})$ contains points a and b, $a \neq b$. Choose a point z_0 for which both $f(z_0) \neq 0$ and $f'(z_0) \neq 0$. Why must such a point exist? Look at the function $g(z) = f(z + z_0)$ to get a contradiction.)

5.96. Use Theorem 4.6 to generalize Schwarz's lemma as follows: if D is a plane domain with at least two boundary points, if $f: D \to D$ is an analytic function, and if f fixes a point z_0 of D, then $|f'(z_0)| \leq 1$. Show by example that the same needn't be true if $D = \mathbb{C}$ or if ∂D has only one point. (*Hint.* For the first part, assume that $|f'(z_0)| > 1$ and obtain a contradiction by examining the sequence $\langle f_n \rangle$, where $f_1 = f, f_2 = f \circ f, f_3 = f \circ f \circ f, \cdots$.)

Chapter VIII

Isolated Singularities of Analytic Functions

Introduction

The scene in the present chapter is dominated by the residue theorem and a number of its many applications. The residue theorem is a general form of Cauchy's theorem. It is concerned with integrals of the type $\int_\gamma f(z)dz$, in which γ is a closed path in a set where the function f is analytic, but in which this contour of integration is permitted to wind about isolated points where the integrand is either undefined or non-differentiable, so-called "isolated singularities" of f. One must not count on the integral of f along such a path being zero. Quite to the contrary, the integral typically picks up a contribution from each of the singularities in question. The residue theorem details the exact character of those contributions, as well as the manner in which they must be combined to evaluate the given integral. Our treatment of the residue theorem presupposes some understanding of the behavior of an analytic function in the vicinity of an isolated singularity, so it is this matter we address first. To make that important topic accessible, however, we must upgrade our knowledge of the local structure of analytic functions. This we do in the initial section of the chapter.

1 Zeros of Analytic Functions

1.1 The Factor Theorem for Analytic Functions

We are well aware that an analytic function f whose first derivative has value zero at each point of a domain D must be constant in the domain. The next theorem allows us to draw the same conclusion if it is known that at some particular point of D every derivative of f vanishes. In this instance extensive information about a function at one point is a fair trade for a small amount of information about it at every point.

Theorem 1.1. *If a function f is analytic in a domain D and if there exists a point ζ_0 of D with the property that $f^{(n)}(\zeta_0) = 0$ for every positive integer n, then f is constant in D.*

Proof. We consider the sets $U = \{z \in D : f^{(n)}(z) = 0 \text{ for every } n \geq 1\}$ and $V = D \sim U$. Quite obviously $D = U \cup V$ and $U \cap V = \phi$. By assumption the point ζ_0 lies in U, rendering this set non-empty. We shall prove that $U = D$. To accomplish this we need only verify that both U and V are open sets, for then we can invoke Theorem II.3.4.

We look first at a point z_0 of U. We choose an open disk Δ centered at z_0 that is contained in the domain D. From Theorem VII.3.4 and from the definition of the set U it follows that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = f(z_0)$$

for every z in Δ, so f is constant in that disk. This implies that $f^{(n)}(z) = 0$ for each $n \geq 1$ and for each z in Δ, which makes Δ a subset of U. The implication that the set U is open becomes clear.

Next, we focus on an element z_0 of V. By the definition of V we can find and fix a positive integer n for which $f^{(n)}(z_0) \neq 0$. Since $f^{(n)}$ is a continuous function in D, there exists an open disk $\Delta = \Delta(z_0, r)$ lying in D with the property that $f^{(n)}(z) \neq 0$ for every z belonging to Δ. This places the disk Δ inside the set V. It, too, is thus seen to be an open set.

Theorem II.3.4 informs us that $V = \phi$ or, equivalently, that $U = D$. In particular, $f'(z) = 0$ for every z in D, a fact already known to entail the constancy of f in D. ∎

An important consequence of the result just established is a generalization for analytic functions of the "Factor Theorem" for polynomial functions. It asserts, albeit in more precise terms, that if z_0 is a root of an analytic function f, then f is the product of the linear function ℓ, $\ell(z) = z - z_0$, and another analytic function. In other words, $z - z_0$ is a factor of $f(z)$ in the class of analytic functions.

Theorem 1.2. *Suppose that a function f is analytic and non-constant in a domain D and that z_0 is a point of D for which $f(z_0) = 0$. Then f can be uniquely represented in D in the fashion*

$$f(z) = (z - z_0)^m g(z) ,$$

where m is a positive integer and $g: D \to \mathbb{C}$ is an analytic function that obeys the condition $g(z_0) \neq 0$.

Proof. Since f is non-constant in D, Theorem 1.1 implies that there is at least one positive integer n for which $f^{(n)}(z_0) \neq 0$. Let m be the smallest such integer. We select and fix an open disk $\Delta = \Delta(z_0, r)$ that lies in D. In Δ we can represent f by its Taylor series expansion about the point z_0: $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$, where $a_n = f^{(n)}(z_0)/n!$. Owing to the definition of m and to the assumption that $a_0 = f(z_0) = 0$, we note that $a_n = 0$ when $0 \leq n \leq m - 1$, whereas $a_m \neq 0$. Consequently, we are entitled to write

$$f(z) = \sum_{n=m}^{\infty} a_n(z - z_0)^n = (z - z_0)^m \sum_{n=m}^{\infty} a_n(z - z_0)^{n-m}$$

for z belonging to Δ. Now the function $g: D \to \mathbb{C}$ defined by

$$g(z) = \begin{cases} \dfrac{f(z)}{(z - z_0)^m} & \text{if } z \neq z_0 , \\ a_m & \text{if } z = z_0 , \end{cases}$$

is plainly analytic in $D \sim \{z_0\}$. Moreover, inside Δ it admits the Taylor series development $g(z) = \sum_{n=m}^{\infty} a_n(z - z_0)^{n-m} = a_m + a_{m+1}(z - z_0) + \cdots$, which according to Theorem VII.3.3 marks g as analytic in Δ. In particular, g is differentiable at z_0. Therefore, g is an analytic function, $g(z_0) = a_m \neq 0$, and $f(z) = (z - z_0)^m g(z)$ holds throughout D.

Could f enjoy a second representation in D of the type described — say $f(z) = (z - z_0)^{\ell} h(z)$, where ℓ is a positive integer and $h: D \to \mathbb{C}$ is an analytic function with $h(z_0) \neq 0$? Assume so. For any z in D we have $(z - z_0)^m g(z) = (z - z_0)^{\ell} h(z)$. Then $m \leq \ell$, for the inequality $m > \ell$ would lead to

$$0 = \lim_{z \to z_0} (z - z_0)^{m-\ell} g(z) = \lim_{z \to z_0} h(z) = h(z_0) \neq 0 ,$$

a contradiction. Similarly, $\ell > m$ can be ruled out. It must then be the case that $\ell = m$. This, in turn, implies that $g(z) = h(z)$ for any z belonging to $D \sim \{z_0\}$. By continuity, $g(z_0) = h(z_0)$ as well. We conclude that $m = \ell$ and $g = h$ — hence, that the alleged alternate representation of f is nonexistent. The uniqueness assertion of the theorem follows. ∎

As a direct corollary of Theorem 1.2, we record:

Corollary 1.3. *Suppose that a function f is analytic and non-constant in a domain D and that z_0 is a point of D. Then f can be uniquely represented in D in the fashion*

$$f(z) = f(z_0) + (z - z_0)^m g(z) ,$$

where m is a positive integer and $g: D \to \mathbb{C}$ is an analytic function that obeys the condition $g(z_0) \neq 0$.

Proof. Apply Theorem 1.2 to the function f_1 defined in D by $f_1(z) = f(z) - f(z_0)$. ∎

1.2 Multiplicity

Let f be a function that is analytic and non-constant in some open disk Δ centered at the point z_0. Corollary 1.3 states that f admits a unique representation in Δ of the form

$$f(z) = w_0 + (z - z_0)^m g(z) ,$$

where $w_0 = f(z_0)$, m is a positive integer, and g is an analytic function in Δ satisfying $g(z_0) \neq 0$. The integer m — which, incidentally, does not depend in any way on the choice of disk Δ — is called the *multiplicity* (or, synonymously, the *order*) *of* f *at* z_0. To be a little more precise, we say that f *takes the value* w_0 *with multiplicity* (or *order*) m *at* z_0. For instance, when $w_0 = 0$ we speak of f having a zero of multiplicity (or order) m at z_0. (The expressions "simple zero," "double zero," "triple zero," etc. are optional terms of reference to zeros of order one, two, three, etc.) The proof of Theorem 1.2 shows that the multiplicity of f at z_0 is the smallest positive integer m for which $f^{(m)}(z_0) \neq 0$, although, as the third in the following series of examples points out, evaluating derivatives is not always the most efficient method to ascertain multiplicity.

EXAMPLE 1.1. Determine the multiplicity of the zero of $f(z) = \cos z$ at the point $z_0 = \pi/2$.

Since $f'(\pi/2) = -\sin(\pi/2) = -1 \neq 0$, we see that the given function has a simple zero at $\pi/2$.

EXAMPLE 1.2. Find the order of the zero of $f(z) = e^z - z - 1$ at the origin.

Again we simply compute: $f'(0) = 0$, $f''(0) = 1$. It follows that $f(z) = e^z - z - 1$ has a double zero at the origin.

EXAMPLE 1.3. Determine the multiplicity with which $f(z) = \cos(z^3)$ assumes its value at the origin.

In this case we use to advantage the Taylor expansion of $\cos z$ about the origin by writing

$$\cos(z^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (z^3)^{2n}}{(2n)!} = 1 - \frac{z^6}{2!} + \frac{z^{12}}{4!} - \cdots$$

$$= 1 + z^6 \left(-\frac{1}{2} + \frac{z^6}{4!} - \cdots \right) = 1 + z^6 g(z) ,$$

where

$$g(z) = \sum_{n=1}^{\infty} \frac{(-1)^n z^{6n-6}}{(2n)!} .$$

The function g is obviously an entire function that has $g(0) = -1/2 \neq 0$. We conclude that $f(z) = \cos(z^3)$ takes the value $w_0 = 1$ with multiplicity $m = 6$ at the origin. Observe that in the present example it would have involved considerably more work to arrive at the multiplicity by computing derivatives.

As an application of Theorem 1.2 and the concept of multiplicity we establish an extension to the complex setting of one of L'Hospital's rules for evaluating limits.

Theorem 1.4. (L'Hospital's Rule) *Let f and g be functions that are analytic and non-constant in a disk $\Delta = \Delta(z_0, r)$. Assume that each of these functions has a zero at the point z_0. Then*

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}\,, \tag{8.1}$$

understood to mean that either both limits exist and are the same, or else neither limit exists.

Proof. Let m and ℓ be the respective orders of the zeros of f and g at z_0. In the disk Δ we can write $f(z) = (z-z_0)^m f_1(z)$ and $g(z) = (z-z_0)^\ell g_1(z)$, where f_1 and g_1 are analytic functions in Δ, neither of which has a zero at z_0. Furthermore, the proof of Theorem 1.2 gives $f_1(z_0) = f^{(m)}(z_0)/m!$ and $g_1(z_0) = g^{(\ell)}(z_0)/\ell!$. We infer that

$$\frac{f(z)}{g(z)} = (z - z_0)^{m-\ell}\frac{f_1(z)}{g_1(z)}$$

at any z in the punctured disk $\Delta^* = \Delta^*(z_0, r)$ for which $g_1(z) \neq 0$. Since g_1 is continuous at z_0 and $g_1(z_0) \neq 0$, this includes all z in some smaller punctured disk centered at z_0. Due to the fact that

$$\lim_{z \to z_0} \frac{f_1(z)}{g_1(z)} = \frac{f_1(z_0)}{g_1(z_0)} = \frac{\ell!\, f^{(m)}(z_0)}{m!\, g^{(\ell)}(z_0)}\,,$$

it then becomes apparent that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \begin{cases} 0 & \text{if } m > \ell\,, \\ f^{(m)}(z_0)/g^{(m)}(z_0) & \text{if } m = \ell\,, \\ \text{fails to exist} & \text{if } m < \ell\,. \end{cases} \tag{8.2}$$

What about the limit of $f'(z)/g'(z)$ as $z \to z_0$? Assume initially that both $m > 1$ and $\ell > 1$. Because the first $m-1$ derivatives of f vanish at z_0 and $f^{(m)}(z_0) \neq 0$, we see that f' and its first $m-2$ derivatives are zero at z_0, whereas $(f')^{(m-1)}(z_0) \neq 0$ — i.e., we see that f' has a zero of order $m-1$ at z_0. Similarly, g' has a zero of order $\ell - 1$ at z_0. The reasoning used to arrive at (8.2) then applies equally well to the pair of functions f' and g', yielding

$$(8.3)\quad \lim_{z\to z_0}\frac{f'(z)}{g'(z)}=\begin{cases}0 & \text{if } m-1>\ell-1\ ,\\ (f')^{(m-1)}(z_0)/(g')^{(m-1)}(z_0) & \text{if } m-1=\ell-1\ ,\\ \text{fails to exist} & \text{if } m-1<\ell-1\ ,\end{cases}$$

at least when $m>1$ and $\ell>1$. Next, suppose that $\ell=1$. Then $g'(z)\to g'(z_0)\neq 0$ as $z\to z_0$, so $f'(z)/g'(z)\to f'(z_0)/g'(z_0)$. Since $m\geq 1$ and since $f'(z_0)=0$ if $m>1$, (8.3) continues to hold when $\ell=1$. Finally, in the case $m=1$ we have $f'(z)\to f'(z_0)\neq 0$ as $z\to z_0$, which implies that $f'(z)/g'(z)\to f'(z_0)/g'(z_0)$ if $g'(z_0)\neq 0$ — this happens only for $\ell=1$ — and that $\lim_{z\to z_0}[f'(z)/g'(z)]$ fails to exist if $g'(z_0)=0$, i.e., if $\ell>1$. Again when $m=1$, (8.3) is valid. Thus, (8.3) summarizes the behavior of $f'(z)/g'(z)$ as $z\to z_0$ for all admissible values of m and ℓ. Because the right-hand side of (8.3) is just a paraphrase of the right-hand side of (8.2), formula (8.1), with the stated interpretation, follows. ∎

The proof of Theorem 1.4 actually demonstrates that

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\lim_{z\to z_0}\frac{f'(z)}{g'(z)}=\cdots=\lim_{z\to z_0}\frac{f^{(m)}(z)}{g^{(m)}(z)}=\frac{f^{(m)}(z_0)}{g^{(m)}(z_0)}\neq 0$$

when $m=\ell$, and that

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=0$$

when $m>\ell$. This implies, naturally, that

$$\lim_{z\to z_0}\frac{g(z)}{f(z)}=0$$

when $m<\ell$. Accordingly, $|f(z)|/|g(z)|\to\infty$ as $z\to z_0$ in case $m<\ell$.

EXAMPLE 1.4. Evaluate $\lim_{z\to 0} z^{-2}(e^z-z-1)$.

We apply Theorem 1.4 twice in doing the calculation

$$\lim_{z\to 0}\frac{e^z-z-1}{z^2}=\lim_{z\to 0}\frac{e^z-1}{2z}=\lim_{z\to 0}\frac{e^z}{2}=\frac{1}{2}\ .$$

Since both $f(z)=e^z-z-1$ and $g(z)=z^2$ have double zeros at the origin, we were assured of a finite limit from the outset.

EXAMPLE 1.5. Evaluate $\lim_{z\to 0}(z^3\cos z-z^3-2z^4)/(e^{z^4}-1)$.

We once again appeal to Theorem 1.4 and compute

$$\lim_{z\to 0}\frac{z^3\cos z-z^3-2z^4}{e^{z^4}-1}=\lim_{z\to 0}\frac{3z^2\cos z-z^3\sin z-3z^2-8z^3}{4z^3e^{z^4}}$$

$$= \lim_{z \to 0} \frac{3\cos z - z\sin z - 3 - 8z}{4z} \lim_{z \to 0} e^{-z^4}$$

$$= \lim_{z \to 0} \frac{-3\sin z - \sin z - z\cos z - 8}{4} = -2 .$$

Notice that we did not mechanically continue to take derivatives here until the limit popped out, as l'Hospital's rule would really permit us to do. Instead, we combined the use of l'Hospital's rule with some simple algebraic manipulation in order to shorten the calculation.

1.3 Discrete Sets, Discrete Mappings

Let U be an open set in the complex plane. A subset E of U is termed a *discrete subset of* U if E has no limit point that belongs to U. (Recall: for z_0 to be a limit point of E means that $E \cap \Delta^*(z_0, r) \neq \phi$ for every $r > 0$ or, equivalently, that there is a sequence $\langle z_n \rangle$ in $E \sim \{z_0\}$ such that $z_n \to z_0$.) Examples of discrete subsets of U include the empty set ϕ, any finite subset of U (a finite set has no limit points, period), and a set of the type $E = \{z_n : n = 1, 2, 3, \cdots\}$, where $\langle z_n \rangle$ is a sequence in U without any accumulation points in U. We emphasize that a discrete subset of U is allowed to have — indeed, is likely to have — limit points in $\mathbb{C} \sim U$.

We record some general observations about a discrete subset E of a plane open set U. First of all, E must be a set of isolated points: if z_0 belongs to E, then there exists an $r > 0$ for which $E \cap \Delta(z_0, r) = \{z_0\}$. (Otherwise, z_0 would itself be a limit point of E in U.) Secondly, the set $U \sim E$ is open and, if U is domain, then $U \sim E$ is also a domain (Exercise 5.4). Finally, if K is an arbitrary compact set in U, then $K \cap E$ has at most finitely many elements. (If $K \cap E$ were an infinite set, then we could construct a sequence $\langle z_n \rangle$ in $K \cap E$ with the property that $z_n \neq z_m$ for $n \neq m$. This sequence would have an accumulation point in K — hence, in U. Any such accumulation point would clearly be a limit point of E in U, contrary to the definition of a discrete subset of U.)

Consider again an open set U in $\mathbb{C}$, and let f be a complex-valued function whose domain-set includes U. We speak of f as a *discrete mapping of* U if for each fixed complex number w the set $E_w = \{z \in U : f(z) = w\}$ is a discrete subset of U. (This does, of course, allow for the possibility of E_w being empty.) In particular, for each fixed w the set of solutions z in U to the equation $f(z) = w$ — assuming any such exist — is a set of isolated points of U.

The reason for introducing the above terminology is supplied by the following important theorem.

Theorem 1.5. (Discrete Mapping Theorem) *If a function f is analytic and non-constant in a plane domain D, then f is a discrete mapping of D.*

Proof. Fix w in $\mathbb{C}$ and set $E = E_w = \{z \in D : f(z) = w\}$. We must prove that E is a discrete subset of D. For this, assume that z_0 is a limit point of E. We are required to demonstrate that z_0 lies in $\mathbb{C} \sim D$. We shall suppose the contrary to be true (i.e., that z_0 belongs to D) and derive a contradiction. Let $\langle z_n \rangle$ be a sequence in $E \sim \{z_0\}$ such that $z_n \to z_0$. The continuity of f at z_0 insures that

$$f(z_0) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} w = w$$

and thus places z_0 in the set E. Invoking Corollary 1.3, we express $f(z)$ for z in D as

$$f(z) = w + (z - z_0)^m \, g(z) \ ,$$

where m is a positive integer and where $g: D \to \mathbb{C}$ is an analytic function for which $g(z_0) \neq 0$. Putting to use the fact that g, too, is continuous at z_0, we choose a disk $\Delta = \Delta(z_0, r)$ in D such that $g(z) \neq 0$ holds for every z in Δ. This implies that $f(z) \neq w$ whenever z is a point of the punctured disk $\Delta^*(z_0, r)$. The result: $E \cap \Delta^*(z_0, r) = \phi$, in contradiction with the definition of z_0 as a limit point of E. The contradiction is traceable directly to the assumption that z_0 lies in D. Therefore, any and all limit points of E must belong to $\mathbb{C} \sim D$. ∎

We take special note of the following corollary of Theorem 1.5: *the set of zeros of a function in a domain D where that function is analytic and non-constant, if non-empty, consists entirely of isolated points.* Two other corollaries of the discrete mapping theorem are frequently cited. The first of these is known as the "Principle of Analytic Continuation." It comes into play in conjunction with the problem of extending (or "continuing") a function known to be analytic in some set to a function that is analytic in a larger set. (For more on this subject see Chapter X.)

Corollary 1.6. (Principle of Analytic Continuation) *If functions f and g are analytic in a domain D and if $f(z) = g(z)$ for all z belonging to some subset A of D that has a limit point in D, then $f(z) = g(z)$ for every z in D.*

Proof. The function $h = g - f$ is analytic in D, and its set of zeros in that domain is not a discrete subset of D, for h has a zero at each point of A, a set that by hypothesis has a limit point in D. In view of Theorem 1.5, h must vanish identically in D or, what is the same, $f(z) = g(z)$ must hold everywhere in D. ∎

The second corollary alluded to above states that the collection of functions which are analytic in a domain D is free of so-called "zero divisors." This term refers to a non-zero element in a multiplicative algebraic system whose product with some other non-zero element is zero. Such elements exist, for instance, in the system of 2×2 real matrices: if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and

$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, the zero in that system, even though both A and B are non-zero.

Corollary 1.7. *If functions f and g are analytic in a domain D and if $f(z)g(z) = 0$ for every z in D, then either $f(z) = 0$ for every z in D or $g(z) = 0$ for every z in D.*

Proof. Suppose that there is a point z_0 of D for which $f(z_0) \neq 0$. We prove that in this situation $g(z) = 0$ throughout the domain. By the continuity of f we can choose an open disk $\Delta = \Delta(z_0, r)$ contained in D such that the condition $f(z) \neq 0$ persists for all z in Δ. But then $g(z) = 0$ is true whenever z lies in Δ, showing that the zeros of g in D are not isolated points. By Theorem 1.5, g must be constant in D and, thus, must vanish identically there. ∎

In a third application of the discrete mapping theorem, we tie up a loose end that we left hanging in Section III.4.1. There it was asserted that a branch of the inverse of an analytic function is automatically analytic. We now have the tools to prove this.

Theorem 1.8. *Suppose that U is an open set in the complex plane, that $f: U \to \mathbb{C}$ is an analytic function, and that g is a branch of f^{-1} in a domain D. Then g is an analytic function.*

Proof. By definition, $g: D \to U$ is nothing but a continuous function endowed with the property that $f[g(z)] = z$ for every z in D. We must show that g is differentiable at every point of D. Let G denote the component of U that contains the connected set $g(D)$. (Recall Theorems II.3.8 and II.3.7.) If $w_1 = g(z_1)$ and $w_2 = g(z_2)$, where z_1 and z_2 are distinct points of D, then $f(w_1) = z_1 \neq z_2 = f(w_2)$, which shows that that f is not constant in G. As a consequence, f' does not vanish identically there. It follows from the discrete mapping theorem, applied to f', that $E = \{w \in G: f'(w) = 0\}$ is a discrete subset of G. Now consider an arbitrary point z_0 of D and write $w_0 = g(z_0)$. If w_0 is not an element of E, then the differentiability of g at z_0 is assured by Theorem III.4.1. If the theorem we are now trying to establish were already known to be true, the chain rule would imply that $f'(w_0)g'(z_0) = 1$, so it could not really happen that w_0 is a point of E! In the present proof, however, we do not have the benefit of such hindsight. Here we must treat the possibility of w_0 being in E as genuine — and somehow get around it. Assuming then that w_0 does belong to E, we profit by the discreteness of that set to choose $r > 0$ for which $E \cap \Delta(w_0, r) = \{w_0\}$. Next, we use the continuity of g at z_0 to select an open disk $\Delta = \Delta(z_0, s)$ in D such that $g(\Delta)$ is a subset of $\Delta(w_0, r)$. Because the function g is univalent and $g(z_0) = w_0$, the image $g(z)$ of any point z in the punctured disk $\Delta^* = \Delta^*(z_0, s)$ lies in $\Delta^*(w_0, r)$ — hence, is not a point of E. As above, Theorem III.4.1 certifies that g is differentiable at every point of

Δ^*; i.e., g is analytic in Δ^*. Since g is continuous in Δ, we can appeal to Theorem V.3.4 and conclude that g is differentiable at z_0 as well. The differentiability of g throughout D is thus demonstrated. ∎

2 Isolated Singularities

2.1 Definition and Classification of Isolated Singularities

We say that a function f has an *isolated singularity* at a point z_0 of the complex plane provided there exists an $r > 0$ with the property that f is analytic in the punctured disk $\Delta^*(z_0, r)$, yet not analytic in the full open disk $\Delta(z_0, r)$. Given that f is analytic in $\Delta^*(z_0, r)$, this situation can come about for one of two reasons: either z_0 does not belong to the domain-set of f from the start or, alternatively, z_0 is a member of that domain-set, but is a point at which f is discontinuous. (Recall: if f is analytic in $\Delta^*(z_0, r)$ and continuous in $\Delta(z_0, r)$, then by Theorem V.3.4 it is actually analytic in $\Delta(z_0, r)$.) The first case is exemplified by the function $f(z) = z^{-1}$ at $z_0 = 0$; at $z_0 = 1$ the function defined by $f(z) = z$ for $z \neq 1$ and $f(1) = 2$ fits the second description. We call a function f *analytic modulo isolated singularities in an open set* U under the following conditions: there is a discrete subset E of U, the *singular set of* f *in* U, with the feature that f is analytic in the open set $U \sim E$, but has a singularity, in the sense just defined, at each point of E. This usage is not intended to bar the possibility that the singular set E is empty, which would simply mean that f is analytic in U.

Assume now that f is a function with an isolated singularity at z_0. Let $\Delta^* = \Delta^*(z_0, r)$ be a punctured disk in which f is analytic. It follows from Theorem VII.3.6 that f can be represented in Δ^* as the sum of a uniquely determined Laurent series centered at z_0: $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$. The singularity of f at z_0 falls into one of three categories, depending on the character of this Laurent expansion. We declare f to have a *removable singularity* at z_0 if $a_n = 0$ for every negative index n; to have a *pole* at z_0 if $a_n \neq 0$ holds for at least one, but for at most finitely many negative values of n; to have an *essential singularity* at z_0 if $a_n \neq 0$ is true for an infinite number of negative integers n. Since the given Laurent series $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ obviously has inner radius of convergence $\rho_I = 0$, we can appeal to Theorem VII.3.5 and affirm that $S(z) = \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ defines an analytic function with domain-set $\mathbb{C} \sim \{z_0\}$. The function S is called the *singular part* (or *principal part*) *of* f *at* z_0. The difference $f - S$ also has an isolated singularity at z_0, and its Laurent series in Δ^* is easily written down, being nothing more than the Taylor series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, the non-singular part of f at z_0. As a consequence, the function $f - S$ has a removable singularity at z_0. The coefficient a_{-1} that appears in the singular function S has special significance, as we shall soon learn. This number is known as the *residue of f at z_0*. The notation $\operatorname{Res}(z_0, f)$ (or, as an option, $\operatorname{Res}[z_0, f(z)]$) represents this quantity.

We proceed next to subject each kind of isolated singularity to closer scrutiny.

2.2 Removable Singularities

Consider a function f that is analytic in a punctured disk $\Delta^* = \Delta^*(z_0, r)$ and has an isolated singularity at z_0. Suppose initially that the singularity is removable. According to our definition this means that the Laurent series expansion of f in Δ^* reduces to the form $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + \cdots$. If we define — or redefine, as the case may be — the value of f at z_0 by setting $f(z_0) = a_0$, we arrive at a function that, as the sum of a Taylor series in the disk $\Delta = \Delta(z_0, r)$, is certainly analytic there. In particular, we obtain a function that is differentiable at z_0. Conversely, if we make no presupposition about the nature of its singularity at z_0, but if we are somehow able to assign to the function f a value at z_0 that causes the resulting function to become differentiable at that point, then the extended function becomes analytic in Δ and so can be expanded there in a Taylor series centered at z_0. The restriction of this Taylor series to Δ^* furnishes the Laurent series development of the original function f in Δ^*, from which fact we infer that f has a removable singularity at z_0. Conclusion: *an isolated singularity of a function f at a point z_0 is removable if and only if $f(z_0)$ can be defined — or redefined — so as to render f differentiable* at z_0. Take, for example, the function $f(z) = z^{-1} \sin z$. It is analytic in its domain-set $D = \{z : 0 < |z| < \infty\}$ — hence, exhibits an isolated singularity at $z_0 = 0$. For z in D we have the expansion

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots ,$$

which shows that the singularity of f at the origin is removable. By defining $f(0) = 1$ we effectively "remove" the singularity and, in the process, create out of f an entire function.

It is possible to characterize removable singularities without reference to Laurent series. One theorem that does so is known as "Riemann's Extension Theorem." This result has already been discussed in Chapter VI (Theorem VI.3.6), but no harm is done by presenting a second version of it here.

Theorem 2.1. (Riemann Extension Theorem) *Let a function f have an isolated singularity at a point z_0. The singularity is removable if and only if f is bounded in some punctured disk centered at z_0.*

Proof. Suppose at first that f is bounded in the punctured disk $\Delta^* = \Delta^*(z_0, r)$, say $|f(z)| \leq m$ for every z in Δ^*. By taking r sufficiently small we may, of course, assume that f is analytic in Δ^*. The coefficient a_n in the Laurent series expansion of f in Δ^* is given by

$$a_n = \frac{1}{2\pi i} \int_{|z-z_0|=s} \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

for any s satisfying $0 < s < r$. Standard estimation procedures lead to

$$|a_n| \leq \frac{1}{2\pi} \int_{|z-z_0|=s} \frac{|f(z)|\,|dz|}{|z-z_0|^{n+1}} \leq \frac{m}{s^n}\,.$$

When $n < 0$, we can let $s \to 0$ in this inequality and conclude that $|a_n| = 0$. Therefore, $a_n = 0$ for every negative value of n, which is exactly what the definition requires in order for the singularity of f at z_0 to be classified as removable.

As to the converse, assume that the singularity of f at z_0 is given to be removable. We are free to suppose that the value $f(z_0)$ has been defined — or adjusted — to make f differentiable at z_0. Naturally, $|f(z_0)| = \lim_{z \to z_0} |f(z)|$ must then be true. We can thus choose $r > 0$ in a way to insure, for instance, that $|f(z)| < |f(z_0)| + 1$ holds for every z in the punctured disk $\Delta^* = \Delta^*(z_0, r)$. This shows that f is bounded in Δ^*. ∎

A corollary of the Riemann extension theorem (or, rather, a corollary of this theorem and the deliberations that preceded it) is the following:

Theorem 2.2. *Let a function f have an isolated singularity at a point z_0. The singularity is removable if and only if $\lim_{z \to z_0} |f(z)|$ exists.*

Needless to say, the condition that $\lim_{z \to z_0} f(z)$ exists could be substituted for the existence of $\lim_{z \to z_0} |f(z)|$ in Theorem 2.2 to produce another characterization of a removable singularity. A key point in Theorems 2.1 and 2.2, however, is that the nature of the singularity is already discernible in the behavior of the magnitude of f, a real quantity, as z approaches z_0.

2.3 Poles

We turn next to a function f that has a pole at a point z_0. Assuming that f is analytic in the punctured disk $\Delta^* = \Delta^*(z_0, r)$, we expand it in a Laurent series there: $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$. The definition of a pole demands the existence of a positive integer m such that $a_{-m} \neq 0$, while $a_{-n} = 0$ whenever $n > m$. In other words, the expansion takes the form

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-1}}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n\,, \tag{8.4}$$

with $a_{-m} \neq 0$. Under these conditions we say that f has a *pole of order* m at z_0. (Paralleling the situation with zeros, we often speak of a pole of order one as a *simple pole*, a pole of order two as a *double pole*, etc.) If we multiply both sides of (8.4) by $(z-z_0)^m$, we obtain for all z in Δ^*

$$(z-z_0)^m f(z) = a_{-m} + a_{-m+1}(z-z_0) + \cdots = \sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n \ .$$

The last series is a Taylor series that converges at every point of the disk $\Delta = \Delta(z_0, r)$. It follows that the function g defined by the formula $g(z) = \sum_{n=0}^{\infty} a_{n-m}(z-z_0)^n$ is analytic in Δ, that $g(z_0) = a_{-m} \neq 0$, and that $f(z) = (z-z_0)^{-m} g(z)$ for every z in Δ^*. This establishes one direction in the following characterization of a pole of order m. The other direction is equally straightforward.

Theorem 2.3. *Let m be a positive integer. A function f that is analytic in a punctured disk $\Delta^* = \Delta^*(z_0, r)$ has a pole of order m at z_0 if and only if f can be represented in Δ^* in the fashion*

$$f(z) = \frac{g(z)}{(z-z_0)^m} \ , \tag{8.5}$$

where g is a function that is analytic in the disk $\Delta(z_0, r)$ and obeys the condition $g(z_0) \neq 0$.

The residue at z_0 of a function f matching the description in (8.5) is just the coefficient of $(z-z_0)^{m-1}$ in the Taylor expansion of g about z_0. That coefficient is given by $g^{(m-1)}(z_0)/(m-1)!$. Since f itself is undefined at z_0, we prefer to express this as $\lim_{z \to z_0} g^{(m-1)}(z)/(m-1)!$. The result of this observation is a frequently applied residue formula: *under the assumption that f has a pole of order m at z_0,*

$$\operatorname{Res}(z_0, f) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \ . \tag{8.6}$$

Formula (8.6) is especially valuable in the case of a single or double pole, for which the task of evaluating the expression on the right-hand side usually remains quite manageable.

An operation that often causes poles to appear is forming quotients of analytic functions. To be specific, let g and h be functions that are analytic in an open disk $\Delta = \Delta(z_0, r)$, neither of them being identically zero there, and let $h(z_0) = 0$. We consider their quotient, $f = g/h$. We assume, as we certainly may by taking r suitably small, that both g and h are free of zeros in the punctured disk $\Delta^* = \Delta^*(z_0, r)$. This insures that the function f is analytic in Δ^*. Since f remains undefined at z_0, it has an isolated singularity at that point. In analyzing the character of this singularity we denote by m and ℓ the orders of the zeros that g and h, respectively, have at z_0. (N.B. We have deliberately not made the assumption that $g(z_0) = 0$. If $g(z_0) \neq 0$, we agree to set $m = 0$ and $g_1 = g$ in what follows.) In Δ we can write $g(z) = (z - z_0)^m g_1(z)$ and $h(z) = (z - z_0)^\ell h_1(z)$, where g_1 and h_1 are zero-free analytic functions defined in that disk. This leads to a representation $f(z) = (z - z_0)^{m-\ell} f_1(z)$ for the function f in Δ^*. Here $f_1 = g_1/h_1$ is both analytic and free of zeros in Δ. With the aid of Theorems 2.2 and 2.3 we conclude that the quotient f has a removable singularity at z_0 when $m \geq \ell$ and a pole of order $\ell - m$ at z_0 when $m < \ell$. Furthermore, when $m > \ell$ the process of removing this singularity endows f with a zero of order $m - \ell$ at z_0. One special instance of the preceding discussion deserves to be highlighted: *if an analytic function f has a zero of order m at z_0, then $1/f$ has a pole of order m at z_0.* Theorem 2.3 implies that something tantamount to a converse is likewise true: *if a function f has a pole of order m at z_0, then $1/f$ has a zero of order m there, in the sense that $1/f$ has a removable singularity at z_0 and that, upon its removal, $1/f$ acquires a zero of order m at that point.*

A look at some concrete examples will shed further light on the subject of poles.

EXAMPLE 2.1. Describe the character of the singularity that $f(z) = (e^z - 1)^{-1}$ has at the origin, and calculate $\text{Res}(0, f)$.

The entire function $g(z) = e^z - 1$ has a simple zero at the origin, so $f = 1/g$ has a simple pole there. We use (8.6) for the case $m = 1$, along with L'Hospital's rule, in computing

$$\text{Res}(0, f) = \lim_{z \to 0} z\, f(z) = \lim_{z \to 0} \frac{z}{e^z - 1} = \lim_{z \to 0} \frac{1}{e^z} = 1 \,.$$

EXAMPLE 2.2. Determine the nature of the singularity of $f(z) = (z - 1)^{-3} \sin \pi z$ at $z_0 = 1$. Compute $\text{Res}(1, f)$.

Since $g(z) = \sin \pi z$ has a simple zero at z_0, while $h(z) = (z - 1)^3$ has a zero of order three there, $f = g/h$ has a pole of order two at z_0. Referring to Example VII.3.13, we write down the Laurent expansion of f in $D = \{z : 0 < |z - 1| < \infty\}$:

$$\frac{\sin \pi z}{(z - 1)^3} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \pi^{2n+1} (z - 1)^{2n-2}}{(2n + 1)!}$$

$$= -\frac{\pi}{(z - 1)^2} + \frac{\pi^3}{3!} - \frac{\pi^5 (z - 1)^2}{5!} + \cdots \,.$$

From this representation we read off directly that $\text{Res}(1, f) = 0$. Alternatively, we can apply (8.6) with $m = 2$ to arrive at the same conclusion:

$$\text{Res}(1, f) = \lim_{z \to 1} \frac{d}{dz} \left(\frac{\sin \pi z}{z - 1} \right) = \lim_{z \to 1} \frac{\pi (z - 1) \cos \pi z - \sin \pi z}{(z - 1)^2}$$

$$= \lim_{z\to 1} \frac{(-\pi^2)(z-1)\sin \pi z}{2(z-1)} = -\frac{\pi^2}{2} \lim_{z\to 1} \sin \pi z = 0 \ .$$

L'Hospital's rule justifies the third step in this computation.

EXAMPLE 2.3. Classify the singularity that $f(z) = [\sin z][\cos(z^3) - 1]^{-1}$ exhibits at the origin. Compute $\mathrm{Res}(0, f)$.

The function $g(z) = \sin z$ shows a simple zero at the origin, and $h(z) = \cos(z^3) - 1$ has a zero of multiplicity six there. The latter assertion is clear from the Taylor expansion

$$\cos(z^3) - 1 = -\frac{z^6}{2!} + \frac{z^{12}}{4!} - \cdots = z^6 \left(-\frac{1}{2} + \frac{z^6}{4!} - \cdots \right) .$$

As a result, $f = g/h$ is seen to have a pole of order five at the origin.

We want now to evaluate $\mathrm{Res}(0, f)$. We could certainly find this residue by applying formula (8.6) with $m = 5$. The amount of time and effort required to carry out the calculation, however, would be, if not prohibitive, at least discouraging. Fortunately, a more efficient method is available here. It involves doing some simple algebraic manipulation of power series in order to display explicitly the singular part of f at the origin, from which the desired residue is instantly retrievable. In the ensuing discussion — and in others later — the symbol $O[(z - z_0)^N]$ will serve as an abbreviation for "an expression containing only terms of degree N and above." More precisely, $O[(z - z_0)^N]$ will indicate a well-determined, but unspecified Taylor series centered at z_0 whose radius of convergence is positive and in which the coefficient of $(z - z_0)^n$ is zero for $n < N$. The notation $O(1) = O[(z - z_0)^0]$ will signify a "generic" Taylor series centered at z_0 with a positive radius of convergence. Just to demonstrate the use of this notation, we might elect to write

$$\sin z = z + O(z^3) \ . \tag{8.7}$$

The right-hand side is read "z plus a Taylor series that contains only terms of order three and above." In this instance $O(z^3)$ stands for the series $-(z^3/3!) + (z^5/5!) - \cdots$. We operate algebraically with expressions of type (8.7) in the manner suggested by the following illustration:

$$(z+1)\sin z = (z+1)\big[z + O(z^3)\big] = z^2 + z\,O(z^3) + z + O(z^3)$$

$$= z + z^2 + O(z^3) \ .$$

We remark that $O(z^3)$ represents a different power series in the final expression than it did in the penultimate one. It is this flexibility that makes the notation so ideal for our present purposes.

Returning to the example at hand, we express $f(z)$ in the form

$$f(z) = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots}{-\frac{z^6}{2!} + \frac{z^{12}}{4!} - \frac{z^{18}}{6!} + \cdots} = \frac{1}{z^5}\left(\frac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots}{-\frac{1}{2} + \frac{z^6}{4!} - \frac{z^{12}}{6!} + \cdots}\right) .$$

The idea is to find a_0, a_1, a_2, a_3, and a_4 so that

$$\frac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots}{-\frac{1}{2} + \frac{z^6}{4!} - \frac{z^{12}}{6!} + \cdots} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + O(z^5) ,$$

for then dividing by z^5 will expose the singular part of f at the origin. In the computations that follow we systematically disregard terms of degree five or more, terms that do not influence the singular part of f, by absorbing them into the $O(z^5)$ term. Thus, we write

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots = 1 - \frac{z^2}{6} + \frac{z^4}{120} + O(z^5)$$

and

$$-\frac{1}{2} + \frac{z^6}{4!} - \frac{z^{12}}{6!} + \cdots = -\frac{1}{2} + O(z^5) = -\frac{1}{2}\Big[1 - O(z^5)\Big] .$$

Now $|O(z^5)| < 1$ when z is near the origin, so for z of small magnitude we can use the geometric series to obtain

$$\left(-\frac{1}{2} + \frac{z^6}{4!} - \frac{z^{12}}{6!}\right)^{-1} = -2\Big[1 - O(z^5)\Big]^{-1}$$

$$= -2\left\{1 + O(z^5) + \Big[O(z^5)\Big]^2 + \cdots\right\} = -2\Big[1 + O(z^5)\Big] = -2 + O(z^5) .$$

This leads to

$$f(z) = \frac{1}{z^5}\left[1 - \frac{z^2}{6} + \frac{z^4}{120} + O(z^5)\right]\Big[-2 + O(z^5)\Big]$$

$$= \frac{1}{z^5}\left[-2 + \frac{z^2}{3} - \frac{z^4}{60} + O(z^5)\right]$$

$$= -\frac{2}{z^5} + \frac{1}{3z^3} - \frac{1}{60z} + O(1) .$$

The singular part S of f at the origin is immediately recognized to be

$$S(z) = -\frac{2}{z^5} + \frac{1}{3z^3} - \frac{1}{60z} ,$$

implying that $\operatorname{Res}(0,f) = -1/60$.

EXAMPLE 2.4. Find the singular part of $f(z) = (e^z - 1)^{-3}$ at the origin, and use it to determine $\operatorname{Res}(0,f)$.

The function we are dealing with in this example has a pole of order 3 at the origin. To exhibit its singular part there we employ the same method as in the preceding example. We first calculate

$$\begin{aligned}(e^z-1)^3 &= \left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right)^3 \\ &= z^3\left[1 + \frac{z}{2} + \frac{z^2}{6} + O(z^3)\right]^3 \\ &= z^3\left[1 + \frac{z}{2} + \frac{z^2}{6} + O(z^3)\right]^2\left[1 + \frac{z}{2} + \frac{z^2}{6} + O(z^3)\right] \\ &= z^3\left[1 + z + \frac{7z^2}{12} + O(z^3)\right]\left[1 + \frac{z}{2} + \frac{z^2}{6} + O(z^3)\right] \\ &= z^3\left[1 + \frac{3z}{2} + \frac{5z^2}{4} + O(z^3)\right] .\end{aligned}$$

An appeal to the geometric series gives

$$\begin{aligned}&\left[1 + \frac{3z}{2} + \frac{5z^2}{4} + O(z^3)\right]^{-1} \\ &= 1 - \left[\frac{3z}{2} + \frac{5z^2}{4} + O(z^3)\right] + \left[\frac{3z}{2} + \frac{5z^2}{4} + O(z^3)\right]^2 + O(z^3) \\ &= 1 - \frac{3z}{2} - \frac{5z^2}{4} + \frac{9z^2}{4} + O(z^3) \\ &= 1 - \frac{3z}{2} + z^2 + O(z^3) .\end{aligned}$$

Consequently, we obtain the expansion

$$f(z) = \frac{1}{(e^z-1)^3} = \frac{1}{z^3} - \frac{3}{2z^2} + \frac{1}{z} + O(1) .$$

The requested singular part is now seen to be

$$S(z) = \frac{1}{z^3} - \frac{3}{2z^2} + \frac{1}{z} ,$$

which reveals that $\text{Res}(0, f) = 1$.

Theorem 2.2 has an analogue characterizing poles.

Theorem 2.4. *Let a function f have an isolated singularity at a point z_0. The singularity is a pole if and only if $\lim_{z\to z_0}|f(z)| = \infty$. Moreover, the singularity is a pole of order m if and only if m is the unique positive exponent for which $\lim_{z\to z_0}|z - z_0|^m|f(z)|$ is a positive real number.*

Proof. Assume first that f has a pole at z_0, say of order m. In some punctured disk $\Delta^* = \Delta^*(z_0, r)$ we can write $f(z) = (z - z_0)^{-m}g(z)$, where g is a function that is analytic in $\Delta = \Delta(z_0, r)$ and meets the condition $g(z_0) \neq 0$. It follows that

$$\lim_{z\to z_0} |f(z)| = \lim_{z\to z_0} \frac{|g(z)|}{|z - z_0|^m} = \infty$$

and that, for any positive real number ℓ,

$$\text{(8.8)} \qquad \lim_{z\to z_0} |z - z_0|^\ell |f(z)| = \begin{cases} \infty & \text{if } \ell < m\ , \\ |g(z_0)| & \text{if } \ell = m\ , \\ 0 & \text{if } \ell > m\ . \end{cases}$$

Only when $\ell = m$ is the limit in (8.8) a positive real number.

Turning to the converses, let it be assumed that $|f(z)| \to \infty$ as $z \to z_0$. We can then fix a punctured disk $\Delta^* = \Delta^*(z_0, r)$ such that f is analytic in Δ^* and satisfies $|f(z)| \geq 1$ throughout Δ^*. From this it is evident that $h = 1/f$ has an isolated singularity at z_0. Since $|h(z)| \leq 1$ for every z in Δ^*, the Riemann extension theorem informs us that the singularity of h at z_0 is removable. We remove it by defining $h(z_0) = \lim_{z\to z_0}h(z) = 0$. As h is obviously not identically zero in Δ^*, it acquires through this extension process a zero of some positive order at z_0, implying that $f = 1/h$ has a pole of that same order there.

Finally, if it is known that for some $m > 0$ the quantity $|z - z_0|^m|f(z)|$ tends to a positive real limit as $z \to z_0$, then certainly

$$\lim_{z\to z_0} |f(z)| = \lim_{z\to z_0} \frac{|z - z_0|^m\, |f(z)|}{|z - z_0|^m} = \infty\ .$$

We infer that f has a pole at z_0. A glance at (8.8) discloses that this pole is necessarily of order m. ∎

We introduce the expression "a function f has no worse than a pole at a point z_0" as a catch-all phrase to describe the following state of affairs: there exists a radius $r > 0$ such that the given function f is either analytic in the full open disk $\Delta = \Delta(z_0, r)$ or else analytic in the punctured disk $\Delta^* = \Delta^*(z_0, r)$ with a pole or removable singularity at its center. Except when f vanishes identically near z_0, it is always possible by choosing the radius r appropriately small to arrange that such a function be represented in Δ^* as $f(z) = (z - z_0)^m f_1(z)$, where m is an integer and f_1 is a function that is both analytic and zero-free in Δ. Partly by taking advantage of this sort of representation, partly by direct examination of Laurent expansions, one deduces the contents of the next theorem. The detailed proof is left as an exercise (Exercise 5.30).

Theorem 2.5. *If neither of two functions f and g has worse than a pole at a point z_0, then none of the functions f', $f+g$, fg, and, unless g vanishes identically in some punctured disk centered at z_0, f/g has worse than a pole at z_0.*

2.4 Meromorphic Functions

Let U be an open set in the complex plane. We characterize a function f as *meromorphic in* U provided f has at no point of U worse than a pole. Put differently, this definition requires f to be analytic modulo isolated singularities in U and insists that any non-removable singularities of f in that set be poles. It does, however, permit f to have removable singularities in U. Many authors prefer to define a function as meromorphic in U if it is analytic in U except for possible poles, with no heed paid to removable singularities. Of course, under the latter definition it is not technically true that the sum of functions meromorphic in U is automatically meromorphic there, for the sum may well have removable singularities — e.g., if $f(z) = z^{-1}$ and $g(z) = z - z^{-1}$, then f and g have poles at the origin, but $f+g$ has a removable singularity there. Whatever definition of a meromorphic function one opts to use, it is necessary sooner or later to deal with the niggling annoyances caused by removable singularities. We handle this problem by adopting the following "removable singularity policy" in conjunction with meromorphic functions: *whenever it is established that a function f is meromorphic (according to our definition of the term) in an open set U, it will always be tacitly assumed that f undergoes immediate modification to rid it of any removable singularities in U, this by defining or redefining the function at each such singular point so as to make it continuous there.*

Any function that is analytic in an open set U certainly qualifies as meromorphic in U. The most obvious example of a function that is meromorphic in the entire plane $\mathbb{C}$ is a rational function of z, for the only singularities such a function has in $\mathbb{C}$ are removable singularities or poles at the zeros of its denominator. The function $f(z) = \tan z$, which has a simple pole at every odd multiple of $\pi/2$ but is otherwise free of singularities in $\mathbb{C}$, is another function that is meromorphic in the whole plane. The function defined by $f(z) = (z^2+1)^{-2}e^{1/z}$ is meromorphic in $U = \mathbb{C} \sim \{0\}$, where its only singularities are double poles at the points i and $-i$. It is not, however, meromorphic in $\mathbb{C}$, for its singularity at the origin turns out to be an essential singularity.

Suppose that functions f and g are both meromorphic in an open set U. Theorem 2.5 implies directly that the functions f', $f+g$, fg, and, unless g is identically zero in some component of U, f/g are likewise meromorphic in this open set. Notice especially that the quotient f/g, where f and g are analytic functions in U and where g does not vanish identically in any component of U, is meromorphic in U. It is true (but not so easy to prove) that every function which is meromorphic in U admits a representation as the quotient of two functions which are analytic in U. We shall confirm this fact in Chapter X.

2.5 Essential Singularities

The last — and by far most interesting — type of isolated singularity that a function f can exhibit at a point z_0 is an essential singularity, one for which the singular part of f at z_0 contains infinitely many non-zero terms. The function $f(z) = e^{1/z} = 1 + z^{-1} + (z^{-2}/2!) + \cdots$ at $z_0 = 0$ provides a simple example of this kind of singularity. Dealing with essential singularities is, in general, a much more delicate matter than working with removable singularities or poles. There is, for instance, no foolproof formula along the lines of (8.6) for computing the residue of a function at an essential singularity. Frequently one is forced to extract such information directly from a Laurent expansion. Here are two examples.

EXAMPLE 2.5. Determine the character of the singularity of $f(z) = z\cos[(z-1)^{-1}]$ at the point $z_0 = 1$, and calculate $\mathrm{Res}(1, f)$.

We do the problem by deriving the Laurent expansion of f in the set $D = \mathbb{C} \sim \{1\}$. To start, we substitute $(z-1)^{-1}$ for z in the Taylor expansion of $\cos z$ about the origin and obtain

$$\cos\left(\frac{1}{z-1}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(z-1)^{2n}} = 1 - \frac{1}{2!(z-1)^2} + \frac{1}{4!(z-1)^4} - \cdots .$$

We are thus led to

$$\begin{aligned} f(z) = z\cos\left(\frac{1}{z-1}\right) &= (z-1)\cos\left(\frac{1}{z-1}\right) + \cos\left(\frac{1}{z-1}\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(z-1)^{2n-1}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!(z-1)^{2n}} \\ = (z-1) + 1 - \frac{1}{2!(z-1)} &- \frac{1}{2!(z-1)^2} + \frac{1}{4!(z-1)^3} + \frac{1}{4!(z-1)^4} - \cdots , \end{aligned}$$

which makes it evident that f has an essential singularity at 1 and that $\text{Res}(1, f) = -1/2$.

EXAMPLE 2.6. Compute the Laurent coefficients of $f(z) = \exp(z + z^{-1})$ in $D = \{z : 0 < |z| < \infty\}$. Use the information to classify the singularity of f at the origin and to determine $\text{Res}(0, f)$.

The Laurent coefficient a_n of f in D is given by

$$a_n = \frac{1}{2\pi i} \int_{|z|=1} z^{-n-1} f(z)\, dz . \tag{8.9}$$

Fixing n, we write

$$z^{-n-1} f(z) = z^{-n-1} e^{1/z} e^z = z^{-n-1} e^{1/z} \sum_{k=0}^{\infty} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{z^{k-n-1} e^{1/z}}{k!} .$$

As the convergence of the last series is uniform on the circle $K(0, 1)$, we are justified in integrating it term by term to get

$$\int_{|z|=1} z^{-n-1} f(z)\, dz = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{|z|=1} z^{k-n-1} e^{1/z}\, dz . \tag{8.10}$$

Since the Laurent expansion of $g(z) = e^{1/z}$ in D is already known to us — namely,

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k! z^k}$$

— we can appeal to the analogue of (8.9) for the function g and deduce that

$$\int_{|z|=1} z^{k-n-1} e^{1/z}\, dz = \begin{cases} \dfrac{2\pi i}{(k-n)!} & \text{if } k - n \geq 0 , \\ 0 & \text{if } k - n < 0 . \end{cases}$$

Coupled with (8.9) and (8.10) this shows that

$$a_n = \sum_{k=\max\{0,n\}}^{\infty} \frac{1}{k!(k-n)!} .$$

Noting that $a_n > 0$ for all negative n, we infer that the singularity of f at the origin is an essential singularity. Taking $n = -1$ yields

$$\text{Res}(0, f) = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} .$$

Although the representation of a_n — and, in particular, of $\text{Res}(1, f)$ — as the sum of an infinite series may not seem altogether satisfactory, it is about the best we can hope for in the present example. This phenomenon is not untypical of residue calculations involving essential singularities.

In view of Theorems 2.2 and 2.4 we arrive by a process of elimination at a characterization of isolated essential singularities.

Theorem 2.6. *Let a function f have an isolated singularity at a point z_0. The singularity is essential if and only if* $\lim_{z\to z_0}|f(z)|$ *fails to exist either in the strict sense or as an infinite limit.*

Theorem 2.6 does not even begin to tell the tale of how truly ill-behaved a function can be in the vicinity of an isolated essential singularity. A better sense of that behavior is conveyed by a result of Felice Casorati (1835-1890) and Weierstrass.

Theorem 2.7. (Casorati-Weierstrass Theorem) *If a function f is analytic in a punctured disk $\Delta^* = \Delta^*(z_0, r)$ and has an essential singularity at its center, then $f(\Delta^*)$ is dense in the complex plane — i.e., the set $\mathbb{C} \sim f(\Delta^*)$ has no interior points.*

Proof. We argue by contradiction: we assume that $\mathbb{C} \sim f(\Delta^*)$ does have interior points and deduce that f can have no worse than a pole at z_0, contrary to hypothesis. Let w_0 be an interior point of $\mathbb{C} \sim f(\Delta^*)$, and let $s > 0$ be such that the disk $\Delta(w_0, s)$ is contained in $\mathbb{C} \sim f(\Delta^*)$. Then $|f(z) - w_0| \geq s$ plainly holds for every z in Δ^*. It follows that the function $g: \Delta^* \to \mathbb{C}$ defined by $g(z) = [f(z)-w_0]^{-1}$ is analytic and satisfies $|g(z)| \leq s^{-1}$ throughout Δ^*. Riemann's extension theorem tells us that the singularity of g at z_0 is removable. Moreover, since g is clearly zero-free in Δ^*, its reciprocal $1/g$ also has an isolated singularity at z_0, a singularity that must be either a pole or a removable singularity, depending on whether $\lim_{z\to z_0}|g(z)|$ is zero or not. This, in turn, insures that the singularity of $f = w_0 + (1/g)$ at z_0 can be no worse than a pole, the contradiction anticipated. Accordingly, the set $\mathbb{C} \sim f(\Delta^*)$ must be devoid of interior points. ∎

The Casorati-Weierstrass theorem states that the image of a punctured disk $\Delta^*(z_0, r)$, no matter how small, under a function f with an isolated essential singularity at z_0 effectively fills up the whole complex plane. Among its consequences is this: corresponding to any given complex number w_0 there exists a sequence $\langle z_n \rangle$ in $\mathbb{C} \sim \{z_0\}$ such that $z_n \to z_0$ and $f(z_n) \to w_0$. (Apply the Casorati-Weierstrass theorem to choose for $n = 1, 2, 3, \cdots$ a point z_n in $\Delta^*(z_0, 1/n)$ with $f(z_n)$ in $\Delta(w_0, 1/n)$.) An even more remarkable fact was announced in 1879 by Émile Picard:

Theorem 2.8. (Picard's Theorem) *If a function f is analytic in a punctured disk $\Delta^* = \Delta^*(z_0, r)$ and has an essential singularity at its center, then the set $\mathbb{C} \sim f(\Delta^*)$ contains at most one point.*

We shall not prove this deep and beautiful theorem, for the methods required to do so go beyond the scope of this book. (One standard proof is based on Theorem VII.4.6. Exercise 5.35 outlines the derivation of Picard's theorem from that result of Montel.) Picard's theorem implies the following fact concerning a function f that has an essential singularity at z_0: with allowance for one possible exceptional value of w_0, there exists for any complex number w_0 a sequence $\langle z_n \rangle$ in $\mathbb{C} \sim \{z_0\}$ such that $z_n \to z_0$ and such that $f(z_n) = w_0$ for every n. The function $f(z) = e^{1/z}$, which has an essential singularity at the origin and which maps each punctured disk centered there to $\mathbb{C} \sim \{0\}$, demonstrates that the "exceptional value" permitted by Picard's theorem may, in fact, exist. The existence of such a value is by no means, however, a foregone conclusion. For instance, the singularity of $f(z) = \sin(1/z)$ at the origin is also essential and, in this case, $f(\Delta^*) = \mathbb{C}$ for every punctured disk $\Delta^* = \Delta^*(0, r)$.

2.6 Isolated Singularities at Infinity

Having treated in some detail the types of isolated singularities a function might experience in the complex plane, we now briefly discuss the character of a function that is analytic in the complement of an arbitrarily large disk centered at the origin. To streamline the discussion we introduce the notation $\Delta^*(\infty, r)$ to represent the set $\{z : |z| > r^{-1}\}$ for $r > 0$. (N.B. This definition is designed to make sure that $\Delta^*(\infty, r)$ will be contained in $\Delta(\infty, s)$ when $r < s$. For further insight into the notation we refer the reader to Section 4 of this chapter.) We speak of a function that is analytic in $\Delta^*(\infty, r)$ for some $r > 0$ as having an *isolated singularity at* ∞. The example which immediately leaps to mind is that of an entire function, which is analytic in $\Delta^*(\infty, r)$ for every $r > 0$. The function $f(z) = \tan z$, by contrast, does not have an isolated singularity at ∞, because for each $r > 0$ the set $\Delta^*(\infty, r)$ includes poles of f.

Assuming that a function f is analytic in $\Delta^*(\infty, r)$, we can expand it there in a Laurent series centered at the origin, $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. The singularity of f at ∞ is subject to classification based on this expansion: f has a *removable singularity at* ∞ if $a_n = 0$ for every positive integer n; f has a *pole at* ∞ if there exists a positive integer m — the *order* of the pole — such that $a_m \neq 0$, whereas $a_n = 0$ for every n greater than m; f has an *essential singularity at* ∞ if $a_n \neq 0$ holds for infinitely many positive integers n. The entire function S defined by $S(z) = \sum_{n=1}^{\infty} a_n z^n$ is called the *singular* (or *principal*) *part of* f at ∞. Just as was true for singularities in the complex plane, the function $f - S$ has a removable singularity at ∞. For technical reasons that we shall not go into here the *residue of* f *at* ∞ is taken to be $-a_{-1}$ — not a_{-1} itself or some quantity associated with the singular part of f at ∞. In the way of simple illustrations we remark that $f(z) = z^{-1}$ has a removable singularity at ∞, that a polynomial function

$f(z) = a_0 + a_1 z + \cdots + a_m z^m$ of positive degree m has a pole of order m at ∞, and that any non-polynomial entire function ($f(z) = e^z$, to name one) has an essential singularity at ∞.

It is evident from the definition of the concept that a function f has an isolated singularity at ∞ if and only if the function g given by $g(z) = f(z^{-1})$ has an isolated singularity at the origin, in which event the type of singularity f exhibits at ∞ is precisely the same as the type of singularity g displays at the origin. For example, $f(z) = (z^4 + 2z^2 + 1)(z^2 - z - 3)^{-1}$ has a double pole at ∞ by reason of the fact that the associated function $g(z) = f(z^{-1}) = z^{-2}(1 + 2z^2 + z^4)(1 - z^3 - 3z^4)^{-1}$ quite obviously shows a double pole at 0. Through this mechanism it is possible to convert virtually any question about an isolated singularity at ∞ to one concerned with an isolated singularity at the origin, where results established earlier in this chapter can be brought to bear on it. To see this principle in action, consider a function f that is known to be analytic and bounded in $\Delta^*(\infty, r)$. The Riemann extension theorem for a singularity at a finite point would suggest that the singularity of f at ∞ ought to be removable. We actually prove this by passing to the function $g(z) = f(z^{-1})$, which is clearly analytic and bounded in $\Delta^*(0, r)$ — hence, which has a removable singularity at the origin. It follows that the singularity of f at ∞ is indeed removable, as expected. Similar reasoning establishes the Casorati-Weierstrass theorem for essential singularities at ∞: if a function f is analytic in $\Delta^* = \Delta^*(\infty, r)$ and has an essential singularity at ∞, then $f(\Delta^*)$ is dense in $\mathbb{C}$. Finally, we remark that the analogue of Theorem 2.5 in which the finite point z_0 gets replaced by ∞ remains a valid theorem.

3 The Residue Theorem and its Consequences

3.1 The Residue Theorem

We have laid sufficient groundwork for the proof of — as well as for many applications of — the result that is the centerpiece of this chapter.

Theorem 3.1. (Residue Theorem) *Suppose that a function f is analytic modulo isolated singularities in an open set U, that $E \neq \phi$ is the singular set of f in U, and that σ is a cycle in $U \sim E$ which is homologous to zero in U (Figure 1). Then*

$$\int_\sigma f(z)\,dz = 2\pi i \sum_{z \in E} n(\sigma, z)\,\mathrm{Res}(z, f) . \tag{8.11}$$

Proof. We first demonstrate that, even though the set E may have infinitely many elements, the condition $n(\sigma, z) \neq 0$ can hold for at most a

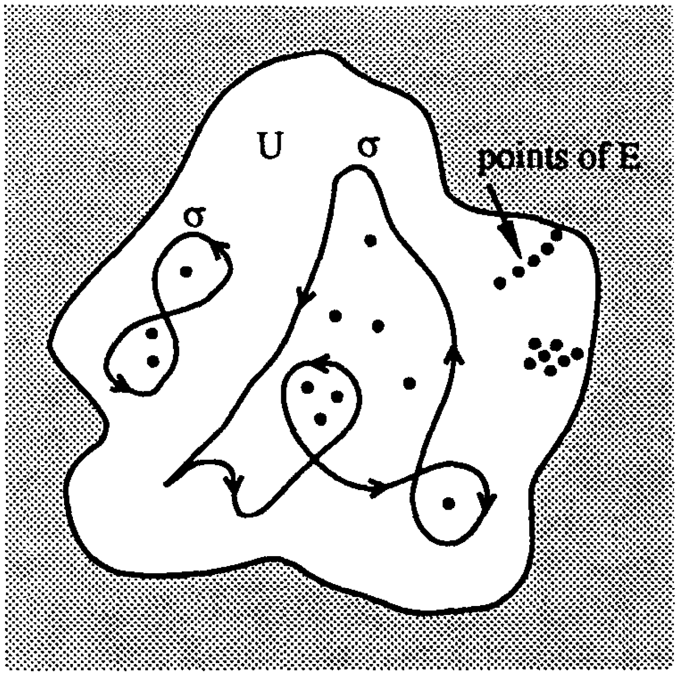

Figure 1.

finite number of those elements, with the result that the sum on the right-hand side of (8.11) reduces to a finite sum. Suppose, to the contrary, that $n(\sigma, z) \neq 0$ happens to be true for infinitely many points z belonging to E. This assumption enables us to choose a sequence $\langle z_k \rangle$ of distinct points of E such that $n(\sigma, z_k) \neq 0$ holds for every k. None of the points z_k can then lie in the unbounded component of $\mathbb{C} \sim |\sigma|$, a fact which implies that $\langle z_k \rangle$ is a bounded sequence. Therefore $\langle z_k \rangle$ must have at least one accumulation point in the complex plane. Choose such a point and label it z_0. The point z_0 is, among other things, a limit point of E. Because E, a discrete subset of U, has no limit points in U, z_0 has to be be located in $\mathbb{C} \sim U$. As σ is homologous to zero in U, we infer that $n(\sigma, z_0) = 0$. Now fix a disk $\Delta = \Delta(z_0, r)$ that does not intersect the set $|\sigma|$. Then $n(\sigma, z)$ remains constant as z varies over Δ; i.e., $n(\sigma, z) = n(\sigma, z_0) = 0$ whenever z is a point of Δ (Lemma V.2.1). On the other hand, since the sequence $\langle z_k \rangle$ accumulates at z_0, we can pick an N for which z_N belongs to Δ and, by definition, $n(\sigma, z_N) \neq 0$. This contradiction leaves us no choice but to conclude that there can exist only finitely many elements z of E for which $n(\sigma, z) \neq 0$. Let $\zeta_1, \zeta_2, \ldots, \zeta_p$ be an enumeration of the points of E enjoying this property, and let V be the open set obtained by starting with U and removing from it every member of E other than $\zeta_1, \zeta_2, \ldots, \zeta_p$. Thus σ is a cycle in $V \sim \{\zeta_1, \zeta_2, \ldots, \zeta_p\}$ and, since $\mathbb{C} \sim V = (\mathbb{C} \sim U) \cup \{z \in E : z \neq \zeta_1, \zeta_2, \ldots, \zeta_p\}$, it is clear that $n(\sigma, z) = 0$ for every z in $\mathbb{C} \sim V$. In other words, σ is homologous to zero in V. Incidentally, it is not out of the question that $n(\sigma, z) = 0$ could hold from the outset for every z in E. In that situation (8.11) would convey no information not already available to us, for the right-hand side of (8.11) would then be zero, the set V would be $U \sim E$, and (8.11) would just re-state the conclusions of Cauchy's theorem, as applied to f and σ in V. We shall proceed, therefore, under the premise that points $\zeta_1, \zeta_2, \ldots, \zeta_p$ really do exist.

Let S_k designate the singular part of f at the point ζ_k. The function S_k is analytic in $\mathbb{C} \sim \{\zeta_k\}$, and $f - S_k$ has a removable singularity at ζ_k. It follows that the function $g = f - S_1 - S_2 \cdots - S_p$ is analytic in V except for removable singularities at $\zeta_1, \zeta_2, \ldots, \zeta_p$. We take the liberty of removing these singularities, so are free to assume that g is analytic in V. Appealing to Cauchy's theorem, we can thus assert that

$$0 = \int_\sigma g(z)\,dz = \int_\sigma f(z)\,dz - \sum_{k=1}^{p} \int_\sigma S_k(z)\,dz$$

or, equivalently, that

$$\int_\sigma f(z)\,dz = \sum_{k=1}^{p} \int_\sigma S_k(z)\,dz\ . \tag{8.12}$$

If $S(z) = \sum_{n=1}^{\infty} a_{-n}(z - \zeta_0)^{-n}$ is the singular part of f at an arbitrary point ζ_0 of E, then the series defining S converges normally in $\mathbb{C} \sim \{\zeta_0\}$. Most importantly, it converges uniformly on $|\sigma|$, which justifies the interchange of integration and summation in the computation

$$\begin{aligned}\int_\sigma S(z)\,dz &= \int_\sigma \left(\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-\zeta_0)^n} \right) dz = \sum_{n=1}^{\infty} a_{-n} \int_\sigma \frac{dz}{(z-\zeta_0)^n} \\ &= a_{-1} \int_\sigma \frac{dz}{z - \zeta_0} = 2\pi i\, n(\sigma, \zeta_0)\mathrm{Res}(\zeta_0, f)\ .\end{aligned} \tag{8.13}$$

(N.B. $\int_\sigma (z-\zeta_0)^{-n}\,dz = 0$ when $n > 1$, for in that case the integrand is blessed with a primitive in $\mathbb{C} \sim \{\zeta_0\}$.) In tandem (8.12) and (8.13) lead to

$$\int_\sigma f(z)\,dz = 2\pi i \sum_{k=1}^{p} n(\sigma, \zeta_k)\mathrm{Res}(\zeta_k, f) = 2\pi i \sum_{z \in E} n(\sigma, z)\mathrm{Res}(z, f)\ ,$$

as stated in (8.11). ∎

One special case of the residue theorem is invoked sufficiently often that it warrants a separate statement.

Corollary 3.2. *Suppose that a function f is analytic modulo isolated singularities in an open set U, that E is the singular set of f in U, and that γ is a Jordan contour in $U \sim E$ with the property that the inside D of the Jordan curve $|\gamma|$ is contained in U. If $D \cap E$ is non-empty, then*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{p} \mathrm{Res}(z_k, f)\ ,$$

where $z_1, z_2, \ldots, z_p$ lists the elements of E that belong to D.

Corollary 3.2 takes for granted the validity of the Jordan curve theorem. That the points of E in D (i.e., those z in E for which $n(\gamma, z) = 1$) are finite in number would follow here directly from the discreteness of E and compactness of $\overline{D}$. Corollary 3.2 can be strengthened slightly if one assumes knowledge of Goursat's theorem (Theorem V.5.7). This is described in Exercise 5.42.

3.2 Evaluating Integrals with the Residue Theorem

The residue theorem is of tremendous importance in complex analysis both on a concrete level as a practical device for evaluating integrals, including many obstinate integrals from ordinary calculus, and on a more abstract plane as a powerful theoretical tool. The following series of examples is aimed at documenting its value in the former capacity. Interspersed with the examples are a couple of technical theorems pertinent to the discussion. The theoretical applications of the residue theorem are held in reserve until the next section.

EXAMPLE 3.1. Evaluate $\int_{|z|=1}(z^2+2z)\csc^2 z\,dz$.

The integrand, $f(z) = (z^2+2z)\csc^2 z$, has its singularities at the integral multiples of π. It has but one singularity in the closed disk $\overline{\Delta}(0,1)$, that being a simple pole at the origin. With the aid of l'Hospital's rule we obtain from formula (8.6)

$$\operatorname{Res}(0,f) = \lim_{z\to 0} zf(z) = \lim_{z\to 0}\frac{z^3+2z^2}{\sin^2 z} = \lim_{z\to 0}\frac{3z^2+4z}{2\sin z\cos z}$$

$$= \lim_{z\to 0}\frac{6z+4}{2\cos^2 z - 2\sin^2 z} = 2 \ .$$

Corollary 3.2 informs us that

$$\int_{|z|=1}(z^2+2z)\csc^2 z\,dz = 4\pi i \ .$$

EXAMPLE 3.2. Evaluate $\int_{\partial Q}[\sin z][\cos(z^3)-1]^{-1}dz$, where Q is the square bounded by the curve with equation $|x|+|y| = 1$.

The singularities of $f(z) = [\sin z][\cos(z^3)-1]^{-1}$ are located at the cube roots of integral multiples of 2π. Each of these points, save one, has magnitude greater than 1 — hence, exerts no influence on the given integral. The sole exception occurs at the origin, where f has a fifth order pole. In Example 2.3 we computed $\operatorname{Res}(0,f)$ and found it to be $-1/60$. We deduce from Corollary 3.2 that

$$\int_{\partial Q}\frac{\sin z\,dz}{\cos(z^3)-1} = -\frac{\pi i}{30} \ .$$

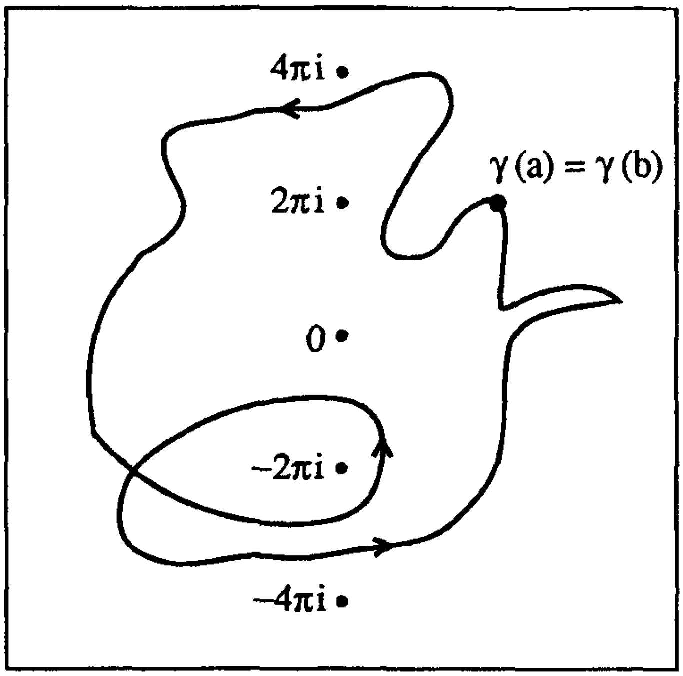

Figure 2.

EXAMPLE 3.3. Evaluate $\int_\gamma z(e^z-1)^{-1}\,dz$, where $\gamma: [a,b] \to \mathbb{C}$ is the path pictured in Figure 2.

The function $f(z) = z(e^z-1)^{-1}$ has isolated singularities at $z_k = 2k\pi i$ for $k = 0, \pm1, \pm2, \cdots$. Since $n(\gamma, z_k) = 0$ for $k = \pm2, \pm3, \cdots$, only the singularities at the points $0, 2\pi i$, and $-2\pi i$ have an effect on the integral. The singularity of f at the origin is removable, so $\mathrm{Res}(0, f) = 0$. At each of the points $2\pi i$ and $-2\pi i$ this function has a simple pole. By (8.6)

$$\mathrm{Res}(2\pi i, f) = \lim_{z\to 2\pi i}(z-2\pi i)f(z) = \lim_{z\to 2\pi i}\frac{z(z-2\pi i)}{e^z-1}$$

$$= \lim_{z\to 2\pi i}\frac{2z-2\pi i}{e^z} = 2\pi i$$

and, similarly, $\mathrm{Res}(-2\pi i, f) = -2\pi i$. The residue theorem yields

$$\int_\gamma \frac{z\,dz}{e^z-1} = 2\pi i \sum_{k=-\infty}^{\infty} n(\gamma, z_k)\mathrm{Res}(z_k, f)$$

$$= 2\pi i\,[(2)(-2\pi i) + 0 + (1)(2\pi i)] = 4\pi^2 .$$

EXAMPLE 3.4. Evaluate $\int_\sigma (1-z)^{-1}e^{1/z}\,dz$ for the cycle $\sigma = (\gamma, \beta)$, where $\gamma(t) = 2e^{it}$ and $\beta(t) = (-1/2) + e^{it}$ for $0 \le t \le 2\pi$ (Figure 3).

The integrand $f(z) = (1-z)^{-1}e^{1/z}$ has two singularities, a simple pole at the point 1, for which $n(\sigma, 1) = 1$, and an essential singularity at the origin, for which $n(\sigma, 0) = 2$. The residue at the first of these is easy to

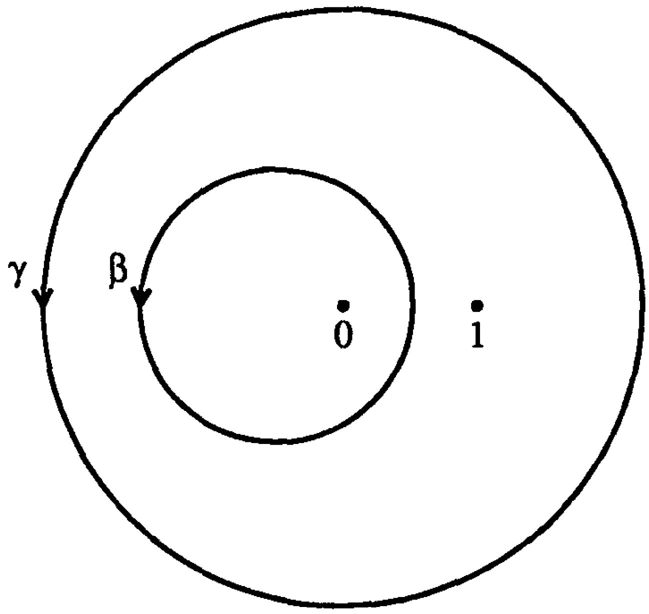

Figure 3.

handle with the assistance of (8.6),

$$\operatorname{Res}(1, f) = \lim_{z\to 1}(z-1)f(z) = -\lim_{z\to 1} e^{1/z} = -e\ .$$

As might be expected, the residue at the essential singularity poses somewhat more of a problem. We take advantage of two familiar series — namely, the geometric series,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n\ ,$$

and the Laurent expansion of $e^{1/z}$ in $D = \mathbb{C} \sim \{0\}$,

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n}$$

— and revert to the definition of a Laurent coefficient in order to ascertain $\operatorname{Res}(0, f)$:

$$\begin{aligned}\operatorname{Res}(0,f) &= \frac{1}{2\pi i}\int_{|z|=1/2} f(z)\,dz = \frac{1}{2\pi i}\int_{|z|=1/2}(1-z)^{-1}\,e^{1/z}\,dz\\ &= \frac{1}{2\pi i}\int_{|z|=1/2}\left(\sum_{n=0}^{\infty} z^n\,e^{1/z}\right)dz = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{|z|=1/2} z^n\,e^{1/z}\,dz\\ &= \sum_{n=0}^{\infty}\frac{1}{(n+1)!} = \sum_{n=1}^{\infty}\frac{1}{n!} = e-1\ .\end{aligned}$$

The exchange of integration and summation is allowable because the series $\sum_{n=0}^{\infty} z^n e^{1/z}$ is uniformly convergent on the circle $K(0, 1/2)$. That

$$\frac{1}{2\pi i}\int_{|z|=1/2} z^n\,e^{1/z}\,dz = \frac{1}{(n+1)!}$$

follows from the fact that this integral expression represents the coefficient of z^{-n-1} in the above Laurent series expansion for $e^{1/z}$. Having calculated the necessary residues, we learn from the residue theorem that

$$\int_\sigma \frac{e^{1/z}\,dz}{1-z} = 2\pi i\,[2\mathrm{Res}(0,f)+\mathrm{Res}(1,f)] = 2\pi i(2e-2-e) = (2e-4)\pi i\ .$$

We were, of course, quite fortunate here in being able to determine explicitly the residue at the essential singularity.

Preparations for the next two examples are met in the following theorem. One piece of notation in the theorem deserves a comment. Given a rational function $R(x,y)$ of the real variables x and y we generate a rational function of z by substituting $(z+z^{-1})/2$ for x and $(z-z^{-1})/2i$ for y in the formula describing R. We indicate these substitutions in the obvious way, by writing $R\left[(z+z^{-1})/2,(z-z^{-1})/2i\right]$. Notice, however, that the substituted quantities are only real when $|z|=1$. (Remember that, in general, $x=(z+\bar{z})/2$ — not $x=(z+z^{-1})/2$. A similar remark applies to y.) If $R(x,y)=y/x$, for example, we obtain

$$R\left[(z+z^{-1})/2,(z-z^{-1})/2i\right] = \frac{(z-z^{-1})/2i}{(z+z^{-1})/2} = \frac{1}{i}\frac{z^2-1}{z^2+1}\ .$$

Theorem 3.3. *Let R be a rational function of x and y whose domain-set includes the circle $K(0,1)$. Then*

$$(8.14)\qquad \int_0^{2\pi} R(\cos\theta,\sin\theta)\,d\theta = 2\pi\sum_{k=1}^{p}\mathrm{Res}(z_k,f)\ ,$$

where $f(z)=z^{-1}R\left[(z+z^{-1})/2,(z-z^{-1})/2i\right]$ and $z_1,z_2,\ldots,z_p$ are the poles of f in the disk $\Delta(0,1)$.

Proof. The function f is a rational function of z. Each of its finitely many singularities is at worst a pole, and by hypothesis it has no singularities on the circle $K(0,1)$. Let $z_1,z_2,\ldots,z_p$ be the poles of f in $\Delta(0,1)$. Because the residue of a function is zero at any removable singularity, we infer from Corollary 3.2 that

$$\int_{|z|=1} f(z)\,dz = 2\pi i\sum_{k=1}^{p}\mathrm{Res}(z_k,f)\ .$$

On the other hand, by the very definition of this complex line integral

$$\int_{|z|=1} f(z)\,dz = \int_0^{2\pi} e^{-i\theta}R\left[(e^{i\theta}+e^{-i\theta})/2,(e^{i\theta}-e^{-i\theta})/2i\right]ie^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} R(\cos\theta,\sin\theta)\,d\theta\ .$$

Formula (8.14) now follows easily. ∎

EXAMPLE 3.5. Given that $a > 1$, evaluate $\int_0^{2\pi}(a+\cos\theta)^{-1}\,d\theta$.

This integral is of the type covered by Theorem 3.3. Here $R(x,y) = (a+x)^{-1}$. We consider the function f given by

$$f(z) = \frac{1}{z}R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) = \frac{1}{z}\frac{1}{a+[(z+z^{-1})/2]} = \frac{2}{z^2+2az+1},$$

a rational function whose only poles are simple ones located at the points $-a+\sqrt{a^2-1}$ and $-a-\sqrt{a^2-1}$. Since $a > 1$, just the first of these, $z_1 = -a+\sqrt{a^2-1}$, finds itself in $\Delta(0,1)$. Formula (8.6) gives

$$\operatorname{Res}(z_1, f) = \lim_{z\to z_1}(z-z_1)f(z) = \lim_{z\to z_1}\frac{2(z-z_1)}{z^2+2az+1}$$

$$= \lim_{z\to z_1}\frac{2}{2z+2a} = \frac{1}{\sqrt{a^2-1}} .$$

Referring to Theorem 3.3 we conclude that

$$\int_0^{2\pi}\frac{d\theta}{a+\cos\theta} = \frac{2\pi}{\sqrt{a^2-1}}$$

for $a > 1$.

EXAMPLE 3.6. Evaluate $\int_0^{\pi}\sin^{2n}\theta\,d\theta$ for n a positive integer.

Since $\sin^{2n}\theta$ has period π, $\int_0^{\pi}\sin^{2n}\theta\,d\theta = (1/2)\int_0^{2\pi}\sin^{2n}\theta\,d\theta$. The latter integral falls within the scope of Theorem 3.3, this time with $R(x,y) = y^{2n}$. We utilize the binomial theorem in computing

$$f(z) = \frac{1}{z}R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) = \frac{1}{z}\left(\frac{z-z^{-1}}{2i}\right)^{2n}$$

$$= \frac{1}{z}\sum_{k=0}^{2n}\binom{2n}{k}(2i)^{-2n}z^k(-z^{-1})^{2n-k}$$

$$= \sum_{k=0}^{2n}(-1)^{n-k}4^{-n}\binom{2n}{k}z^{2k-2n-1} .$$

The only singularity of f in $\Delta(0,1)$ is a pole at the origin, where the residue can be extracted directly from the expression derived for f:

$$\operatorname{Res}(0, f) = 4^{-n}\binom{2n}{n} .$$

By Theorem 3.3,

$$\int_0^{\pi} \sin^{2n}\theta\, d\theta = \frac{1}{2}\int_0^{2\pi} \sin^{2n}\theta\, d\theta = 4^{-n}\binom{2n}{n}\pi$$

for any positive integer n.

It goes without saying that there are means other than the residue theorem for evaluating integrals of the type $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$, where R is a rational function of x and y. To name but one of these, the substitution $t = \tan(\theta/2)$ transforms such an integral into an integral to which the method of partial fractions is applicable. What the residue theorem can provide in this kind of problem is a labor-saving device of sometimes major proportions.

The next theorem addresses itself to a class of integrals more resistant to standard elementary techniques of integration than the integrals in the two preceding examples.

Theorem 3.4. *If $f(z) = (a_0 + a_1 z + \cdots + a_n z^n)/(b_0 + b_1 z + \cdots + b_m z^m)$ is a rational function in which $m \geq n + 2$ and in which the denominator has no real roots, then for $c \geq 0$*

(8.15)
$$\int_{-\infty}^{\infty} f(x)e^{icx}\, dx = 2\pi i \sum_{k=1}^{p} \text{Res}\left[z_k, f(z)e^{icz}\right] ,$$

where $z_1, z_2, \ldots, z_p$ are the poles of f in the half-plane $H = \{z : \text{Im}\, z > 0\}$. Furthermore, if all the coefficients in f are real numbers, then

(8.16)
$$\int_{-\infty}^{\infty} f(x)\cos(cx)\, dx = \text{Re}\left\{2\pi i \sum_{k=1}^{p} \text{Res}\left[z_k, f(z)e^{icz}\right]\right\}$$

and

(8.17)
$$\int_{-\infty}^{\infty} f(x)\sin(cx)\, dx = \text{Im}\left\{2\pi i \sum_{k=1}^{p} \text{Res}\left[z_k, f(z)e^{icz}\right]\right\} .$$

Proof. Because $m \geq n + 2$ it is clear that $|z^2 f(z)|$ tends to a finite limit L as $|z| \to \infty$. This fact permits us to fix a number $M > |L|$ and then to choose $r_0 > 0$ with the property that

(8.18)
$$|f(z)| \leq \frac{M}{|z|^2}$$

for every z having $|z| \geq r_0$. Together with the assumption that f is free of poles on the real axis, (8.18) makes certain that the improper integral $\int_{-\infty}^{\infty} f(x)e^{icx} dx$ is convergent; i.e.,

$$\int_{-\infty}^{\infty} f(x)e^{icx} dx = \lim_{a\to-\infty} \int_a^0 f(x)e^{icx} dx + \lim_{b\to\infty} \int_0^b f(x)e^{icx} dx$$

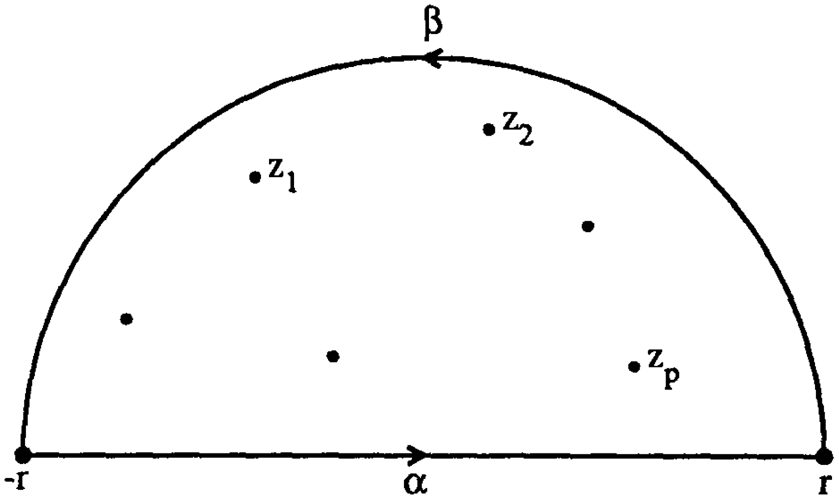

Figure 4.

exists.

Let $r \geq r_0$. Inequality (8.18) implies that the disk $\Delta(0,r)$ contains all the poles of f. We consider the integral of f along the contour $\gamma = \alpha + \beta$, where $\alpha(t) = t$ for $-r \leq t < r$ and $\beta(t) = re^{it}$ for $0 \leq t \leq \pi$ (Figure 4). By Corollary 3.2,

$$\int_\gamma f(z)e^{icz}\,dz = 2\pi i \sum_{k=1}^{p} \operatorname{Res}\left[z_k, f(z)e^{icz}\right] \ . \tag{8.19}$$

Here $z_1, z_2, \ldots, z_p$ is a list of the poles of f in $H = \{z : \operatorname{Im} z > 0\}$. The left-hand side of (8.19) can be expressed in the form

$$\int_\gamma f(z)e^{icz}\,dz = \int_{-r}^{r} f(t)e^{ict}\,dt + \int_\beta f(z)e^{icz}\,dz \ .$$

We now let $r \to \infty$. The right-hand side of (8.19) is unaffected by the passage to the limit, whereas on the left we have

$$\int_{-r}^{r} f(t)e^{ict}\,dt \to \int_{-\infty}^{\infty} f(t)e^{ict}\,dt \ ,$$

since the latter integral is known to be convergent, and

$$\int_\beta f(z)e^{icz}\,dz \to 0 \ .$$

Justification for the last assertion is found in (8.18) and the by now familiar method of estimation:

$$\left|\int_\beta f(z)e^{icz}\,dz\right| \leq \int_\beta |f(z)|\,|e^{icz}|\,|dz| \leq \frac{M}{r^2}\int_\beta e^{\operatorname{Re}(icz)}|dz|$$

$$= \frac{M}{r}\int_0^{\pi} e^{-cr\sin t}\,dt \le \frac{M\pi}{r} \to 0$$

as $r \to \infty$. Putting these observations together, we obtain

$$\int_{-\infty}^{\infty} f(t)e^{ict}\,dt = 2\pi i \sum_{k=1}^{p} \operatorname{Res}\left[z_k, f(z)e^{icz}\right] ,$$

as asserted in (8.15). Formulas (8.16) and (8.17) follow by taking real and imaginary parts in (8.15) — provided the function f has real coefficients and is, as a result, real-valued on the real axis. ∎

If $c > 0$, then the conclusions of Theorem 3.4 are valid even in the case $m = n + 1$. However, a different argument is needed to confirm them, for inequality (8.18) becomes $|f(z)| \le M/|z|$ when $m = n + 1$, an estimate which is insufficient on its own to force the convergence of $\int_{-\infty}^{\infty} f(x)e^{icx}dx$. Exercise 5.51 indicates a method of circumventing this difficulty.

EXAMPLE 3.7. Evaluate $\int_{-\infty}^{\infty}(x^2+1)^{-1}(x^2+4)^{-1}\,dx$.

We apply Theorem 3.4 to $f(z) = (z^2+1)^{-1}(z^2+4)^{-1}$ with $c = 0$. The poles of f in H are simple poles at the points i and $2i$. The corresponding residues are easily calculated with the help of (8.6): $\operatorname{Res}(i,f) = -i/6, \operatorname{Res}(2i,f) = i/12$. Consequently,

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = 2\pi i\left(-\frac{i}{6}+\frac{i}{12}\right) = \frac{\pi}{6} .$$

We would be quick to agree that the integral in Example 3.7 could have been worked out using partial fractions, but only at a much greater expenditure of effort than was involved in the solution presented. The integral in the next example would be even more difficult to evaluate without recourse to residue theorem methods.

EXAMPLE 3.8. Evaluate $\int_0^{\infty} x(x^2+1)^{-2}\sin x\,dx$.

In this instance we employ Theorem 3.4 with $f(z) = z(z^2+1)^{-2}$ and $c = 1$. The only pole of f in H is a double pole at the point i. We take $m = 2$ in (8.6) and compute

$$\operatorname{Res}\left[i, f(z)e^{iz}\right] = \lim_{z\to i}\frac{d}{dz}\left[(z-i)^2 f(z)e^{iz}\right] = \lim_{z\to i}\frac{d}{dz}\left[\frac{(z-i)^2 ze^{iz}}{(z^2+1)^2}\right]$$

$$= \lim_{z\to i}\frac{d}{dz}\left[\frac{ze^{iz}}{(z+i)^2}\right] = \lim_{z\to i}\frac{(z+i)^2(1+iz)e^{iz} - 2z(z+i)e^{iz}}{(z+i)^4} = \frac{1}{4e} .$$

A glance at (8.17) reveals that

$$\int_{-\infty}^{\infty} \frac{x \sin x \, dx}{(x^2+1)^2} = \frac{\pi}{2e} .$$

Since the integrand is an even function we deduce that

$$\int_{0}^{\infty} \frac{x \sin x \, dx}{(x^2+1)^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x \, dx}{(x^2+1)^2} = \frac{\pi}{4e} .$$

Only a small fraction of the integrals to which the residue theorem is relevant fit convenient standard patterns like the ones exemplified in Theorems 3.3 and 3.4. More typically the use of this theorem to evaluate a definite integral calls for a bit of ingenuity, especially in finding a contour appropriate to the integral under consideration and in establishing necessary estimates. (Even experienced practitioners of complex analysis can find their resourcefulness put to the test in coming up with a "correct" path of integration!) Although it may at first seem that the contours which appear in the next two examples are plucked from thin air, these examples are quite representative of the kind of inventiveness that enters into making the residue theorem a truly effective tool.

An important concept in harmonic analysis is the so-called *Hilbert transform* Hf of a function $f: \mathbb{R} \to \mathbb{R}$, which for the sake of the present discussion we assume to be continuous. The value of Hf at the real number x is given by

$$(8.20) \qquad \begin{aligned} Hf(x) &= -\frac{1}{\pi} \lim_{\substack{s \to 0+ \\ r \to \infty}} \int_{s \le |t| \le r} \frac{f(x+t)\,dt}{t} \\ &= -\frac{1}{\pi} \lim_{\substack{s \to 0+ \\ r \to \infty}} \left\{ \int_{-r}^{-s} \frac{f(x+t)\,dt}{t} + \int_{s}^{r} \frac{f(x+t)\,dt}{t} \right\} , \end{aligned}$$

provided the limit exists. (The interpretation of the double limit becomes clearer when it is rewritten as the sum of

$$\lim_{s \to 0+} \left\{ \int_{-1}^{-s} \frac{f(x+t)\,dt}{t} + \int_{s}^{1} \frac{f(x+t)\,dt}{t} \right\}$$

and

$$\lim_{r \to \infty} \left\{ \int_{-r}^{-1} \frac{f(x+t)\,dt}{t} + \int_{1}^{r} \frac{f(x+t)\,dt}{t} \right\} .$$

The requirement is that both limits should exist.) We have intentionally avoided writing

$$Hf(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x+t)\,dt}{t}$$

here — and for good reason. The normal improper integral,

$$\int_{-\infty}^{\infty} \frac{f(x+t)\,dt}{t} = \lim_{a\to-\infty}\int_a^{-1} \frac{f(x+t)\,dt}{t} + \lim_{b\to 0^-}\int_{-1}^{b} \frac{f(x+t)\,dt}{t}$$

$$+ \lim_{c\to 0^+}\int_c^{1} \frac{f(x+t)\,dt}{t} + \lim_{d\to\infty}\int_1^{d} \frac{f(x+t)\,dt}{t} ,$$

is frequently divergent in situations where the extremely symmetric limits involved in (8.20) exist. One often finds the Hilbert transform expressed as

$$Hf(x) = -\frac{1}{\pi}\,\mathrm{P.V.}\int_{-\infty}^{\infty} \frac{f(x+t)\,dt}{t} .$$

The symbol P.V. $\int$ stands for "principal value integral" and distinguishes this object from a standard improper integral. In our next example we use the residue theorem to compute the Hilbert transforms of two elementary functions.

EXAMPLE 3.9. Compute the Hilbert transforms of the functions $f(x) = \cos x$ and $g(x) = \sin x$.

We shall demonstrate that

$$\lim_{\substack{s\to 0^+\\ r\to\infty}} \int_{s\le|t|\le r} \frac{e^{i(x+t)}\,dt}{t} = e^{ix} \lim_{\substack{s\to 0^+\\ r\to\infty}} \int_{s\le|t|\le r} \frac{e^{it}\,dt}{t} = \pi i e^{ix}$$

for any real x. Multiplying by $-1/\pi$ and taking real and imaginary parts, we obtain $Hf(x) = \sin x$ and $Hg(x) = -\cos x$, respectively. We are thus led naturally to look at the meromorphic function $h(z) = z^{-1}e^{iz}$. The only singularity of h is a simple pole at the origin, and $\mathrm{Res}(0,h) = 1$. What we need is an expression for the integral $\int_{s\le|t|\le r} h(t)\,dt$, say with $0 < s < 1$ and $r > 1$, that makes transparent its behavior when $s \to 0^+$ and $r \to \infty$. It is here that, by making a shrewd choice of contour, one can bring the residue theorem into the picture. The trick is to integrate h along the path $\gamma = \alpha_1 + \beta_1 + \alpha_2 + \beta_2$ depicted in Figure 5. The paths that compose γ are: $\alpha_1(t) = t$ for $s \le t \le r$, $\beta_1(t) = re^{it}$ for $0 \le t \le \pi$, $\alpha_2(t) = t$ for $-r \le t \le -s$, and $\beta_2(t) = se^{it}$ for $\pi \le t \le 2\pi$. Owing to the residue theorem, integrating h along γ gives

$$\int_\gamma h(z)\,dz = 2\pi i\,\mathrm{Res}(0,h) = 2\pi i .$$

It follows that

$$\int_{s\le|t|\le r} \frac{e^{it}\,dt}{t} = \int_{-r}^{-s} h(t)\,dt + \int_s^r h(t)\,dt = \int_{\alpha_2} h(z)\,dz + \int_{\alpha_1} h(z)\,dz$$

$$= 2\pi i - \int_{\beta_1} h(z)\,dz - \int_{\beta_2} h(z)\,dz .$$

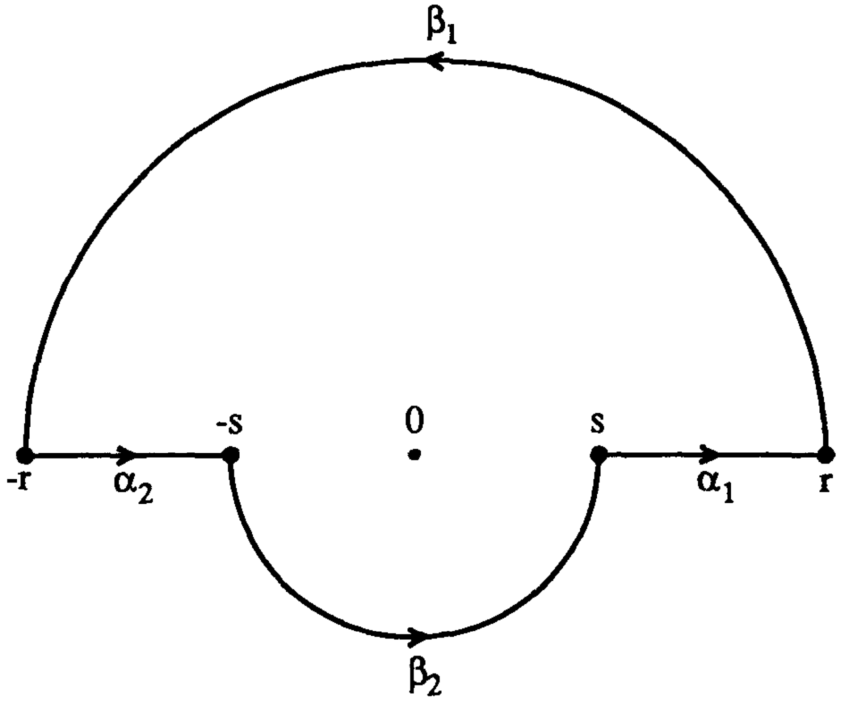

Figure 5.

The behavior of $\int_{\beta_1} h(z)\,dz$ as $r \to \infty$ and $\int_{\beta_2} h(z)\,dz$ as $s \to 0^+$ is not hard to decipher. First,

$$\left|\int_{\beta_1} \frac{e^{iz}\,dz}{z}\right| \leq \frac{\pi(1-e^{-r})}{r} \to 0$$

as $r \to \infty$. (A similar estimate was derived in the proof of Theorem 3.4.) Secondly,

$$\int_{\beta_2} \frac{e^{iz}\,dz}{z} = \int_{\beta_2} \frac{(e^{iz}-1)\,dz}{z} + \int_{\beta_2} \frac{dz}{z} = \int_{\beta_2} \frac{(e^{iz}-1)}{z} + \pi i \to \pi i$$

as $s \to 0^+$, since

$$\left|\int_{\beta_2} \frac{(e^{iz}-1)\,dz}{z}\right| \leq \int_{\beta_2} \frac{|e^{iz}-1|\,|dz|}{|z|} \leq \pi \max\left\{|e^{iz}-1| : z \in K(0,s)\right\} \to 0$$

as $s \to 0^+$. We conclude that

$$\lim_{\substack{s\to 0^+ \\ r\to\infty}} \int_{s\leq|t|\leq r} \frac{e^{it}\,dt}{t} = \pi i\ ,$$

as desired.

The final example in this section is another improper Riemann integral, an integral whose innocent appearance is misleading. Despite being armed with the residue theorem, we shall still not have a terribly easy time carrying out its evaluation.

EXAMPLE 3.10. Assuming that $0 < \lambda < 1$ and $b > 0$, show that

$$\int_0^\infty \frac{dt}{t^\lambda(t+b)} = \frac{\pi}{b^\lambda \sin(\lambda\pi)} . \tag{8.21}$$

Our job is to verify that

$$\lim_{\substack{s\to 0^+ \\ r\to\infty}} \int_s^r \frac{dt}{t^\lambda(t+b)} = \frac{\pi}{b^\lambda \sin(\lambda\pi)} .$$

We shall do this by using the residue theorem to derive the estimate

$$\left|\pi b^{-\lambda} - \sin(\lambda\pi)\int_s^r \frac{dt}{t^\lambda(t+b)}\right| \le \frac{\pi r^{1-\lambda}}{r-b} + \frac{\pi s^{1-\lambda}}{b-s} , \tag{8.22}$$

valid when $0 < s < b$ and $r > b$. Once (8.22) is established, (8.21) follows almost immediately: since $0 < \lambda < 1$, the right-hand side of (8.21) tends to zero as $s \to 0^+$ and $r \to \infty$, implying that

$$\sin(\lambda\pi) \lim_{\substack{s\to 0^+ \\ r\to\infty}} \int_s^r \frac{dt}{t^\lambda(t+b)} = \frac{\pi}{b^\lambda} .$$

We now fix s in $(0,b)$ and r in (b,∞). Even the clue that the residue theorem plays a role in the verification of (8.22) does not quickly suggest a way to proceed. To what function are we supposed to apply the theorem? The initial guess is very likely to be $f(z) = z^{-\lambda}(z+b)^{-1}$, but there is a problem with it: there are no isolated singularities of f in sight! This function is discontinuous at every point of the negative real axis, which means, in particular, that an apparent pole at $-b$ is not an isolated singularity at all. It is just the point $-b$, as it so happens, that is the key source of information in the derivation of (8.22). The secret to gaining access to that information lies in passing to a different branch of the λ-power function, one that is analytic near $-b$. Specifically, we set $D = \mathbb{C} \sim [0,\infty)$ and let h denote the branch of the λ-power function in D associated with the branch of $\arg z$ in D whose range is the interval $(0, 2\pi)$; i.e., if $z = te^{i\theta}$ with $t > 0$ and $0 < \theta < 2\pi$, then $h(z) = t^\lambda e^{i\lambda\theta}$. The function g defined by $g(z) = 1/[(z+b)h(z)]$ is then meromorphic in D, its only singularity in this domain is a simple pole at $-b$, and, since $h(-b) = h(be^{\pi i}) = b^\lambda e^{\lambda\pi i}$,

$$\operatorname{Res}(-b, g) = \lim_{z\to -b}(z+b)g(z) = \frac{1}{h(-b)} = b^{-\lambda}e^{-\lambda\pi i} .$$

Having settled on the function g as a reasonable candidate for the integrand in our application of the residue theorem, we are now faced with the problem of selecting a suitable contour in D along which to integrate it. We want to pick a path that manages to link the singularity of g at $-b$

to the integral $\int_s^r t^{-\lambda}(t+b)^{-1}\,dt$. The critical observation involved in our eventual choice is this: for each fixed $t>0$,

$$\lim_{\theta\to 0+}\left[g(te^{i\theta})e^{i\theta}\right] = \lim_{\theta\to 0+}\frac{e^{i(1-\lambda)\theta}}{t^\lambda(te^{i\theta}+b)} = \frac{1}{t^\lambda(t+b)}$$

and

$$\lim_{\theta\to 2\pi-}\left[g(te^{i\theta})e^{i\theta}\right] = \lim_{\theta\to 2\pi-}\frac{e^{i(1-\lambda)\theta}}{t^\lambda(te^{i\theta}+b)} = \frac{e^{-2\lambda\pi i}}{t^\lambda(t+b)}\ .$$

Furthermore, in both cases the convergence is seen without difficulty to be uniform on the interval $[s,r]$, which permits us to conclude that

$$\text{(8.23)}\qquad \begin{aligned}\lim_{\theta\to 0+}\int_s^r g(te^{i\theta})e^{i\theta}\,dt &= \int_s^r \frac{dt}{t^\lambda(t+b)}\ ,\\ \lim_{\theta\to 2\pi-}\int_s^r g(te^{i\theta})e^{i\theta}\,dt &= e^{-2\lambda\pi i}\int_s^r \frac{dt}{t^\lambda(t+b)}\ .\end{aligned}$$

It is now important to recognize that

$$\text{(8.24)}\qquad \int_s^r g(te^{i\theta})e^{i\theta}\,dt = \int_\alpha g(z)\,dz\ ,$$

where $\alpha(t)=te^{i\theta}$ for $s\le t\le r$. Coupled with (8.23), (8.24) and the need to encircle the point $-b$ in order to benefit from the residue information there make it not totally farfetched to consider, for any fixed θ in $(0,\pi/2)$, the path $\gamma=\alpha_1+\beta_1-\alpha_2-\beta_2$ represented in Figure 6. To be precise: $\alpha_1(t)=te^{i\theta}$ and $\alpha_2(t)=te^{i(2\pi-\theta)}$ for $s\le t\le r$; $\beta_1(t)=re^{it}$ and $\beta_2(t)=se^{it}$ for $\theta\le t\le 2\pi-\theta$.

The residue theorem tells us that

$$\int_{\alpha_1} g(z)\,dz+\int_{\beta_1} g(z)\,dz-\int_{\alpha_2} g(z)\,dz-\int_{\beta_2} g(z)\,dz$$

$$=\int_\gamma g(z)\,dz = 2\pi i\operatorname{Res}(-b,g) = 2\pi i b^{-\lambda}e^{-\lambda\pi i}\ .$$

Therefore, for θ in $(0,\pi/2)$ we can appeal to (8.24) and assert that

$$\left|2\pi i b^{-\lambda}e^{-\lambda\pi i}-\int_s^r g(te^{i\theta})e^{i\theta}\,dt+\int_s^r g(te^{i\theta})e^{i(2\pi-\theta)}\,dt\right|$$

$$=\left|\int_{\beta_1} g(z)\,dz-\int_{\beta_2} g(z)\,dz\right| \le \int_{\beta_1}|g(z)|\,|dz|+\int_{\beta_2}|g(z)|\,|dz|$$

$$\le \frac{(2\pi-2\theta)r}{r^\lambda(r-b)}+\frac{(2\pi-2\theta)s}{s^\lambda(b-s)}\ ,$$

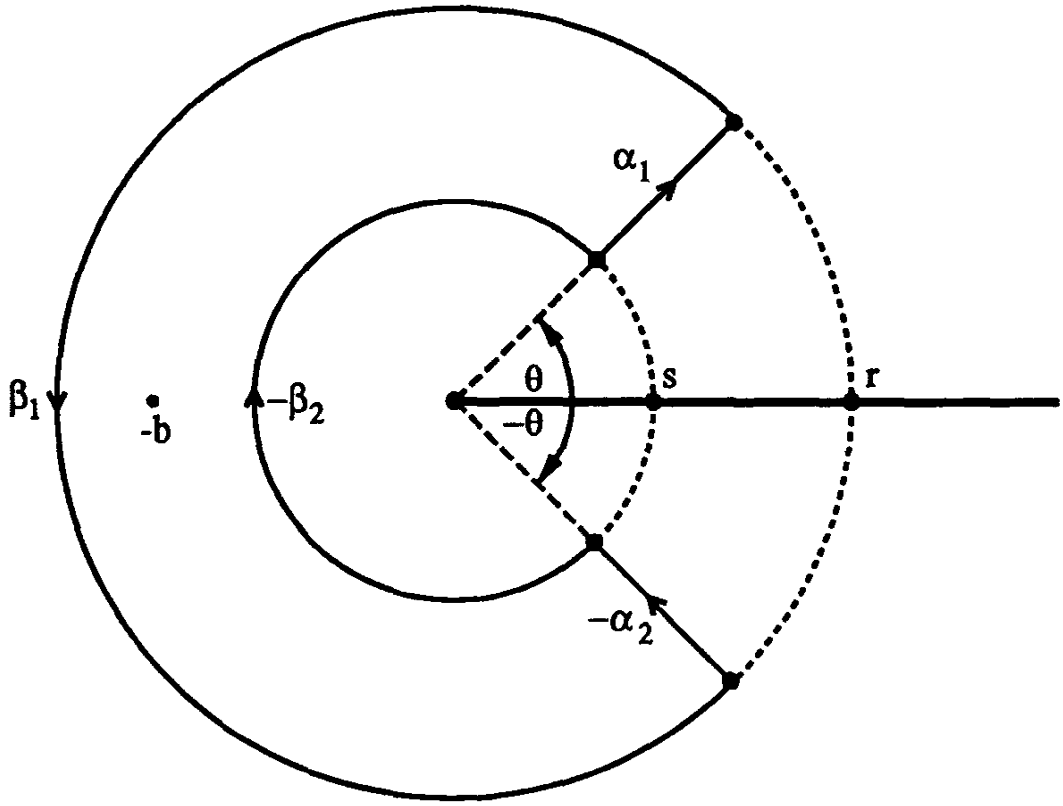

Figure 6.

since $|g(z)| \le r^{-\lambda}(r-b)^{-1}$ for z in $|\beta_1|$ and $\ell(\beta_1) = (2\pi - 2\theta)r$, while $|g(z)| \le s^{-\lambda}(b-s)^{-1}$ on $|\beta_2|$ and $\ell(\beta_2) = (2\pi - 2\theta)s$. In view of (8.23), letting $\theta \to 0^+$ results in the inequality

$$\left| 2\pi i b^{-\lambda} e^{-\lambda\pi i} - \left(1 - e^{-2\lambda\pi i}\right) \int_s^r \frac{dt}{t^\lambda (t-b)} \right| \le \frac{2\pi r^{1-\lambda}}{r-b} + \frac{2\pi s^{1-\lambda}}{b-s} .$$

Dividing through by $|2ie^{-\lambda\pi i}|$, which is 2 in disguise, we arrive at (8.22).

3.3 Consequences of the Residue Theorem

Having seen the residue theorem at work in the evaluation of integrals, we move on to consider some of its theoretical implications for analytic functions. To say that the influence of this theorem on complex analysis extends far beyond the few ideas that will be treated in the forthcoming pages would be a vast understatement.

The information recoverable from the residue theorem is typified by a result known as the "Argument Principle." This proposition admits many different formulations. The version we present is based on Corollary 3.2 and, as such, involves a temporary departure from our usual policy of avoiding appeal to results that depend on the Jordan curve theorem. It is in the setting of Corollary 3.2 that the argument principle has its simplest — and, some might argue, most elegant — formulation. Variants of this basic

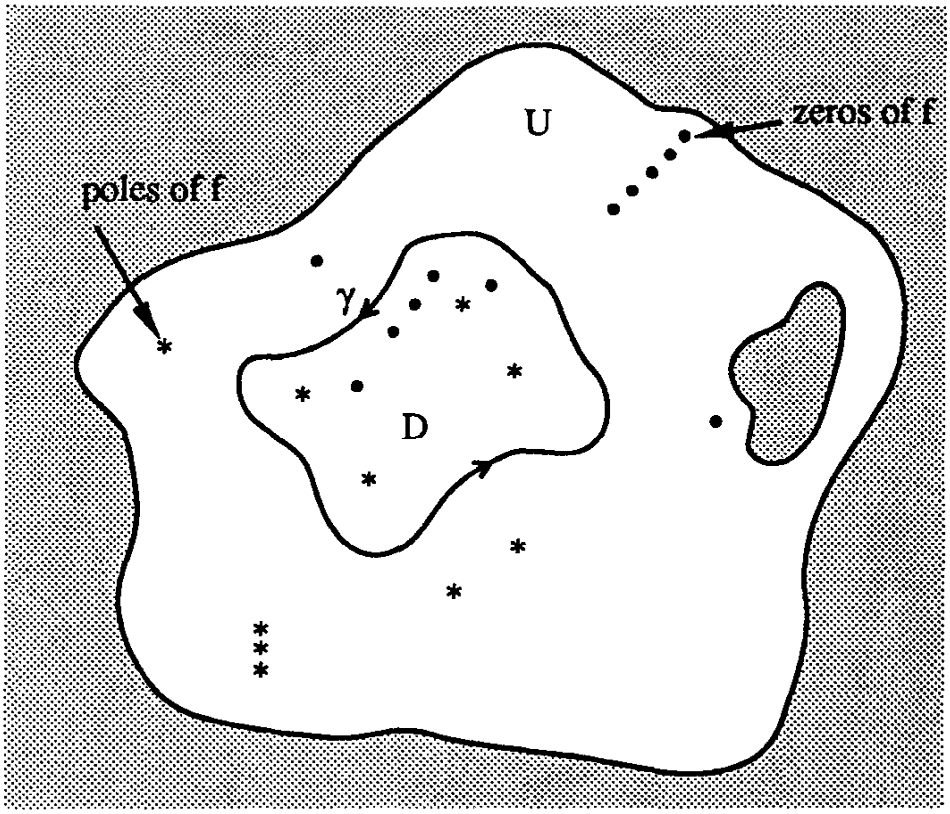

Figure 7.

form of the principle can be found in the exercises. Phrases like "taking multiplicity into account" and "counted with due regard for multiplicity" appear regularly in the discussion that follows. They refer to the way in which zeros and poles of a function are tallied: a zero of order m counts as m zeros, a pole of order m as m poles.

Theorem 3.5. (Argument Principle) *Assume that a function f is meromorphic in an open set U. Let γ be a Jordan contour in U such that the Jordan curve $|\gamma|$ does not pass through any zero or pole of f and such that the inside D of $|\gamma|$ is contained in U (Figure 7). Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)} = Z - P\ ,$$

where Z and P indicate the number of zeros and the number of poles, respectively, that f has in D, multiplicity being taken into account.

Proof. The hypotheses imply that f does not vanish identically in G, the component of U that contains $\overline{D}$. This means that the function f'/f is meromorphic in G, its only singularities appearing at zeros and poles of f. Consider, first, a point z_0 of G at which f has a zero of order m. In some open disk $\Delta = \Delta(z_0, r)$ we can express f in the form $f(z) = (z - z_0)^m g(z)$, where $g\colon \Delta \to \mathbb{C}$ is a function that is both analytic and zero-free. This gives rise to the representation $f'(z) = m(z - z_0)^{m-1}g(z) + (z - z_0)^m g'(z)$ in Δ

and has the consequence that

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}$$

for z belonging to the punctured disk $\Delta^*(z_0, r)$. Because g'/g is analytic in Δ, it becomes evident that f'/f has a simple pole at z_0 and that $\operatorname{Res}(z_0, f'/f) = m$. A completely analogous computation reveals that, at a point z_0 of G where f has a pole of order m, f'/f has a simple pole with $\operatorname{Res}(z_0, f'/f) = -m$. On the authority of Corollary 3.2 we can assert that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)} = \sum_{k=1}^{p} \operatorname{Res}(z_k, f'/f) ,$$

where $z_1, z_2, \ldots, z_p$ are the (distinct) poles of f'/f in D. (Intrepret an "empty" sum to mean 0.) Our analysis of the singularities of f'/f makes clear that the residue sum reduces to the number of zeros of f in D less the number of poles of f in this set, assuming that both are computed with multiplicity taken into consideration. Thus, the residue sum is $Z - P$. ∎

Suppose that the path γ in Theorem 3.5 is parametrized on the interval $[a, b]$. Then, as we noted in the comments after Theorem V.4.1,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)} = n(\beta, 0) ,$$

where β is the image of γ under f, meaning the path defined on $[a, b]$ by $\beta(t) = f[\gamma(t)]$. In supplying a geometric interpretation of winding numbers we observed that $2\pi n(\beta, 0)$ can be thought of as the net change in the argument of $\beta(t)$ when t increases from a to b; i.e., the net change in the argument of $w = f(z)$ as $z = \gamma(t)$ moves around $J = |\gamma|$. This explains the name "argument principle": the theorem establishes a precise relationship between the number of zeros and poles of f inside the Jordan curve J and the change in the argument of $f(z)$ as z traverses J once in the positive direction.

The argument principle has consequences galore. Prominent among these is a theorem of Eugène Rouché (1832-1910). We state Rouché's theorem in the context of analytic functions, the setting where the result is most frequently applied. Its straightforward generalization for meromorphic functions is relegated to the exercises (Exercise 5.62).

Theorem 3.6. (Rouché's Theorem) *If D is the domain inside the trajectory of a Jordan contour in the complex plane, if f and g are functions that are analytic in some open set which contains $\overline{D}$, and if the inequality*

$$|f(z) - g(z)| < |f(z)| + |g(z)| \tag{8.25}$$

holds at every point z of ∂D, then f and g have the same number of zeros in D, provided that zero-counts are made with due regard for multiplicity.

Proof. Inequality (8.25) prevents either f or g from having a zero on the Jordan curve $J = \partial D$. In particular, J passes through no zero or pole of $h = f/g$, a function that is meromorphic in some open set containing $\overline{D}$. Furthermore, dividing both sides of (8.25) by $|g(z)|$ leads to

$$(8.26) \qquad |1 - h(z)| < 1 + |h(z)|$$

for every z on J. Since $|1 - h(z)| = 1 + |h(z)|$ would plainly be true of any z for which $h(z)$ is real and non-positive, (8.26) implies that $h(J)$ is disjoint from the real interval $(-\infty, 0]$. It follows that the origin lies in the unbounded component of $\mathbb{C} \sim h(J)$ and, thus, that the path $\beta = h \circ \gamma$, where γ is any Jordan contour with trajectory J, has $n(\beta, 0) = 0$. Recalling the remarks pursuant to the proof of the argument principle and noting, as a simple calculation verifies, that $h'/h = (f'/f) - (g'/g)$, we conclude that

$$(8.27) \quad \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)} - \frac{1}{2\pi i}\int_\gamma \frac{g'(z)\,dz}{g(z)} = \frac{1}{2\pi i}\int_\gamma \frac{h'(z)\,dz}{h(z)} = n(\beta, 0) = 0.$$

In light of the argument principle and the assumption that neither f nor g has poles in D, the first term in (8.27) translates to $Z_f - Z_g$, where Z_f and Z_g designate the zero-totals for f and g in D, computed with allowance for multiplicity. Therefore, $Z_f = Z_g$. ∎

Condition (8.25) is definitely met if the inequality $|f(z) - g(z)| < |g(z)|$ is satisfied at every point of ∂D. It is this inequality, rather than (8.25), that appears in the classical statement of Rouché's theorem. The theorem in the form we have presented is of more recent vintage. (For extra commentary on this subject we refer the reader to the article "A remark on Rouché's theorem" by Irving Glicksberg in *The American Mathematical Monthly*, Vol. 83, No. 3, 1976.) The great value of Rouché's theorem stems from its capacity to provide information on the existence, number, and location of the zeros of an analytic function f solely on the basis of comparisons between $|f - g|$ and $|f| + |g|$ for suitably chosen "test functions" g, whose zeros are known from the start, along properly selected Jordan curves. Two concrete examples will demonstrate how this can work.

EXAMPLE 3.11. Let $f(z) = z^5 + 5z^3 + z - 2$. Show that f has three of its roots in the disk $\Delta(0, 1)$ and all of its roots in $\Delta(0, 5/2)$.

There is faint hope of actually finding the zeros of f, but Rouché's theorem will enable us to get some fix on their locations. When $|z|$ is near 1 the dominant term in $f(z)$ is $5z^3$. The suggestion is that this term might govern the number of zeros of f in the disk $\Delta(0, 1)$. If we take $g(z) = 5z^3$ we find that

$$|f(z) - g(z)| = |z^5 + z - 2| \leq |z|^5 + |z| + 2 = 4 < 5 = |g(z)|$$

when $|z| = 1$, so by Rouché's theorem f and g have an equal number of zeros in $\Delta(0, 1)$. The only zero of g in that disk is a zero of order three at the origin. Therefore, making allowances for multiple roots, f has three of its roots in $\Delta(0,1)$. Once $|z|$ exceeds 2 the term z^5 begins to take over as the dominant one in $f(z)$. If we switch to $h(z) = z^5$ as our "comparison function" in Rouché's theorem and notice that

$$|f(z) - h(z)| = |5z^3 + z - 2| \leq 2644/32 < 3125/32 = |h(z)|$$

when $|z| = 5/2$, we learn that f must have all five of its zeros in $\Delta(0,5/2)$. (N.B. Other sources of information can, of course, be tapped in order to pinpoint further the zeros of f. For instance, since $f(0) = -2 < 0$ and $f(1) = 5 > 0$, the intermediate value theorem guarantees the presence of a root of f in the interval $(0,1)$. As $f'(z) = 5z^4 + 15z^2 + 1 > 0$ for every real z, f has no other real roots. Because the non-real roots of f occur in conjugate pairs, we conclude that exactly two of these roots have positive imaginary parts. It follows that f has no multiple roots. Finally, since the sum of the roots of f — i.e., the negative of the coefficient of z^4 in $f(z)$ — is zero, since three of these roots lie in $\Delta(0,1)$, and since $z + \bar{z} = 2\,\mathrm{Re}\,z$, we discover that the two roots of f outside $\Delta(0,1)$ have their shared real part in the interval $(-3/2, 3/2)$.)

EXAMPLE 3.12. Assuming that c is a complex number with $|c| > e$, show that the equation $e^z = cz$ has exactly one solution in the half-plane $H = \{z : \mathrm{Re}\,z < 1\}$.

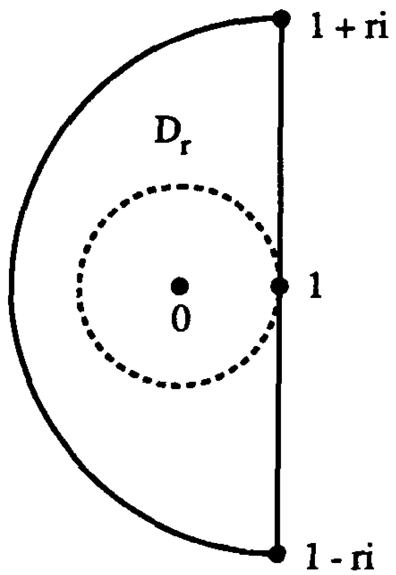

Figure 8.

It is enough to verify that for every $r \geq 2$ the given equation has one and only one solution located in the domain $D_r = H \cap \Delta(1,r)$. (See Figure 8.) We fix $r \geq 2$ and compare the function $f(z) = cz - e^z$ with $g(z) = cz$ on the Jordan curve $J = \partial D_r$. Since $r \geq 2$, D_r encompasses the disk $\Delta(0,1)$, which forces $|z| \geq 1$ to hold whenever z lies on J. For such z, therefore,

$$|f(z) - g(z)| = |-e^z| = e^{\mathrm{Re}\,z} \leq e < |c| \leq |c||z| = |g(z)|\,.$$

By Rouché's theorem f and g have an equal number of zeros in D_r. Because the only zero of g there is a simple zero at the origin, f has exactly one zero in D_r as well. Accordingly, $e^z = cz$ has precisely one solution in D_r. As this is true for every $r \geq 2$, this equation has a unique solution belonging to H. (N.B. We have not ruled out the possibility that $e^z = cz$ has additional solutions in the set $\mathbb{C} \sim H$. Such solutions do, in fact, exist.)

The establishment of Rouché's theorem has put us in a position to assemble a qualitatively accurate picture of the local mapping properties of analytic functions. Roughly speaking, what this picture reveals is that, in the vicinity of a point where an analytic function assumes the value w_0 with multiplicity m, its behavior very much mimics that of the function $f(z) = w_0 + z^m$ in the proximity of the origin. We give this statement a precise formulation under the title "Branched Covering Principle," but we warn the reader that this is by no means standard terminology in the literature.

Theorem 3.7. (Branched Covering Principle) *Suppose that a function f is analytic in an open set U, that z_0 is a point of U, and that f takes the value w_0 with multiplicty m at z_0. Let $r > 0$ be any number sufficiently small that the following conditions prevail: the closed disk $\overline{\Delta} = \overline{\Delta}(z_0, r)$ is contained in U, and the statements $f(z) \neq w_0$, $f'(z) \neq 0$ are true for every z in the set $\overline{\Delta} \sim \{z_0\}$. Define $s = s(r) > 0$ by $s = \min\{|f(z) - w_0| : z \in K(z_0, r)\}$. Then $G = \{z \in \Delta(z_0, r) : f(z) \in \Delta(w_0, s)\}$ is a domain. Moreover, for each point w of the punctured disk $\Delta^*(w_0, s)$ the set $E_w = \{z \in \Delta(z_0, r) : f(z) = w\}$ consists of exactly m points of G, at each of which f assumes the value w with multiplicity one.*

Proof. (Figure 9 illustrates the situation for $m = 3$.) The definition of f having a multiplicity at z_0 includes the assumption that f is not constant in

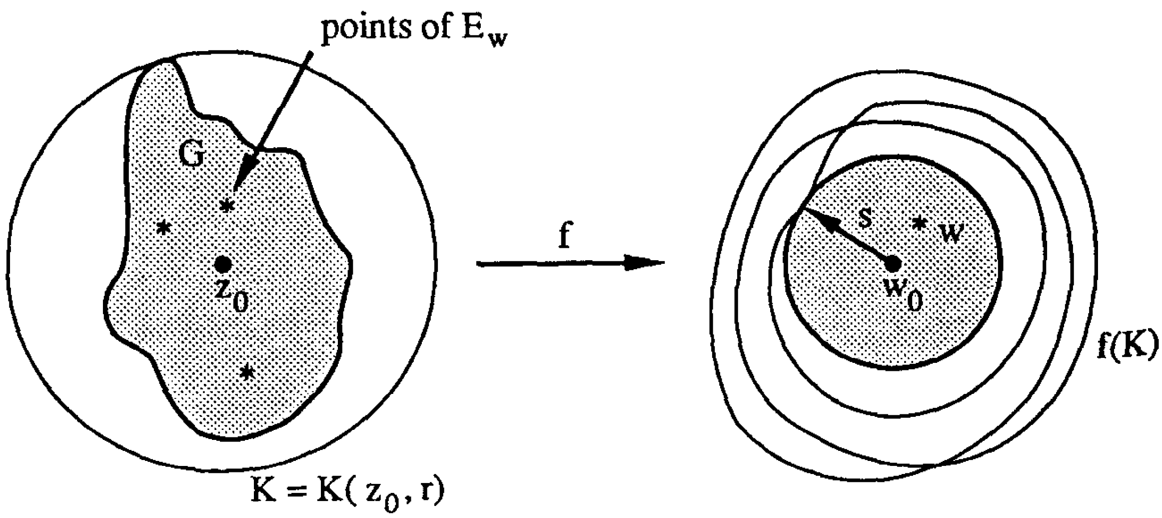

Figure 9.

D, the component of U which contains z_0 — hence, that f' does not vanish identically in this domain. The discrete mapping theorem thus certifies that both $\{z \in D: f(z) = w_0\}$ and $\{z \in D: f'(z) = 0\}$ are discrete subsets of D. In particular, it is the case that for every suitably small $r > 0$ the disk $\overline{\Delta} = \overline{\Delta}(z_0, r)$ lies in D and the conditions $f(z) \neq w_0$ and $f'(z) \neq 0$ are satisfied at every point z of $\overline{\Delta} \sim \{z_0\}$. We focus on such an r. Since the function sending z to $|f(z) - w_0|$ is continuous and positive on the circle $K = K(z_0, r)$, it has a positive minimum value on K, a value we have labeled s. The continuity of f in $\Delta(z_0, r)$ implies that $G = \{z \in \Delta(z_0, r): f(z) \in \Delta(w_0, s)\}$ describes an open set (Theorem II.2.5).

We now fix a point w belonging to the punctured disk $\Delta^*(w_0, s)$. We apply Rouché's theorem to the pair of functions $g(z) = f(z) - w_0$ and $h(z) = f(z) - w$ in the disk $\Delta = \Delta(z_0, r)$. Since

$$|g(z) - h(z)| = |f(z) - w_0 - (f(z) - w)| = |w - w_0| < s \leq |g(z)|$$

for every point z of $\partial\Delta = K$, we infer that, once multiplicity is accounted for, g and h have an equal number of zeros in Δ. By construction the function g has exactly m zeros there, all of them concentrated in a zero of order m at z_0. Consequently, h also has precisely m zeros in Δ — these necessarily located in $\Delta^*(z_0, r)$, for $w \neq w_0$. The fact that $h'(z) = f'(z) \neq 0$ for z in $\Delta^*(z_0, r)$ means that each of the m zeros of h in $\Delta(z_0, r)$ is a simple zero. In other words, $E_w = \{z \in \Delta(z_0, r): f(z) = w\}$ contains exactly m points, at each of which f assumes the value w with multiplicity one. By the definition of G, E_w is a subset of G.

It remains only to demonstrate that the open set G is connected. For this, let V be any component of G. We show that z_0 must be a point of V. If this is true, then G has as its sole component the unique one containing z_0; i.e., G is connected. We shall assume, to the contrary, that z_0 does not belong to V, and then argue to a contradiction. Consider the function $k: \overline{V} \to \mathbb{C}$ given by $k(z) = [f(z) - w_0]/s$. This function is continuous, it is analytic in V, and, since z_0 is not a point of V and V lies in $\overline{\Delta}$, k is free of zeros in V. Owing to the definition of G, $|k(z)| < 1$ holds for all z in V. Thus, the continuity of k implies that $|k(z)| \leq 1$ throughout $\overline{V}$. It now follows from the fact that V is a component of G that $|k(z)| = 1$ whenever z is a point of ∂V. (N.B. If $|k(z)| < 1$ were to hold for some z on ∂V, then z would have to lie in Δ — z is certainly an element of $\overline{\Delta}$ and, in view of the definition of s, $|k(z)| \geq 1$ on the circle $K = K(z_0, r)$ — and by continuity there would be an open disk $\Delta(z, \rho)$ contained in Δ with the property that $|k(\zeta)| < 1$ for every ζ in $\Delta(z, \rho)$. This would stamp $\Delta(z, \rho)$ as a subset of G and so place this disk inside some component of G, a state of affairs incompatible with z being a boundary point of a component of G.) We conclude that the function $1/k$ is continuous on $\overline{V}$, analytic in V, and of modulus one everywhere on ∂V. The maximum principle — or, more precisely, Corollary V.3.11 — informs us that $1/|k(z)| \leq 1$ for every

z in V; i.e., $|k(z)| \geq 1$ in V. This obvious contradiction renders untenable the assumption that z_0 is not a member of V. Accordingly, G has but one component — hence, is a domain. ∎

The paradigm for Theorem 3.7 is found in the function $f(z) = z^m$ at $z_0 = 0$. In this example the conditions on r are met for every $r > 0$, we have $s = r^m$ and $G = \Delta(0,r)$, and for w in $\Delta^*(0,s)$ the points of E_w are just the m^{th}-roots of w. In the general case described by the theorem the arrangement of points in the sets E_w need not be so regular, but f still maps the set $G \sim \{z_0\}$ in an m-to-one fashion onto the punctured disk $\Delta^*(w_0,s)$.

The restriction of the function f in Theorem 3.7 to the domain G belongs to a class of mappings that has great significance both for complex analysis and for topology, the class of "branched covering mappings." Although there are slight discrepancies in the way the term "branched covering" is used in the two subjects, complex analysts and topologists would concur here in describing the restriction of f to G as an *m-sheeted branched covering* of the disk $\Delta(w_0,s)$, *branched* (or *ramified*) in the case $m > 1$ over the point w_0. If $m > 1$, z_0 is the unique *branch point* of this particular covering, meaning the only point of G where the multiplicity of f exceeds one. The machinery of covering space theory, a part of topology, leads to an interesting refinement in the conclusion of the branched covering principle: *if $\Delta_0 = \Delta(0, \sqrt[m]{s})$, then it is possible to construct a univalent analytic function $\psi: \Delta_0 \to \mathbb{C}$ that satisfies $\psi(\Delta_0) = G, \psi(0) = z_0$, and $f[\psi(\zeta)] = w_0 + \zeta^m$ for every ζ in Δ_0* (Figure 10).

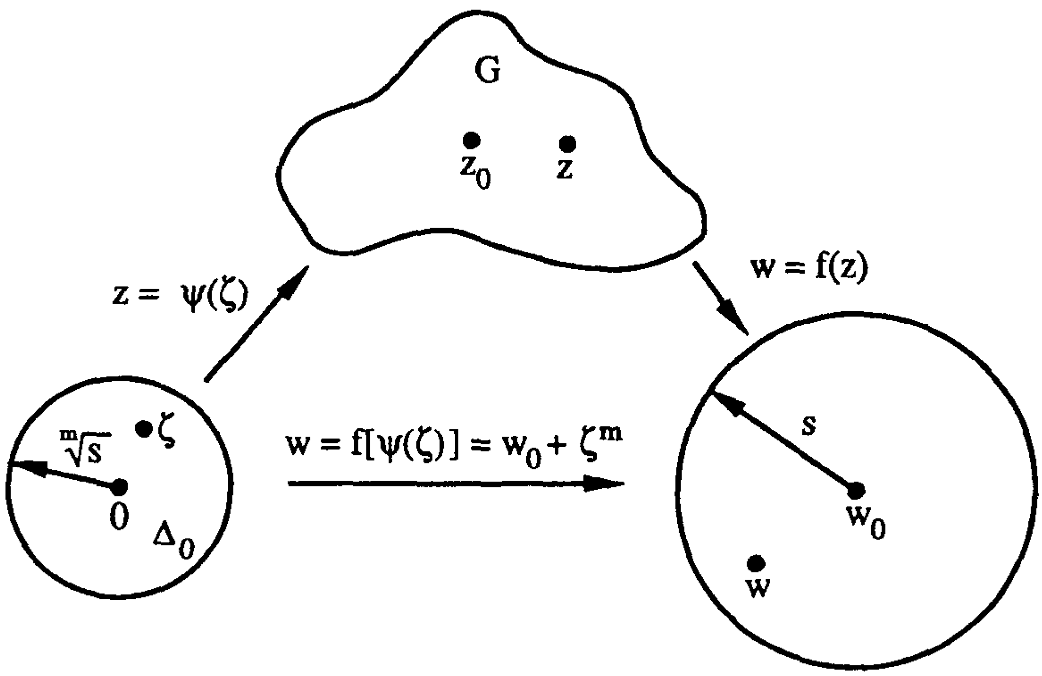

Figure 10.

It is not uncommon in mathematics to simplify a discussion by making an appropriate change of variables or, to put it differently, by choosing a coordinate system suited to that discussion. For instance, a correspondence $w = f(z) = f(x,y)$ might be more easily understood when viewed in polar coordinates, $w = f(r,\theta)$, following the change of variables $x = r\cos\theta$, $y = r\sin\theta$. The message of Figure 10 is that the mapping ψ can be used to install a new coordinate system in the domain G by assigning to a point z of G the coordinates of the point $\zeta = \xi + i\eta$ in Δ_0 for which $\psi(\zeta) = z$ and that, when expressed in terms of this transplanted coordinate system, the mapping $w = f(z)$ assumes the extremely simple form $w = f(\zeta) = w_0 + \zeta^m$. The mapping ψ is a particularly nice one — ψ is a "conformal mapping," the type of mapping that provides the subject matter for the next chapter — so nice, in fact, that almost every important qualitative geometric feature exhibited by f in G is inherited by the function $f \circ \psi$. Accepting this statement at face value and realizing that the geometry of $f \circ \psi$, which just maps ζ to $w_0 + \zeta^m$, is completely transparent, we recognize in the mapping ψ, whose existence is assured by covering space theory, the key to a total understanding of the local geometric structure of f near z_0.

A complex-valued function f whose domain-set includes the open set V in the complex plane is called an *open mapping of* V under the condition that $f(U)$ is an open set whenever U is an open subset of V. Not the least of the corollaries of the branched covering principle is:

Theorem 3.8. (Open Mapping Theorem) *If a function f is analytic and non-constant in a plane domain D, then f is an open mapping of D. In particular, $f(D)$ is a domain.*

Proof. Let U be an open set that is contained in D. We must verify that the set $f(U)$ is open. Given a point w_0 of $f(U)$ — say $w_0 = f(z_0)$, where z_0 belongs to U — we must produce an open disk $\Delta(w_0, s)$ that is a subset of $f(U)$. To this end, we simply choose $r > 0$ such that $\overline{\Delta} = \overline{\Delta}(z_0, r)$ lies in U and such that the requirements $f(z) \neq w_0$, $f'(z) \neq 0$ in the branched covering principle are fulfilled at every point z of $\overline{\Delta} \sim \{z_0\}$. The latter is possible because f is assumed to be analytic and non-constant in D. The aforementioned proposition certifies that every point of the disk $\Delta(w_0, s)$, where $s = \min\{|f(z) - w_0| : z \in K(z_0, r)\}$, is in the image of $\Delta(z_0, r)$ under f — hence, lies in $f(U)$. Thus, $f(U)$ is an open set. In particular, $f(D)$ is an open set. As this set is also connected (Theorem II.3.8), it is seen to be a domain. ∎

Another interesting observation based on the branched covering principle will be required for our discussion of conformal mappings in the succeeding chapter.

Theorem 3.9. *Let f be a function that is analytic in a domain D. If f is univalent in D, then $f'(z) \neq 0$ holds for every z in D.*

Proof. Suppose that $f'(z_0) = 0$ for some point z_0 of D. Being univalent in D, f is certainly non-constant there. Therefore, f assumes the value

$w_0 = f(z_0)$ with some multiplicity m at z_0. Because $f'(z_0) = 0$, $m \geq 2$. The branched covering principle shows that any point w sufficiently close to w_0, but different from it, has at least two preimages in D. This contradicts the univalence hypothesis. The conclusion: $f'(z) \neq 0$ must hold at each z in D. ∎

Theorem 3.9 should be contrasted with the situation in ordinary calculus where, for instance, the function $f(x) = x^3$ is a univalent differentiable function from $\mathbb{R}$ to itself, and yet $f'(0) = 0$. The converse of Theorem 3.9 is false. As an example, $f(z) = e^z$ is an entire function, $f'(z) = e^z$ is never zero, but f is far from univalent. On a local level, however, Theorem 3.9 does admit a converse.

Theorem 3.10. *Let f be a function that is analytic in a domain D. If z_0 is a point of D for which $f'(z_0) \neq 0$, then there is a subdomain G of D containing z_0 in which f is univalent.*

Proof. Since $f'(z_0) \neq 0$, f assumes the value $w_0 = f(z_0)$ with multiplicity $m = 1$ at z_0. Referring to the statement of Theorem 3.7 and taking $U = D$, we see that in the case $m = 1$ the domain G contains z_0 and is mapped by f in a univalent fashion onto the disk $\Delta(w_0, s)$. ∎

A final contribution to the circle of ideas surrounding the branched covering principle concerns inverses of analytic functions.

Theorem 3.11. (Inverse Function Theorem) *Suppose that D is a domain in the complex plane and that $f: D \to \mathbb{C}$ is a univalent analytic function. Then its inverse function $f^{-1}: f(D) \to D$ is also analytic.*

Proof. The open mapping theorem insures that $D' = f(D)$ is a domain. We first prove that f^{-1} is a continuous function. Let z_0 be a point of D', and let $w_0 = f^{-1}(z_0)$. Given $\epsilon > 0$ we shall produce a $\delta > 0$ for which $f^{-1}[\Delta(z_0, \delta)]$ is contained in $\Delta(w_0, \epsilon)$. Now $U = D \cap \Delta(w_0, \epsilon)$ is an open set in D, one that includes the point w_0, so by the open mapping theorem $U' = f(U)$ is an open subset of D' containing z_0. Plainly $f^{-1}(U') = U$. We select $\delta > 0$ so that $\Delta = \Delta(z_0, \delta)$ is contained in U'. Then $f^{-1}(\Delta)$ is a subset of U — hence, of $\Delta(w_0, \epsilon)$. This demonstrates that f^{-1} is continuous at each point of D'. It follows that $g = f^{-1}$ satisfies the hypotheses of Theorem 1.8 in the domain D'. We conclude on the basis of that result that f^{-1} is an analytic function. ∎

Rouché's theorem is pertinent to topics other than the local behavior of analytic functions. Evidence of this fact surfaces in the following convergence theorem of Adolf Hurwitz (1859-1919).

Theorem 3.12. (Hurwitz's Theorem) *Suppose that each function in a sequence $\langle f_n \rangle$ is analytic and zero-free in a domain D and that $f_n \to f$ normally in D. Then either f is free of zeros in D or it is identically zero there.*

Proof. The function f is analytic in D. Assume that $f(z_0) = 0$ for some point z_0 of D. We show that $f(z) = 0$ for every z in D. Suppose this not to be the case. The alternative is for z_0 to be an isolated zero of f. We can, accordingly, choose an $r > 0$ such that the closed disk $\overline{\Delta} = \overline{\Delta}(z_0, r)$ is contained in D and such that $f(z) \neq 0$ holds for every z on the circle $K = K(z_0, r)$. The quantity $|f(z)|$ attains a positive minimum value on K, call that value ϵ. Since $f_n \to f$ uniformly on K, there exists an index n with the property that $|f_n(z) - f(z)| < \epsilon \leq |f(z)|$ is true for every point z of K. Rouché's theorem dictates that f_n and f have the same number of zeros in $\Delta(z_0, r)$. Therefore, f_n has at least one zero there, contrary to hypothesis. The contradiction compels the conclusion that f does, indeed, vanish identically in D. ∎

The function $f_n(z) = z/n$ is analytic and zero-free in the domain $D = \mathbb{C} \sim \{0\}$. Quite clearly, $f_n \to 0$ normally in D. The second possibility allowed by the conclusion of Hurwitz's theorem is thus seen to be a real one. In the next chapter we shall have need of a noteworthy consequence of Hurwitz's theorem, a result with which we close this section.

Theorem 3.13. *Suppose that each function in a sequence $\langle f_n \rangle$ is analytic and univalent in a domain D and that $f_n \to f$ normally in D. Then either f is univalent in D or it is constant there.*

Proof. The function $f: D \to \mathbb{C}$ is analytic. Under the assumption that f is non-constant we verify that it is univalent in D. Fix z_0 in D. We show that $f(z) \neq f(z_0)$ when $z \neq z_0$. For this, consider the sequence of functions g_n defined in the domain $D_0 = D \sim \{z_0\}$ by $g_n(z) = f_n(z) - f_n(z_0)$. Because f_n is univalent in D, g_n is zero-free in D_0. Of course, $g_n \to g$ normally in D_0, where $g(z) = f(z) - f(z_0)$. Furthermore, g is not the zero function in D_0. (If it were, then f would be constant in D.) Hurwitz's theorem confirms that g is free of zeros in D_0; i.e., $f(z) \neq f(z_0)$ for $z \neq z_0$. As z_0 was an arbitrary point of D, the univalence of f is established. ∎

4 Function Theory on the Extended Plane

4.1 The Extended Complex Plane

Expressions like

$$\lim_{x \to x_0} f(x) = \infty \quad , \quad \lim_{x \to -\infty} g(x) = \ell \quad , \quad \lim_{x \to \infty} h(x) = -\infty$$

are commonplace in calculus. Such "limits involving infinity" provide us with valuable knowledge about the behavior of functions in situations that are not strictly covered by the primary definition of a limit, in which both the limit itself and the point at which it is taken are required to be real numbers. That definition can be modified, of course, so as to turn each of the above limit expressions into a meaningful and informative mathematical statement. A standard way to unify the treatments of ordinary limits and limits involving infinity is to work in the context of the "extended real line" obtained by attaching two "ideal boundary points," the points $-\infty$ and ∞, to the field $\mathbb{R}$ of real numbers. These added points are not to be thought of as "extra" real numbers. In fact, $-\infty < x < \infty$ holds by definition for all real numbers x. Moreover, the algebraic structures in $\mathbb{R}$ do not extend fully to $[-\infty, \infty]$: certain sums, products, and quotients containing the points at infinity can be consistently defined — e.g., $\infty + \infty = \infty$, $\infty \cdot (-\infty) = -\infty$, $1/\infty = 0$ — while others remain indeterminate — e.g., $\infty + (-\infty)$, $0 \cdot \infty$, $1/0$. Nevertheless, this expanded number system furnishes an excellent framework in which to discuss simultaneously all the variants of the limit concept suggested above.

An extended system similar to the one just described for the real numbers can be introduced for the complex numbers. There is, however, a crucial difference between the real and complex cases: the "extended complex plane" is formed by adjoining to $\mathbb{C}$ not two, but a single point at infinity. The disparity is tied up with the fact that $\mathbb{C}$ is an unorderable field, whereas $\mathbb{R}$ comes equipped with its standard ordering. (See Appendix A.) Thus, the *extended complex plane*, a set we symbolize by $\widehat{\mathbb{C}}$, consists of the complex plane $\mathbb{C}$ and an ideal point ∞ which, as part of its definition, is required not to be an element of $\mathbb{C}$. The plane $\mathbb{C}$ and the point ∞ fit together to form $\widehat{\mathbb{C}}$ in such a fashion that any sequence $\langle z_n \rangle$ of complex numbers without an accumulation point in $\mathbb{C}$ must tend to ∞ as a limit in $\widehat{\mathbb{C}}$. (This will be made precise when we discuss the topology of $\widehat{\mathbb{C}}$ several paragraphs hence.) The following algebraic rules are postulated for $\widehat{\mathbb{C}}$ to supplement those already in effect in $\mathbb{C}$:

$$(8.28) \qquad \begin{cases} \infty \pm z = z \pm \infty = \infty & \text{for } z \text{ in } \mathbb{C} \ ; \\ \infty \cdot z = z \cdot \infty = \infty & \text{for } z \text{ in } \widehat{\mathbb{C}} \sim \{0\} \ ; \\ z/\infty = 0 & \text{for } z \text{ in } \mathbb{C} \ ; \\ z/0 = \infty & \text{for } z \text{ in } \widehat{\mathbb{C}} \sim \{0\} \ . \end{cases}$$

Observe that none of the expressions $\infty + \infty$, $\infty - \infty$, ∞/∞, $0/0$, or $0 \cdot \infty$ is assigned meaning in $\widehat{\mathbb{C}}$.

4.2 The Extended Plane and Stereographic Projection

There is a traditional way of representing the extended complex plane $\widehat{\mathbb{C}}$ as a concrete geometric object. To describe it, we start with the sphere $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ in three-dimensional euclidean space $\mathbb{R}^3$. By identifying the complex number $x_1 + ix_2$ with the point $(x_1, x_2, 0)$

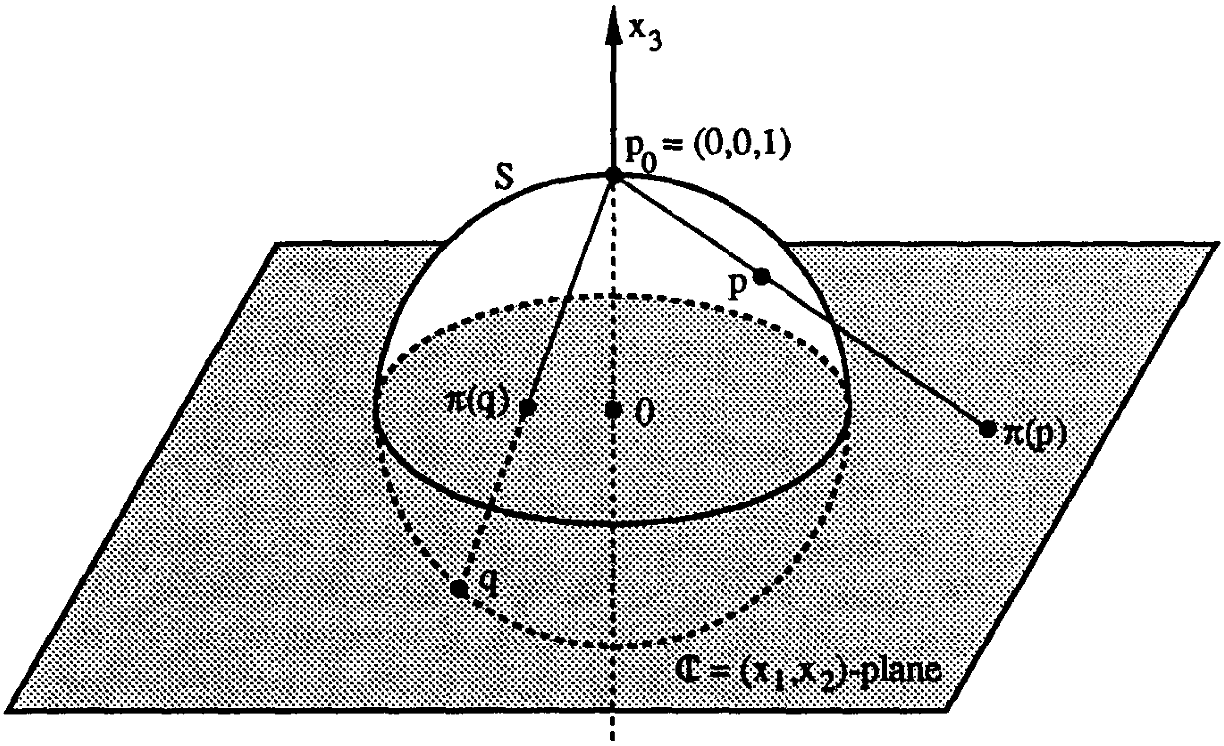

Figure 11.

we are free to think of $\mathbb{C}$ as sitting inside $\mathbb{R}^3$, masquerading as the (x_1, x_2)-plane. Having made this identification, we set $p_0 = (0,0,1)$ and define a function $\pi: S \sim \{p_0\} \to \mathbb{C}$ as follows: for p in S, $p \neq p_0$, $\pi(p)$ is the point of intersection of $\mathbb{C}$ with the ray in $\mathbb{R}^3$ that issues from p_0 and passes through p (Figure 11). The function π is called the *stereographic projection of $S \sim \{p_0\}$ onto $\mathbb{C}$*.

To determine an explicit formula for $\pi(p)$, notice that by definition $\pi(p) = p_0 + t(p - p_0)$, where t is the unique positive real number that forces the third coordinate of $p_0 + t(p - p_0)$ to vanish. Writing $p = (x_1, x_2, x_3)$, we obtain

$$p_0 + t(p - p_0) = (0,0,1) + t(x_1, x_2, x_3 - 1) = (tx_1, tx_2, 1 + t(x_3 - 1))$$

and see that $t = (1 - x_3)^{-1}$ is the number in question. Consequently, for $p = (x_1, x_2, x_3)$ in $S \sim \{p_0\}$ we have

$$\text{(8.29)} \qquad \pi(p) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}, 0\right) = \frac{x_1}{1 - x_3} + i\left(\frac{x_2}{1 - x_3}\right).$$

Given a complex number $z = x + iy$, one checks easily that $z = \pi(p)$ for

$$p = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right)$$

and arrives at the fact that π has an inverse function $\pi^{-1}: \mathbb{C} \to S \sim \{p_0\}$, the function whose rule of correspondence is

$$\text{(8.30)} \qquad \pi^{-1}(z) = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right).$$

The upshot of the preceding comments is that π establishes a one-to-one correspondence between $S \sim \{p_0\}$ and $\mathbb{C}$. Furthermore, formulas (8.29) and (8.30) make it evident that both π and π^{-1} are continuous functions. In the standard language of topology a continuous function with a continuous inverse is known as a *homeomorphism*. The domain-set and range of a homeomorphism are from a purely topological point of view indistinguishable sets. In our case, $S \sim \{p_0\}$ and $\mathbb{C}$ are thus seen via π to be *homeomorphic* (or *topologically equivalent*) sets. Remarking that $|\pi(p)| \to \infty$ as $p \to p_0$ and that $\pi^{-1}(z) \to p_0$ as $|z| \to \infty$, it seems only natural to extend π to a univalent function on S with range $\widehat{\mathbb{C}}$ by setting $\pi(p_0) = \infty$. Once this is done (and once the appropriate topological structure on $\widehat{\mathbb{C}}$ is in place), π becomes a continuous mapping from S to $\widehat{\mathbb{C}}$ that has a continuous inverse; i.e., π becomes a homeomorphism from S onto $\widehat{\mathbb{C}}$. Because of this beautiful correspondence between S and $\widehat{\mathbb{C}}$ under stereographic projection, it is topologically correct to visualize $\widehat{\mathbb{C}}$ as a sphere. Indeed, one of the common terms of reference to $\widehat{\mathbb{C}}$ describes it as the "Riemann sphere."

The stereographic projection mapping $\pi: S \to \widehat{\mathbb{C}}$ has many interesting geometric properties. At least a couple of these deserve a mention in passing. Thinking of p_0 as the north pole of S, we note that latitudinal circles on S project under π to circles in $\mathbb{C}$ centered at the origin, whereas longitudinal circles on S are mapped by π to lines in $\mathbb{C}$ passing through the origin — or, rather, to such lines with the point ∞ appended to them. In fact, it is a general phenomenon that a circle K situated on S will be transformed by π to a *circle in* $\widehat{\mathbb{C}}$, by which we mean either a true circle in $\mathbb{C}$ or a line in $\mathbb{C}$ with the point ∞ adjoined, and that circles in $\widehat{\mathbb{C}}$ are carried by π^{-1} to circles on S (Exercise 5.81). In addition, if two different circles on S intersect at a point p of $S \sim \{p_0\}$ at an angle θ — in order to be definite, use the non-obtuse angle of intersection — it can be demonstrated that the images of these circles under π intersect at the point $\pi(p)$ at the same angle θ. Consequently, we speak of π as a "conformal" (= angle preserving) mapping.

4.3 Functions in the Extended Setting

Just as we have declared that, when nothing is said to the contrary, the notation $f: A \to \mathbb{C}$ always indicates a function whose domain-set A lies in $\mathbb{C}$, we now establish the convention that, unless otherwise stated, $f: A \to \widehat{\mathbb{C}}$ will invariably signify a function having a subset A of $\widehat{\mathbb{C}}$ as its domain-set. The latter class of functions includes the former one, but it also embraces functions with ∞ located in their domain-sets or ranges or both. We have earlier agreed to take as the domain-set of a function f that is given by a formula, with no domain-set specified, the set of all points in $\mathbb{C}$ for which the formula expressing f has meaning. The analogous convention will apply in $\widehat{\mathbb{C}}$. Bearing in mind the extra algebraic rules we've adopted for $\widehat{\mathbb{C}}$, however, we remark that a formula which fails to make sense relative to the complex number system at a given point may well become meaningful at that point when viewed in the context of $\widehat{\mathbb{C}}$. Thus, a function presented by a formula typically has associated with it two distinct domain-sets, the first arising when the formula is interpreted as describing a complex-valued function of a complex variable, the second when ∞ is admitted as a candidate for the domain-set or range of the function in question. In order to distinguish between the two situations we shall refer to the domain-set in $\widehat{\mathbb{C}}$ of a formula-generated function as its *extended domain-set.* Take, for example, the function $f(z) = z^{-1}$. By our previous convention, its domain-set is $\mathbb{C} \sim \{0\}$. Its extended domain-set is $\widehat{\mathbb{C}}$, since according to (8.28) both $f(0) = 1/0 = \infty$ and $f(\infty) = 1/\infty = 0$ are well-defined in $\widehat{\mathbb{C}}$. In fact, $f(z) = z^{-1}$ defines a univalent mapping of $\widehat{\mathbb{C}}$ onto itself. The domain-set and the extended domain-set of the function $g(z) = e^z$, on the other hand, coincide (each is the plane $\mathbb{C}$) for e^{∞} is an undefined expression in $\widehat{\mathbb{C}}$.

Let f be a rational function of z, say

$$f(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m} \tag{8.31}$$

with $a_n \neq 0$ and $b_m \neq 0$. We assume that f is in lowest terms, which demands that its numerator and denominator have no common polynomial factor of positive degree or, equivalently, that these functions have no common zero. (N.B. A rational function of z not in lowest terms can always be reduced to lowest terms and, in fact, there is an algorithm for carrying out the reduction that does not depend on finding the roots of its numerator and denominator.) As a formula defining a function in $\widehat{\mathbb{C}}$, (8.31) then has meaning for every z in $\mathbb{C}$. It assigns the value ∞ at each root of the denominator. Furthermore, by rewriting $f(z)$ for $z \neq 0$ in the form

$$f(z) = z^{n-m} \frac{\dfrac{a_0}{z^n} + \dfrac{a_1}{z^{n-1}} + \cdots + a_n}{\dfrac{b_0}{z^m} + \dfrac{b_1}{z^{m-1}} + \cdots + b_m},$$

we can also make sense of $f(\infty)$; namely, $f(\infty) = a_n/b_n$ if $n = m$, $f(\infty) = 0$ if $n < m$, and $f(\infty) = \infty$ if $n > m$. In other words, we can assert: *allowing for reduction to lowest terms, the extended domain-set of a rational function of z is $\widehat{\mathbb{C}}$.* As a special case, any polynomial function of z has extended domain-set $\widehat{\mathbb{C}}$ and, if it has positive degree, it takes the value ∞ at ∞.

Given functions $f: A \to \widehat{\mathbb{C}}$ and $g: B \to \widehat{\mathbb{C}}$ we can form the functions cf with a complex constant c, $f + g$, fg, and f/g, just as we did for complex-valued functions. In the extended case it is necessary to exercise a bit more caution with regard to the extended domain-sets of such algebraic combinations of f and g, for any one of these may be different from $A \cap B$.

The point is that in forming these functions we must avoid elements of $A \cap B$ at which indeterminacies such as $\infty + \infty$, $0 \cdot \infty$, etc., crop up. Assuming that $f(A)$ is contained in B, the composition $g \circ f$ is defined exactly as it was earlier. In particular, the notion of an inverse function carries over directly to the setting of $\widehat{\mathbb{C}}$.

4.4 Topology in the Extended Plane

Pains were taken in Chapter II to phrase each of the basic topological definitions in terms that would make the transition from the topology of $\mathbb{C}$ to that of $\widehat{\mathbb{C}}$ as smooth as possible. We now briefly retrace the main lines of development in Chapter II and indicate how to transfer the key ideas found there to the extended plane. Since most of the pertinent definitions were formulated using the open disks $\Delta(z,r)$ for z in $\mathbb{C}$ and $r > 0$, the first thing we must do is introduce the analogous "open disks centered at ∞." This presents no problem: for $r > 0$, we simply define

$$\Delta(\infty, r) = \{z : |z| > r^{-1}\} \cup \{\infty\} = \widehat{\mathbb{C}} \sim \overline{\Delta}(0, r^{-1}) .$$

The closed disk $\overline{\Delta}(\infty, r)$ and punctured disk $\Delta^*(\infty, r)$ are defined similarly.

As in the finite plane, a point z of $\widehat{\mathbb{C}}$ is called an *interior point* of a subset A of $\widehat{\mathbb{C}}$ if there exists an $r > 0$ such that the open disk $\Delta(z,r)$ is contained in A. A set U in $\widehat{\mathbb{C}}$ all of whose points are interior points of U is pronounced an *open subset* of $\widehat{\mathbb{C}}$. Thus, any open set in $\mathbb{C}$ is an open set in $\widehat{\mathbb{C}}$ as well. We declare a subset A of $\widehat{\mathbb{C}}$ to be *closed in* $\widehat{\mathbb{C}}$ provided $\widehat{\mathbb{C}} \sim A$ is an open set. A closed set A in $\mathbb{C}$ need not be closed in $\widehat{\mathbb{C}}$; in fact, a closed subset of $\mathbb{C}$ is closed in $\widehat{\mathbb{C}}$ if and only if the set in question is also bounded. Theorems II.1.1 and II.1.2 have obvious parallels in $\widehat{\mathbb{C}}$.

To say that a point z in $\widehat{\mathbb{C}}$ is a *boundary point* of a subset A of the extended plane means that for every $r > 0$ the disk $\Delta(z,r)$ has non-empty intersection with both A and $\widehat{\mathbb{C}} \sim A$. The *extended boundary* of A, a set we represent by $\widehat{\partial} A$, is made up of all such boundary points. If A is a set in $\mathbb{C}$, then $\widehat{\partial} A = \partial A$ when A is bounded and $\widehat{\partial} A = \partial A \cup \{\infty\}$ otherwise. The *extended closure* $\widehat{A}$ of a set A in $\widehat{\mathbb{C}}$ is defined by $\widehat{A} = A \cup \widehat{\partial} A$. Again, for a subset A of $\mathbb{C}$ we have $\widehat{A} = \overline{A}$ when A is bounded and $\widehat{A} = \overline{A} \cup \{\infty\}$ in the unbounded case. Both of the sets $\widehat{\partial} A$ and $\widehat{A}$ are closed in $\widehat{\mathbb{C}}$.

Let $\langle z_n \rangle$ be a sequence of $\widehat{\mathbb{C}}$. The statement that $\langle z_n \rangle$ is *convergent in* $\widehat{\mathbb{C}}$ with the point z_0 of $\widehat{\mathbb{C}}$ as its *limit* means this: corresponding to each $\epsilon > 0$ there exists an index $N = N(\epsilon)$ such that z_n belongs to $\Delta(z_0, \epsilon)$ for every $n \geq N$. Similarly, $\langle z_n \rangle$ has the point z_0 of $\widehat{\mathbb{C}}$ as an *accumulation point* if for every $\epsilon > 0$ the disk $\Delta(z_0, \epsilon)$ contains z_n for infinitely many values of n. For a sequence $\langle z_n \rangle$ in $\mathbb{C}$ we emphasize the distinction between "$\langle z_n \rangle$ is convergent" and "$\langle z_n \rangle$ is convergent in $\widehat{\mathbb{C}}$ ": the former requires that $\lim_{n\to\infty} z_n$ belong to $\mathbb{C}$, whereas the latter recognizes $\lim_{n\to\infty} z_n = \infty$ as an acceptable alternative. Observe, too, that a sequence $\langle z_n \rangle$ in $\mathbb{C}$ which has no accumulation points in $\mathbb{C}$ necessarily satisfies $\lim_{n\to\infty} z_n = \infty$ in $\widehat{\mathbb{C}}$. The entire discussion of complex sequences in Chapter II demands little more than "fine tuning" when we transport it to $\widehat{\mathbb{C}}$. For instance, Theorem II.1.2, which describes the algebra of limits for convergent sequences, must undergo a slight adjustment in order to accomodate the rules of arithmetic peculiar to $\widehat{\mathbb{C}}$. One especially useful remark is that a sequence $\langle z_n \rangle$ in $\widehat{\mathbb{C}}$ has $\lim_{n\to\infty} z_n = \infty$ if and only if $\lim_{n\to\infty} 1/z_n = 0$.

A function $f: A \to \widehat{\mathbb{C}}$ is *continuous at a point* z_0 provided z_0 belongs to A and corresponding to each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, z_0) > 0$ with the property that

$$f[A \cap \Delta(z_0, \delta)] \subset \Delta(f(z_0), \epsilon) .$$

The only difference between this definition and the definition of continuity recorded in Chapter II is that we now admit $z_0 = \infty$ or $f(z_0) = \infty$ into the realm of possibilities. If f is continuous at every point of A, then f is proclaimed a *continuous function*. The rational functions of z, for example, constitute an important subclass of the continuous functions from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$. By simply writing $\widehat{\mathbb{C}}$ in place of $\mathbb{C}$ and substituting the words "extended complex plane" for "complex plane" we can transfer the bulk of the discussion of continuity from Chapter II to the present setting. A few points, these relating primarily to the algebraic conventions in $\widehat{\mathbb{C}}$ and their effects on the domain-sets of functions, demand minor cosmetic attention before they become valid in $\widehat{\mathbb{C}}$. This comment applies, for instance, to the counterpart of Theorem II.2.2 for the extended plane.

Modulo small technical details, again associated with the algebraic rules operative in $\widehat{\mathbb{C}}$, the treatment of limits of functions in the extended complex plane is also a retelling of the tale related in Chapter II concerning limits in the finite plane. Thus, if $f: A \to \widehat{\mathbb{C}}$ and if z_0 is a limit point of A in $\widehat{\mathbb{C}}$ (as in $\mathbb{C}$, this means that $A \cap \Delta^*(z_0, r) \neq \phi$ for every $r > 0$) we speak of f possessing a *limit in* $\widehat{\mathbb{C}}$ *at* z_0 if there exists a (necessarily unique) point w_0 of $\widehat{\mathbb{C}}$ with the following property: corresponding to each $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that

$$f[A \cap \Delta^*(z_0, \delta)] \subset \Delta(w_0, \epsilon) .$$

We continue to refer to w_0 as the *limit of* f *at* z_0 and to write $\lim_{z\to z_0} f(z) = w_0$ or $f(z) \to w_0$ as $z \to z_0$. In line with an earlier comment about sequences, we make a distinction, when dealing with a function $f: A \to \mathbb{C}$ and a finite limit point z_0 of A, between the statements "$\lim_{z\to z_0} f(z)$ exists" and "$\lim_{z\to z_0} f(z)$ exists in $\widehat{\mathbb{C}}$." In the first case, $\lim_{z\to z_0} f(z)$ must by definition be a complex number; in the second instance, $\lim_{z\to z_0} f(z) = \infty$

is an admissible possibility. Here are three simple examples:

$$\lim_{z\to 1}\frac{z^2-1}{(z-1)^2}=\lim_{z\to 1}\frac{z+1}{z-1}=\frac{2}{0}=\infty\ ;$$

$$\lim_{z\to\infty}(z^4+z^2+1)=\lim_{z\to\infty}\left[z^4\left(1+\frac{1}{z^2}+\frac{1}{z^4}\right)\right]$$

$$=\lim_{z\to\infty}z^4\lim_{z\to\infty}\left(1+\frac{1}{z^2}+\frac{1}{z^4}\right)=\infty\cdot 1=\infty\ ;$$

$$\lim_{z\to\infty}z\sin(z^{-1})=\lim_{z\to 0}\frac{\sin z}{z}=1\ .$$

The first and second examples, which illustrate the algebraic rules governing limits $\widehat{\mathbb{C}}$, also exploit the continuity of rational functions of z; the third applies the useful fact that $\lim_{z\to\infty}f(z)=\lim_{z\to 0}f(z^{-1})$ when one of these limits is known to exist in $\widehat{\mathbb{C}}$.

The definitions of *connected set*, *domain*, and *compact set* carry over verbatim from $\mathbb{C}$ to $\widehat{\mathbb{C}}$. A non-empty open set U in $\widehat{\mathbb{C}}$ is again the disjoint union of domains, the *components* of U. An important remark is that any closed subset K of $\widehat{\mathbb{C}}$ — this includes $K=\widehat{\mathbb{C}}$ — is compact. Indeed, let $\langle z_n\rangle$ be a sequence in K. Then $\langle z_n\rangle$ has an accumulation point in $\widehat{\mathbb{C}}$, for either it has an accumulation point in $\mathbb{C}$ or $z_n\to\infty$. Being closed in $\widehat{\mathbb{C}}$, K contains all accumulation points of $\langle z_n\rangle$ in $\widehat{\mathbb{C}}$. The sequence $\langle z_n\rangle$ must, therefore, have an accumulation point in K. Theorems II.3.8 and II.4.6 have analogues in $\widehat{\mathbb{C}}$: if $f\colon A\to\widehat{\mathbb{C}}$ is a continuous function, if C is a connected subset of A, and if K is a compact subset of A, then $f(C)$ is a connected set and $f(K)$ is a compact set.

Finally, the notions of a *discrete subset of an open set*, a *discrete mapping*, and an *open mapping* are defined in the obvious ways that generalize the definitions given for the corresponding ideas in the finite plane situation.

4.5 Meromorphic Functions and the Extended Plane

Let U be an open set in $\widehat{\mathbb{C}}$. A complex-valued function f is called *meromorphic in U* subject to the condition that f have at each point z_0 of U no worse than a pole. The only difference between this and the previous definition offered for the concept is that the set U is now allowed to include the point ∞, in which event the new definition adds the requirement that f have an isolated singularity at ∞ in the sense of Section 2.6 and that this singularity be non-essential. For example: any rational function of z is meromorphic in $\widehat{\mathbb{C}}$; $f(z)=\tan z$ is meromorphic in $\mathbb{C}$, but not in $\widehat{\mathbb{C}}$, since it does not have an isolated singularity at ∞; $g(z) = (z^2+1)^{-1}e^z$ is likewise meromorphic in the finite plane, but not the extended plane, for g has an essential singularity at ∞.

As soon as it is confirmed that a function f is meromorphic in an open subset U of $\widehat{\mathbb{C}}$, we shall enforce our announced policy of summarily ridding f of any removable singularities in U: when f has a removable singularity at a point z_0 of U — $z_0 = \infty$ is a possibility now — $f(z_0)$ automatically finds itself defined or redefined so as to insure that $f(z_0) = \lim_{z \to z_0} f(z)$. If we insist on working only with complex-valued functions, then no similar policy can be implemented for poles of f. If, however, we are willing to broaden our perspective and to admit functions taking values in $\widehat{\mathbb{C}}$, then something along these lines can be done. Namely, it follows from Theorem 2.4 that when z tends to a pole z_0 of f — $z_0 = \infty$ not excluded — $f(z) \to \infty$ with respect to the topology of $\widehat{\mathbb{C}}$. As a consequence, by defining $f(z_0) = \infty$ at each pole z_0 of f in the set U we extend the already modified function to a continuous function from U into $\widehat{\mathbb{C}}$. Motivated by these thoughts we issue a revamped "removal of singularities" policy: *whenever it is established that a function is meromorphic in an open subset U of $\widehat{\mathbb{C}}$, it is assumed that this function undergoes immediate modification to make it continuous in U as a function whose target-set is $\widehat{\mathbb{C}}$.*

We henceforth reserve the title *meromorphic function* for a function of the type $f: U \to \widehat{\mathbb{C}}$, where U is an open set in $\widehat{\mathbb{C}}$ and f is both continuous and meromorphic in U. Embedded in the foregoing discussion is the following characterization of a meromorphic function.

Theorem 4.1. *Let U be an open set in $\widehat{\mathbb{C}}$, and let $f: U \to \widehat{\mathbb{C}}$ be a continuous function. Then f is a meromorphic function if and only if the set $E = \{z \in U : z = \infty \text{ or } f(z) = \infty\}$ is a discrete subset of U and f is analytic in the open subset $U \sim E$ of $\mathbb{C}$.*

It is occasionally convenient to have a special designation for a function that is meromorphic in an open set U in $\widehat{\mathbb{C}}$ but has no poles in U. We refer to such a function as *holomorphic in U*. (If U lies in $\mathbb{C}$, then because of our removable singularity policy the phrase "holomorphic in U" is essentially synonymous with "analytic in U.") A *holomorphic function* is a meromorphic function without poles. There is one open set in which the holomorphic functions are especially easy to describe.

Theorem 4.2. *If $f: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a holomorphic function, then f is constant in $\widehat{\mathbb{C}}$.*

Proof. Since f is continuous in $\widehat{\mathbb{C}}$ and since $\widehat{\mathbb{C}}$ is a compact set, $f(\widehat{\mathbb{C}})$ is a compact set. Because f does not take the value ∞ anywhere, $f(\widehat{\mathbb{C}})$ is actually a compact set in $\mathbb{C}$ — hence, a bounded set. The restriction of f to $\mathbb{C}$ is, therefore, a bounded entire function. Liouville's theorem states that f is constant in $\mathbb{C}$. Owing to its continuity, f is then constant in $\widehat{\mathbb{C}}$. ∎

As was the case in the finite plane, when functions f and g are both meromorphic in an open subset U of the extended complex plane, so too are the functions f', $f+g$, fg, and, unless g is identically zero in some component of U, f/g. Along with Theorem 4.2 this observation comes into play in identifying all the functions that are meromorphic in $\widehat{\mathbb{C}}$.

Theorem 4.3. *If $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a meromorphic function, then f is a rational function of z.*

Proof. (We are already aware, of course, that rational functions of z are meromorphic.) By definition, $E = \{z \in \widehat{\mathbb{C}} : z = \infty \text{ or } f(z) = \infty\}$ is a discrete subset of $\widehat{\mathbb{C}}$. As $\widehat{\mathbb{C}}$ is compact, this can only happen if E is a finite set. Let $z_1, z_2, \ldots, z_p = \infty$ list the points of E, and let S_k denote the singular part of f at z_k. Then S_k is a rational function of z whose only singularities appear at z_k and ∞. For $k < p$ the singularity of S_k at ∞ is clearly removable: for such k, $S_k(z) \to 0$ as $z \to \infty$. The function $g = f - S_1 - S_2 \cdots - S_p$ is thus seen to be meromorphic in $\widehat{\mathbb{C}}$, where its only singularities are removable ones at $z_1, z_2, \ldots, z_p$; i.e., g is holomorphic in $\widehat{\mathbb{C}}$. Theorem 4.2 reveals that, once its singularities are removed, g is a constant function in $\widehat{\mathbb{C}}$ — say $g(z) = c$ for all z in $\widehat{\mathbb{C}}$, where c is a complex number. Consequently, $f = c + S_1 + S_2 + \cdots + S_p$ is seen to be a rational function of z. ∎

Many properties of analytic functions, especially those of a topological character, are enjoyed by meromorphic functions as well. The next theorem, which generalizes Theorem 1.5, affords a prime example of such a property.

Theorem 4.4. (Discrete Mapping Theorem) *If D is a domain in $\widehat{\mathbb{C}}$ and if $f\colon D \to \widehat{\mathbb{C}}$ is a non-constant meromorphic function, then f is a discrete mapping of D.*

Proof. Let w be a point of $\widehat{\mathbb{C}}$. We claim that $E = \{z \in D : f(z) = w\}$ is a discrete subset of D. If $w = \infty$, this follows from Theorem 4.1. We proceed under the assumption that w is a finite point. The set E does not have a limit point in the domain $G = \{z \in D : z \neq \infty, f(z) \neq \infty\}$, where f is analytic: if it did, then the discrete mapping theorem for analytic functions would imply that f is constant in G — hence, by continuity, constant in D, contrary to hypothesis. The only way left in which E could conceivably have a limit point in D would be for ∞ to belong to D and for E to have ∞ as a limit point. Assuming this to be so, we choose $r > 0$ with the property that the punctured disk $\Delta^* = \Delta^*(\infty, r)$ is contained in the domain G and select a sequence $\langle z_n \rangle$ of points in $E \cap \Delta^*$ converging to ∞. The continuity of f tells us that $f(\infty) = \lim_{n\to\infty} f(z_n) = w$, so f does not have a pole at ∞. It follows that f takes only finite values in $\Delta(\infty, r)$. The function $g\colon \Delta(0, r) \to \mathbb{C}$ given by $g(z) = f(z^{-1})$ is continuous, and due to the choice of r it is analytic in $\Delta^*(0, r)$. We infer using Riemann's extension theorem that g is analytic in $\Delta(0,r)$. On the other hand, $g(\zeta_n) = w$ for $\zeta_n = z_n^{-1}$, and $\langle \zeta_n \rangle$ is a sequence in $\Delta^*(0,r)$ satisfying $\zeta_n \to 0$. In view of the discrete mapping property of non-constant analytic functions, it can only be the case that $g(z) = w$ for every z in $\Delta(0,r)$ or, equivalently, that $f(z) = w$ for every z in $\Delta(\infty,r)$. In particular, any point of $\Delta^*(\infty,r)$ is a limit point of the set E in G, something we have already ruled out. We conclude on the grounds of this contradiction that ∞ could not be a limit point of E in D. As a set in D without limit points in this domain, E is a discrete subset of D. ∎

Theorem 4.4 has the following corollary, which is a generalization of the principle of analytic continuation (Exercise 5.82).

Corollary 4.5. *If meromorphic functions f and g are both defined in a domain D in $\widehat{\mathbb{C}}$ and if $f(z) = g(z)$ for all z belonging to some subset A of D that has a limit point in D, then $f(z) = g(z)$ for every z in D.*

Theorem 4.4 also makes it easy to see that, under mild technical restrictions, the composition of meromorphic functions is again meromorphic.

Theorem 4.6. *Let $f: D \to \widehat{\mathbb{C}}$ be a non-constant meromorphic function, where D is a domain in $\widehat{\mathbb{C}}$, and let $g: U \to \widehat{\mathbb{C}}$ be a meromorphic function such that U contains $f(D)$. Then the function $g \circ f$ is meromorphic.*

Proof. Write $h = g \circ f$. Certainly h maps D continuously into $\widehat{\mathbb{C}}$. Given a point z_0 in D, we shall exhibit a punctured disk $\Delta^*(z_0,r)$ in which h is analytic. Because $\lim_{z\to z_0} h(z) = h(z_0)$ exists in $\widehat{\mathbb{C}}$, the function h could not have an essential singularity at z_0. It follows that h has no worse than a pole at z_0, an arbitrary point of D. By definition, this makes h a meromorphic function. Set $w_0 = f(z_0)$. Since the function g is meromorphic, we can select $s > 0$ so that g is analytic in $\Delta^*(w_0,s)$. Then, as f is meromorphic, we can pick $r > 0$ with the property that f is analytic in $\Delta^*(z_0,r)$. Furthermore, the fact that f is a continuous, discrete mapping of D into $\widehat{\mathbb{C}}$ means that r can be so chosen that f maps $\Delta(z_0,r)$ into $\Delta(w_0,s)$ and that $f(z) \neq w_0$ holds for every z in $\Delta^*(z_0,r)$. For such a choice of r the set $f[\Delta^*(z_0,r)]$ lies in $\Delta^*(w_0,s)$. The analyticity of the composition h in $\Delta^*(z_0,r)$ is then clear. ∎

If the function f in Theorem 4.6 were constant in D, then $g \circ f$ would also be constant there. This would trivially make $g \circ f$ meromorphic in D, save in one case: should the value of f be a pole of g, $g \circ f$ would be constantly infinite in D — and definitely not meromorphic.

Another significant feature that meromorphic functions inherit from analytic functions is the open mapping property.

Theorem 4.7. (Open Mapping Theorem) *If D is a domain in $\widehat{\mathbb{C}}$ and if $f: D \to \widehat{\mathbb{C}}$ is a non-constant meromorphic function, then f is an open mapping of D. In particular, $f(D)$ is a domain in $\widehat{\mathbb{C}}$.*

Proof. Let U be an open set in D. We must prove that $f(U)$ is an open set in $\widehat{\mathbb{C}}$: given z_0 in U, we must show that there is an $s > 0$ with the property that the disk $\Delta(w_0, s)$ is contained in $f(U)$, where $w_0 = f(z_0)$. For this we assume initially that $w_0 \neq \infty$. Choose $r > 0$ such that $\Delta = \Delta(z_0, r)$ is contained in U and such that f is analytic in the punctured disk $\Delta^*(z_0, r)$. If $z_0 \neq \infty$, then f is just a non-constant analytic function in Δ. In this event the open mapping theorem for analytic functions implies that $f(\Delta)$, a subset of $f(U)$, is an open set in $\mathbb{C}$ and insures the existence of the requisite s. If $z_0 = \infty$, then the function $g: \Delta(0, r) \to \mathbb{C}$ defined by $g(z) = f(z^{-1})$ is non-constant and analytic in $\Delta(0, r)$. Furthermore, $f(\Delta) = g[\Delta(0, r)]$. The open mapping theorem applied to g confirms that $f(\Delta)$ is open in this case, too, again implying that an s with the desired property exists. Finally, suppose that $w_0 = \infty$. Then $h = 1/f$ is a non-constant meromorphic function in D and $h(z_0) = 0$. Applying the previous considerations to h, we are guaranteed the existence of a radius s for which $\Delta(0, s)$ lies in $h(U)$. This, of course, places $\Delta(\infty, s)$ within $f(U)$. We are thus able to conclude that $f(U)$ is an open set. In particular, the set $f(D)$ is open. As the continuous image of a connected set, this set is also connected — i.e., $f(D)$ is a domain. ∎

With the aid of the open mapping theorem we can verify that the inverse of a univalent meromorphic function is meromorphic.

Theorem 4.8. (Inverse Function Theorem) *Suppose that D is a domain in $\widehat{\mathbb{C}}$ and that $f: D \to \widehat{\mathbb{C}}$ is a univalent meromorphic function. Then its inverse function $f^{-1}: f(D) \to D$ is also meromorphic.*

Proof. By Theorem 4.7, $D' = f(D)$ is a domain. The identical argument used in the proof of Theorem 3.11 to demonstrate the continuity of f^{-1} works to establish the continuity of the inverse in the present situation. The function f maps the domain $G = \{z \in D : z \neq \infty, f(z) \neq \infty\}$ in an analytic and univalent fashion to $G' = \{w \in D' : w \neq \infty, f^{-1}(w) \neq \infty\}$. Theorem 3.11 implies that f^{-1} is analytic in G'. Thus f^{-1} is analytic except for two eventual isolated singularities in D', the point ∞ if it belongs to D' and the point $f(\infty)$ if ∞ should lie in D. The continuity of f^{-1} rules out any chance that either of these potential singularities might be essential, so f^{-1} can have no worse than a pole at any point of D' — hence, is meromorphic in D'. ∎

A further quotable consequence of the open mapping theorem is a maximum principle for functions holomorphic in subdomains of $\widehat{\mathbb{C}}$.

Theorem 4.9. (Maximum Principle) *Let $f: D \to \mathbb{C}$ be a holomorphic function, where D is a domain in $\widehat{\mathbb{C}}$. Suppose that there exists a point z_0 of D with the property that $|f(z)| \leq |f(z_0)|$ for every z in D. Then f is constant in D.*

Proof. The point $f(z_0)$ is not an interior point of the set $f(D)$, a bounded subset of $\mathbb{C}$: if $f(z_0)$ were an interior point of $f(D)$, there would obviously be elements w in $f(D)$ for which $|w| > |f(z_0)|$, in conflict with the hypothesis. Accordingly, $f(D)$ is not an open set. The open mapping theorem leaves only one alternative: f is constant in D. ∎

The branched covering principle can likewise be generalized to the extended plane setting. We state the theorem to round out the discussion in this section, but we leave its proof as an exercise (Exercise 5.83). Even the statement demands a few words of explanation, these concerning the interpretation of "multiplicity." Consider a meromorphic function f that is defined and non-constant in some disk $D = \Delta(z_0, r)$ in $\widehat{\mathbb{C}}$. Set $w_0 = f(z_0)$. Suppose first that $z_0 \neq \infty$. If $w_0 \neq \infty$, then we already know what it means to say that f assumes the value w_0 with multiplicity m at z_0, for f is then analytic and non-constant in a perhaps smaller open disk centered at z_0. Next, if $w_0 = \infty$, then f has a pole of some order m at z_0. In this case we say that f assumes the value ∞ with multiplicity m at z_0. Lastly, when $z_0 = \infty$ we say that f assumes the value w_0 with multiplcity m at z_0 if and only if the function g defined by $g(z) = f(z^{-1})$ assumes the value w_0 with multiplicity m at the origin. In terms of the Laurent expansion of f in a punctured disk $\Delta^*(\infty, r)$, the definition of f assuming the value w_0 with multiplicity m at ∞ translates to the following: when $w_0 \neq \infty$, f takes the value w_0 with multiplicity m at ∞ if and only if the Laurent expansion in question has the form

$$f(z) = w_0 + \frac{a_{-m}}{z^m} + \frac{a_{-m-1}}{z^{m+1}} + \cdots$$

with $a_{-m} \neq 0$; f takes the value ∞ with multiplicity m at ∞ if and only if this expansion looks like

$$f(z) = a_m z^m + \cdots + a_1 z + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots$$

with $m \geq 1$ and $a_m \neq 0$.

Theorem 4.10. (Branched Covering Principle) *Suppose that $f: U \to \widehat{\mathbb{C}}$ is a meromorphic function, that z_0 is a point of U, and that f takes the value w_0 with multiplicity m at z_0. Let $r > 0$ be any number sufficiently small that the following conditions prevail: the closed disk $\overline{\Delta} = \overline{\Delta}(z_0, r)$ is contained in U, and the statements $f(z) \neq w_0$, $f(z) \neq \infty$, $f'(z) \neq 0$ are true for every z in the set $\overline{\Delta} \sim \{z_0\}$. Define $s = s(r) > 0$ to be the largest number for which $\Delta(w_0, s) \cap f(K) = \phi$, where K is the circle that bounds $\overline{\Delta}$. Then $G = \{z \in \Delta(z_0, r) : f(z) \in \Delta(w_0, s)\}$ is a domain. Moreover, for each point w of the punctured disk $\Delta^*(w_0, s)$ the set $E_w = \{z \in \Delta(z_0, r) : f(z) = w\}$ consists of exactly m points of G, at each of which f assumes the value w with multiplicity one.*

5 Exercises for Chapter VIII

5.1 Exercises for Section VIII.1

5.1. Determine the multiplicity with which f takes its value at z_0: (i) $f(z) = e^{z\cos z - z}, z_0 = 0$; (ii) $f(z) = z^{\mathrm{Log}\, z}, z_0 = 1$; (iii) $f(z) = \mathrm{Log}^2(\cos z)$, $z_0 = 2\pi$; (iv) $f(z) = \tan^2(1 + 2z^2 + z^4), z_0 = i$; (v) $f(z) = (1 + z^2 - e^{z^2})^3, z_0 = 0$; (vi) $f(z) = z^2 - \mathrm{Arctan}(z^2), z_0 = 0$.

5.2. Let f be a function that is analytic and non-constant in a domain D, and let g be a function with those same properties in a domain D' that contains the range of f. Show that the multiplicity of the composition $g \circ f$ at any point z_0 of D is the product of the multiplicities of f at z_0 and g at $w_0 = f(z_0)$.

5.3. Evaluate the following limits: (i)$\lim_{z\to 0} z^{-2}(1 - \cos z)$; (ii) $\lim_{z\to 0} z^{-2}[1 - \cos(z^{3/2})]$; (iii) $\lim_{z\to 1}(z - 1)^{-2}(z^{1/z} - z)$; (iv) $\lim_{z\to 2\pi}(1 - e^{iz})^{-2}\,\mathrm{Log}(\cos z)$; (v) $\lim_{z\to 0}(1 + cz)^{1/z}$ for any complex number c; (vi) $\lim_{z\to 0}[(e^z - 1)^{-1} - z^{-1}]$.

5.4. Let E be a discrete subset of a plane domain D. Show that $D \sim E$ is a domain.

5.5. A function f is analytic in a domain D. It is known that for each point z of D there is a non-negative integer n_z with the property that $f^{(n_z)}(z) = 0$. Show that f must be a polynomial function of z in D. (*Hint.* Choose and fix a closed disk K in D. Feel free to use the information that K cannot be written as a union $K = \cup_{n=0}^{\infty} A_n$, where $A_0, A_1, A_2, \cdots$ are finite sets. Exercise III.6.16 is also pertinent.)

5.6. Let f be a function that is continuous on the closed disk $\overline{\Delta}(0,r)$ and analytic in $\Delta(0,r)$. Assuming that $f(z) = 0$ for every point z of the circle $K(0,r)$ having $\mathrm{Im}\, z \geq 0$, prove that $f(z) = 0$ for every z in $\overline{\Delta}(0,r)$. (*Hint.* Consider $f(-z)$ along with $f(z)$.)

5.7. Let Q be a closed square centered at the origin, and let S be a side of Q (including the endpoints). If $f\colon Q^0 \to \mathbb{C}$ is a bounded analytic function with the property that $\lim_{z\to\zeta} f(z) = 0$ for every point ζ of S, then $f(z) = 0$ for every z in Q^0. Prove this.

5.8. If $f\colon \overline{\Delta}(0,1) \to \mathbb{C}$ is a non-constant continuous function that is analytic in $\Delta(0,1)$ and satisfies $|f(z)| = 1$ for every z on the circle $K(0,1)$, demonstrate that f has the form

$$f(z) = c\prod_{k=1}^{r}\left(\frac{z - a_k}{1 - \bar{a}_k z}\right)^{m_k}$$

for z in $\overline{\Delta}(0,1)$, where $a_1, a_2, \ldots, a_r$ are distinct points of the disk $\Delta(0,1)$, $m_1, m_2, \ldots, m_r$ are positive integers, and c is a constant of unit modulus.

(*Hint.* Begin by showing the f has at least one zero, but at most finitely many zeros, in $\Delta(0,1)$. Remember Exercise V.8.38 and also Exercise I.4.21.)

5.9. If a non-constant entire function f obeys the condition $|f(z)| = 1$ whenever $|z| = 1$, show that f must be of the type $f(z) = cz^m$, where m is a positive integer and c is a constant with $|c| = 1$. (*Hint.* Use Exercise 5.8 and the principle of analytic continuation.)

5.10. Let f be a non-constant entire function with the following property: there exist a pair of circles K and K' in the complex plane such that $f(K)$ is a subset of K'. Establish that f has the structure $f(z) = c(az+b)^m + d$, where m is a positive integer and where $a \neq 0, b, c \neq 0$, and d are constants.

5.11. Suppose that a function f is analytic in a domain D. Under the assumption that a branch g of the p^{th}-root of f exists in D, prove that there are exactly p distinct branches of the p^{th}-root of f in D, each having the form cg for some p^{th}-root of unity c. This result completes a cycle of ideas that started with Theorem III.4.2 and was continued in Exercise V.8.57.

5.12. Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a Taylor series with a finite radius of convergence $\rho > 0$, and let f denote the function that is its sum in the disk $\Delta = \Delta(z_0, \rho)$. Show that there must be at least one point ζ of the circle $K(z_0, \rho)$ about which the following is true: for no $r > 0$ can f be extended to a function that is analytic in the set $\Delta \cup \Delta(\zeta, r)$. (A point ζ of this type is called a "singular point" for the given Taylor series. *Hint.* Assume that no such ζ exists and derive a contradiction.)

5.13. Assume that a function f is analytic in an open set U and that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ is its Taylor series expansion about a point z_0 of U. Let Δ denote the disk of convergence of this series. Given that $U \cap \Delta$ is connected, show that one defines an analytic function $h: U \cup \Delta \to \mathbb{C}$ by setting $h(z) = f(z)$ for z in U and $h(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ for z in Δ. (Recall the comments following Theorem VII.3.4.)

5.14. If D is a bounded plane domain that contains the origin and if $f: D \to D$ is an analytic function that obeys the conditions $f(0) = 0$ and $f'(0) = 1$, then $f(z) = z$ for every z in D. Justify this statement. (*Hint.* Let $\Delta = \Delta(0,r)$ be contained in D. Show first that $f(z) = z$ for every z in Δ. If not, the Taylor series expansion of f about the origin would assume the form $f(z) = z + a_m z^m + O(z^{m+1})$ for some $m \geq 2$, where $a_m \neq 0$. Cauchy's estimate gives the bound $|a_m| \leq cr^{-m}$, with $c = \max\{|z| : z \in \overline{D}\}$. What would the Taylor series development in Δ of the k-fold composition $f_k = f \circ f \circ \cdots \circ f$ (k factors) look like? The correct answer to this question will produce a contradiction when $k \to \infty$.)

5.15. Let f be a branch of the inverse secant function in a domain D — i.e., $f: D \to \mathbb{C}$ is a continuous function and $\sec[f(z)] = z$ for every z in D. Show that f is an analytic function, that $f'(z) = \pm z^{-1}(z^2 - 1)^{-1/2}$

for every z in D (the sign need not remain constant in D) and that any branch g of the inverse secant in D has either the form $g = f + 2k\pi$ for some integer k or the form $g = -f + 2k\pi$ for some integer k.

5.16. Establish the following result of Guisseppe Vitali (1875-1932) concerning a family $\mathcal{F}$ of analytic functions which is normal in a domain D: if $\langle f_n \rangle$ is a sequence from $\mathcal{F}$ and if $\langle f_n(\zeta) \rangle$ converges for every point ζ belonging to a subset A of D that has a limit point in D, then $\langle f_n \rangle$ converges normally in D. (*Hint.* Recall Exercise VII.5.87.)

5.2 Exercises for Section VIII.2

5.17. Classify the singularity of f at z_0 and determine $\text{Res}(z_0, f)$: (i) $f(z) = \cot z, z_0 = k\pi$ for an integer k ; (ii) $f(z) = (z-1)^{-3}\cos(\pi z/2), z_0 = 1$; (iii) $f(z) = (z-\pi)^{-6}\sin^2 z, z_0 = \pi$; (iv) $f(z) = z^2 e^{-1/z^3}, z_0 = 0$; (v) $f(z) = (z+1)^4 \sin[\pi(z+1)^{-1}], z_0 = -1$; (vi) $f(z) = \text{Arctan}^{-2} z, z_0 = 0$; (vii) $f(z) = (z-1)^{-5}\text{Log}^2 z, z_0 = 1$; (viii) $f(z) = (z^2+z)\cos(z^{-1}), z_0 = 0$.

5.18. Compute the singular part of f at z_0, and use it to obtain $\text{Res}(z_0, f)$: (i) $f(z) = z^2(z-1)^{-2}, z_0 = 1$; (ii) $f(z) = (z^2+1)^{-3}, z_0 = -i$; (iii) $f(z) = (1-\cos z)^{-3}\sin(z^3), z_0 = 0$; (iv) $f(z) = z\,\text{Log}^{-3}(1+z), z_0 = 0$; (v) $f(z) = (z^2+1)(e^{\pi z}+1)^{-4}, z_0 = i$; (vi) $f(z) = [z^2 - \text{Arctan}(z^2)]^{-1}, z_0 = 0$; (vii) $f(z) = (z^z - 1)^{-3}, z_0 = 1$.

5.19. Suppose that a function f is analytic in the punctured plane $\mathbb{C} \sim \{0\}$, where it obeys the estimate $|f(z)| \le c|z||\text{Log}\, z|$ for some constant $c > 0$. Demonstrate that $f(z) = 0$ for every z in $\mathbb{C} \sim \{0\}$. (*Hint.* Recall Exercise V.8.29.)

5.20. Let f have a pole of order m at a point z_0, and let g have a pole of order n at the same point. Show (i) that $f+g$ has either a removable singularity at z_0 or a pole of order not greater than $\max\{m, n\}$ at z_0; (ii) that fg has a pole of order $m+n$ at z_0; (iii) that g/f has a removable singularity at z_0 if $m \ge n$ and a pole of order $n-m$ at z_0 if $m < n$.

5.21. Given that a function f has a pole of order m at a point z_0, check that its derivative f' has a pole of order $m+1$ at that point.

5.22. Let $f\colon \Delta(z_0, r) \to \mathbb{C}$ be a non-constant analytic function. If f takes the value w_0 with multiplicity m at z_0 and if g is a function with a pole of order n at w_0, verify that the composition $g \circ f$, which is well-defined in $\Delta^*(z_0, r)$ for sufficiently small r, has a pole of order mn at z_0.

5.23. Assuming that a function f is both analytic and even in a punctured disk $\Delta^* = \Delta^*(0, r)$, show that $\text{Res}(0, f) = 0$.

5.24. Let $r > 1$. If a function f is analytic in the disk $\Delta(0, r)$ except for a simple pole at the point $z_0 = 1$, verify that $\lim_{n\to\infty}(n!)^{-1}f^{(n)}(0) = -\text{Res}(1, f)$.

5.25. If a function f is analytic in $\Delta^*(z_0, r)$ with a simple pole at z_0 and if $I(s)$ is defined for $0 < s < r$ by $I(s) = \int_{\gamma_s} f(z)dz$, where $\gamma_s(t) = z_0 + se^{it}$ for $a \le t \le b$, then $I(s) \to i(b-a)\text{Res}(z_0, f)$ as $s \to 0$. Justify this statement.

5.26. Functions f and g are both analytic modulo isolated singularities in a domain D. Let E_f and E_g denote the singular sets of these functions in D, and let all the singularities in question be non-removable — i.e., assume that any removable singularities have been "removed." From the information that $f(z) = g(z)$ for all z belonging to a subset A of the domain $G = D \sim (E_f \cup E_g)$ and that A has a limit point in G, deduce that $E_f = E_g$ and that $f(z) = g(z)$ for every z in $D \sim E_f$.

5.27. If f and g are entire functions and if $|g(z)| \le |f(z)|$ for every z in $\mathbb{C}$, demonstrate that $g(z) = cf(z)$ throughout the complex plane for some constant c.

5.28. Assuming that a function f is meromorphic in $\mathbb{C}$ and satisfies $|f(z)| = 1$ whenever $|z| = 1$, show that f admits a representation

$$f(z) = c\prod_{k=1}^{r}\left(\frac{z - a_k}{1 - \bar{a}_k z}\right)^{m_k}\prod_{\ell=1}^{s}\left(\frac{1 - \bar{b}_\ell z}{z - b_\ell}\right)^{n_\ell},$$

where c is a constant of modulus one, where $a_1, a_2, \ldots, a_r$ are the zeros of f in the disk $\Delta = \Delta(0, 1)$ and $m_1, m_2, \ldots, m_r$ their respective orders, and where $b_1, b_2, \ldots, b_s$ are the poles of f in Δ and $n_1, n_2, \ldots, n_s$ their respective orders. (N.B. If f has no zeros in Δ, just leave out the first product; if f has no poles there, omit the the second product. *Hint.* Begin by showing that the above representation is valid in Δ. For this make use of Exercise V.8.38.)

5.29. If a function f is meromorphic in the whole complex plane and if there exist circles K and K' in the plane such that $f(K)$ is a subset of K', then f is necessarily a rational function of z. Support this claim.

5.30. Prove Theorem 2.5.

5.31. Suppose that a function f is analytic in some open set which contains the punctured closed disk $\overline{\Delta}(0, \delta) \sim \{0\}$ and that f has an isolated singularity at the origin. For $0 < p \le 2$, set $I(p) = \lim_{\epsilon\to 0}\iint_{\epsilon\le|z|\le\delta}|f(z)|^p\,dxdy$. Certify that: (i) the singularity of f at 0 is removable if and only if $I(2) < \infty$; (ii) if the singularity is a pole of order m, then $I(p) < \infty$ for $0 < p < 2/m$ and $I(p) = \infty$ for $2/m \le p \le 2$; (iii) if $I(p) = \infty$ for every p in $(0, 2]$, then the singularity is essential. (*Hint for* (i). Show that $\iint_{\epsilon\le|z|\le\delta}|f(z)|^2dxdy = 2\pi\sum_{n=-\infty}^{\infty}|a_n|^2\int_\epsilon^\delta r^{2n+1}dr$, $f(z) = \sum_{n=-\infty}^{\infty}a_n z^n$

being the Laurent expansion of f in $\Delta^*(0,\delta)$.)

5.32. Let a function f be analytic and odd in $\Delta^* = \Delta^*(0,r)$ with an essential singularity at the origin. Assuming that Picard's theorem is true, check that either $f(\Delta^*) = \mathbb{C}$ or $f(\Delta^*) = \mathbb{C} \sim \{0\}$.

5.33. A function f is analytic in $\Delta^* = \Delta^*(0,r)$, has an essential singularity at the origin, and takes a real value at every point of $\Delta^* \cap \mathbb{R}$. Given that $f(\Delta^*)$ contains $\mathbb{R}$, prove by means of Picard's theorem that $f(\Delta^*) = \mathbb{C}$.

5.34. In each of the following cases the function f experiences an essential singularity at the origin. Granted the validity of Picard's theorem, decide whether $f[\Delta^*(0,1)] = \mathbb{C}$ or not: (i) $f(z) = (z^2 - z)e^{1/z}$; (ii) $f(z) = z\cos(z^{-1})$; (iii) $f(z) = e^{2/z} + 2e^{1/z} + 1$; (iv) $f(z) = z^{-1}\sin(z^{-1})$; (v) $f(z) = e^{1/z} - e^{-1/z}$; (vi) $f(z) = z^3 e^{1/z}\sin(z^{-1})$.

5.35. An analytic function $f\colon \Delta^*(0,1) \to \mathbb{C}$ obeys the conditions $f(z) \neq a$ and $f(z) \neq b$ for every z in $\Delta^*(0,1)$, where a and b are distinct complex numbers. Confirm that f can have no worse than a pole at the origin. (*Hint.* We may assume $a = 0$. (Why?) For $n = 1,2,3,\cdots$ define f_n in $\Delta^*(0,1)$ by $f_n(z) = f(n^{-1}z)$. Apply Theorem VII.4.6 to the sequence $\langle f_n \rangle$ in $D = \Delta^*(0,1)$. Remember that the terms of any sequence from $C(D)$ which converges normally in D are uniformly bounded on each compact set in D — in particular, are so on the circle $K(0,1/2)$.)

5.36. Deduce Picard's theorem from Exercise 5.35.

5.37. If a function f has an isolated singularity at ∞, then $\mathrm{Res}[\infty, f(z)] = -\mathrm{Res}[0, z^{-2}f(z^{-1})]$. Corroborate this fact.

5.38. Assuming that a function f has no worse than a pole at ∞, show that $\mathrm{Res}(\infty, f'/f) = \mathrm{Res}(0, g'/g)$, where $g(z) = f(z^{-1})$.

5.39. Classify the singularity of f at ∞, and compute $\mathrm{Res}(\infty, f)$: (i) $f(z) = z(1-z)^{-2}$; (ii) $f(z) = (1+z^2)(1-z^2)^{-1}$; (iii) $f(z) = z^3\cos(z^{-1})$; (iv) $f(z) = z^{-3}\cos(\sqrt{z})$; (v) $f(z) = z^2(2z+1)(z^2+1)^{-1}$; (vi) $f(z) = \exp(z - z^{-1})$.

5.40. Let $f(z) = [1 + \cos(\sqrt{z})]e^z$ and let $\Delta^* = \Delta^*(\infty, r)$. Appealing to Picard's theorem if need be, determine $f(\Delta^*)$.

5.41. If f is an entire function with the property that $|f(z)| \to \infty$ as $|z| \to \infty$, verify that $f(\mathbb{C}) = \mathbb{C}$.

5.3 Exercises for Section VIII.3

5.42. Let γ be a Jordan contour in the complex plane, let D be the inside of the Jordan curve $|\gamma|$, and let f be a function that is analytic modulo isolated singularities in D and continuous in $\overline{D} \sim E$, where E is the singular set of f in D. Assuming Goursat's theorem (Theorem V.5.7), prove that

$\int_\gamma f(z)dz = 2\pi i \sum_{z\in E} \operatorname{Res}(z, f)$. (*Hint.* Show first that the number of non-removable singularities of f in D must be finite. Then adapt the proof given in the text for the residue theorem.)

5.43. A function f is analytic modulo isolated singularities in the complex plane and has an isolated singularity at ∞. Let E be the singular set of f in $\mathbb{C}$, let γ be a Jordan contour in $\mathbb{C} \sim E$, and let D^* denote the outside of the Jordan curve $|\gamma|$. Verify that

$$\int_\gamma f(z)\, dz = -2\pi i \left[\operatorname{Res}(\infty, f) + \sum_{z\in E\cap D^*} \operatorname{Res}(z, f)\right] .$$

Use this information to deduce that $\operatorname{Res}(\infty, f) + \sum_{z\in E} \operatorname{Res}(z, f) = 0$ for any function f of the type described. (*Hint.* Choose $r > 0$ so that $\Delta(0, r)$ contains both E and $|\gamma|$. Consider the cycle $\sigma = (\gamma, -\beta)$, where $\beta(t) = re^{it}$ for $0 \le t \le 2\pi$.)

5.44. A function f is analytic modulo isolated singularities in a simply connected domain D, where its singular set is E. Show that f has a primitive in the domain $G = D \sim E$ if and only if $\operatorname{Res}(z, f) = 0$ for all points z of E.

5.45. Evaluate the integrals: (i) $\int_{|z|=3} z^{-1}(z-1)^{-1}(z-2)^{-1}\, dz$; (ii) $\int_{|z-e|=2}[(z-1)\operatorname{Log} z]^{-1}\, dz$; (iii) $\int_{|z|=1}[(2z+1)^2 \operatorname{Arctan}(iz/2)]^{-1}\, dz$; (iv) $\int_{\partial Q} \csc(z^3)\, dz$, where Q is the square with vertices $1+i, -1+i, -1-i$, and $1-i$; (v) $\int_\sigma z^{-2}\tan z\, dz$, where $\sigma = (2\gamma, 3\beta)$ is the cycle in which $\gamma(t) = 1+2e^{it}$ and $\beta(t) = 2+e^{-it}$ for $0 \le t \le 2\pi$.

5.46. Evaluate $\int_\gamma z\csc^2 z\, dz$ with γ given by $\gamma = \gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5$, where $\gamma_1(t) = e^{it}$ for $0 \le t \le 3\pi$, $\gamma_2(t) = 2+3e^{it}$ for $\pi \le t \le 2\pi$, $\gamma_3(t) = 3+2e^{it}$ for $0 \le t \le 6\pi$, $\gamma_4(t) = -1+6e^{it}$ for $0 \le t \le \pi$, and $\gamma_5(t) = -3+4e^{it}$ for $\pi \le t \le 4\pi$.

5.47. Compute $\int_\gamma (z+\pi)(z-\pi)^{-1}(e^z+1)^{-1}\, dz$ with $\gamma = \beta + [-5\pi, 0]$, in which $\beta(t) = te^{it}$ for $0 \le t \le 5\pi$.

5.48. Evaluate: (i) $\int_{|z|=1} z^2 e^{i/z}\, dz$; (ii) $\int_{|z-1|=2} z^5 \sin(z^{-2})\, dz$; (iii) $\int_{|z-i|=3/2}(z^2+1)^{-1}e^{\pi/z}\, dz$; (iv) $\int_\gamma z^{-1}\exp(z+z^{-1})\, dz$, where $\gamma(t) = \cos t + 2i\sin t$ for $0 \le t \le 4\pi$. (N.B. The answer in (iv) involves an infinite series.)

5.49. Certify that:

(i) $\int_0^{2\pi}(a+b\cos\theta)^{-1}\cos\theta\, d\theta = 2\pi b^{-1}[1 - a(a^2-b^2)^{-1/2}]$ if $0 < b < a$;

(ii) $\int_0^{2\pi}(a+b\cos\theta)^{-2}\, d\theta = 2\pi a(a^2-b^2)^{-3/2}$ if $0 < b < a$;

(iii) $\int_0^{2\pi}(ai+b\sin\theta)^{-1}\, d\theta = -2\pi i(a^2+b^2)^{-1/2}$ if $a > 0$ and $b > 0$;

(iv) $\int_0^{2\pi}(a^2+b^2\sin^2\theta)^{-1}\, d\theta = 2\pi a^{-1}(a^2+b^2)^{-1/2}$ if $a > 0$ and $b > 0$;

(vi) $\int_0^{2\pi}(a^2\cos^2\theta+b^2\sin^2\theta)^{-2}\,d\theta=\pi(ab)^{-3}(a^2+b^2)$ if $a>0$ and $b>0$.
(*Hint.* In each instance start with the case $b=1$.)

5.50. Evaluate: (i) $\int_{-\infty}^{\infty}(x^2+\pi^2)^{-2}\cos x\,dx$; (ii) $\int_0^{\infty}(x^2+9)^{-1}\sin^2(\pi x)\,dx$; (iii) $\int_{-\infty}^{\infty}(x^2+a^2)^{-1}(x^2+b^2)^{-1}\sin(\mu x)\sin(\nu x)\,dx$, $0<\mu\le\nu$, $0<a\le b$; (iv) $\int_0^{\infty}x^2(x^2+1)^{-2}\cos^2(\pi x)\,dx$; (v) $\int_{-\infty}^{\infty}(x^2+1)^{-1}(x^2+4)^{-2}\cos(\pi x)\,dx$; (vi) $\int_{-\infty}^{\infty}x(1+x^4)^{-1}\sin x\,dx$.

5.51. Let f be a function that is analytic modulo isolated singularities in the complex plane and has no singularities on the real axis. Assume that $|zf(z)|$ tends to a (finite) limit as $|z|\to\infty$. Demonstrate that for $c>0$

$$\int_{-\infty}^{\infty}f(x)e^{icx}\,dx=2\pi i\sum_{k=1}^{p}\operatorname{Res}[z_k,f(z)e^{icz}]\ , \tag{8.32}$$

where $z_1,z_2,\ldots,z_p$ are the non-removable singularities of f in the half-plane $H=\{z:\operatorname{Im}z>0\}$. Part of the problem is to establish the convergence of the integral in (8.32). The stated assumptions imply the existence of positive constants r and M such that $|f(z)|\le M/|z|$ whenever $|z|\ge r$, but this inequality does not guarantee the integral's convergence. (*Hint.* All the non-removable singularities of f must lie in the disk $\Delta(0,r)$. Consider $\int_{\partial R}f(z)e^{icz}\,dz$, where R is the rectangle with vertices $a,b,b+i(b-a)$, and $a+i(b-a)$ for $a<-r$ and $b>r$.)

5.52. Let f be a proper rational function of z (i.e., the degree of the denominator in f exceeds that of the numerator) without any real poles. Certify that for $c>0$

$$\int_{-\infty}^{\infty}f(x)\cos(cx)\,dx=2\pi i\sum_{k=1}^{p}\operatorname{Res}[z_k,f(z)e^{icz}]$$

if f is an even function, while

$$\int_{-\infty}^{\infty}f(x)\sin(cx)\,dx=2\pi\sum_{k=1}^{p}\operatorname{Res}[z_k,f(z)e^{icz}]$$

if f is an odd function. Here $z_1,z_2,\ldots,z_p$ are the poles of f in the half-plane $H=\{z:\operatorname{Im}z>0\}$. (*Hint.* Apply Exercise 5.51.)

5.53. Find: (i) $\int_{-\infty}^{\infty}x(x^2+1)^{-1}\sin(\pi x)\,dx$; (ii) $\int_0^{\infty}x^3(x^2+\pi^2)^{-2}\sin x\,dx$; (iii) $\int_{-\infty}^{\infty}x^3(x^2+1)^{-1}(x^2+4)^{-1}\sin(2\pi x)\,dx$; (iv) $\int_0^{\infty}(x^2-i)^{-1}\cos x\,dx$.

5.54. By first integrating the function $f(z)=z^{-2}(1-e^{2iz})$ along the path γ that was introduced to work Example 3.9, calculate $\int_0^{\infty}t^{-2}\sin^2 t\,dt$. (*Hint.* Recall Exercise 5.25.)

5.55. Use the method of Example 3.10 to demonstrate that

$$\int_0^{\infty}t^{-\lambda}(t+b)^{-2}\,dt=\lambda\pi b^{-\lambda-1}\csc(\lambda\pi)$$

and

$$\int_0^\infty t^{-\lambda}(t+b)^{-1}\,\mathrm{Log}\,t\,dt = \pi b^{-\lambda}\csc(\lambda\pi)[\pi\cot(\lambda\pi)+\mathrm{Log}\,b]$$

when $0<\lambda<1$ and $b>0$. (N.B. The first of these formulas can also be obtained by differentiating both sides of (8.21) with respect to the parameter b; differentiation of (8.21) with respect to λ will yield the second.)

5.56. Show that $\int_0^\infty (1+x^n)^{-1}\,dx = \pi n^{-1}\csc(\pi/n)$ for any integer $n\geq 2$. Conclude that $\int_{-\infty}^\infty (1+x^{2n})^{-1}\,dx = \pi n^{-1}\csc(\pi/2n)$ for every positive integer n. (*Hint.* Make a change of variable in Example 3.10 for some λ.)

5.57. Derive the formula $\int_0^\infty t^\lambda(t^2+b^2)^{-1}\,dt = (\pi/2)b^\lambda\sec(\lambda\pi/2)$, valid when $-1<\lambda<1$ and $b>0$. (*Hint.* Let $0<s<b$ and $b<r<\infty$. Integrate $f(z)=z^\lambda(z^2+b^2)^{-1}$ along $\gamma=[s,r]+\alpha+[-r,-s]-\beta$, where α and β are the paths defined on $[0,\pi]$ by $\alpha(t)=re^{it}$ and $\beta(t)=se^{it}$. Feel free to invoke Exercise 5.42.)

5.58. Verify that $\int_{-\infty}^\infty e^{\lambda t}\mathrm{sech}\,t\,dt = \pi\sec(\lambda\pi/2)$ for any complex number λ satisfying $-1<\mathrm{Re}\,\lambda<1$. (*Hint.* The estimate $|e^z+e^{-z}|\geq(1/2)e^{|x|}$ when $|x|\geq 1$ insures the convergence of the integral. Consider $\int_{\partial R} f(z)\,dz$, where $f(z)=e^{\lambda z}\mathrm{sech}\,z$ and R is the rectangle with vertices $-c, c, c+\pi i$ and $-c+\pi i$ for $c\geq 1$.)

5.59. Let p be a positive integer, and let $f(z)=z^{-2p}(e^z-1)^{-1}$. If $Q=\{z: |x|\leq(2n+1)\pi, |y|\leq(2n+1)\pi\}$ for a positive integer n, show that

$$\int_{\partial Q} f(z)\,dz = 2\pi i\left(\frac{(-1)^{p-1}B_p}{(2p)!}+\frac{2(-1)^p}{(2\pi)^{2p}}\sum_{k=1}^n\frac{1}{k^{2p}}\right),$$

where B_p is the p^{th} Bernoulli number (Exercise VII.5.62). Make use of this information to sum the series $\sum_{k=1}^\infty k^{-2p}$: $\sum_{k=1}^\infty k^{-2p}=2^{-1}[(2p)!]^{-1}(2\pi)^{2p}B_p$. (*Hint.* For the last part notice that $|e^z-1|\geq 1-e^{-3\pi}$ for every point z of the set ∂Q.)

5.60. Prove the following generalized argument principle: if a function f is meromorphic in an open set U and if γ is a closed, piecewise smooth path in U that is homologous to zero in U and finds no zeros or poles of f on its trajectory, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z)} = \sum_{j=1}^r m_j n(\gamma,z_j) - \sum_{k=1}^s \ell_k n(\gamma,w_k),$$

where $z_1, z_2, \ldots, z_r$ lists all zeros of f for which $n(\gamma,z_j)\neq 0$, $m_1, m_2, \ldots, m_r$ being the respective orders of these zeros, while $w_1, w_2, \ldots, w_s$ are all the poles of f for which $n(\gamma,w_k)\neq 0$, $\ell_1,\ell_2,\ldots,\ell_s$ indicating their respective orders. Interpret an "empty" sum to mean zero.

5.61. If f, U, γ, and D meet all the conditions spelled out in Theorem 3.5 and if g is a function that is analytic in some open set which contains $\overline{D}$, where $z_1, z_2, \ldots, z_r$ lists all zeros of f for which $n(\gamma, z_j) \neq 0$, $m_1, m_2, \ldots, m_r$ being the respective orders of these zeros, while $w_1, w_2, \ldots, w_s$ are all the poles of f for which $n(\gamma, w_k) \neq 0$, $\ell_1, \ell_2, \ldots, \ell_s$ indicating their respective orders. Interpret an "empty" sum to mean zero.

5.61. If f, U, γ, and D meet all the conditions spelled out in Theorem 3.5 and if g is a function that is analytic in some open set which contains $\overline{D}$, verify that

$$\frac{1}{2\pi i}\int_\gamma \frac{g(z)f'(z)\,dz}{f(z)} = \sum_{j=1}^{r} m_j g(z_j) - \sum_{k=1}^{s} \ell_k g(w_k)\ ,$$

where $z_1, z_2, \ldots, z_r$ are the zeros of f in D and $m_1, m_2, \ldots, m_r$ their respective orders, and where $w_1, w_2, \ldots, w_s$ are the poles of f in D and $\ell_1, \ell_2, \ldots, \ell_s$ the orders of these poles. An empty sum is taken to mean zero.

5.62. Rouché's theorem for meromorphic functions reads: if D is the domain inside the trajectory of a Jordan contour in the complex plane, if f and g are functions that are meromorphic in some open set which contains $\overline{D}$, if both of these functions are free of poles on ∂D, and if the inequality $|f(z) - g(z)| < |f(z)| + |g(z)|$ holds at every point z of ∂D, then $Z_f - P_f = Z_g - P_g$. Here Z_f and Z_g are the numbers of zeros that f and g display in D, while P_f and P_g are the corresponding totals for poles, all zeros and poles being counted with consideration for multiplicity. Establish this result.

5.63. It follows from elementary considerations in calculus that the real solutions of the equation $\tan z = z$ can be listed as $z_0 = 0, \pm z_1, \pm z_2, \cdots$, where $k\pi < z_k < (2k+1)\pi/2$. Show that this equation has no other solutions in the complex plane. Then find the sum of the series $\sum_{k=1}^{\infty} z_k^{-2}$. (*Hint.* Let $Q = \{z : |x| < n\pi, |y| < n\pi\}$, where n is an arbitrary positive integer. For the first part compare zero-pole differences for the functions $f(z) = z - \tan z$ and $g(z) = z$ in the interior of Q. For the second part look at $\int_{\partial Q}[(z - \tan z)^{-1} - z^{-1}]dz$. Useful in both parts is the observation that $|\tan z| \leq 2$ on ∂Q. Verify this.)

5.64. Let c be a complex number with $|c| > e$. Confirm that for every positive integer n the equation $cz^n = e^z$ has n different solutions in the disk $\Delta(0,1)$ and has no solutions apart from these in the half-plane $H = \{z : \operatorname{Re} z < 1\}$. (*Hint.* Recall Example 3.12.)

5.65. Let $f(z) = z^5 + 5z^3 - 1$. Verify that f has five simple zeros, three of these lying in the disk $\Delta(0,1)$ and the remaining two located in the set $\Delta(0, 7/3) \sim \{z : |z| \leq 2, |\operatorname{Re} z| > 3/2\}$.

5.66. Let $f(z) = z^3 + 3a^2 z - 1$, where $a > 0$. Then f has a simple zero in the interval $(0,1)$, but no additional real roots. (Why?) The two non-real roots of f are complex conjugates of one another. Let z_2 denote the non-real root of f whose imaginary part is positive. By combining Rouché's theorem with the Gauss-Lucas theorem (Exercise V.8.35) and using other elementary facts about polynomials, demonstrate that z_2 finds itself in the set $\{z: -(1/2) < x < 0, y > a(1-x), \max\{1,a\} < |z| < \sqrt{1+3a^2}\}$.

5.67. For $n = 1,2,3,\cdots$ set $f_n(z) = \sum_{k=0}^{n}(k!)^{-1}z^k$. If $r > 0$ is fixed, prove that there is an index $N = N(r)$ such that f_n is free of zeros in the disk $\Delta(0,r)$ once $n \geq N$.

5.68. Confirm that the roots of a polynomial depend continuously on its coefficients in the following sense: given $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, a polynomial of degree $n \geq 1$ whose different roots are $z_1, z_2, \ldots, z_r$, and given $\epsilon > 0$, there exists a corresponding $\delta > 0$ such that any polynomial $g(z) = b_0 + b_1 z + \cdots + b_n z^n$ with coefficients which satisfy $|b_k - a_k| < \delta$ for $k = 0,1,\ldots,n$ will have at least one root in each of the disks $\Delta(z_\ell,\epsilon)$ and all of its roots in the set $\cup_{\ell=1}^{r}\Delta(z_\ell,\epsilon)$.

5.69. Let $f(z) = a_0 + a_1 z + \cdots + a_N z^N$ be a polynomial of degree $N \geq 2$ with the property that $\sum_{n=2}^{N} n|a_n| \leq |a_1|$. Prove that f is univalent in the disk $\Delta = \Delta(0,1)$. (*Hint.* Noting that $a_1 = 0$ is not compatible with these assumptions, fix z_0 in Δ and compare the number of zeros that the functions $g(z) = a_1(z - z_0)$ and $h(z) = f(z) - f(z_0)$ have in Δ.)

5.70. Suppose that a function f is analytic and non-constant in the disk $\Delta = \Delta(0,1)$, where it has a Taylor series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ whose coefficients obey the condition $\sum_{n=2}^{\infty} n|a_n| \leq |a_1|$. Prove that f is univalent in Δ. (*Hint.* Use the preceding exercise.)

5.71. A fairly general version of Rouché's theorem states: if D is a bounded, simply connected domain in the complex plane, if f and g are functions that are continuous on $\overline{D}$ and analytic in D, and if the inequality $|f(z) - g(z)| < |f(z)| + |g(z)|$ is satisfied for every point z of ∂D, then f and g have the same number of zeros in D, always presuming that zeros are counted with due regard for multiplicity. Prove this under the added assumption that D can be represented in the fashion $D = \varphi(\Delta)$, where $\Delta = \Delta(0,1)$ and $\varphi: \Delta \to \mathbb{C}$ is a univalent analytic function. Conclude that, in particular, this form of Rouché's theorem is true when D is a disk. (N.B. The "Riemann Mapping Theorem," to be established in the next chapter, insures that every domain D of the type under consideration here can be represented in the manner indicated. *Hint.* Use a uniform continuity argument to show that Theorem 3.6 is applicable to f and g in the domain $D_r = \varphi[\Delta(0,r)]$ for all r sufficiently close to 1.)

5.72. Assume that a function f is continuous on the closed disk $\overline{\Delta}(0,1)$, is analytic in $\Delta(0,1)$, and satisfies $|f(z)| \leq 1$ when $|z| = 1$. Show that f has at least one fixed point in $\overline{\Delta}(0,1)$ and that, if f fixes no point of the circle $K(0,1)$, it has exactly one fixed point in $\overline{\Delta}(0,1)$.

5.73. Let $f: \Delta(z_0,r) \to \mathbb{C}$ be a non-constant analytic function. If a function

g has an essential singularity at the point $w_0 = f(z_0)$, demonstrate that the composition $g \circ f$, defined in $\Delta^*(z_0, r)$ for small r, exhibits an essential singularity at z_0.

5.74. Identify all the entire functions f with the property that $f \circ f = f$.

5.75. Let D and D' be bounded domains in the complex plane, and let $f: \overline{D} \to \mathbb{C}$ be a continuous function that is analytic in D. Under the assumptions that $f(D)$ is a subset of D' and that $f(\partial D)$ is a subset of $\partial D'$, verify that $f(D) = D'$, $f(\partial D) = \partial D'$, and $f(\overline{D}) = \overline{D}'$. (*Hint.* To start, assume that $G = f(D) \neq D'$, and argue to a contradiction. Recall Exercise II.5.25.)

5.76. Show that a univalent entire function f must be a sense-preserving similarity transformation — i.e., $f(z) = az + b$ for constants $a \neq 0$ and b. (*Hint.* Begin by ruling out the possibility that f has an essential singularity at ∞.)

5.77. Let f be an entire function with the property that $f(z)$ is real for every point z of the real axis, but for no other z. Prove that $f'(z) \neq 0$ holds for all real z. Infer from this that f is univalent on the real axis. Conclude with the aid of Picard's theorem — or otherwise — that $f(z) = az + b$ for some real constants $a \neq 0$ and b. (*Hint.* Assuming that $f'(z_0) = 0$ for some point z_0 of the real axis, put the branched covering principle to work and obtain a contradiction.)

5.78. Let $f: U \to \mathbb{C}$ be a univalent analytic function, and let γ be a Jordan contour in U with the property that the inside D of the Jordan curve $|\gamma|$ is contained in U. Demonstrate that for every point w of the domain $D' = f(D)$ the value of the inverse of f at w is given by the formula

$$f^{-1}(w) = \frac{1}{2\pi i} \int_\gamma \frac{z f'(z)\, dz}{f(z) - w} .$$

5.79. Suppose that $f: \Delta(z_0, r) \to \mathbb{C}$ is a univalent analytic function, that $w_0 = f(z_0)$, and that the disk $\Delta(w_0, s)$ is contained in the range of f. Verify that $|f'(z_0)| \geq s/r$ and that equality holds if and only if f takes the form $f(z) = w_0 + (cs/r)(z - z_0)$ in $\Delta(z_0, r)$, where c is a constant of unit modulus. (*Hint.* Consider the function $g: \Delta(0,1) \to \mathbb{C}$ defined by $g(z) = r^{-1}[f^{-1}(w_0 + sz) - z_0]$.)

5.80. Let D be a plane domain, let $f: D \to \mathbb{C}$ be an analytic function, and let v be a continuous function that is subharmonic in some open set which contains the range of f. Demonstrate that the composition $w = v \circ f$ is subharmonic in D. (*Hint.* Assuming f to be non-constant — the assertion follows trivially otherwise — begin by showing that w is subharmonic in the domain $G = \{z \in D : f'(z) \neq 0\}$. Consult Exercises VI.4.36 and VI.4.39.)

5.4 Exercises for Section VIII.4

5.81. Confirm that under the stereographic projection $\pi: S \to \widehat{\mathbb{C}}$ circles on S are transformed to circles in $\widehat{\mathbb{C}}$ and that under π^{-1} circles in $\widehat{\mathbb{C}}$ go to circles on S. (*Hint.* A circle on S is the intersection of S with a plane in $\mathbb{R}^3$. Make use of Exercise I.4.19.)

5.82. Prove Corollary 4.5.

5.83. Prove Theorem 4.10.

5.84. Let $f: \mathbb{C} \to \widehat{\mathbb{C}}$ be a non-constant meromorphic function with the feature that $f(\lambda z) = f(z)$ for every z in $\mathbb{C}$, where λ is a constant. Prove that there must be a positive integer p for which $\lambda^p = 1$.

5.85. Let $f: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a non-constant rational function of z. Prove that $f(\widehat{\mathbb{C}}) = \widehat{\mathbb{C}}$. (*Hint.* It is enough to show that each such function must have a pole in $\widehat{\mathbb{C}}$. Why?)

5.86. Refine the preceding exercise as follows: if $f: \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a non-constant rational function of z, then f assumes every value in $\widehat{\mathbb{C}}$ the same number of times, provided multiplicity is taken into account. To be more precise, if w is a point of $\widehat{\mathbb{C}}$ and if $z_1, z_2, \ldots, z_r$ are the different solutions in $\widehat{\mathbb{C}}$ of the equation $f(z) = w$, then the sum $m_1 + m_2 + \cdots + m_r$, where m_k is the multiplicity of f at z_k, is independent of w. (*Hint.* It suffices to check that, for arbitrary f of this type, these multiplicity sums are the same for $w = 0$ and $w = \infty$. Why?)

5.87. Let $f: \Delta(z_0, r) \to \widehat{\mathbb{C}}$ be a non-constant meromorphic function. If a function g has an isolated singularity at the point $w_0 = f(z_0)$, prove that the composition $g \circ f$, which makes sense in $\Delta^*(z_0, r)$ when r is sufficiently small, has the same kind of singularity at z_0 that g does at w_0. Of course, we are allowing $z_0 = \infty$ or $w_0 = \infty$ — or both — here.

5.88. A function f has an isolated singularity at a point z_0 of $\widehat{\mathbb{C}}$. Assuming that $\operatorname{Re} f$ is bounded above (or below) in some punctured disk $\Delta^*(z_0, r)$, show that the singularity of f at z_0 is removable. (*Hint.* Consider e^f.)

5.89. A function f is meromorphic in a punctured disk $\Delta^* = \Delta^*(z_0, r)$ in $\widehat{\mathbb{C}}$. Furthermore, it is known that the set of poles of f in Δ^* has z_0 as a limit point. Prove that the set $\widehat{\mathbb{C}} \sim f(\Delta^*)$ can have no interior points. (*Warning.* The function f does not have an isolated singularity at z_0, so the terms "removable singularity," "pole," and "essential singularity" are not applicable there.)