Φυλλάδιο 10 - Λύσεις

1. $f(x)=(1+x)^{-1}$, $f'(x)=-(1+x)^{-2}$, $f''(x)=2(1+x)^{-3}$, $f'''(x)=-2\cdot 3\,(1+x)^{-4}$

$f^{(4)}(x)=2\cdot 3\cdot 4\,(1+x)^{-5}$, $f^{(5)}(x)=-2\cdot 3\cdot 4\cdot 5\,(1+x)^{-6}$

$$P_{5,0}(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(x)}{3!}x^3+\frac{f^{(4)}(x)}{4!}x^4+\frac{f^{(5)}(x)}{5!}x^5$$

$$=1-x+\frac{2}{2!}x^2-\frac{2\cdot 3}{3!}x^3+\frac{2\cdot 3\cdot 4}{4!}x^4-\frac{2\cdot 3\cdot 4\cdot 5}{5!}x^5$$

$$=1-x+x^2-x^3+x^4-x^5$$

$$P_{5,1}(x)=f(1)+\frac{f'(1)}{1!}(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3+\frac{f^{(4)}(1)}{4!}(x-1)^4+\frac{f^{(5)}(1)}{5!}(x-1)^5$$

$$=\frac{1}{2}-\frac{1}{2^2}(x-1)+\frac{1}{2^3}(x-1)^2-\frac{1}{2^4}(x-1)^3+\frac{1}{2^5}(x-1)^4-\frac{1}{2^6}(x-1)^5$$

(γιατί $f(1)=\frac{1}{2}$, $f'(1)=-\frac{1}{2^2}$, $f''(1)=\frac{2}{2^3}$, $f'''(1)=-\frac{2\cdot 3}{2^4}$, $f^{(4)}(1)=\frac{2\cdot 3\cdot 4}{2^5}$, $f^{(5)}(1)=-\frac{2\cdot 3\cdot 4\cdot 5}{2^6}$)

2. Βρίσκουμε το πολυώνυμο Taylor βαθμού 4 στο σημείο $x_0=1$

$p'(x)=2-2x+15x^2-4x^3$ | $p(1)=1+2-1+5-1=6$

$p''(x)=-2+30x-12x^2$ | $p'(1)=2-2+15-4=11$

$p'''(x)=30-24x$ | $p''(1)=-2+30-12=16$

$p^{(4)}(x)=-24$ | $p'''(1)=30-24=6$

$p^{(4)}(1)=-24$

$$P(x)=P(1)+\frac{P'(1)}{1!}(x-1)+\frac{P''(1)}{2!}(x-1)^2+\frac{P'''(1)}{3!}(x-1)^3+\frac{P^{(4)}(1)}{4!}(x-1)^4$$

$$=6+11(x-1)+\frac{16}{2!}(x-1)^2+\frac{6}{3!}(x-1)^3-\frac{24}{4!}(x-1)^4$$

$$=6+11(x-1)+8(x-1)^2+(x-1)^3-(x-1)^4.$$

$$=6+11(x-1)+$$

3. $P_{8,0}(x)=\frac{x^8}{8!}$

4. $f(x)=e^{x^2}$: $f(0)=1$, $f'(0)=0$, $f''(0)=2$, $f'''(0)=0$

$$P_{3,0}(x)=1+x^2$$

$g(x) = \frac{1}{1+x^2}$: $g(1) = \frac{1}{2}$ $g'(1) = -\frac{1}{2}$, $g''(1) = \frac{1}{2}$, $g'''(1) = 0$, $g^{(4)}(1) = -3$

$$P_{4,1}(x) = \frac{1}{2} - \frac{1}{2}(x-1) + \frac{1}{4}(x-1)^2 - \frac{1}{8}(x-1)^4$$

$h(x) = \sqrt{3+\cos x}$: $h(0) = 2$, $h'(0) = 0$, $h''(0) = -\frac{1}{4}$

$$P_{2,0}(x) = 2 - \frac{1}{8}x^2$$

$k(x) = \sin x$: $k(\pi/6) = \frac{1}{2}$, $k'(\pi/6) = \frac{\sqrt{3}}{2}$, $k''(\pi/6) = -\frac{1}{2}$, $k'''(\pi/6) = -\frac{\sqrt{3}}{2}$
$k^{(4)}(\pi/6) = \frac{1}{2}$ $k^{(5)}(\pi/6) = \frac{\sqrt{3}}{2}$, $k^{(6)}(\pi/6) = -1/2$

$$P_{6,\frac{\pi}{6}}(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x-\pi/6) - \frac{1}{4}(x-\pi/6)^2 - \frac{\sqrt{3}}{12}(x-\pi/6)^3 + \frac{1}{48}(x-\pi/6)^4 + \frac{\sqrt{3}}{240}(x-\pi/6)^5 - \frac{1}{1440}(x-\pi/6)^6.$$

$\varphi(x) = \log\left(\frac{1}{1-x}\right)$: $\varphi(0) = 0$, $\varphi'(0) = 1$, $\varphi''(0) = 1$, $\varphi'''(0) = 2!$, $\varphi^{4}(0) = 3!$
$\varphi^{(5)}(0) = 4!$

$$P_{5,0}(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}$$

$\psi(x) = e^x - (1 + x + \frac{x^2}{2})$: $P_{5,0}(x) = \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$

$s(x) = \sqrt[3]{1+x}$: $P_{3,0}(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3.$

5. $$e^x = e^{x-1}e^1 = e\,e^{x-1} = e\left(1 + x-1 + \frac{1}{2!}(x-1)^2 + \frac{1}{3!}(x-1)^3 + \dots\right)$$
$$= e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \frac{e}{4!}(x-1)^4 + \dots$$
$$= \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n, \qquad -\infty < x < \infty.$$

$$\frac{1}{x} = \frac{1}{1+x-1} = \frac{1}{1-(-(x-1))} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \dots$$
$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n, \qquad 0 < x < 2.$$

$$\log\frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n}, \quad -1 \le x < 1$$

$\varphi(x) = \cos(x)$: $\varphi(\pi) = -1$, $\varphi'(\pi) = 0$, $\varphi''(\pi) = 1$, $\varphi'''(\pi) = 0$, $\varphi^{(4)}(\pi) = -1$

$\varphi^{(5)}(\pi) = 0$, $\varphi^{(6)}(\pi) = 1$, $\varphi^{(7)}(\pi) = 0$, ...

σειρά: $-1 + \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \frac{1}{6!}(x-\pi)^6 \ldots \quad -\infty < x < \infty$.

2ος τρόπος: $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$, ισχύει $\forall x \in \mathbb{R}$, άρα

$$\Rightarrow \cos(x-\pi) = 1 - \frac{1}{2!}(x-\pi)^2 + \frac{1}{4!}(x-\pi)^4 - \frac{1}{6!}(x-\pi)^6 + \cdots$$

$$\Rightarrow \cos x \cos\pi + \sin x \sin\pi = \text{-----}$$

$$\Rightarrow -\cos x = 1 - \frac{1}{2!}(x-\pi)^2 + \frac{1}{4!}(x-\pi)^4 - \frac{1}{6!}(x-\pi)^6 + \cdots$$

$$\Rightarrow \cos x = -1 + \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \frac{1}{6!}(x-\pi)^6 - \cdots$$

6. Υπολογίζω: $R_{4,0}(x) = \frac{f^{(5)}(\xi)}{5!} x^5$ με ξ ανάμεσα στο x και στο 0.

$$|R_{4,0}(x)| \le \frac{1}{5!}|x|^5 \le \frac{1}{5!}\left(\frac{1}{2}\right)^5 = \frac{1}{5!\,2^5} = \frac{1}{120 \cdot 32} = \frac{1}{3840}$$

Δηλ. $|\cos x - P_{4,0}(x)| \le \frac{1}{3840} \quad \forall x \in [-\frac{1}{2}, \frac{1}{2}]$.

7. Υπολογίζω $|R_{n,0}(x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}\right| = \frac{e^{-\xi}}{(n+1)!}|x|^{n+1}$

$$\le \frac{e^{|x|}}{(n+1)!}|x|^{n+1} \le \frac{e}{(n+1)!}$$

Θέλουμε $\frac{e}{(n+1)!} \le \frac{1}{10^4} \Rightarrow (n+1)! > e \cdot 10^4$.

Βρίσκουμε $8! = 40320 > e\,10^4$, άρα $n+1 = 8 \Rightarrow n = 7$

Το ζητούμενο πολυώνυμο είναι $P_{7,0}(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!}$.

9. Θεωρούμε την $f(x) = \sqrt{x}$

$$P_{1,n^2}(x) = f(n^2) + \frac{f'(n^2)}{1!}(x-n^2)$$

$$= n + \frac{1}{2n}(x-n^2),$$ προσέγγιση της f κοντά στο n^2.

Για $x = n^2+1$,

$$\sqrt{n^2+1} \sim n + \frac{1}{2n}(n^2+1-n^2) = n + \frac{1}{2n}$$

10. $$P_{2,9}(x) = f(9) + \frac{f'(9)}{1!}(x-9) + \frac{f''(9)}{2!}(x-9)^2$$

$$= \sqrt{9} + \frac{1}{2\sqrt{9}}(x-9) - \frac{1}{216}(x-9)^2$$

$$= 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2$$

Υπόλοιπο: $$R_{2,9}(x) = \frac{f^{(3)}(\xi)}{3!}(x-9)^3,$$

$$|R_{2,9}(x)| \leq \frac{|f^{(3)}(\xi)|}{3!}|x-9|^3$$

$$f^{(3)}(x) = \frac{3}{8}x^{-5/2} = \frac{3}{8}\frac{1}{x^{5/2}}$$

$$|f^{(3)}(\xi)| \leq \frac{3}{8}\frac{1}{8^{5/2}}$$ για $8 < \xi < 10$

$$\Rightarrow |R_{2,9}(x)| \leq \frac{3}{8^{7/2}\cdot 3!}$$ για $8<x<10$.

11. Υπολογισμός των $g(0), g'(0) \ldots g^{(4)}(0)$ κλπ.

12. Η $f(x)$ είναι άπειρες φορές παραγωγίσιμη στο 0 και
$f^{(n)}(0) = 0$ $\forall n$. (Άσκηση 17, Φυλλάδιο 7)

Άρα $$P_{4,0}(x) = 0 + 0x + \frac{0}{2!}x^2 + \frac{0}{3!}x^3 + \frac{0}{4!}x^4 = 0$$