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The role of a conventional transformation model for vertical reference frames

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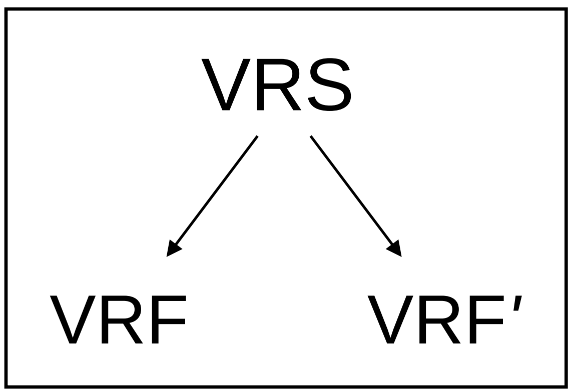
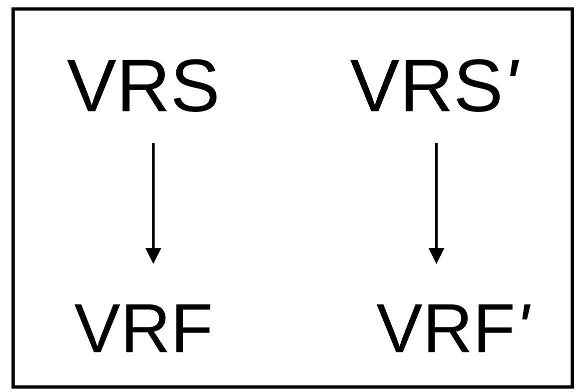
Introduction

The "problem"…





How much two different realizations of a vertical reference system differ from each other ?

Rationale

Conventional comparison of **3D spatial TRFs**
(linearized similarity transformation)

$$\begin{bmatrix} x'-x \\ y'-y \\ z'-z \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} + \begin{bmatrix} \delta s & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & \delta s & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & \delta s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Conventional comparison of **1D spatial VRFs**
for physically meaningful heights

$$H' - H = f(datum\ perturbation\ parameters) \quad ?$$

Height transformation in practice

- "Corrector surfaces" for GPS-aided leveling within a local vertical datum

$$H^{GPS,N} - H^{LVD} = \mathbf{a}^T\mathbf{x} + s + v$$

- Estimation of Earth's mean equatorial radius and CoM from heterogeneous height data

$$N(h, H) - N(C_{nm}, S_{nm}, \Delta g) = f(\delta a, \delta f, t_x, t_y, t_z)$$

- Altimeter calibration at GPS/TG stations
- Other auxiliary tasks
  - normal-to-orthometric hgt conversion (and vice versa)
  - $h^{\mathrm{TRF1}} \rightarrow h^{\mathrm{TRF2}}$ & $N^{\mathrm{TRF1}} \rightarrow N^{\mathrm{TRF2}}$
  - geodynamical corrections (permanent tide, PGR uplift, etc.)

However…

- A conventional transformation/comparison model for VRFs is **not presently in common use**
- It should employ specific parameters to quantify the (actual + apparent) inconsistencies in the realization of 1D vertical reference systems
- Why is it needed?
  - ….
  - ….
  - ….

basically, for the same reasons that the conventional 3D similarity transformation is useful in TRF studies

Datum perturbation parameters

	TRF → TRF′	VRF → VRF′
Shift	t_x, t_y, t_z	δW_o
Rotation	$\varepsilon_x, \varepsilon_y, \varepsilon_z$	–
Scale	δs	δs (*)

(*) The TRF scale change factor is not “equivalent” with the VRF scale change factor !

Anatomy of a conventional VRF transformation model

Forward effect of δW_o

	VRF	VRF′
Zero-height level	$W(\cdot) = W_o$	$W(\cdot) = W_o + \delta W_o$
Geopot. number	$c(P_i)$	$c(P_i) + \delta W_o$
Ortho height	$H(P_i)$	$H(P_i) + \frac{\delta W_o}{g_i} - \frac{1}{2}\frac{\partial g}{\partial H}\frac{\delta W_o^2}{g_i^3} + \dots$
Normal height	$\tilde{H}(P_i)$	$\tilde{H}(P_i) + \frac{\delta W_o}{\gamma_i} - \frac{1}{2}\frac{\partial \gamma}{\partial h}\frac{\delta W_o^2}{\gamma_i^3} + \dots$

Ortho/normal heights are affected in a **nonlinear** and **spatially inhomogeneous** way by δW_o

Forward effect of δW_o

	VRF	VRF′
Zero-height level	$W(\cdot) = W_o$	$W(\cdot) = W_o + \delta W_o$
Geopot. number	$c(P_i)$	$c(P_i) + \delta W_o$
Ortho height	$H(P_i)$	$H(P_i) + \frac{\delta W_o}{g_i} - \frac{1}{2}\frac{\partial g}{\partial H}\frac{\delta W_o^2}{g_i^3} + \ldots$
Normal height	$\tilde{H}(P_i)$	$\tilde{H}(P_i) + \frac{\delta W_o}{\gamma_i} - \frac{1}{2}\frac{\partial \gamma}{\partial h}\frac{\delta W_o^2}{\gamma_i^3} + \ldots$

Second (and higher) order terms are **negligible** (< 1 mm) even for δW_o up to 100 $m^2\ s^{-2}$

Forward effect of δW_o

	VRF	VRF′
Zero-height level	$W(\cdot) = W_o$	$W(\cdot) = W_o + \delta W_o$
Geopot. number	$c(P_i)$	$c(P_i) + \delta W_o$
Ortho height	$H(P_i)$	$H(P_i) + \frac{\delta W_o}{g_i} - \frac{1}{2}\frac{\partial g}{\partial H}\frac{\delta W_o^2}{g_i^3} + \ldots$ ($g_i \to \gamma_i$)
Normal height	$\tilde{H}(P_i)$	$\tilde{H}(P_i) + \frac{\delta W_o}{\gamma_i} - \frac{1}{2}\frac{\partial \gamma}{\partial h}\frac{\delta W_o^2}{\gamma_i^3} + \ldots$

Replacing g_i by γ_i causes a negligible error (< 1 mm) when $|\delta W_o| < 20\ m^2\ s^{-2}$, even for $\Delta g = g_i - \gamma_i$ up to 500 mgal

Forward effect of δs

	VRF	VRF′
Zero-height level	$W(\cdot) = W_o$	$W(\cdot) = W_o$
Geopotential number difference	$\Delta c_{ij} = c(P_j) - c(P_i)$	$\Delta c'_{ij} = (1+\delta s) \cdot \Delta c_{ij}$
Ortho height difference	$\Delta H_{ij} = H(P_j) - H(P_i)$	$\Delta H'_{ij} = (1+\delta s) \cdot \Delta H_{ij}$
Normal height difference	$\Delta \tilde{H}_{ij} = \tilde{H}(P_j) - \tilde{H}(P_i)$	$\Delta \tilde{H}'_{ij} = (1+\delta s) \cdot \Delta \tilde{H}_{ij}$

Uniform **scale change** along a certain spatial direction that is used for physical height determination

(*) with respect to a fixed reference surface

Forward effect of δs

	VRF	VRF′
Zero-height level	$W(\cdot) = W_o$	$W(\cdot) = W_o$
Geopotential number difference	$\Delta c_{ij} = c(P_j) - c(P_i)$	$\Delta c'_{ij} = (1 + \delta s) \cdot \Delta c_{ij}$
Ortho height difference	$\Delta H_{ij} = H(P_j) - H(P_i)$	$\Delta H'_{ij} = (1 + \delta s) \cdot \Delta H_{ij}$
Normal height difference	$\Delta \tilde{H}_{ij} = \tilde{H}(P_j) - \tilde{H}(P_i)$	$\Delta \tilde{H}'_{ij} = (1 + \delta s) \cdot \Delta \tilde{H}_{ij}$

(*) the scale factor is not identical among the different types of VRF vertical coordinates !

Conventional VRF transformation

$$c'(P_i) = (1+\delta s)\cdot c(P_i) \; + \; \delta W_o$$

Geopotential numbers

$$H'(P_i) = (1+\delta s)\cdot H(P_i) \; + \; \frac{\delta W_o}{\gamma_i}$$

Orthometric heights

$$\tilde{H}'(P_i) = (1+\delta s)\cdot \tilde{H}(P_i) \; + \; \frac{\delta W_o}{\gamma_i}$$

Normal heights

Should we use such a “vertical similarity transformation” to infer VRF inconsistencies over a terrestrial network ?

Optimal LS inversion

Given VRF (**d**) and VRF′ (**d**′), and a weight matrix **P** for their differences, the relative 'datum perturbations' can be estimated as:

$$\begin{bmatrix} \delta\hat{W}_o \\ \delta\hat{s} \end{bmatrix} = \begin{bmatrix} \mathbf{q}^T\mathbf{P}\mathbf{q} & \mathbf{q}^T\mathbf{P}\mathbf{d} \\ \mathbf{d}^T\mathbf{P}\mathbf{q} & \mathbf{d}^T\mathbf{P}\mathbf{d} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{q}^T\mathbf{P}(\mathbf{d}'-\mathbf{d}) \\ \mathbf{d}^T\mathbf{P}(\mathbf{d}'-\mathbf{d}) \end{bmatrix}$$

(*) If **d** & **d**′ are geopotential numbers

$$\mathbf{q}^T = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$$

(*) If **d** & **d**′ are normal or orthometric heights

$$\mathbf{q}^T = \begin{bmatrix} 1/\gamma_1 & \cdots & 1/\gamma_N \end{bmatrix}$$

Example

Network	d	d′	$\delta\hat{W}_o$ (gpu)	$\delta\hat{s}$ (ppm)
20 EUVN_DA points (in Switzerland)	EVRF00	EVRF07	0.025 ± 0.001	2.9 ± 0.8
	GPS/EGG08	EVRF07	0.044 ± 0.012	-76.6 ± 10.7
	GPS/EGG97	EVRF07	-0.159 ± 0.035	-107.7 ± 30.8
	LN02	EVRF07	-0.251 ± 0.030	35.7 ± 26.8
	LHN95	EVRF07	-0.060 ± 0.026	-220.7 ± 22.9

(*) based on normal heights

Example (cont'd)

$\sigma_{d'-d}$

Network	d	d′	Before transformation (cm)	After transformation (cm)
20 EUVN_DA points (in Switzerland)	EVRF00	EVRF07	0.3	0.2
	GPS/EGG08	EVRF07	5.2	2.6
	GPS/EGG97	EVRF07	9.9	7.6
	LN02	EVRF07	6.9	6.6
	LHN95	EVRF07	14.0	5.6

Example (cont'd)

Network	d	d′	$\delta\hat{W}_o$ (gpu)	$\delta\hat{s}$ (ppm)
20 EUVN_DA points (in Switzerland)	LN02	EVRF07	-0.251 ± 0.030	35.7 ± 26.8
	LHN95	EVRF07	-0.060 ± 0.026	-220.7 ± 22.9
222 EUVN points (in Switzerland)	LN02	EVRF07	-0.232 ± 0.008	13.3 ± 9.6
	LHN95	EVRF07	-0.080 ± 0.005	-220.9 ± 5.8

(*) based on normal heights

Another example

Network	d	d′	$\delta\hat{W}_o$ (gpu)	$\delta\hat{s}$ (ppm)
13 ‘core datum’ UELN points over EU (see Sacher et al.)	EVRF00	EVRF07	0.002 ± 0.004	-25.5 ± 27.7

It should be exactly zero (theoretically)

The effect of parameter correlation

$$\begin{bmatrix} \delta\hat{W}_o \\ \delta\hat{s} \end{bmatrix} = \begin{bmatrix} \mathbf{q}^T\mathbf{P}\mathbf{q} & \mathbf{q}^T\mathbf{P}\mathbf{d} \\ \mathbf{d}^T\mathbf{P}\mathbf{q} & \mathbf{d}^T\mathbf{P}\mathbf{d} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{q}^T\mathbf{P}(\mathbf{d}'-\mathbf{d}) \\ \mathbf{d}^T\mathbf{P}(\mathbf{d}'-\mathbf{d}) \end{bmatrix}$$

Equivalently,

$$\delta\hat{W}_o = \boxed{\frac{\mathbf{q}^T\mathbf{P}(\mathbf{d}'-\mathbf{d})}{\mathbf{q}^T\mathbf{P}\mathbf{q}}} + \rho_{\delta\hat{W}_o,\delta\hat{s}} \left(\frac{\mathbf{d}^T\mathbf{P}\mathbf{d}}{\mathbf{q}^T\mathbf{P}\mathbf{q}}\right)^{1/2} \delta\hat{s}$$

$$\rho_{\delta\hat{W}_o,\delta\hat{s}} = -\frac{\mathbf{q}^T\mathbf{P}\mathbf{d}}{(\mathbf{d}^T\mathbf{P}\mathbf{d})^{1/2}\cdot(\mathbf{q}^T\mathbf{P}\mathbf{q})^{1/2}} \approx -\frac{mean[\mathbf{d}]}{rms[\mathbf{d}]}$$

Example

			(*) from combined estimation	(*) from individual estimation
Network	**d**	**d′**	$\delta\hat{W}_o$ (gpu)	$\delta\hat{W}_o$ (gpu)
20 EUVN_DA points (in Switzerland)	EVRF00	EVRF07	0.025 ± 0.001	0.028 ± 0.001
	GPS/EGG08	EVRF07	0.044 ± 0.012	-0.032 ± 0.011
	GPS/EGG97	EVRF07	-0.159 ± 0.035	-0.265 ± 0.022
	LN02	EVRF07	-0.251 ± 0.030	-0.216 ± 0.015
	LHN95	EVRF07	-0.060 ± 0.026	-0.277 ± 0.031

(*) based on normal heights

Example

	(*) from combined estimation	(*) from individual estimation

Network	d	d′	$\delta\hat{s}$ (ppm)	$\delta\hat{s}$ (ppm)
20 EUVN_DA points (in Switzerland)	EVRF00	EVRF07	2.9 ± 0.8	22.2 ± 2.5
	GPS/EGG08	EVRF07	-76.6 ± 10.7	-42.9 ± 6.7
	GPS/EGG97	EVRF07	-107.7 ± 30.8	-230.2 ± 21.7
	LN02	EVRF07	35.7 ± 26.8	-157.4 ± 28.1
	LHN95	EVRF07	-220.7 ± 22.9	-266.9 ± 12.5

(*) based on normal heights

Remark

Alternative procedure

The vertical scale factor δs can be also determined directly from the comparison of **height differences** formed **within each VRF** (for a suitable number of baselines)

- selection of VRF baselines
- choice of weight matrix for $\Delta H' - \Delta H$
- δW_o can be estimated **after** reducing the VRFs into a common spatial scale

Vertical S-transformation

$$\mathbf{x} = \mathbf{x}^o + \mathbf{E}^T \boldsymbol{\theta}$$

$$\hat{\boldsymbol{\theta}} = (\mathbf{E}\mathbf{E}^T)^{-1}\mathbf{E}(\mathbf{x} - \mathbf{x}^o)$$

Forward

$$\mathbf{d} = \mathbf{d}^o + \begin{bmatrix} \mathbf{q} & \mathbf{d}^o \end{bmatrix} \begin{bmatrix} \delta W_o \\ \delta s \end{bmatrix}$$

Inverse

$$\begin{bmatrix} \delta \hat{W}_o \\ \delta \hat{s} \end{bmatrix} = \begin{bmatrix} \mathbf{q}^T\mathbf{q} & \mathbf{q}^T\mathbf{d}^o \\ \mathbf{d}^{oT}\mathbf{q} & \mathbf{d}^{oT}\mathbf{d}^o \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{q}^T(\mathbf{d} - \mathbf{d}^o) \\ \mathbf{d}^{oT}(\mathbf{d} - \mathbf{d}^o) \end{bmatrix}$$

Important tool $\rightarrow$ **inner constraints**

(*) development of an optimal VRF from different techniques and/or heterogeneous data sources (leveling, GPS/GGM, tide-gauge data, altimetry, SSTs, etc.)

Conclusions

- δW_o and δs are the basic parameters of a conventional transformation model for (static) VRFs
- Useful tool for evaluating the consistency between different VRS realizations (e.g. global GNSS/GGM-based vs. regional leveling-based)
- Useful tool for determining an “optimal” combined VRF solution using different techniques and/or heterogeneous height data
- Generalization to time-dependent cases is necessary (temporal evolution of a VRF)

Thanks for your attention !

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