The term "Geometrical Algebra", target of a contemporary epistemological debate.

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1. Introduction

In the traditional historiography of ancient Greek mathematics, the term "Geometrical Algebra" indicates an interpretative approach and a way of " reading" a number of propositions from Euclid's "Elements", of Pythagorean, as it is generally believed, origin.

A characteristic example is proposition 4 of Book II [1], which is interpreted as "a geometrical formulation and proof of algebraic identity $(a+b)^2=a^2+b^2+2ab$ " [2]. Similarly, proposition II, 11 [3] which is considered to be "a geometrical solution of the quadratic equation $a(a-x)=x^2$ or $x^2+ax=a^2$ " [4].

The term "geometrical algebra" was introduced towards the end of the 19th century by P. Tannery [5] and H. Zeuthen [6] and was established in historiography after the classic edition of "Elements" by Heath, who made an extensive and systematic use of the term in the analysis of Euclid's work. [7]

The hypothesis of the existence of the "geometrical algebra" of the Ancient Greeks was reinforced when the archaeological finds that for the first time gave a sufficiently satisfactory picture of the Babylonian Mathematics, came to light. In the early 1930's it was suggested that the content of the "geometrical algebra" of the Greeks was nothing more than the geometrical formulation and proof of the Babylonian arithmetical rules for the solution of second degree problems. It was claimed that the reason behind this conversion was the need to overcome the problem that been generated by the discovery of incommensurability during the Pythagorean period.[8] These views were projected particularly by B. van der Waerden, and were incorporated in the historiography of Greek Mathematics. As a result, "geometrical algebra" was established as a definitive and firm conclusion of historical research. [9]

The first deviation from the established historical interpretation, took place in the late 1960's when A. Szabo argued that the propositions in Book II of "Elements" were of a purely geometrical nature, and rejected both the algebraic interpretation and the babylonian origin hypothesis. More specifically, Szabo showed that all these propositions belong to a pre-Euclidean stage of development of the Greek geometry, characterized by the disregard for the propotions and the problem of incommensurability. [10]

Szabo's views and critique went rather unnoticed and did not create a more general controversy over the matter of "geometrical algebra". However, since the mid-1970's a debate between historians of mathematics has been taking place-one that has, at times, taken open-conflict propositions. The main point of this debate is "geometrical algebra". This debate has led to the weakening of the methodological basis of traditional historiography and cautiousness behind the current research into ancient Greek Mathematics. This instability and critical examination releases a rich problematique on the nature of the mathematical thought of the Ancient Greeks.

The aim of this paper is to trace the beginning and development of this debate and to attempt to decipher the epistemological approaches it expresses.

2. S. Unguru's polemic.

During the 14th World Conference on the History of Science, in August 1974, which took place in Japan, Sabetai Unguru spoke about "the need to re-write the History of Greek Mathematics". This speech had a very aggressive tone towards the representatives of traditional historiography of Ancient Greek Mathematics, P. Tannery. H. Zeuthen and B. van der Waerden, claiming that the use of the term "geometrical algebra" led to a distorted understanding of the mathematical thought of the ancient Greeks. He emphatically stressed that:

> "The historiographical position embodied in the term "geometric algebra" is offensive, naive, and historically untenable. To read ancient mathematical texts with modern mathematics in mind is the safest method for misunderstanding the character of ancient mathematics, in which philosophical presuppositions and metaphysical commitments played a much more fundamental and decisive role than they play in modern mathematics. To assume that one can apply automatically and indiscriminately to any mathematical content the modern manipulative techniques of algebraic symbols is the surest way to fail to understand the inherent differences built in the mathematics of bygone eras. Geometry is NOT Algebra!
>
> > Geometry is thinking about space and its properties. Geometrical thinking is embodied in diagrammatic representation, accompanied by a rhetorical component, the proof. Algebraic thinking is characterized by operational symbolism, by the preoccupation with mathematical relations rather than with mathematical objects, by freedom from any ontological commitments, and by supreme abstractness. "Geometrical algebra", is not only a logical impossibility it is also a historical impossibility" [11].

In this extract from Unguru's speech, it is not difficult to distinguish the point of view of his criticism, which is condensed in the phrase: we cannot ignore "the character of ancient mathematics" without the danger of misunderstanding. What is also clear is his request for the introspection into the nature of the mathematical thought of Ancient Greeks, when he compares the ontological infrastructure of the Ancient Greek Mathematics to the relational one of modern mathematics. And this is because not only the understanding of Ancient Greek Mathematics cannot be achieved through the traditional approach, which aims at revealing the "modern ideas and procedures under the ungainly cloak of antiquity ways of expression" [12] but the "unique and (ontologically) idiosyncratic...way in which ancient Greek mathematicians went about their proofs" [13] is also distorted.

On the other hand, in this speech, mention is made of the reasons behind this historiographical discrepancy of using the term "geometrical algebra" in the analysis of Ancient Greek Mathematics. It is implied that this situation is due to the fact that "the history of mathematics has typically been written by mathematicians" [14] whose aim was "to unravel past mathematical texts and transcribe them into the modern language of mathematics, making them thus easily available to all interested" [15].

The same subject appears in 1975, in an extended form, under the same title, by the same author in the journal "Archive for History of Exact Sciences." [16] Compared to the speech at the 14th World Conference on the History of Science, Unguru has enriched his argumentation with supporting bibliography and his interpretative presentation of a few examples from Euclid's "Elements" that particularly represent "geometrical algebra". At the same time he adds to further aspects to his critique. The first one examines the historiographical substratum of the formulation and adoption of the term "geometrical algebra" by Tannery-Zeuthen-Heath on the one hand and Neugebauer-Waerden on the other. The latter's book "Science Awakening", which constitutes the bastion of this historiographical selection, bears the brunt of Unguru's critical stance [17]. In the second part of his critique, Unguru hints at what could be the alternative approach to the historical interpretation of "geometrical algebra".

In the last paragraph of this article those studies that have, directly on indirectly, expressed a diametrically opposite view are contrasted with the historical mentality connected with the term "geometrical algebra".

Abel Rey and Michael Mahoney who were the first to question, in 1939 [18] and 1971 [19] respectively, the algebraic interpretation of ancient mathematics, and to stress the difference between that and the geometrical way of thinking are mentioned. Even though they never questioned the legitimacy of the term "geometrical algebra", they could both be regarded, according to Unguru, [20] as the forerunners of this historiographical debate. The one who expressly rejected the historical value of the "geometrical algebra" concept, notes Unguru, was Arpad Szabo in 1969.

We should note here, that this like-minded position, even though it favours the critical stance of the particular article, does not act as its epistemological backbone.

On the contrary, the historical analysis of the emergence of algebra, carried out by Jacob Klein in the mid-1930's, [21] based on the conceptual evolution of the Number from the Pythagoreans till Wallis, is praised by Unguru [22] as the most prominent contribution to the historiography of mathematics.

From the preceding material we can discern the perspective of Unguru's fight against the historiographical establishment of "geometrical algebra". His position is that: "Mathematics is a reflection of culture" and he adds "it is, clearly, not immune to the intellectual and cultural environment in which it grows" [23].

3. The traditional school retorts

The first reply to this highly polemic article appeared, as it was natural, without any delay.

Van der Waerden, who had taken most of the flak, reacted instantly with an article-reply in the same edition of the same journal, only a few pages after Unguru's article [24]. His arguments concentrate on two points:

a) The historical dispute of the nature of algebraic reasoning projected and exploited in Unguru's critique. According to van der Waerden, those features that determine the algebraic way of thinking, as cited by Unguru, from Mahoney, concern the modern perception of algebra. On the contrary, when he refers to the Babylonian, Greek or Arabic algebra, he means Al-Khwarizmi's, Cardano's or even our school algebra. As far as he is concerned algebra is

> "the art of handling algebraic expressions like $(a+b)^2$
> and of solving equations like $x^2+ax=b$ " [25].

b) The refutation of the reasoning behind the adoption of the term "geometrical algebra", which Unguru imputed to the representatives of this historiographical standpoint. According to van der Waerden, Zeuthen and his followers did not prove, as Unguru claims, the algebraic core of the ancient mathematical thinking by making the understanding of the 2nd book of Euclid's "Elements" easier through the use of modern algebraic notation in order to interpret its theorems. "Our line of thought", notes van der Waerden, "was quite different. We studied the wording of the theorems and tried to reconstruct the original ideas of the author. We found it evident that these theorems did not arise out of geometrical problems**.** We were not able to find any interesting geometrical problem that would give rise to theorems like II 1-4. On the other hand, we found that the explanation of these theorems as arising from algebra worked well. Therefore we adopted the latter explanation" [26].

While confronting criticism, van der Waerden stages a counteroffensive. He writes accordingly:

> "Unguru, like many non-mathematicians, grossly overestimates the importance of symbolism in mathematics. These people see our papers full of formulae, and they think that these formulae are an essential part of mathematical thinking. We, working mathematicians, know that in many cases the formulae are not at all essential, only convenient" [27].

In the following edition of the same journal, another article-reply to Unguru, by Hans Freudenthal, is published [28]. The reasoning he develops to confute Unguru's criticism is no different than that of van der Waerden. We might even say that it is of a complementary nature and at times even more assertive. This is an indicative extract:

> "...Elements V is algebra, and nothing else, and I cannot imagine anyone's challenging this statement. To be sure, it is not geometric algebra - exorcised by Unguru - since there is no geometry at all in Elements V, where terms like line, square, rectangle are entirely absent... In order to distinguish the pure algebra of V from the matter of II and VI, which necessarily depends on lines and rectangles, the latter has been christened geometric algebra" [29].

Following van der Waerden and Freudenthal, yet another distinguished mathematician, Andre Weil, will side with them in the defense of "geometrical algebra". He joined the debate by publishing an extract from a letter he had sent to the editor of the "Archive for History of Exact Sciences" journal [30]. The exclusive purpose of that letter was to castigate the attempt to discredit Euclid. Quite bluntly he crushes the one who dares touch on matters considered settled:

> "...it is well to know mathematics before concerning oneself with its history" [31].

The nature of his arguments and his attitude towards the matter under debate are no different to that of his predecessors. The following lines from his letter are indicative:

> "In Euclid's books VII, VIII and IX, there is no trace of geometry, nor even of so-called "geometrical algebra". According to our modern classifications, those books are mostly algebra pure and simple (the algebra of the ring of integers); the balance, which is far deeper and more interesting, is pure number-theory. Of course it is more practical to carry out algebraic operations as we do, with the help of our algebraic symbolism, than in words as Euclid did; just as it is more practical to perform arithmetical operations in the decimal (or, as computers do, in the dyadic) system, rather than as Archimedes did; this does not affect the substance of the matter" [32].

4. The debate goes on

Unguru's counter-reply was rather belated [33], since it came only after the "old guard's" reactions had ceased. The largest part of this text is consumed by a "philological" commentary on van der Waerden's and Freudenthal's diagnostic insistence on the algebraic syndrome symptomatology of ancient Greek Mathematics and Euclid in particular. He writes for example:

> "...van der Waerden's assertions represent an unconscious but nevertheless clear-cut vindication of the argument that the real roots of the methodological position embodied in the concept "geometric algebra" lie in the modern mathematician's ability to read geometric texts algebraically without any historical qualms" [34].

Further down, referring to Freudenthal's view, that the distinctive characteristic of algebraic thought is the ability to describe relationships, solution procedures and relevant techniques in a general manner, he writes:

"However, it is precisely the inability of the Babylonian mathematician "to describe relations and solving procedures, and the techniques involved in a general way" that warrants his disqualification as algebraist. What the Babylonian mathematician lacks is precisely the ability to dispense with specific, definite numbers, and it is this deficiency that dictates the particular form of his approach. What he can produce is recipes, not general formulas. With respect to the Greek mathematician (geometer), on the other hand, though it is legitimate to see his approach as a general approach (the so-called theorem of Pythagoras is true of any right-angled triangle, etc), the language he uses is the geometric language and the generality involved is an outgrowth of dealing with geometrical and not with algebraic entities. Consequently, by Freudenthal's own criteria of "algebraic thinking", Babylonian and Greek mathematics are non-algebraic" [35].

As well as attempting to refute the juxtaposed views, Unguru proceeded, in his article, to determine epistemologically the views of the traditional historiography of mathematics.

Using as a starting point the discovery, from his first reaction to "geometric algebra", that the majority of contemporary historians of mathematics are in fact mathematicians, he deems that the substratum of this historiographical mentality is impregnated with the Platonic perception of mathematics.

Mathematical concepts are thus treated as "eternal, unchanging, unaffected by the idiosyncratic features of the culture in which they appear, each one clearly identifiable in its various historical occurrences, since these occurrences represent different clothings of the same Platonic hypostasis" [36]. In reference to the function of this perception in the methodology of the History of Mathematics, he writes:

> "Various forms of the same mathematical concept or operation are not considered merely mathematically equivalent but also historically equivalent. Indeed mathematical equivalence is taken to represent historical equivalence. Since the mathematical Forms are eternal and since in their works mathematicians of all ages share in the expression of the same Forms, the specific mathematical idiom used by a mathematician has no bearing on the content of his thought. Mathematical language is at best a secondary appurtenance of the mathematical culture of any epoch. The mathematical kernel is untouched by the peculiar language used, since all mathematical languages lead back to the same ideal Forms. This makes the various casts in which the same mathematical truth has been expressed throughout the centuries completely equivalent... Under such an ontology, the object of the history of mathematics becomes the task of identifying the ideal forms present in the work of each historical author and apportioning out proper credit to that mathematician who first gave expression to one of these eternal forms, i.e. who first brought it out of the eternal Platonic realm into the world of human consciousness. This is precisely the task performed traditionally by the historian of mathematics" [37].

Further down he contrasts this historiographical establishment to the modern dynamics in the following words:

> "Entrenched as it is, the traditional interpretation of the history of ancient mathematics must give way to a new, more sympathetic, and historically responsive interpretation, simply because the old interpretation has outlived its usefulness and is now an obstacle on the road to a sensitive historical understanding of ancient mathematical texts. After all, like scientific theories, historical theories are tentative attempts to make sense of the past; they are provisional by their very nature, and consequently their authors should not be dreaming hopelessly of endowing them, in God-like fashion, with eternal life and immaculate beatitude" [38].

Following this revealing, from an epistemological point of view article Unguru, in collaboration with David Rowe, sum up the former' s counter-reply in an article published in two parts in 1981 and 1982 in the first two volumes of the new "Libertas Mathematica" journal [39].

In contrast to its previous historical-philosophical direction, the criticism in this article focuses of the mathematical substratum of the positions adopted by the supporters of "geometrical algebra". The following extract is quite revealing of Unguru's and Rowe' s approach to the matter:

> "... there are important distinctions between algebra and the concrete arithmetical relationships appearing in Babylonian and some Greek materials. For there is a vast mathematical gap involved between having a general knowledge of concrete number facts on the one hand, and

being able to abstract that knowledge and manipulate it symbolically without any reference to the concrete, on the other. Ignoring these distinctions, representative as they are of a wide gulf in mathematical outlook and technique, has been one of the main ways that confusion has arisen over the use of the term "algebra". It follows, then, that arithmetic precedes algebra, i.e. the existence of a coherent arithmetic system is required in order to have an algebraic superstructure, and without a firm arithmetical foundation, the attempt to do algebra (or, in our case, to find algebra) collapses like a house of cards. It is our contention that this is exactly the predicament of "geometric algebra", the arithmetical foundations of which turn out, at close scrutiny, to be rather shaky. To show this convincingly will require an extended discussion of the various operations that comprise the "geometric arithmetic", together with an assessment of their proper place in what we feel is a viable interpretation of classical Greek Mathematics. What will emerge very clearly from this discussion is the fact that there are insuperable difficulties inherent, in this, the very foundation of "geometrical algebra", and that these difficulties peremptorily show that the foundation is much too weak to support the weight of the required algebraic superstructure" [40].

At this point we should note that the positions of traditional historiography concerning the substratum of "geometrical algebra" (i.e. the geometrical equivalent of algebraic operations) have been expressed by Heath as follows:

"The addition and subtraction of quantities represented in the geometrical algebra by lines is of course effected by producing the line to the required extend or cutting off a portion of it. The equivalent of multiplication is the construction of the rectangle of which the given lines are adjacent sides. The equivalent of the division of one quantity represented by a line by another quantity represented by a line is simply the statement of a ratio between lines on the principles of Books V. and VI. The division of a product of two quantities by a third is represented in the geometrical algebra by the finding of a rectangle with one side of a given length and equal to a given rectangle or square. This is the problem of application of areas solved in I. 44, 45. The addition and subtraction of products is, in the geometrical algebra, the addition and subtraction of rectangles or squares; the sum or difference can be transformed into a single rectangle by means of the application of areas to any line of given length, corresponding to the algebraical process of finding a common measure. Lastly, the extraction of the square root is, in the geometrical algebra, the finding of a square equal to a given rectangle, which is done in 11.14 with the help of I.47" [41].

Unguru and Rowe counter this with a detailed analysis of the operations mentioned in "Elements", noting their main characteristics, particularly the fact that these operations are applied only to homogenous magnitudes (i.e. magnitudes of the same dimension). This fundamental characteristic leads to the conclusion that multiplication, as found in "Elements" (i.e. as a repeated addition, which when applied to two one-dimensional magnitudes, produces another one-dimensional magnitude) is not compatible with the generalised multiplication introduced by Heath (the construction of a rectangle, i.e. the production of a two-dimensional magnitude). The same, of course, applies to division, between the formation of a ratio ("Elements") and the application of areas (Heath). This is how the ensuing problem can be solved according to Unguru and Rowe:

> "It is our contention that the dilemma posed by the incompatibility between rectangle formation as generalised multiplication and the homogeneity relation underlying Greek magnitude can be resolved in but one reasonable way, namely by taking rectangle formation at face value, precisely the way find it throughout Greek geometry, and viewing it as a geometric operation and not as part of a system of generalized arithmetic. Thus, while recognizing the operational character of this construction, as well as the close analogy it shares with the geometric properties of numbers, we cannot accept that this construction was part of an "algebraicized" geometry which sought to extend the ordinary "arithmetic" operations found in the <u>Elements.</u> These two systems (geometrical and "arithmetical") cannot be fused into one without altering altogether the character of Greek Geometry, since the "arithmetic" system found in the <u>Elements</u> cannot withstand the type of surgery required to support an algebra for general magnitudes. It follows that rectangle formation, "application of areas", etc. should be viewed as geometric operations and not as part of an integrated, generalized system of "arithmetic". Thus the only "division" in Greek mathematics is ratio-formation, and we can quite confidently assert that no Greek would have confused this operation with the application of a rectangle to a given line by thinking of the two as but different forms of a more general "division" operation" [42].

Having concluded that there does not exist a system of "geometrical arithmetic", which would support "geometrical algebra", the writers move onto a critical examination, in the sequel of their article, of the notion that the Greeks could solve second-degree equations. Analyzing those chains of propositions in "Elements" which supposedly support this notion, they point out the contradictions in the traditional interpretation of the use of the principles of "geometrical arithmetic" and underline the particular geometrical character of those propositions [43]. Their lay-out in "Elements" and the demonstrative techniques that Euclid employs (such as, for example, in 1.47 i.e. Pythagoras's theorem) corroborate the view that their existence is associated with a deliberate delay in introducing and using the general theory of proportions of book V [44].

This article by Unguru and Rowe suggests that the direct confrontation has come full circle. From the representatives of traditional historiography, van der Waerden comes back to the matter of this confrontation in 1983, with a brief mention in his book "Geometry and Algebra in Ancient Civilizations" where supporting the usual arguments for "geometrical algebra" and mentioning Unguru's polemic, he writes:

> "However, I must admit that there is an element of truth in Unguru's criticism. In my book "Science Awakening", I had considered Greek geometric algebra just as a translation of Baby Ionian algebra into the language of geometry. I now see that this view is only partly

correct, and that Greek geometric algebra was a result of a synthesis between earlier geometrical traditions and Babylonian algebra" [45].

Later on, he sums up this revised view on the origin of "geometrical algebra" as follows:

"... the Greeks combined two traditions, which both originated in the Neolithic Age: one tradition of teaching mathematics be means of problems with numerical solutions, and one of geometrical constructions and proofs. The algebraic tradition was mainly transmitted by the Babylonians, and the geometrical tradition probably reached Greece by way of Egypt. Also in the realm of Indo-European languages, a tradition of geometrical altar constructions existed, part of which may well have been preserved by the Greeks in Mycenae and Crete. Traces of this tradition were preserved in Greek stories about the "doubling of the cube". Remember that the origin was ascribed to King Minos of Crete, who wanted to double the size of an altar without destroying its beautiful form. So we have three possible roads of transmission, namely: by way of Babylon, of Egypt, and of Mycenae and Crete" [46].

We should also note that in 1977, Szabo had come back to this controversial matter with his article "Zum der sogenaunten Geometrischen Algebra in Euklids Elementen", [47] where he claims that the term "geometrical algebra" cannot be supported and all those using it are making an historical error. We will note, however, that even though Szabo, by adopting this stance, seems to belong to the "warring" parties, in fact he does not, not only because he makes no mention of the articles on this debate, but mainly because he does not touch on the metatheoretical core of the matter, which is where the substance of this confrontation lies.

5. The epistemological backstage

Following this wide review on the controversial matter of the historiographical legitimacy of the term "geometrical algebra", certain questions naturally come to mind:

Why did this debate start and why is it so stubborn;

Why at this time;

Who supports the one view and who does the other;

Does it have deeper roots and how far do they extend;

Let us attempt to penetrate the backstage of this debate to try and reveal its "instigators" and thus understand its meaning.

It is true that the first impression we get from this debate is that a quarrel has broken out between (historiographers) mathematicians and historians of mathematics. This however is only the tip of the iceberg and is neither sufficient to explain the substance of the matter nor to bring about a critical feedback of the epistemological and the methodological substratum of the historiography of ancient Greek Mathematics. To overcome this apparent picture we must identify those signs from "what has been said" that will allow us to venture into "what has not been said."

Looking then for Ariadne's clue which will lead us to an interpretative outlet, with a little care we locate the first common point between the conflicting texts: Michael Mahoney was instrumental in this debate. [48] Combining this with the fact that he was a student and still is a follower of Thomas Kuhn [49], we are justifiably led to a comparative examination of the viewpoint of the one "warring" side and that of T. Kuhn.

Unguru's first text makes evident the two points that determine the basis of his "attack". The first one appears as a proposal: "To read ancient mathematical texts with modern mathematics in mind is the safest method for misunderstanding the character of ancient mathematics, in which philosophical presuppositions and metaphysical commitments played a much more fundamental and decisive role than they play in modern mathematics" [50].

The second point is expressed in the belief that "Mathematics is a reflection of culture" and that "it is, clearly, not immune to the intellectual and cultural environment in which it grows" [51]. Kuhn, on the other hand, in his article "The History of Science", which was published in the 14th volume of the International Encyclopedia of the Social Sciences in 1968, presents a few characteristics of this point of view. The following are two of those characteristics:

1st**.** "The objective of these older histories of science was to clarify and deepen an understanding of contemporary scientific methods or concepts by displaying their evolution. Committed to such goals, the historian characteristically chose a single established science or branch of science.... and described when, where and how the elements that in his day constituted its subject matter and presumptive method had come into being. Observations, laws, or theories which contemporary science had set aside as error or irrelevancy were seldom considered unless they pointed a methodological moral or explained a prolonged period of apparent sterility" [52].

2nd. "Early in the development of a new field,..., social needs and values are a major determinant of the problems on which its practitioners concentrate. Also during this period, the concepts they deploy in solving problems are extensively conditioned by contemporary common sense, by a prevailing philosophical tradition or by the most prestigious contemporary sciences" [53].

The resemblance between these extracts and the previous ones by Unguru is evident.

At this point we should make it clear that the points we have chosen determine Kuhn's and Unguru's general consideration of the history of science in its first stages of development, as seen from their critical stance against the traditional historiography of science.

These elements allow us to consider that Unguru's fire did, in fact, come from the Kuhn's "camp".

The other school of thought is represented today by van der Waerden and 0. Neugebauer*. The former defended the positions of the traditional historiography of ancient Greek Mathematics, assisted by H. Freudenthal and A. Weil. Neugebauer on the other hand, who had been in charge of that "camp" since the 1930's and was the first to receive the enemy fire, neither reacted nor took any part in the debate. Considering that their predecessor, Sir Thomas Heath, a distinguished historian of

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* This paper was written before 0. Neugebauer's death in February 1990.

ancient Greek Mathematics, used the term "geometrical algebra" and consequently the underlying interpretative "refraction", we understand that the root of the matter goes even deeper. Indeed, it was Unguru who revealed in his critical stance that H. Zeuthen and P. Tannery were the ones who devised and introduced the term in the modern historiography of mathematics [54]. This can be substantiated not only be the presence of the controversial term in the studies of these historians of mathematics in particular, but also by its absence from other histories of ancient Greek Mathematics of the time, as for example from the book "A Short History of Greek Mathematics" (1884) by James Grow. Naturally then, our interest is focused on those two famous historians of mathematics.

Hieronymous G. Zeuthen (1839-1920), the Danish researcher of mathematics, was one of the most prominent [55] historians at the turn of 19th century. As a student of Michel Chasles (1793-1880), he received his influence not only in his research orientation in mathematics and mechanics but in the overall shaping of his mathematical culture. This exemplary teacher, who combined the research into mathematics with the history of mathematics, could not have left his student uninterested. Indeed, the student did adopt the appropriate views.

With reference to Chasles's form of historical understanding, D. Struik quite characteristically wrote for his famous book "Apercu historique sur l'origine et le development des methodes en geometrie." (1837):

> "It is a history seen through the eyes of an outstanding mathematician, seeing his own work and that of his contemporaries in projective and algebraic geometry in the light of past achievements..." [56].

With respect to Chasles's historiographical point of view, something that particularly interests us but rather ambiguously touches on the previous extract, it is very revealing the introduction of his book:

> "Nous nous proposons, dans cet Apercu, de presenter une analyse rapide des principales decouvertes qui ont porte la Geometrie pure au degre d'extension ou elle est parvenue de nos jours, et particulierement de celles qui ont prepare les methodes recentes. Nous indiquerons ensuite, parmi ces methodes, celles auxquelles nous paraissent pouvoir se rattacher la plupart des innombrables theoremes nouveaux dont s'est enrichie la science dans ces derniers temps" [57].

And further down:

> "Celles-ci ne paraltraient pas indispensables si l'on n'envisageait que le but historique de notre travail. Mais nous avons eu en vue surtout, en retracant la marche de la Geometrie, *et* en presentant l'etat de ses decouvertes et de ses doctrines recentes, de montrer, par quelques exemples, que le caractere de ces doctrines est d'apporter, dans toutes les parties de la science de l'etendue, une facilite nouvelle et les moyens d'arriver a une generalisation, jusqu'ici inconnue, de toutes les verites geometriques;..."[58].

His very words, therefore, make obvious the adoption of a inductivistic perception of the history of mathematics, where the predominant element is the reduction of current mathematical accomplishments to previous, more elementary, achievements. This type of historiography attempts to support the normative forms of the current rationality in historical reviews of the scientific thought. This aspect of the history of science was originally promoted by Condorcet and Comte, through their overall philosophical perception and constituted the predominant specification of historiographical activity in the 19th century.

This is the environment where Zeuthen spent the years of his scientific maturing. It might not be easy to establish the degree to which this environment influenced his problematique and his way of thinking, but it would be difficult to claim that it left him unaffected, since the points he has in common with his teacher's scientific horizon and style are not few.

In addition to this indirect and coincidental influence, Paul Tannery (1843-1904) [59] had a more direct effect on his historical perception and activities. They first met soon after 1881, when Tannery visited Germany and the Scandinavian countries with the aim to establish scientific relations with M. Cantor, J.L. Heiberg, H.G. Zeuthen and G. Enestrom, in other words the most dynamic elements in the domain of the history of mathematics [60]. That was the time of Tannery's [61] and Zeuthen's [62] burst of historical activity. They both appeared on the stage of the history of mathematics in 1876. However, it was after 1880 when they produced their profuse scientific works.

Paul Tannery, a French engineer and civil servant working for the national tobacco industry, entered the domain of the history of mathematics without any specialist education, not even in related topics. He was in other words self-taught. The motives that led him to this choice were the study of August Comte's "Cours de philosophie positive" soon after his graduation from the Ecole Polytechnique (1863) and his brother Jules who had studied mathematics and followed an academic career. A determining factor that deeply influenced his thought was the positivist philosophy, which he did not simply accept but used as a means of looking at things and as a methodological tool in his historical activity [63]. We should also note that his brother's appointment as editor of the "Bulletin des Sciences Mathematiques" was catalytic to the development and promotion of his historical work.

The focal point of his interest was the mathematical thought in ancient Greece, where the largest part of his work was devoted. He also contributed, to a considerable degree, to the historical treatment and publication of historical material on the mathematics of the late Middle Ages and the Renaissance. Worthy of mention is the fact that Zeuthen too, systematically dealt with exactly these two periods. And this coincidence becomes even deeper and more complete at the methodological level. According to Zeuthen:

> "P. Tannery war ein genugend tiefer und feiner Geometer, urn sein Denken unabhangig zu machen von den aktuellen Formen der Mathematik und es den antiken Formen so anzupassen, das er aus eigener Erfahrung den Wert der Hilfsmittel und die Tragweite der Verfahrungsweisen beurteilen konnte, uber die man in den vergangenen Zeiten verfugte. Daher wusste er sich auch von jeder klassifikation freizumachen, die der modernen Mathematik entlehnt ware; so verstand er z.B. in der antiken Geometrie die Grundlagen einer Algebra aufzufinden, die unmittelbar anwendbar war auf arithmetische Fragen" [64].

And as M. Noether testifies in his obituary for Zeuthen, these words express Zeuthen himself, since he shared Tannery's views, whose historical works had been his models [65]. In the same text M. Noether notes that:

> "Diese Auffassung einer geometrischen Algebra hatte ubrigens schon

P. Tannery vorbereitet und sie wurde in Zeuthen durch Beschaftigung mit Tannery's Arbeiten angereqt" [66] (underlined by the authors N.K./Y.T.).

The question now arises:

Is Paul Tannery at the bottom of the historiographical adoption of the term "geometrical algebra"?

Our retrospection into the works of the historians of mathematics who used the term stops here. Yet, there is one point that leaves some room for extrapolation. It has to do with the fact that the term "geometrical algebra" constitutes a methodological element of the historical understanding of ancient Greek Mathematics and therefore belongs to the metatheoretical arsenal of the historiography of mathematics. Justifiably then our suspicions can turn to the sources of Tannery's metatheoretical arsenal.

Taking now into account the common view [67] that Tannery was one of the first representative of the positivist historiography of science, we are quite naturally led to Auguste Comte, the founder of positivism. It is only logical then, that our search aims primarily at the heart of positivism, i.e. his books Systeme de Politique Positive (1824) and Cours de Philosophie Positive (1830-1842). Thus in page 300 of the third volume of Systeme de Politique Positive we find the following view: Thales can quite rightly be considered to have laid the foundations of Algebra, since his first theorem constitutes an equation with the full meaning of the word, whereas his second one constitutes a proportion [68]. It is becoming clear here that Comte looks at Thales through an algebraic perspective, in other words, through the prism of modern mathematical perception. This is a peculiar way of historical understanding of ancient Greek Mathematics, which was adopted and cultivated by the followers of the respective epistemological tradition.

Having revealed the genealogy of the historiographic vehicle of "geometrical algebra" as well as the operations camp of its current opponents, we could say that the debate we described, has taken the form of a frontal clash in the field of the methodology of the history of mathematics between the Kuhn's "school" and the positivist tradition camp.

6. The attitude of the current historical research

As we noted in the introduction of this paper, the current research into Greek Mathematics is characterized by reluctance, which has replaced the previous, universal acceptance of the Neugebauer-Waerden interpretation of the existence of the "geometrical algebra" of the ancient Greeks.

This reluctance had already been expressed by W. Knorr in his book "The evolution of the Euclidean Elements" (1975), before the "geometrical algebra" conflict broke out. In analyzing methodology problems of the historical research into ancient Greek Mathematics, Knorr pointed out some of the dangers inherent in interpreting the latter in terms of the modern mathematical thought. He went on to criticize strongly the supporters of "geometrical algebra" and the claims (T. Heath's in particular) about the "solution of quadratic equations." He rounded off his critique with the following characteristic warning to the readers of his book:

"While I shall continue to use the term 'geometrical algebra', I will never mean by it anything other than “Book-II-type-geometry”" [69].

In addition to this reluctance, however, the study of the modern historiography of Greek Mathematics reveals a new trend in the methodological approach and interpretation of Euclid's "Elements", particularly of the 2nd book which is considered to be the core of the "geometrical algebra". A fundamental feature of this trend is that the interpretation is now attempted using terms of ancient mathematics, whereas the use of algebraic concepts and notation is being brushed aside.

Waerden**,** in his reply to Unguru, had expressed the following arguments for "geometrical algebra".

a. There is no interesting geometrical problem that would justify some of the theorems of the 2nd book of "Elements".

b. There is a step by step correspondence between the arithmetical methods of the Babylonians and certain theorems of the 2nd book of "Elements".

c. There are, generally, many points common to Babylonian and Greek mathematics that are indicative of a transfer of knowledge from the former to the latter[70].

These, according to Waerden, satisfactorily support the hypothesis that the Greeks started with algebraic problems which - for reasons related to the problem of incommensurability among others - they translated into the geometrical language.

With reference to Waerden's first argument, D. Fowler's research, which reveals a new role for the 2nd book of "Elements", is of special interest.

Fowler maintains that, before the introduction of the general theory of proportions of Eudoxus, there had been in use a definition of the ratio of two magnitudes that made use of the process of anthyphairesis (the latter being a form of Euclid's algorithm for the determination of the common measure of two magnitudes). By reconstructing the techniques of working out ratios using the anthyphairetic method, Fowler has shown that the propositions contained in the 2nd book of "Elements" play a key role [71]. Thus, this contentious book could be regarded as an organized collection of auxiliary propositions on this particular problem. This fact is, of course, in sharp contrast with Waerden's argument that not one interesting geometrical problem does support the existence of the propositions in the 2nd book.

The second fundamental argument for "geometrical algebra" concerns, as we have already noted, the similarities between the arithmetical methods of the Babylonians and certain propositions in the 2nd book of "Elements". This is in fact true, in the sense that the Babylonian methods on the one hand and Euclid's propositions on the other, could be regarded as two different representations of the same modern algebraic formula or as two methods of solution of the same quadratic equation. This modern representation, however, is not at the least enlightening with reference to the real meaning of the ancient mathematical text. Waerden himself writes on this matter:

"The fact that a theorem can be translated into another notation does not prove a thing about what the author ofthe theorem had in mind" [72].

This matter refers us directly to the problem of interpreting the algebraic way of thinking and the question of whether such a way of thinking could be attributed to Euclid. Among the protagonists of this debate there exist, as we have seen, a radical disagreement about the characteristics of the "algebraic thought" attributed to Euclid, which can work both for and against "geometrical algebra."

I. Mueller has analyzed this particular point in a thorough and extensive study of the structure of Euclid's "Elements" (the first one to be published after the dispute about "geometrical algebra" had broken out). Mueller examined the content as well as the usage of the propositions in the 2nd book and came to the conclusion that they function as geometrical properties of specific figures and not as abstract relationships between quantities or as formal relationships between expressions. The lack of these particular characteristics of algebraic reasoning shows that there is no reason why anyone should assume that Euclid thinks in an algebraic manner [73].

With reference to the matter of the Greeks "translating" the Babylonian methods, modern researchers are adamant: No such hypothesis can be valid when there is no historical evidence of the time or the way such a transfer of mathematical knowledge from Mesopotamia to Greece took place. J. Berggren notes:

> "Some Babylonian ideas - for example, the gnomon - seem to have been transmitted at an early date, whereas other notions - that of degree measurement of angles and the sexagesimal system, even in the modified form in which the Greeks used it, for example - seem to have arrived after Euclid wrote. When, in this interval, one is to date the importation and geometrization of Babylonian algebra is a historical question to be settled not by conjecture but by research. If the event cannot be located historically one must recognize the possibility that it may not have occurred" [74].

We bring this break review of the recent historical research to a close with a reference to W. Knorr, who appears to be adopting a compromising stance on the matter of the debate between the supporters and the adversaries of "geometrical algebra". In his book "The Ancient Tradition of Geometric Problems" (1986), he writes on this matter:

> "Certainly, the ancient geometry never had access to the special advantages afforded by algebraic notations and conceptions in the full modern sense. On the other hand, one often finds that the ancients introduced geometric diagrams whose purpose was merely to represent given quantitative relations, to provide a vehicle for the investigation of these relations with the intent to apply the results to other contexts... In these instances, the Greek geometers have perceived the structural relations which unite ostensibly unrelated diagrams. A principal virtue of the algebraic manner is its abstraction of such quantitative relations for separate examination, so that results may be transferred from one context to another. Of course, it is in the strict sense fallacious to argue that these ancient techniques <u>are</u> a form of algebra because they serve the same function of separating quantitative relations from their special contexts. Nevertheless, the term "geometric algebra" can be useful in alerting us to the fact that, in these instances, diagrams fulfill this function; they are not of intrinsic geometric interest here, but serve only as auxiliaries to other propositions" [75].

Notes

1. Translated by T.L. Heath (*EUCLID. The Thirteen Books of THE ELEMENTS*, Dover Edition, 1956, Vol. 1, p. 379), proposition II,4 has as follows:

   If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.
2. See for example, T.L. Heath op. cit. p. 372, or B.L. van der Waerden *SCIENCE AWAKENING*, J. Wiley - Science Editions 1963 p. 118.
3. Translated by T.L. Heath op. cit. p. 402, proposition II,11 has as follows:

   To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.
4. See for example, T.L. Heath op. cit. p.403, or B.L. van der Waerden op. cit., p. 121.
5. P. Tannery: De la solution geometrique des problemes du second degre avant Euclide, (1882), *Mem. scient* 1, Paris 1912, p. 254-280.
6. H. Zeuthen: *Die Lehre von den Kegelschnitten im Altertum*, Kopenhagen 1886 p. 1-38. By the same author: Geschichte der Mathematik im Altertum und Mittelalter, Kopenhagen 1896 p.32-64.
7. "Elements" was first published by Heath in 1908. The Dover edition has been used for the purposes of this paper is a reprint of the second edition of 1925.
8. See, 0. Neugebauer: Zur geometrischen Algebra. Studien zur Geschichte der antiken Algebra III, *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik Abteilung B: Studien*, 3, (1936), p. 245-259. By the same author: *THE EXACT SCIENCES IN ANTIQUITY*, Dover Edition, 1969, p.l47ff (First edition 1952, Princeton University Press).
9. Waerden's "Science Awakening" which projects the historical interpretation of "geometric algebra" was originally published in Dutch in 1950. The first English edition appeared in 1954, followed by a German edition in 1956. The wide use of the term "geometric algebra" is a common fact in the literature of recent decades. See for example: P. Dedron - J. Itard**:** *Mathematics and Mathematicians*, Open University Press 1978 Vol. 2 p. 76ff (First French Edition 1959, Magnard). C. Boyer: *A History of Mathematics*, J. Wiley 1968 p. 85ff , 120ff. M. Kline: *Mathematical Thought from Ancient to Modern Times*, Oxford University Press 1972 p. 64ff. H. Gericke: *Mathematik in Antike und Orient*, Springer Verlag 1984 p. 127ff.
10. See, A. Szabo: *Anfange der Griechischen Mathematik*, R. Oldenbourg 1969. Anhang: Wie kamen die Pythagoreer zu dem Satz Eucl., Elem. II,5 ?, s. 455-488.
11. See, S. Unguru; On the need to rewrite the history of Greek Mathematics, *Proceedings of XIVth International Congress of the History of Science*, Science Council of Japan, No 2, 1974, p. 171.
12. op. cit. p. 169.
13. op. cit. p. 169.
14. op. cit. p. 169.
15. op. cit. p. 169.
16. Vol. 15 p.p. 67-114.

17. op. cit. p. 71-74, 80-81, 83-86.

18. In the book: *La Science dans l'Antiquite*, 3 (La Maturi-fce de la Pensee Scientifique en Grece), Paris 1939.

19. In his articles: i) "Die Anfange der algebraischen Denkweise im 17 Jahrhundert" Rate 1 (1971) p.p.15-31. ii) "Babylonian Algebra: Form vs Content", *Studies in History and Philosophy of Science,* 1, 1971, p.p. 369-380.

20. In "On the need to rewrite..." (1975) p.109.

21. See, J. Klein: "Die griechische Logistik und die Entstehung der Algebra", *Quellen und Studien zur Geschichte der Mathematik, Astronomic und Physik. Abteilung B: Studien,* Bd 3, fasc 1, 1934, p.p. 18-105, fasc. 2, 1936, p.p. 122-235.

22. In "On the need to rewrite...." (1975) p.109.

23. op. cit. p. 86.

24. See.B.L. van der Waerden: "Defense of a "Shocking" Point of View", *A.H.E.S.,* 15, 1975 p.p. 199-200.

25. op. cit. p. 199.

26. op. cit. p. 203-204.

27. op. cit. p. 205.

28. See, H. Freudenthal: "What is Algebra and What has it been in History?", *A.H.E.S.*, 16, 1976-77 p.p. 189-200.

29. op. cit. p. 193.

30. See, A. Weil: "Who Betrayed Euclid?", *A.H.E.S.,* 19, 1978, p.p. 91-93.

31. op. cit. p. 91.

32. op. cit. p. 93.

33. See, S. Uuguru: "History of Ancient Mathematics: Some Reflections on the State of the Art", *ISIS,* 70 (No 254), 1979, p.p. 555-565.

34. op. cit. p. 558.

35. op . cit. p. 5 61.

36. op. cit. p. 555.

37. op. cit. p. 555.

38. op. cit. p. 563.

39. See, S. Unguru - D.E. Rowe: "Does the quadratic equation have Greek roots? A Study on "Geometric Algebra", "Application of Areas" and related problems." *Libertas Mathematica,* 1, 1981 p.p. 1-49 and 2, 1982 p.p. 1-62.

40. See, S. Unguru - D.E. Rowe 1981 p.p. 11-12.

41. See, T. L. Heath: *EUCLID. The Thirteen Books of THE ELEMENTS*, Vol. 1, p. 374.

42. See, S. Unguru - D.E. Rowe 1981 p. 33.

43. See, S. Unguru - D.E. Rowe 1982 p.p. 19-20.

44. op. cit. p. 12.

45. See, B.L. van der Waerden: *Geometry and Algebra in Ancient Civilizations*, Springer 1983 p. 77.

46. op. cit. p. 89.

47. See, A. Szabo: "Zum Problem der sogenaunten Geometrischen Algebra in Euklids Elementen" in *Y. Maeyama und W.G. Saltzer (Hers): ΠΡΙΣΜΑΤΑ. Naturwissenschafs-geschichtliche Studien*, Franz Steiner Verlag GmbH 1977,p.p. 373-393.

48. See, S. Unguru: "On the need to rewrite...." (1975), p.77, 109. Also see, B.L. van der Waerden: "Defence of a "shocking" point of view" p. 199.

49. See, M.S. Mahoney: *The Mathematical Career of Pierre de FERMAT (1601-1655),* Princeton Univ. Press 1973, p. xv.

50. See, S. Unguru: "On the need to rewrite the history of Greek Mathematics", *Proceedings of XIVth International Congress of the History of Science*, Science Council of Japan, No 2 1974 p. 171.

51. See, S. Uunguru: "On the need to rewrite the history of Greek Mathematics" *A.H.E.S.,* Vol. 15 1975 p. 86.

52. See,T. Kuhn: *The Essential Tension*, The Univ. of Chicago Press, 1977, p.107.

53. op. cit. p. 118.

54. See, S**.** Unguru: "On the need to rewrite..." (1975) p. 69.

55. See, K. Haas: "Zeuthen, Hieronymus Georg" in *C.C. Gillispie (ed): Dictionary of Scientific Biography,* 1980, Vol. 14, p. 619.

56. See, D.J. Stuik: "The Historiography of Mathematics from Proklos to Cantor, *NTM: Zeitschrift fur Geschichte der Naturwissenschaft, Technik und Medizin,* 17, 1980, p.11.

57. See, M. Chasles: *Apercu historique sur l'origine et le development des methodes en geometrie*, Troisieme Edition, Gauthier - Villars, 1889, p. 1.

58. op. cit. p. 2.

59. See, M. Noether: "Hieronymus Georg Zeuthen", *Mathematische Annalen*, 83, 1921. p. 12.

60. See, R. Taton: "Tannery, Paul" in *C.C. Gillispie (ed): Dictionary of Scientific Biography,* 1980, Vol. 13, p. 252.

61. op. cit.

62. See, H.W.R.: "Hieronymus Georg Zeuthen", *Proceedings of the London Math Soc.*, 2 Ser. 19, 1921 p. xxxix.

63. See, G. Sarton: "The History of Science", *The Monist 26*, 1916 p. 331.

64. See, M. Noether: "Hieronymus Georg Zeuthen", *Mathematische Annalen,* 83, 1921, p. 12.

65. op. cit.

66. op. cit.

67. See, for example, G. Sarton: The History of Science, *The Monist* 26, 1916 p. 331 and A.R. Hall: "Can the History of Science be History?", *The British Journal for the History of Science*, 4, 1969 p. 214.

68. See, G.J. Allman: *Greek Geometry from Thales to Euclid Arno Press*, 1976 (First edition 1889) p. 16.

69. See, W.R. Knorr: *The Evolution of the Euclidean Elements,* D. Reidel, 1975, p.11.

70. See, B.L. van den Waerden: "Defence of a "shocking" Point of Veiw" p. 203,209.

71. See, his articles: i) "Patio in Early Greek Mathematics", *Bulletin of the American Mathematical Society*, Vol. 1 No6,1979, p.p. 807-846.

ii) "Book II of Euclid's Elements and a pre-Eudoxean Theory of Ratio", *Archive for History of Exact Sciences*, Vol. 22, 1980, p.p. 5-36.

iii) "Book II of Euclid's Elements and a pre-Eudoxean Theory Ratio. Part 2: Sides and Diameters", *Archive for History Exact Sciences*, Vol. 26, 1982, p. p. 193-209

The results of Fowler's research can also be found in his book: The Mathematics of Plato's Academy. Clarendon Press 1987.

72. See, B.L. van der Waerden: "Defence of..." p. 203.

73. See, I. Mueller: *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*, The M.I.T. Press 1981, especially p.p. 41-52.

74. See, J.L. Berggren: "History of Greek Mathematics: A.Survey of Recent Research", *Historia Mathematica* 11, 1984, p.p. 394-410.

75. See, W.R. Knorr: *Ancient Tradition of Geometric Problems*, Birkhauser 1986, p. 203 n. 94.