Homotopy Continuation Solution Method in Nonlinear Model Predictive Control Applications

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Abstract

A new fast and efficient algorithm for the solution of the dynamic optimization problem resulted from the implementation of a model predictive control (MPC) framework in highly nonlinear dynamic systems is presented. The sequence of the optimal control actions is obtained through the solution of the parameterized set of the Karush-Kuhn-Tucker (KKT) optimality conditions for the nonlinear program as resulted from a constructed homotopy with respect to the initial point of the dynamic optimization problem. A predictor-corrector continuation method tailored for large scale, sparse systems enables the quick calculation of the optimal solution as exhibited in challenging engineering problems such as the control of a cart with a double pendulum.

Keywords: Nonlinear model predictive control, Homotopy, Continuation method

1. Introduction

Model predictive control is a powerful control algorithm that enables the satisfaction of complex control objectives through the systematic consideration of all process dynamic interactions for multivariable control systems. The implementation of nonlinear model predictive control (NMPC) in dynamic systems that possess fast dynamics is prohibitive as the solution to the associated dynamic optimization problem is comparable to the duration of the control interval suitable for adequate control performance. The computational delay appears to be a considerable factor for the deterioration of the NMPC performance \cite{1}. Therefore, either a delayed control action is applied to the system or a simplified model, usually a linear realization of the nonlinear dynamic model is used in the model predictive control algorithm. In both cases, however, the performance of the controller diminishes as crucial process dynamics may be ignored through the utilization of a simplified (linear) model or the adoption of a delayed control action.

Over the years a large number of efficient real-time strategies have been proposed such as explicit NMPC \cite{2}, Newton-type controllers \cite{3, 4} and nonlinear program (NLP) sensitivity-based controllers. Recently, the advanced-step NMPC \cite{5} achieved significant improvement in the speed for the solution of the optimization problem as the control interval is used in order to guide the solution of the problem towards the real optimum using the prediction model before the measurements become available. Despite all efforts the need for fast and efficient NMPC solution techniques is still a challenging problem.

2. Nonlinear model prediction control problem formulation

The NMPC problem can be formulated as follows:
Min \( J = \Phi(x_N, y_N) + \sum_{i=0}^{N-1} \rho(x_i, y_i, u_i) \)

s.t. \( h(x_i, y_i, u_i) = 0 \)
\( g(x_i, y_i, u_i) \leq 0 \quad i = 0, K, N \)
\( x_0 = p \)

Vector \( x \) is the real-valued state vector of dimension \( n \), \( y \) is the real-valued output vector of dimension \( l \), \( u \) is the real-valued input (manipulated variables) piecewise constant vector of dimension \( m \), and \( x_0 \) is the initial values vector for the state variables of dimension \( n \). Functional \( J \) represents the objective of the control problem consisted of a terminal term, \( \Phi(\cdot) \) and a time varying term expressed using nonlinear functions \( \varphi(\cdot) \). Vector \( h \) represents the set of nonlinear modeling equations describing the dynamic behavior of the system, whereas \( g \) represents the set of inequality constraints. The model predictions extend into the future for \( N \) time intervals, whereas the optimal sequence of manipulated variables extends to \( M \) time intervals with \( M < N \).

According to the NMPC algorithm, given an initial point for the system the sequence of optimal control actions is calculated from (1). After implementation of the first control action, a new set of measurements is acquired for the output variables. Provided that the system is observable, the set of measurements is then used to update the model states/parameters through a suitable state estimator method. The updated vector of states thus becomes the initial conditions for problem (1) in the next time interval. Therefore, the solution of problem (1) can be viewed as a parameterized optimization problem with respect to the initial conditions vector, \( p \). Such property has been exploited by Zavala and Biegler [5] in order to improve the efficiency of NMPC solution strategies.

2.1. Dynamic model discretization

In the present work, a direct solution method is applied through the discretization of the differential equations using an orthogonal collocation on finite elements (OCFE) approximation technique. The prediction horizon is partitioned in \( NE \) finite elements and within each finite element the state profiles are approximated by Lagrange polynomials, \( W_{ij}(t) \), or order \( n \). The state profiles and their time derivatives are then expressed as functions of the state variables defined at specified time instances; namely the collocation points as follows:

\[
\tilde{x}_i(t) = \sum_{j=0}^{n} W_{ij}(t) x_i(t_{ij}), \quad \frac{d\tilde{x}_i(t)}{dt} = \sum_{j=0}^{n} W_{ij}(t) \frac{dx_i(t_i)}{dt} \tilde{t}_{ij} \leq t \leq \tilde{t}_{ij+1}, i = 1, \ldots, NE
\]

The collocation points are selected as the roots of orthogonal Legendre polynomials of order equal to the number of collocation points within each finite element. Naturally, the element boundaries coincide with the time intervals in the digital implementation of the NMPC algorithm but also integer multiples of finite elements can form one time (control) interval. Constraints on the state variable at various time instances can be best accommodated within the approximation scheme by either placing a collocation point or a finite element breakpoint at the specific time instance of the constraint.

OCFE approximation requires that the modeling equation residuals vanish only at the collocation points. Zero-order continuity for the state profiles is imposed at the element boundaries. The discretization of the dynamic equations using OCFE transforms the NMPC problem to the following nonlinear program (the tilde denotes approximated
variable profiles).

\[
\min_{\bar{x}, \bar{y}, \bar{u}} J = \Phi(\bar{x}, \bar{y}, \bar{u}) + \sum_{i=0}^{N-1} \Phi_i(\bar{x}, \bar{y}, \bar{u}) \\
s.t. \quad h(\bar{x}, \bar{y}, \bar{u}) = 0 \\
\quad g(\bar{x}, \bar{y}, \bar{u}) \leq 0 \quad i = 1, \ldots, N \\
\quad x_0 = p
\]  

(3)

2.2. Parameterized NMPC algorithm

The discretized NMPC problem (3) can be solved using conventional NLP solution algorithms that may suffer from long solution times and occasionally from convergence failures. An alternative method for the solution of (3) relies on the formulation of the problem as a set of nonlinear equations that correspond to the Karush-Kuhn-Tucker optimality conditions parameterized with respect to the initial conditions [6].

\[
F = \begin{bmatrix}
\nabla J(\bar{x}, \bar{y}, \bar{u}) + \lambda^T \nabla h(\bar{x}, \bar{y}, \bar{u}) + \mu^T \nabla g(\bar{x}, \bar{y}, \bar{u}) \\
\lambda h(\bar{x}, \bar{y}, \bar{u}) \\
\mu g(\bar{x}, \bar{y}, \bar{u}) \\
(P_i + P_{i+1})^{-1} \theta \zeta 
\end{bmatrix} = 0
\]

(4)

The first entry in equation (4) corresponds to the gradient of the Lagrangian with respect to \( \bar{x}, \bar{y} \) and \( u \) of problem (3). Vectors \( \lambda \) and \( \mu \) denote the equality and active inequality constraints Lagrange multipliers, respectively. The second and third entries correspond to the feasibility condition under the assumption that the linear independence constraint qualification (LICQ) holds (i.e. the gradients of the equality and active inequality constraints are linearly independent). The strict complementarity condition is ensured by the positiveness of the Lagrange multipliers corresponding to the active inequalities. Finally, the last entry determines the transition from the initial point \( p_i \) (at current time interval, \( i \)) to the target initial point \( p_{i+1} \) (time interval \( i+1 \)) along the direction \( \theta \) in the multi-dimensional space defined by the system states. Symbol \( \zeta \) denotes the independent continuation parameter for the problem. Problem (4) can be viewed as a homotopy connecting the known optimal solution at initial vector \( p_i \) for \( \zeta \) equal to zero to the unknown solution at initial vector \( p_{i+1} \) for \( \zeta \) equal to unity.

Equation set (4) is an under-determined set of nonlinear equations as the overall number of variables exceeds by one the overall number of equations. Provided that model constraints are twice continuously differentiable the equation set is solved using a predictor-corrector continuation method as implemented in PITCON [7]. The sparse numerical solver UMFPACK [8] has been employed for improved solution speed. The Jacobian of equation (4) involves the Hessian for the modeling equations \( h \) and \( g \), which has been analytically evaluated for enhanced accuracy and computational performance. The solution method evaluates a series of continuation points for increasing values of the continuation parameter \( \zeta \). At every continuation point the algorithm performs rigorous checks for possible violation of inactive inequality constraints including state and control variable bounds. Additionally, the satisfaction of the strict complementarity condition, \( \mu > 0 \), is ensured through the inspection of the sign of the Lagrange
multipliers that correspond to the active inequalities. Any violation of inequality constraints and variable bounds as well as sign changes for Lagrange multiplier corresponding to active inequalities results in an active set change for the problem. In such a case, equation set (4) is suitably modified by either including a new active inequality or removing an inequality that ceased to be active in the equation set. In addition, nonlinear effects such as turning or bifurcation points in the optimal solution path may result to optimality loss, violation of the LICQ and multiple solution branches. The solution of the parameterized KKT conditions for varying initial point conditions enables the robust and accurate identification of the optimal solution of the associated dynamic optimization problem.

3. Case Study

3.1. Background

The position control with minimum sway of a cart with a double pendulum as shown in Figure 1 using NMPC is considered.

![Cart with double pendulum.](image)

Using Lagrange’s method the following 2nd order nonlinear differential equations of motion are derived for the system:

\[
(M + m_1 + m_2) \ddot{\theta}_1 + m_1 m_2 l_2 \left( \ddot{\theta}_1 \cos \theta_2 \sin \theta_1 - \ddot{\theta}_2 \sin \theta_2 \right) + m_2 l_2 \left( \ddot{\theta}_2 \cos \theta_2 \sin \theta_2 + \ddot{\theta}_1 \sin \theta_2 \right) + (m_1 + m_2) g l_1 \sin \theta_1 = F
\]

\[
(m_1 + m_2) \dot{\theta}_2 + (m_1 + m_2) \dot{\theta}_1 + m_1 m_2 l_2 \left( \ddot{\theta}_1 \cos \theta_2 \sin \theta_1 - \ddot{\theta}_2 \sin \theta_2 \right) + m_2 l_2 \left( \ddot{\theta}_2 \cos \theta_2 \sin \theta_2 + \ddot{\theta}_1 \sin \theta_2 \right) + (m_1 + m_2) g l_1 \sin \theta_1 = 0
\]

\[
m_2 l_2 \dddot{\theta}_1 + m_2 l_2 \dddot{\theta}_2
\]

\[
+m_2 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin \theta_2
\]

\[
+m_2 l_2 \dot{\theta}_2 \sin \theta_2 = 0
\]

The performance criterion for the system involves a penalty term for the deviation of the final position and velocity of the cart from the desired values in addition to time varying terms penalizing deviations for the sway angles \( \theta_1 \) and \( \theta_2 \) and the control effort.

\[
J = w_1 \left( x_n - x_f \right)^2 + w_2 \left( y_n - y_f \right)^2 + \sum_{i=0}^{N-1} \left[ w_3 \left( \theta_{1,i} - \theta_{1,i+1} \right)^2 + w_4 \left( \theta_{2,i} - \theta_{2,i+1} \right)^2 + w_5 u_i^2 \right]
\]

The prediction horizon is selected equal to one second. The discretization of the modeling equations is achieved through twenty finite elements utilizing 5th order Lagrange interpolating polynomials, therefore, each finite element represents a control interval with duration of 50ms. The OCFE discretization model exhibited a maximum absolute error of 5.7 \( 10^{-4} \) on the state vector profiles for a given control sequence when compared with results from numerical integration of the full order system. The solution of the control problem as defined in problem (3) has been performed using the augmented Lagrangian solver MINOS 5.5 [9] and a sparse version of PITCON for
the parameterized KKT conditions of equation (4). The two problem formulations involve 721 (equation 3) and 1415 (equation 4) variables, respectively. Two hundred simulation time steps have been calculated with actual control interval equal to 50ms, thus leading to an overall simulation time span of 10s. No computational delay has been considered in the implementation of the NMPC algorithm. However, the computational time has been recorded in order to compare the performance of the two procedures in terms of solution effort. Numerical integration of a plant model is used to simulate the plant behavior whereas a model with mismatch has been utilized in the NMPC algorithm. The update of the state vector is based on the estimation of a bias term equivalent to an integration disturbance model. The control profiles and the state profiles for the NMPC are shown in Figure 2.

The NMPC algorithm successfully placed the cart close to the new target position while maintained the cart velocity and the two suspended masses swaying within acceptable limits. The computational time required for the solution of the NMPC problem with MINOS 5.5 was equal to 16.5ms on average for each control interval, relatively large compared to the interval duration, whereas the respective solution time for the solution of problem (4) with a sparse implementation of PITCON was on average 4.4ms (4GHz Intel processor). The proposed solution approach exhibited significantly better robustness as it fully converged at all simulation steps. On the contrary, the NLP solver MINOS 5.5 failed at a small number of simulation steps to converge to an optimal solution.

4. Conclusions

This paper presents a fast and efficient solution approach of the dynamic optimization problem for nonlinear model predictive control applications. The solution is obtained from the parameterized optimality conditions resulted from a homotopy with respect to the control problem initial point. The speed and robustness of the proposed method enables the efficient implementation of NMPC in challenging engineering problems.

References