Helmert transformation and the intra-frame covariance effect in frame alignment problems

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Introduction

Unconstr. NEQs → **?** → **Estimated positions in a desired frame**

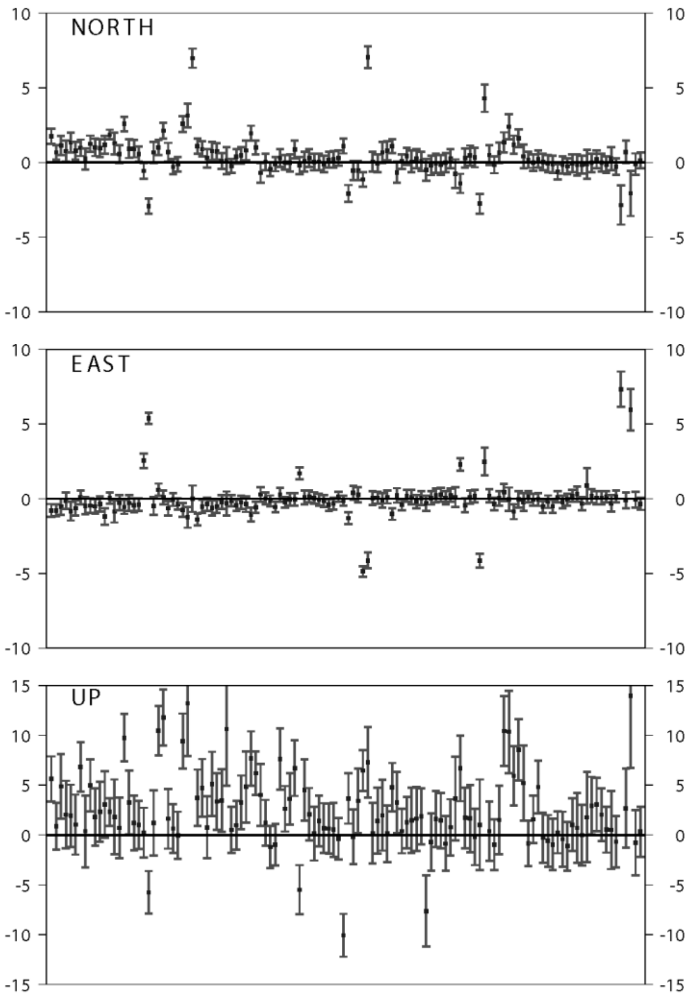
Methods for frame alignment in geodetic networks:

- Constraining known reference stations
- Minimum constraints (MC) approach
- Helmert transformation (HT) approach

The HT approach suffers by the so-called network effect which often causes apparent biases in the transformed coordinates to the desired frame.

Example

Altamimi Z (2003) Discussion on how to express a regional GPS solution in the ITRF. EUREF Publ No. 12, pp. 162-167.



Coordinate differences (mm) between the (unconstrained & transformed) and the originally constrained EUREF solution

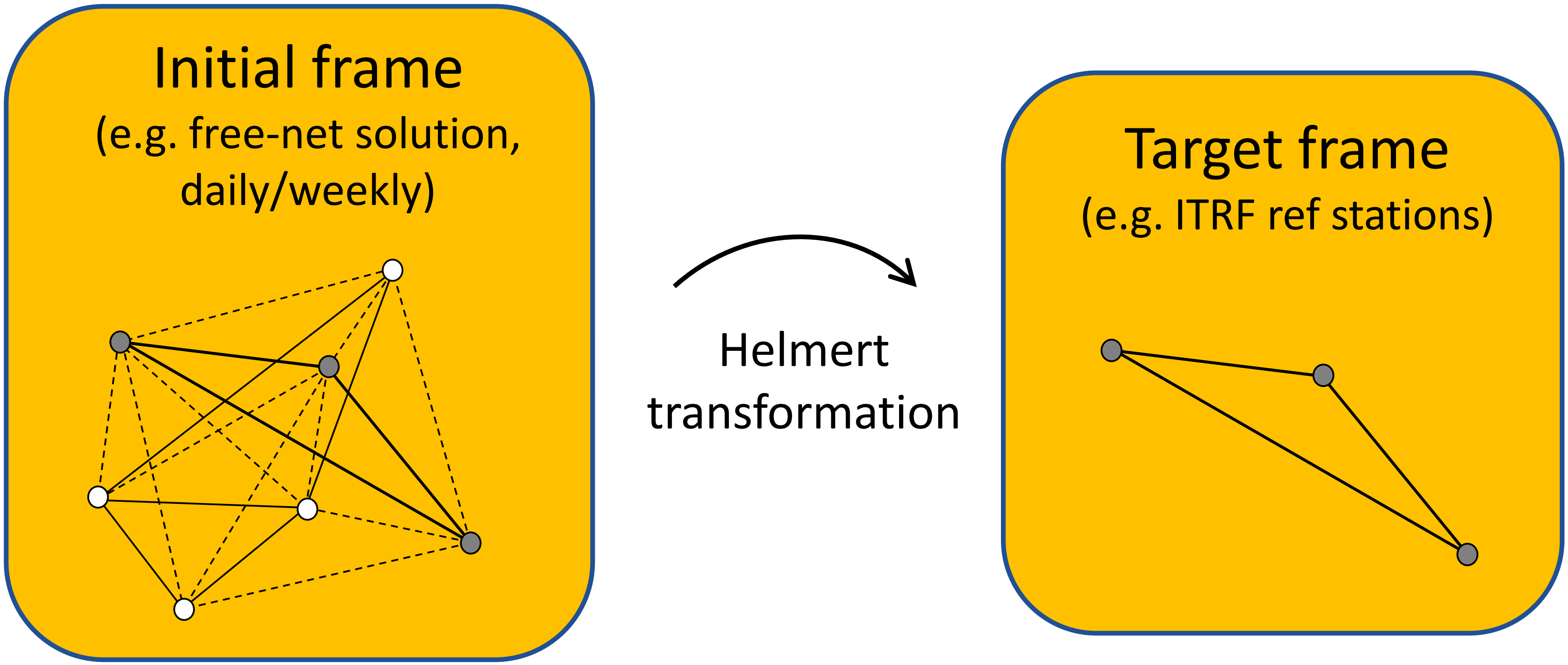
Study objectives

- Retrace the capability of the HT approach for the frame alignment in regional geodetic networks

- Expose the fact that its apparent deficiency originates by its sub-optimal implementation (due to mishandling of the data stochastic model)

- Verify its equivalence with the constrained network adjustment directly to the desired frame

General data setting



$$\begin{bmatrix} \mathbf{X}' \\ \mathbf{Z}' \end{bmatrix}, \quad \begin{bmatrix} \Sigma_{\mathbf{X}'} & \Sigma_{\mathbf{X}'\mathbf{Z}'} \\ \Sigma_{\mathbf{Z}'\mathbf{X}'} & \Sigma_{\mathbf{Z}'} \end{bmatrix} \qquad \mathbf{X}\ ,\ \Sigma_{\mathbf{X}}$$

Intra-frame covariances: neglected in HT-based frame alignment!

Standard stepwise approach

1. LS estimation of Helmert transformation parameters

$$\hat{\boldsymbol{\theta}}^{\mathrm{st}} = \left(\mathbf{G}^T (\boldsymbol{\Sigma}_{\mathbf{X}} + \boldsymbol{\Sigma}_{\mathbf{X}'})^{-1} \mathbf{G}\right)^{-1} \mathbf{G}^T (\boldsymbol{\Sigma}_{\mathbf{X}} + \boldsymbol{\Sigma}_{\mathbf{X}'})^{-1} (\mathbf{X} - \mathbf{X}')$$

2. Forward computation of transformed coordinates

$$\hat{\mathbf{x}}^{\mathrm{st}} = \mathbf{X}' + \mathbf{G}\,\hat{\boldsymbol{\theta}}^{\mathrm{st}}$$ *common stations*

$$\hat{\mathbf{z}}^{\mathrm{st}} = \mathbf{Z}' + \tilde{\mathbf{G}}\,\hat{\boldsymbol{\theta}}^{\mathrm{st}}$$ *non-common stations*

This is not an optimal solution from the available data.

(Kotsakis et al., JGeod, 2014)

Optimal one-step approach

Observations equations

$$\mathbf{X} = \mathbf{x} + \mathbf{v}_{\mathbf{X}}$$

Common stations (target frame)

$$\mathbf{X}' = \mathbf{x} - \mathbf{G}\boldsymbol{\theta} + \mathbf{v}_{\mathbf{X}'}$$

$$\mathbf{Z}' = \mathbf{z} - \tilde{\mathbf{G}}\boldsymbol{\theta} + \mathbf{v}_{\mathbf{Z}'}$$

Common & non-common stations (initial frame)

Weight matrix for LS inversion

$$\mathbf{P} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{X}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{\mathbf{X}'} & \boldsymbol{\Sigma}_{\mathbf{X}'\mathbf{Z}'} \\ \mathbf{0} & \boldsymbol{\Sigma}_{\mathbf{Z}'\mathbf{X}'} & \boldsymbol{\Sigma}_{\mathbf{Z}'} \end{bmatrix}^{-1}$$

Intra-frame covariances considered

Optimal one-step approach

Final solution for transformed coordinates

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}}^{\mathrm{st}} \\ \hat{\mathbf{z}}^{\mathrm{st}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{X}'} \\ \boldsymbol{\Sigma}_{\mathbf{Z}'\mathbf{X}'} \end{bmatrix} (\boldsymbol{\Sigma}_{\mathbf{X}} + \boldsymbol{\Sigma}_{\mathbf{X}'})^{-1} (\mathbf{X} - \hat{\mathbf{x}}^{\mathrm{st}})$$

- BLUE (minimum error variance) estimates
- Kalman-like updating formula,
  LSC-type coordinate corrections
- Note that $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}^{\mathrm{st}}$

Frame consistency

Nullification of Helmert transformation parameters:

Standard (stepwise) approach

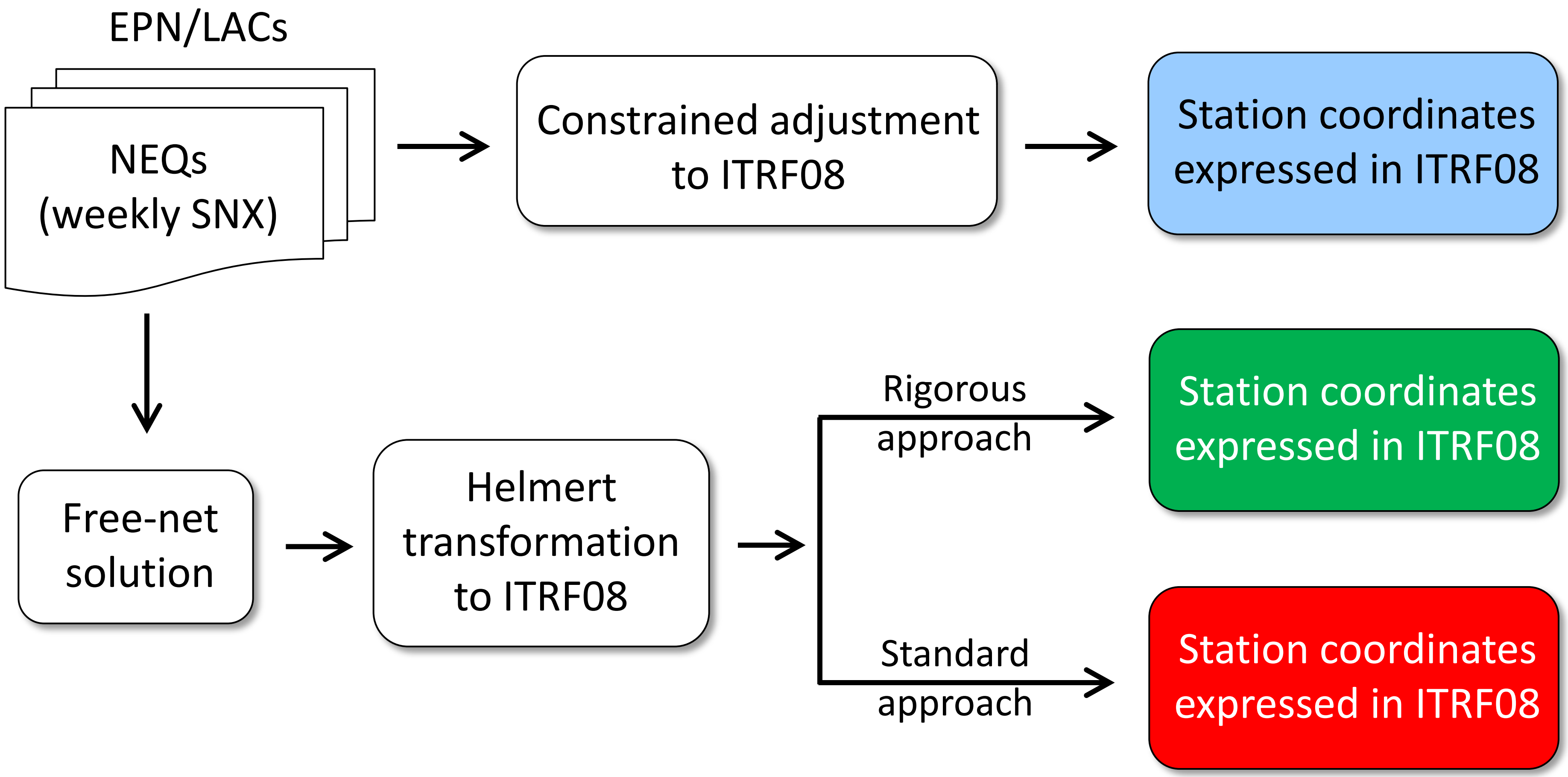
$$\hat{\boldsymbol{\theta}}^{\text{post}} = \left(\mathbf{G}^T(\boldsymbol{\Sigma}_{\mathbf{X}} + \boldsymbol{\Sigma}_{\mathbf{X}'})^{-1}\mathbf{G}\right)^{-1}\mathbf{G}^T(\boldsymbol{\Sigma}_{\mathbf{X}} + \boldsymbol{\Sigma}_{\mathbf{X}'})^{-1}(\mathbf{X} - \hat{\mathbf{x}}^{\text{st}}) = \mathbf{0}$$

Optimal (one-step) approach

$$\hat{\boldsymbol{\theta}}^{\text{post}} = \left(\mathbf{G}^T\boldsymbol{\Sigma}_{\mathbf{X}}^{-1}\mathbf{G}\right)^{-1}\mathbf{G}^T\boldsymbol{\Sigma}_{\mathbf{X}}^{-1}(\mathbf{X} - \hat{\mathbf{x}}) = \mathbf{0}$$

Also: $E\{\hat{\mathbf{x}}\} = E\{\hat{\mathbf{x}}^{\text{st}}\} = E\{\mathbf{X}\} = \mathbf{x}$ $\quad E\{\hat{\mathbf{z}}\} = E\{\hat{\mathbf{z}}^{\text{st}}\} = \mathbf{z}$

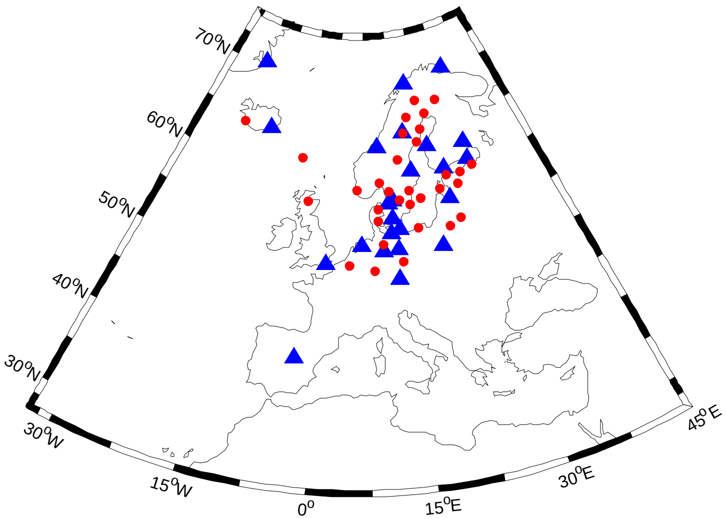
Numerical tests



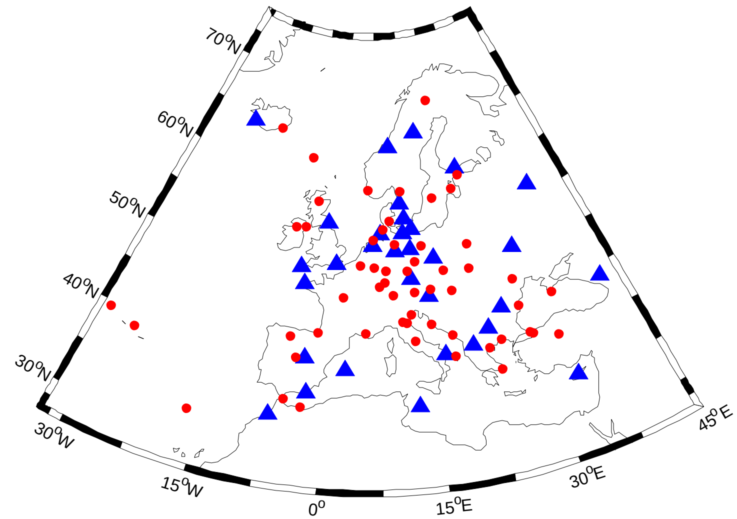
(*) The same reference stations are employed in the constrained network adjustment and the Helmert transformation approach

EPN/LAC subnets

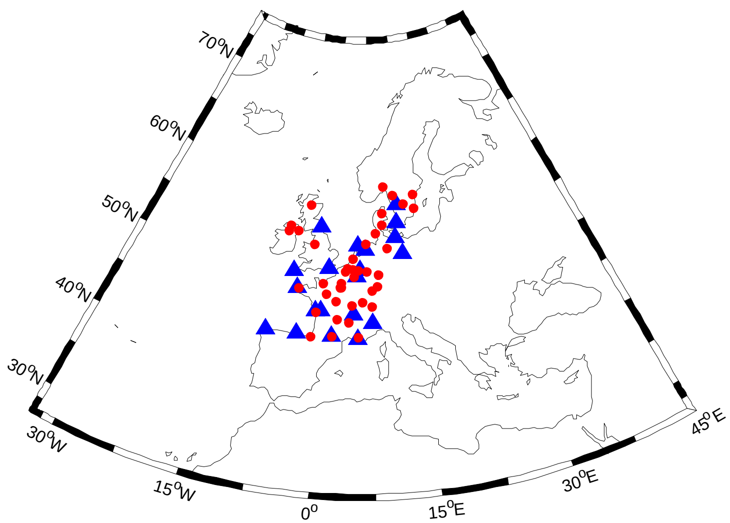
NKG



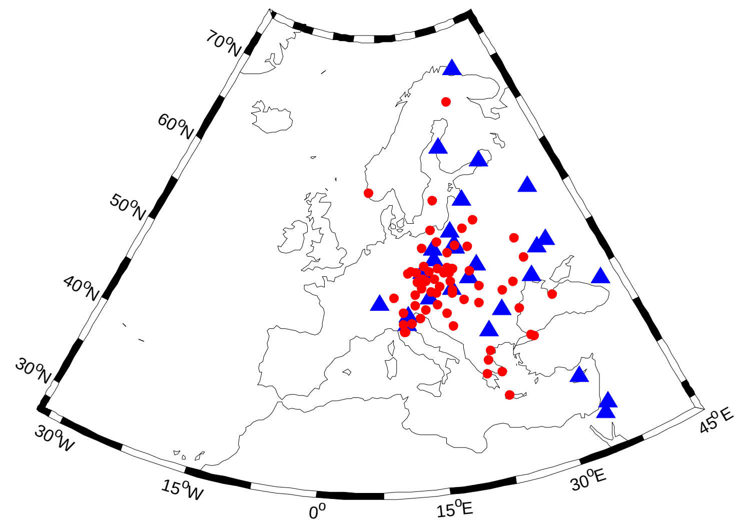
BKG



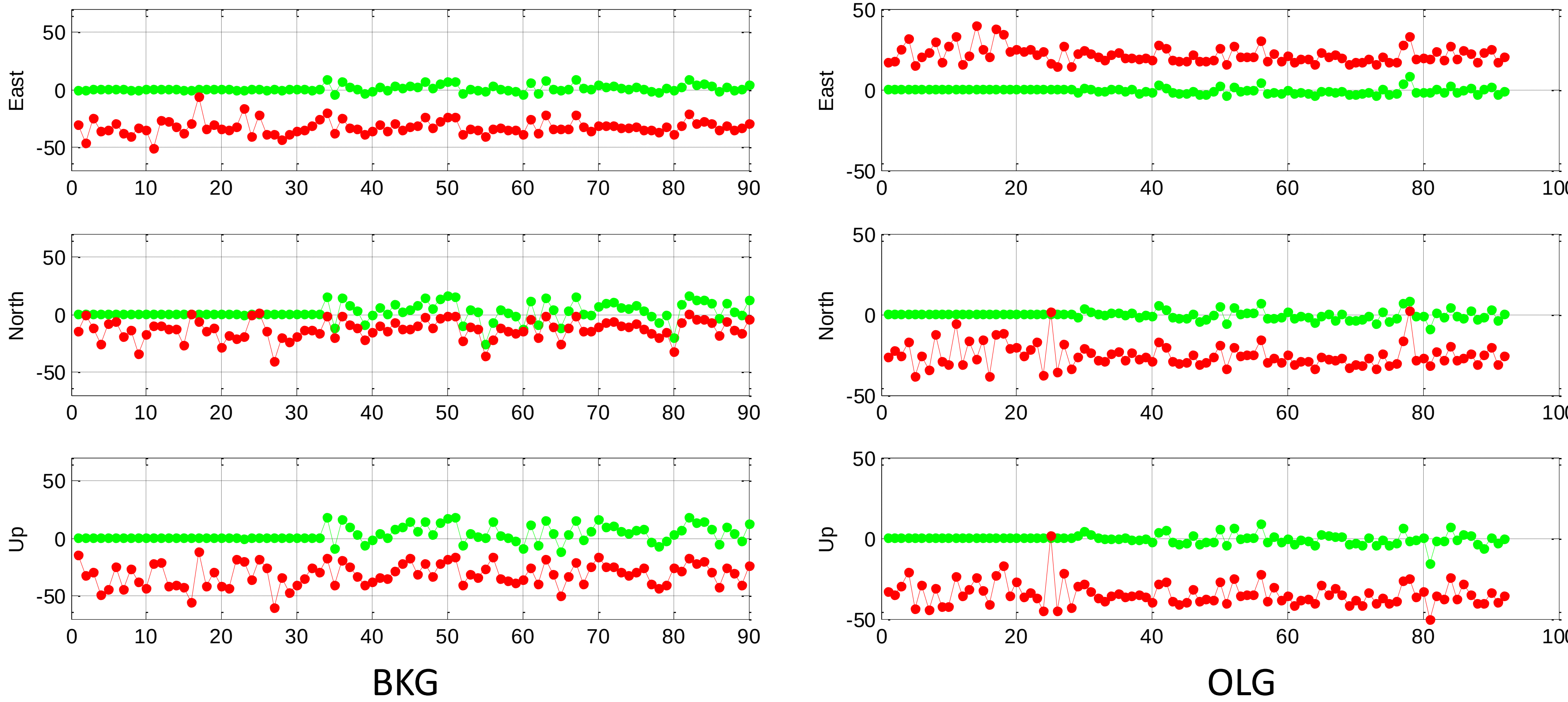
ROB



OLG

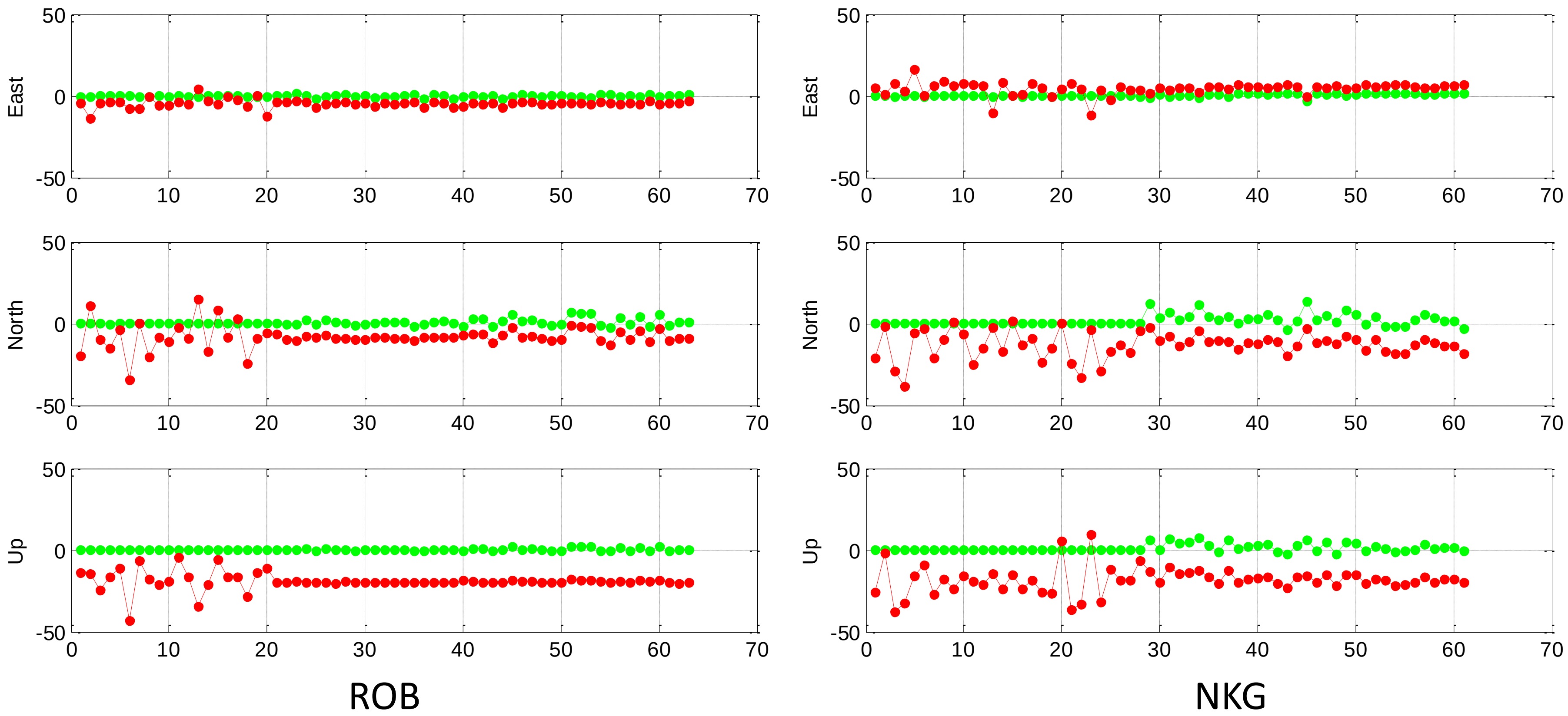


Differences (mm) between the constrained weekly solution and Helmert-transformed solutions (ITRF08, GPS week 1805)



● Optimal (one-step) approach ● Standard (stepwise) approach

Differences (mm) between the constrained weekly solution and Helmert-transformed solutions (ITRF08, GPS week 1805)



● Optimal (one-step) approach ● Standard (stepwise) approach

RMS of coordinate differences btw constrained weekly solutions and Helmert-transformed solutions (ITRF08)

LAC subnet: BKG

	Standard HT approach			**Optimal HT approach**		
	N	E	U	N	E	U
GPS week 1805	15.4	33.0	32.7	7.9	3.0	7.7
GPS week 1806	13.8	32.3	30.8	8.0	2.9	7.8
GPS week 1807	13.2	30.3	26.0	7.5	2.6	7.3
GPS week 1808	15.0	28.0	22.2	8.0	2.8	7.4
GPS week 1809	16.6	27.1	22.5	7.4	2.6	7.0

all values in mm

RMS of coordinate differences btw constrained weekly solutions and Helmert-transformed solutions (ITRF08)

LAC subnet: <u>OLG</u>

	Standard HT approach			**Optimal HT approach**		
	N	E	U	N	E	U
GPS week 1805	26.4	22.2	35.2	2.7	1.9	3.0
GPS week 1806	27.4	27.4	37.4	2.8	2.0	3.6
GPS week 1807	37.2	24.9	45.0	2.6	2.1	3.5
GPS week 1808	30.6	26.1	36.7	2.3	1.7	3.1
GPS week 1809	30.4	22.9	38.0	2.6	1.8	3.3

all values in mm

RMS of coordinate differences btw constrained weekly solutions and Helmert-transformed solutions (ITRF08)

LAC subnet: ROB

	Standard HT approach			Optimal HT approach		
	N	E	U	N	E	U
GPS week 1805	10.6	5.2	19.5	2.2	0.6	0.8
GPS week 1806	6.7	2.4	16.7	1.7	0.7	0.8
GPS week 1807	10.2	2.7	18.7	1.7	0.7	0.6
GPS week 1808	7.3	2.6	16.5	1.8	0.8	0.8
GPS week 1809	8.1	3.1	19.7	1.8	0.9	0.8

all values in mm

RMS of coordinate differences btw constrained weekly solutions and Helmert-transformed solutions (ITRF08)

LAC subnet: NKG

	Standard HT approach			Optimal HT approach		
	N	E	U	N	E	U
GPS week 1805	15.1	6.1	19.9	3.9	1.1	2.6
GPS week 1806	20.4	7.3	16.4	3.6	0.9	2.4
GPS week 1807	35.2	7.9	20.9	3.6	0.6	2.2
GPS week 1808	42.0	7.8	21.8	3.8	0.5	2.2
GPS week 1809	41.2	9.7	20.8	3.9	0.5	2.3

all values in mm

Why is the rigorous approach better than the standard approach?

- Smaller error variances for the Helmert transformed coordinates.

- Reduction of apparent biases in the Helmert transformed coordinates (“network effect”).

- Similar results with the constrained network adjustment directly to the desired frame.

- Does not require any extra matrix inversion!

Is an “abridged” form better than the full Helmert form in frame transformation problems (?)

- Omit certain parameters (e.g. scale factor) from the frame transformation procedure

- This (may) improve the coordinate consistency with the directly constrained network solution in the desired frame!

Example (NKG subnet)

Using full Helmert transformation

	Standard HT approach			Optimal HT approach		
	N	E	U	N	E	U
GPS week 1807	35.2	7.9	20.9	3.6	0.6	2.2
GPS week 1808	42.0	7.8	21.8	3.8	0.5	2.2
GPS week 1809	41.2	9.7	20.8	3.9	0.5	2.3

Using shift-only transformation

	Standard HT approach			Optimal HT approach		
	N	E	U	N	E	U
GPS week 1807	45.3	33.8	14.9	1.4	0.5	1.0
GPS week 1808	32.3	33.5	10.4	1.1	0.4	0.8
GPS week 1809	31.5	39.3	12.0	1.2	0.4	0.9

all values in mm

Example (BKG subnet)

Using full Helmert transformation

	Standard HT approach			Optimal HT approach		
	N	E	U	N	E	U
GPS week 1807	13.2	30.3	26.0	7.5	2.6	7.3
GPS week 1808	15.0	28.0	22.2	8.0	2.8	7.4
GPS week 1809	16.6	27.1	22.5	7.4	2.6	7.0

Using shift-only transformation

	Standard HT approach			Optimal HT approach		
	N	E	U	N	E	U
GPS week 1807	17.0	53.1	36.5	2.2	1.7	2.4
GPS week 1808	47.3	48.5	8.0	2.1	1.6	2.3
GPS week 1809	29.9	46.3	15.5	2.0	1.5	2.3

all values in mm

Conclusions

- Demonstration of a rigorous implementation of the HT approach for reference frame realization
- Elimination of coordinate biases and improved consistency with the constrained network solution directly to the desired frame
- Future work: influence of the rigorous HT approach on the generation of coordinate time series