Reference station weighting and frame optimality in minimally constrained networks

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Introduction

- **Reference frame densification through network adjustment:** a key geodetic task
- Different solution schemes exist for solving this problem
- This paper is concerned with the use of minimal constraints (MCs) over a number of reference stations

Study’s rationale

Usual form of MCs $\mathbf{E}(\mathbf{x}-\mathbf{x}^{\text{ext}}) = \mathbf{0}$

Generalized form of MCs $\mathbf{E}\mathbf{P}_{x}(\mathbf{x}-\mathbf{x}^{\text{ext}}) = \mathbf{0}$

Un-resolved issue: choice of the weight matrix

Example

Usual form of NNT/NNR conditions

$$\sum_i (\mathbf{x}_i - \mathbf{x}_i^{\mathrm{ext}}) = \mathbf{0}$$

$$\sum_i \mathbf{x}_i^{\mathrm{o}} \times (\mathbf{x}_i - \mathbf{x}_i^{\mathrm{ext}}) = \mathbf{0}$$

Weighted form of NNT/NNR conditions

$$\sum_i p_i\,(\mathbf{x}_i - \mathbf{x}_i^{\mathrm{ext}}) = \mathbf{0}$$

$$\sum_i \mathbf{x}_i^{\mathrm{o}} \times p_i\,(\mathbf{x}_i - \mathbf{x}_i^{\mathrm{ext}}) = \mathbf{0}$$

This is a simplified scheme assuming a diagonal weight matrix

(*) p_i=0 or p_i=1 is often considered as a ‘weighting’ choice

Example

Usual form of NNT/NNR conditions

$$\sum_i (\mathbf{x}_i - \mathbf{x}_i^{\text{ext}}) = \mathbf{0}$$

$$\sum_i \mathbf{x}_i^{\text{o}} \times (\mathbf{x}_i - \mathbf{x}_i^{\text{ext}}) = \mathbf{0}$$

Weighted form of NNT/NNR conditions

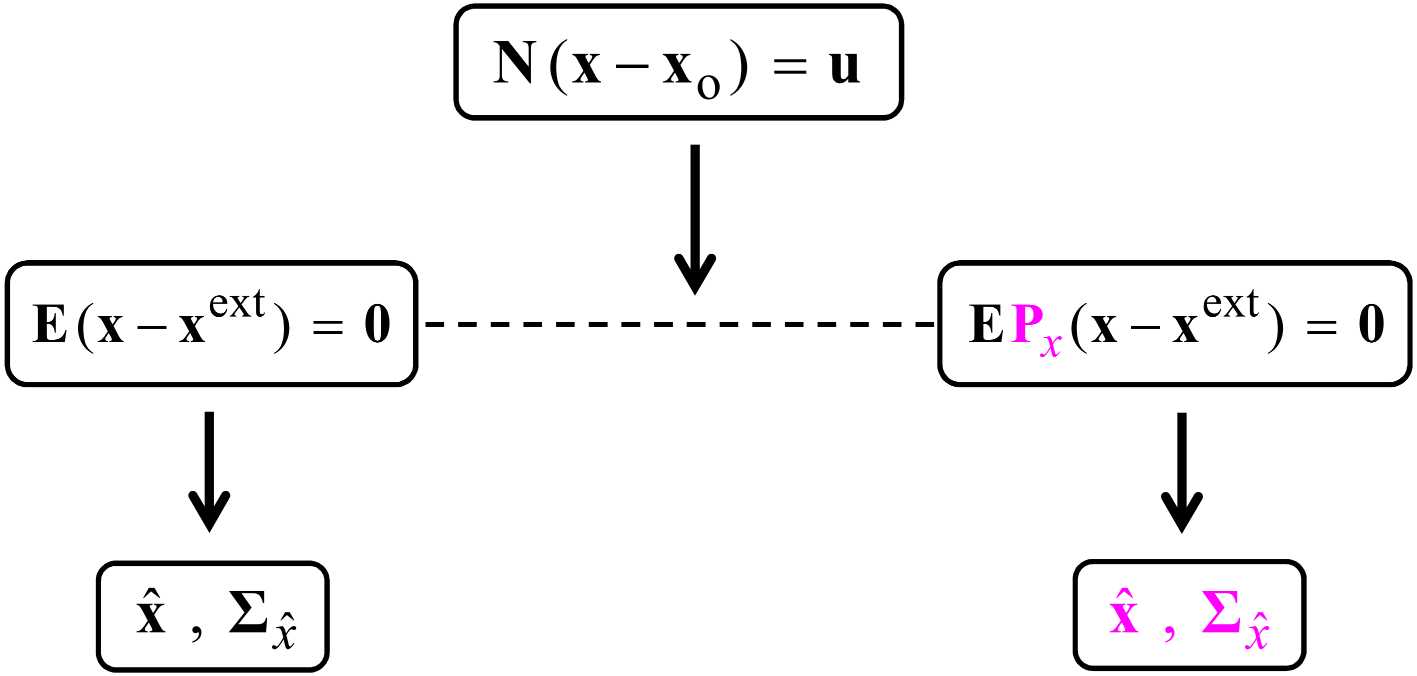
$$\sum_i \mathbf{P}_i \, (\mathbf{x}_i - \mathbf{x}_i^{\text{ext}}) = \mathbf{0}$$

$$\sum_i \mathbf{x}_i^{\text{o}} \times \left(\mathbf{P}_i \, (\mathbf{x}_i - \mathbf{x}_i^{\text{ext}})\right) = \mathbf{0}$$

This is a simplified scheme assuming a block-diagonal weight matrix

(*) $\mathbf{P}_i = (\text{diag}\boldsymbol{\Sigma}_{i,ext})^{-1}$ or $\mathbf{P}_i = (\boldsymbol{\Sigma}_{i,ext})^{-1}$ has been suggested but never rigorously justified as a weighting choice

“Classic” vs. “generalized” MCs



$$\mathbf{N}(\mathbf{x}-\mathbf{x}_o) = \mathbf{u}$$

$$\mathbf{E}(\mathbf{x}-\mathbf{x}^{ext}) = \mathbf{0}$$

$$\mathbf{E}\mathbf{P}_x(\mathbf{x}-\mathbf{x}^{ext}) = \mathbf{0}$$

$$\hat{\mathbf{x}}\ ,\ \mathbf{\Sigma}_{\hat{x}}$$

$$\hat{\mathbf{x}}\ ,\ \mathbf{\Sigma}_{\hat{x}}$$

Can we get an improved MC solution through a justified choice of the weight matrix?

Crucial restrictions in MC-based frame optimality

(*under current un-weighted scheme*)

(1) The realized frame through a network adjustment is optimized only at the stations **participating** in the MCs

(what about the other network stations?)

(2) The optimality of the realized frame considers only the **data noise effect** in the estimated coordinates

(what about the “datum noise” effect?)

What do “classic” MCs optimize?

Rank-deficient NEQs

$$\boxed{\mathbf{N}\,(\mathbf{x}-\mathbf{x}_{\mathrm{o}})=\mathbf{u}}$$

$$\mathbf{x}=\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}$$

Reference stations

MCs applied to the reference stations: $\boxed{\mathbf{E}_1(\mathbf{x}_1-\mathbf{x}_1^{\mathrm{ext}})=\mathbf{0}}$

Optimized covariance matrix of MC solution

$$\mathbf{\Sigma}_{\hat{x}} = \mathbf{N}^{-} = \left[\begin{array}{c|c}\mathbf{\Sigma}_{\hat{x}_1} & \mathbf{\Sigma}_{\hat{x}_1,\hat{x}_2}\\ \hline \mathbf{\Sigma}_{\hat{x}_2,\hat{x}_1} & \mathbf{\Sigma}_{\hat{x}_2}\end{array}\right]$$

Minimum trace

↑

Data noise effect

Consider the generalizations: (1/2)

Rank-deficient NEQs

$$\boxed{\mathbf{N}(\mathbf{x}-\mathbf{x}_o)=\mathbf{u}} \qquad \mathbf{x}=\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}$$

Reference stations

MCs applied to the reference stations: ? $\boxed{\mathbf{E}_1(\mathbf{x}_1-\mathbf{x}_1^{\mathrm{ext}})=\mathbf{0}}$

Optimized covariance matrix of MC solution

$$\boldsymbol{\Sigma}_{\hat{x}} = \mathbf{N}^{-} = \left[\begin{array}{c|c}\boldsymbol{\Sigma}_{\hat{x}_1} & \boldsymbol{\Sigma}_{\hat{x}_1,\hat{x}_2}\\ \hline \boldsymbol{\Sigma}_{\hat{x}_2,\hat{x}_1} & \boldsymbol{\Sigma}_{\hat{x}_2}\end{array}\right]$$

$\uparrow$

Data noise effect

Minimum trace

or

Minimum trace

Consider the generalizations: (2/2)

Rank-deficient NEQs

$$\boxed{\mathbf{N}\,(\mathbf{x}-\mathbf{x}_{\mathrm{o}})=\mathbf{u}}$$

$$\mathbf{x}=\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{bmatrix}$$

Reference stations

MCs applied to the reference stations: $\boxed{?\,\mathbf{E}_1(\mathbf{x}_1-\mathbf{x}_1^{\mathrm{ext}})=\mathbf{0}}$

Optimized covariance matrix of MC solution

$$\boldsymbol{\Sigma}_{\hat{x}} = \mathbf{N}^{-} + \tilde{\boldsymbol{\Sigma}}_{\hat{x}} = \left[\begin{array}{c|c}\boldsymbol{\Sigma}_{\hat{x}_1} & \boldsymbol{\Sigma}_{\hat{x}_1,\hat{x}_2}\\ \hline \boldsymbol{\Sigma}_{\hat{x}_2,\hat{x}_1} & \boldsymbol{\Sigma}_{\hat{x}_2}\end{array}\right]$$

Data noise effect

Datum noise effect

Minimum trace

or

Minimum trace

or

Minimum trace

Formulation of the problem

Rank-deficient NEQs

$$\boxed{\mathbf{N}\,(\mathbf{x}-\mathbf{x}_{\mathrm{o}}) \;=\; \mathbf{u}} \qquad \mathbf{x} \;=\; \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

Reference stations

Prior information for reference stations: $\boxed{\mathbf{x}_1^{\mathrm{ext}},\ \boldsymbol{\Sigma}_{x_1}^{\mathrm{ext}}}$

Datum definition by an arbitrary set of MCs

$$\boxed{\mathbf{H}_1(\mathbf{x}_1-\mathbf{x}_1^{\mathrm{ext}}) \;=\; \mathbf{0}}$$

Optimization problem to be solved $\boxed{\min_{\mathbf{H}_1} trace\left(\mathbf{S}\boldsymbol{\Sigma}_{\hat{x}}\,\mathbf{S}^{\mathrm{T}}\right)}$

$\boldsymbol{\Sigma}_{\hat{x}} \longrightarrow$ CV matrix of MC solution

CV matrix of MC solution

Irrelevant weight matrix $\mathbf{W}$

$$\boxed{\begin{aligned} \mathbf{N}(\mathbf{x}-\mathbf{x}_{\mathrm{o}}) &= \mathbf{u} \\ \mathbf{H}(\mathbf{x}-\mathbf{x}_{\mathrm{o}}) &= \mathbf{c} \end{aligned}} \longrightarrow \boxed{\hat{\mathbf{x}} = \mathbf{x}_{\mathrm{o}} + \left(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{H}\right)^{-1}\left(\mathbf{u}+\mathbf{H}^{\mathrm{T}}\mathbf{c}\right)}$$

General form

$$\boxed{\mathbf{\Sigma}_{\hat{x}} = \mathbf{\Sigma}_{\hat{x}}^{\mathrm{obs}} + \mathbf{\Sigma}_{\hat{x}}^{\mathrm{mc}}}$$

Data noise effect

(due to errors within **u**)

Datum noise effect

(due to errors within **c**)

CV matrix of MC solution

$$\begin{aligned} \mathbf{N}\,(\mathbf{x}-\mathbf{x}_o) &= \mathbf{u} \\ \mathbf{H}\,(\mathbf{x}-\mathbf{x}_o) &= \mathbf{c} \quad ^{(*)} \end{aligned} \quad \longrightarrow$$

Irrelevant weight matrix $\mathbf{W}$

$$\hat{\mathbf{x}} = \mathbf{x}_o + \left(\mathbf{N}+\mathbf{H}^T\mathbf{H}\right)^{-1}\left(\mathbf{u}+\mathbf{H}^T\mathbf{c}\right)$$

(*) in this study we have that:

$$\mathbf{H} = [\mathbf{H}_1 \;\; \mathbf{0}]$$

$$\mathbf{c} = \mathbf{H}_1\,(\mathbf{x}_1^{ext} - \mathbf{x}_1^{o})$$

General form

$$\Sigma_{\hat{x}} = \Sigma_{\hat{x}}^{obs} + \Sigma_{\hat{x}}^{mc}$$

Data noise effect

(due to errors within **u**)

Datum noise effect

(due to errors within **c**)

Basic equations

$$\begin{aligned} \mathbf{N}(\mathbf{x}-\mathbf{x}_o) &= \mathbf{u} \\ \mathbf{H}(\mathbf{x}-\mathbf{x}_o) &= \mathbf{c} \end{aligned} \quad \rightarrow \quad \hat{\mathbf{x}} = \mathbf{x}_o + (\mathbf{N}+\mathbf{H}^T\mathbf{H})^{-1}(\mathbf{u}+\mathbf{H}^T\mathbf{c})$$

error CV propagation &
well-known algebraic properties
from MC theory

General form

$$\boldsymbol{\Sigma}_{\hat{x}} = \boxed{\boldsymbol{\Sigma}_{\hat{x}}^{\text{obs}}} + \boxed{\boldsymbol{\Sigma}_{\hat{x}}^{\text{mc}}}$$

$$\boldsymbol{\Sigma}_{\hat{x}}^{\text{obs}} = (\mathbf{N}+\mathbf{H}^T\mathbf{H})^{-1} - \mathbf{E}^T(\mathbf{H}\mathbf{E}^T)^{-1}(\mathbf{E}\mathbf{H}^T)^{-1}\mathbf{E}$$

$$\boldsymbol{\Sigma}_{\hat{x}}^{\text{mc}} = \mathbf{E}^T\underbrace{(\mathbf{H}\mathbf{E}^T)^{-1}\boldsymbol{\Sigma}_c(\mathbf{E}\mathbf{H}^T)^{-1}}_{\boxed{\boldsymbol{\Sigma}_\theta}}\mathbf{E}$$

(*) $\boldsymbol{\Sigma}_c = \mathbf{H}_1\boldsymbol{\Sigma}_{x_1}^{\text{ext}}\mathbf{H}_1^T$

Basic equations

$$\boxed{\begin{aligned} \mathbf{N}\,(\mathbf{x}-\mathbf{x}_o) &= \mathbf{u} \\ \mathbf{H}\,(\mathbf{x}-\mathbf{x}_o) &= \mathbf{c} \end{aligned}} \;\longrightarrow\; \boxed{\hat{\mathbf{x}} = \mathbf{x}_o + (\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1}(\mathbf{u}+\mathbf{H}^{\mathrm{T}}\mathbf{c})}$$

error CV propagation &
well-known algebraic properties
from MC theory

General form

$$\boxed{\mathbf{\Sigma}_{\hat{x}} = \mathbf{\Sigma}_{\hat{x}}^{\mathrm{obs}} + \mathbf{\Sigma}_{\hat{x}}^{\mathrm{mc}}}$$

$$\mathbf{\Sigma}_{\hat{x}}^{\mathrm{obs}} = (\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{\Sigma}_c^{-1}\mathbf{H})^{-1} - \mathbf{E}^{\mathrm{T}}(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{\Sigma}_c(\mathbf{E}\mathbf{H}^{\mathrm{T}})^{-1}\mathbf{E}$$

$$\mathbf{\Sigma}_{\hat{x}}^{\mathrm{mc}} = \mathbf{E}^{\mathrm{T}}\underbrace{(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{\Sigma}_c(\mathbf{E}\mathbf{H}^{\mathrm{T}})^{-1}}_{\mathbf{\Sigma}_\theta}\mathbf{E}$$

(*) $\mathbf{\Sigma}_c = \mathbf{H}_1\,\mathbf{\Sigma}_{x_1}^{\mathrm{ext}}\mathbf{H}_1^{\mathrm{T}}$

Problem solution

Frame/network optimality principle

$$\boxed{\min_{\mathbf{H}_1} \; trace\left(\boldsymbol{\Sigma}_{\hat{x}}^{\text{obs}} + \lambda\,\boldsymbol{\Sigma}_{\hat{x}}^{\text{mc}}\right)}$$

Optimal MC matrix (applied to the reference stations)

$$\boxed{\mathbf{H}_1 = \mathbf{E}_1\left(\mathbf{Q}_{11} + \lambda\,\boldsymbol{\Sigma}_{x_1}^{\text{ext}}\right)^{-1}}$$

$$(\mathbf{N}+\mathbf{E}^{\mathrm{T}}\mathbf{E})^{-1} = \begin{array}{c c} \begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array} & \\ \left[\begin{array}{c|c} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \hline \mathbf{Q}_{12}^{\mathrm{T}} & \mathbf{Q}_{22} \end{array}\right] & \begin{array}{c} \mathbf{x}_1 \\ \mathbf{x}_2 \end{array} \end{array}$$

see Kotsakis (2013, JGeod) for proofs

Problem solution

Frame/network optimality principle

$$\boxed{\min_{\mathbf{H}_1} \; trace\left(\mathbf{S}\boldsymbol{\Sigma}_{\hat{x}}^{\text{obs}}\mathbf{S}^{\text{T}} + \lambda\,\mathbf{S}\boldsymbol{\Sigma}_{\hat{x}}^{\text{mc}}\mathbf{S}^{\text{T}}\right)}$$

S is a ‘station selection matrix’

Optimal MC matrix (applied to the reference stations)

$$\boxed{\mathbf{H}_1 = \mathbf{E}_1\left(\mathbf{Q}_{11} + \lambda\,\boldsymbol{\Sigma}_{x_1}^{\text{ext}}\right)^{-1}}$$

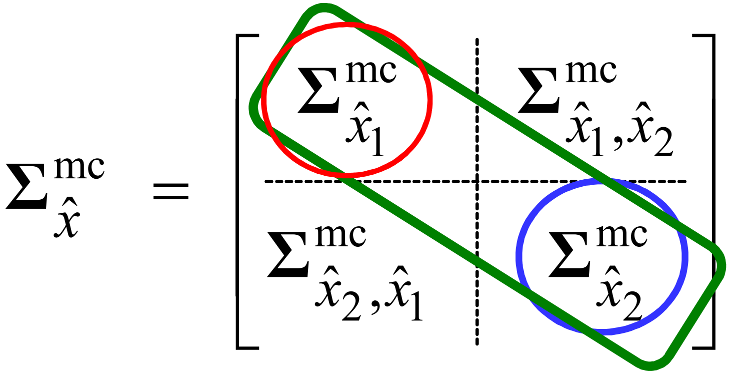
$$(\mathbf{N}+\mathbf{S}^{\text{T}}\mathbf{S}\mathbf{E}^{\text{T}}\mathbf{E}\mathbf{S}^{\text{T}}\mathbf{S})^{-1} = \begin{array}{c} \begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array} \\ \left[\begin{array}{c:c} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \hdashline \mathbf{Q}_{12}^{\text{T}} & \mathbf{Q}_{22} \end{array}\right] \end{array} \begin{array}{c} \mathbf{x}_1 \\ \mathbf{x}_2 \end{array}$$

see Kotsakis (2013, JGeod) for proofs

Minimization of the **datum noise** effect

Frame/network optimality principle	Objective	Optimal MC matrix
$\min_{\mathbf{H}_1} trace\left(\boldsymbol{\Sigma}_{\hat{x}}^{\mathrm{mc}}\right)$	over all network stations	$\mathbf{H}_1 = \mathbf{E}_1\left(\boldsymbol{\Sigma}_{x_1}^{\mathrm{ext}}\right)^{-1}$
$\min_{\mathbf{H}_1} trace\left(\mathbf{S}\boldsymbol{\Sigma}_{\hat{x}}^{\mathrm{mc}}\mathbf{S}^{\mathrm{T}}\right)$	over a subset of network stations	$\mathbf{H}_1 = \mathbf{E}_1\left(\boldsymbol{\Sigma}_{x_1}^{\mathrm{ext}}\right)^{-1}$ Independent of **S** !

Minimization of the **datum noise** effect



$$\boldsymbol{\Sigma}_{\hat{x}}^{\mathrm{mc}} = \left[\begin{array}{c:c} \boldsymbol{\Sigma}_{\hat{x}_1}^{\mathrm{mc}} & \boldsymbol{\Sigma}_{\hat{x}_1,\hat{x}_2}^{\mathrm{mc}} \\ \hdashline \boldsymbol{\Sigma}_{\hat{x}_2,\hat{x}_1}^{\mathrm{mc}} & \boldsymbol{\Sigma}_{\hat{x}_2}^{\mathrm{mc}} \end{array}\right]$$

Minimum trace

or

Minimum trace

or

Minimum trace

Equivalent to the **optimal 'frame alignment' principle**

$$\boldsymbol{\Sigma}_{\hat{x}}^{\mathrm{mc}} = \mathbf{E}^{\mathrm{T}} \boldsymbol{\Sigma}_{\theta} \mathbf{E} \quad \longrightarrow \quad \boxed{\min_{\mathbf{H}_1} \mathit{trace}\left(\boldsymbol{\Sigma}_{\theta}\right)}$$

where $\boldsymbol{\Sigma}_{\theta}$ expresses the accuracy of the (non-estimable) frame parameters from the adopted MCs and reference stations

$$\boldsymbol{\Sigma}_{\theta} = (\mathbf{H}_1 \mathbf{E}_1^{\mathrm{T}})^{-1} \mathbf{H}_1 \boldsymbol{\Sigma}_{x_1}^{\mathrm{ext}} \mathbf{H}_1^{\mathrm{T}} (\mathbf{E}_1 \mathbf{H}_1^{\mathrm{T}})^{-1}$$

Minimization of the **data noise** effect

Frame/network optimality principle	Objective	Optimal MC matrix
$\min_{\mathbf{H}_1} trace\left(\boldsymbol{\Sigma}_{\hat{x}}^{\mathrm{obs}}\right)$	over all network stations	$(\mathbf{N}+\mathbf{E}^{\mathrm{T}}\mathbf{E})^{-1} = \left[\begin{array}{c\|c} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \hline \mathbf{Q}_{12}^{\mathrm{T}} & \mathbf{Q}_{22} \end{array}\right]$ $\mathbf{H}_1 = \mathbf{E}_1\left(\mathbf{Q}_{11}\right)^{-1}$
$\min_{\mathbf{H}_1} trace\left(\mathbf{S}\boldsymbol{\Sigma}_{\hat{x}}^{\mathrm{obs}}\mathbf{S}^{\mathrm{T}}\right)$	over a subset of network stations	$(\mathbf{N}+\mathbf{S}^{\mathrm{T}}\mathbf{S}\mathbf{E}^{\mathrm{T}}\mathbf{E}\mathbf{S}^{\mathrm{T}}\mathbf{S})^{-1} = \left[\begin{array}{c\|c} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \hline \mathbf{Q}_{12}^{\mathrm{T}} & \mathbf{Q}_{22} \end{array}\right]$

Minimization of the **data noise** effect

Frame/network optimality principle	Optimal MC matrix	Objective
$\min_{\mathbf{H}_1} \mathit{trace}\left(\mathbf{S}\boldsymbol{\Sigma}_{\hat{x}}^{\text{obs}}\mathbf{S}^{\text{T}}\right)$ $\boxed{\mathbf{H}_1 = \mathbf{E}_1(\mathbf{Q}_{11})^{-1}}$ $(\mathbf{N}+\mathbf{S}^{\text{T}}\mathbf{S}\mathbf{E}^{\text{T}}\mathbf{E}\mathbf{S}^{\text{T}}\mathbf{S})^{-1} = \begin{array}{c} \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} \mathbf{x}_1 & \mathbf{x}_2 \end{array} \\ \left[\begin{array}{c:c} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \hdashline \mathbf{Q}_{12}^{\text{T}} & \mathbf{Q}_{22} \end{array}\right] \end{array}$	$\boxed{\mathbf{H}_1 = \mathbf{E}_1}$	over the reference stations $\mathbf{S} = [\,\mathbf{I} \quad \mathbf{0}\,]$
	$\boxed{\mathbf{H}_1 = \mathbf{E}_1\mathbf{N}_{12}\mathbf{N}_{22}^{-2}\mathbf{N}_{12}^{\text{T}}}$	over the ‘new’ stations $\mathbf{S} = [\,\mathbf{0} \quad \mathbf{I}\,]$

Conclusions

- Different MC weighting schemes lead to different types of frame optimality

- Reference station weighting can be used to optimize the accuracy of a MC solution in terms of
  - the data and/or datum noise effects
  - the network stations over which these effects are considered

- Numerical testing is still required to assess the differences from MC weighting schemes in geodetic network adjustment problems

Thanks for your attention !