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A rigorous approach for aligning regional networks to global reference frames based on the Helmert transformation

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1. Introduction

Two alternative strategies are mainly used to optimally express the station positions of a regional network into the ITRF, namely: **(a)** constraining a subset of stations to known ITRF coordinates during the network adjustment, or **(b)** aligning an existing solution of the regional network to ITRF based on known reference stations using the Helmert transformation (HT) model.

The latter approach is considered to be vulnerable due to the so-called network effect on the estimated transformation parameters and its resulting bias in the transformed coordinates of regional networks (e.g. Altamimi 2003).

In this paper we expose the fact that the apparent deficiency of the HT approach originates mostly by its suboptimal implementation due to mishandling of the full stochastic model for the available data. Following a rigorous treatment for the frame alignment problem (Sect. 3) it is shown that the HT approach is able to give comparable results with the constrained adjustment of a regional network directly to the ITRF. Various numerical tests using EPN/LACs weekly solutions (Sect. 4) corroborate the validity and the importance of our theoretical analysis.

NOTATION USED IN THE PAPER

Initial frame data

$$\begin{bmatrix} \mathbf{X}' \\ \mathbf{Z}' \end{bmatrix} \quad \begin{bmatrix} \Sigma_{X'} & \Sigma_{X'Z'} \\ \Sigma_{Z'X'} & \Sigma_{Z'} \end{bmatrix}$$

known coordinates and their full covariance matrix (reference + other stations)

Target frame data

$$\mathbf{X} \quad \Sigma_X$$

known coordinates and their full covariance matrix (reference stations only)

Helmert transformation

$$\mathbf{X} = \mathbf{X}' + \mathbf{G}\boldsymbol{\theta}$$

Linearized form of the 3-D similarity transformation model

θ contains three translations, three small rotation angles and one differential scale factor

G (or $\tilde{\mathbf{G}}$) is the design matrix of the 3-D similarity transformation with respect to the reference (or other) stations

2. The standard approach for Helmert frame transformation

The transformation problem of a coordinate set from a reference frame to another is usually solved in two steps by combining both forward and inverse treatment of the linearized similarity transformation model (the so-called Helmert model).

In general, the related procedure can be summarized as follows:

Step 1
Weighted LS adjustment of the Helmert model to estimate the transformation parameters between the two frames.

Step 2
Forward implementation of the Helmert model to compute the transformed coordinates in the target frame.

Formulae for steps 1 & 2:

$$\hat{\boldsymbol{\theta}}^{st} = \left(\mathbf{G}^T(\Sigma_X+\Sigma_{X'})^{-1}\mathbf{G}\right)^{-1}\mathbf{G}^T(\Sigma_X+\Sigma_{X'})^{-1}(\mathbf{X}-\mathbf{X}')$$
transformation parameters

$$\hat{\mathbf{x}}^{st} = \mathbf{X}' + \mathbf{G}\hat{\boldsymbol{\theta}}$$
transformed coordinates of the reference stations

$$\hat{\mathbf{z}}^{st} = \mathbf{Z}' + \tilde{\mathbf{G}}\hat{\boldsymbol{\theta}}$$
transformed coordinates of other stations

This is a straightforward scheme which has provided the standard framework for coordinate transformation and frame alignment problems in geodetic networks. It is not optimal as it **overlooks the intra-frame correlation of the initial coordinate sets X' & Z'**, a fact that degrades their transformation accuracy in the target frame.

The above stepwise approach is not based on any optimal criterion for the transformed coordinates in the target frame. Hence, it cannot assure their best estimation accuracy from the available data!

3. A rigorous approach for Helmert frame transformation

The Helmert transformation of a geodetic network from an initial frame to a target frame can be formulated, in a more rigorous way, according to the observation equations system:

Reference stations (target frame)

$$\mathbf{X} = \mathbf{x} + \mathbf{v}_X \qquad \mathbf{v}_X \sim (\mathbf{0}, \Sigma_X)$$

Reference & other stations (initial frame)

$$\mathbf{X}' = \mathbf{x} - \mathbf{G}\boldsymbol{\theta} + \mathbf{v}_{X'} \qquad \mathbf{v}_{X'} \sim (\mathbf{0}, \Sigma_{X'})$$
$$\mathbf{Z}' = \mathbf{z} - \tilde{\mathbf{G}}\boldsymbol{\theta} + \mathbf{v}_{Z'} \qquad \mathbf{v}_{Z'} \sim (\mathbf{0}, \Sigma_{Z'})$$

where **x** and **z** are the true coordinate vectors of the network stations in the target frame of interest, and **θ** is the vector of the frame transformation parameters.

From the least-squares (LS) adjustment of the above system we obtain, in a single step, all quantities of interest to the frame transformation problem. The weight matrix that should be used for the optimal LS inversion of the above system has the general form:

$$\mathbf{P} = \begin{bmatrix} \Sigma_X & 0 & 0 \\ 0 & \Sigma_{X'} & \Sigma_{X'Z'} \\ 0 & \Sigma_{Z'X'} & \Sigma_{Z'} \end{bmatrix}^{-1}$$

→ It considers not only the prior accuracy of all datasets in their respective frames, but also the intra-frame coordinate correlation within the initial frame between the reference and other stations.

LS optimal solution for the frame transformation problem

It is easily proven that the solution of the aforementioned adjustment problem can be analytically expressed by the following equations:

Transformed coordinates of the reference and other stations

$$\begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{z}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}}^{st} \\ \hat{\mathbf{z}}^{st} \end{bmatrix} + \begin{bmatrix} \Sigma_{X'} \\ \Sigma_{Z'X'} \end{bmatrix}(\Sigma_X+\Sigma_{X'})^{-1}(\mathbf{X}-\hat{\mathbf{x}}^{st})$$

Frame transformation parameters

$$\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}^{st}$$

The optimal estimate of the transformation parameters is identical to the "standard" estimate, as expected since the inclusion of the new (non-common) stations in the LS adjustment does not contribute any actual information for **θ**. On the other hand, the optimally transformed coordinates differ from the "standard" estimates by small correction terms similar to those found in least-squares collocation and Kalman filtering theory.

Why is the rigorous approach better than the standard approach

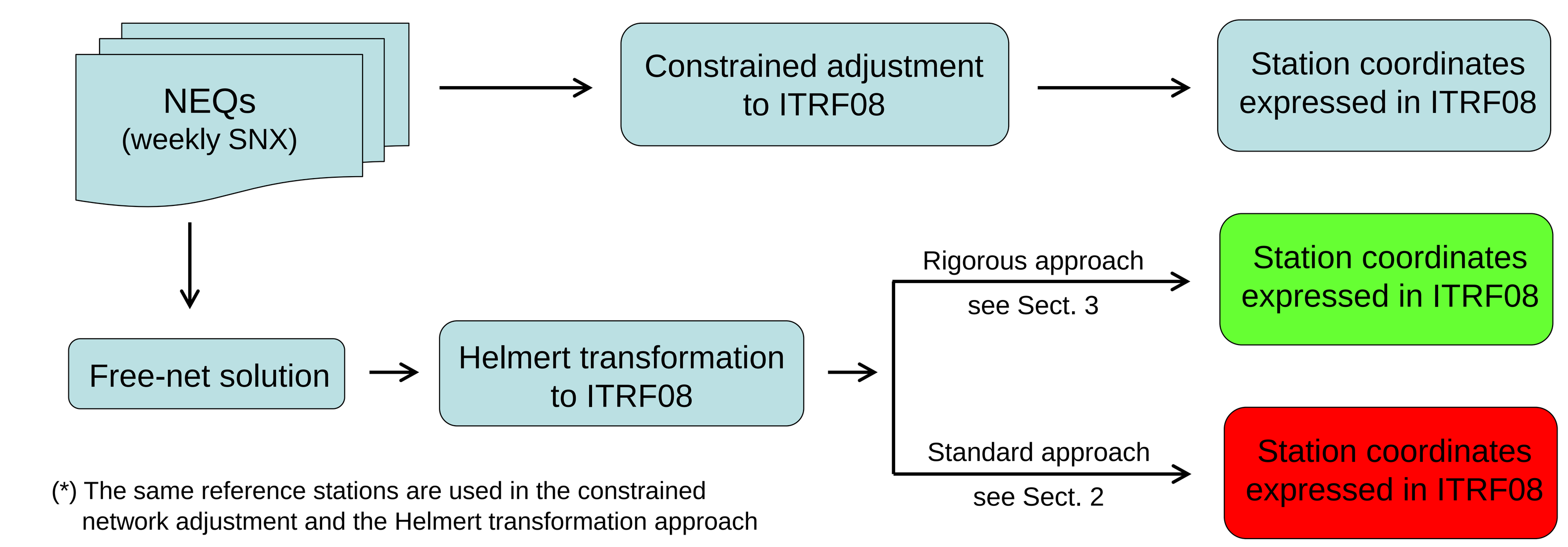
- The effect of data noise is partially minimized in the standard transformation approach (only the part contained in the frame transformation parameters).
- The standard approach neglects some prior stochastic characteristics of the geodetic network in the initial frame (i.e. Σ_{ZX}) and thus it implies a "sloppy" treatment of its spatial configuration under the transformation procedure.
- The computation of the required corrections in the rigorous approach does not involve any additional matrix inversion other than the one already used in the standard approach.

How significant is the difference between the rigorous and the standard approach for frame transformation problems ?

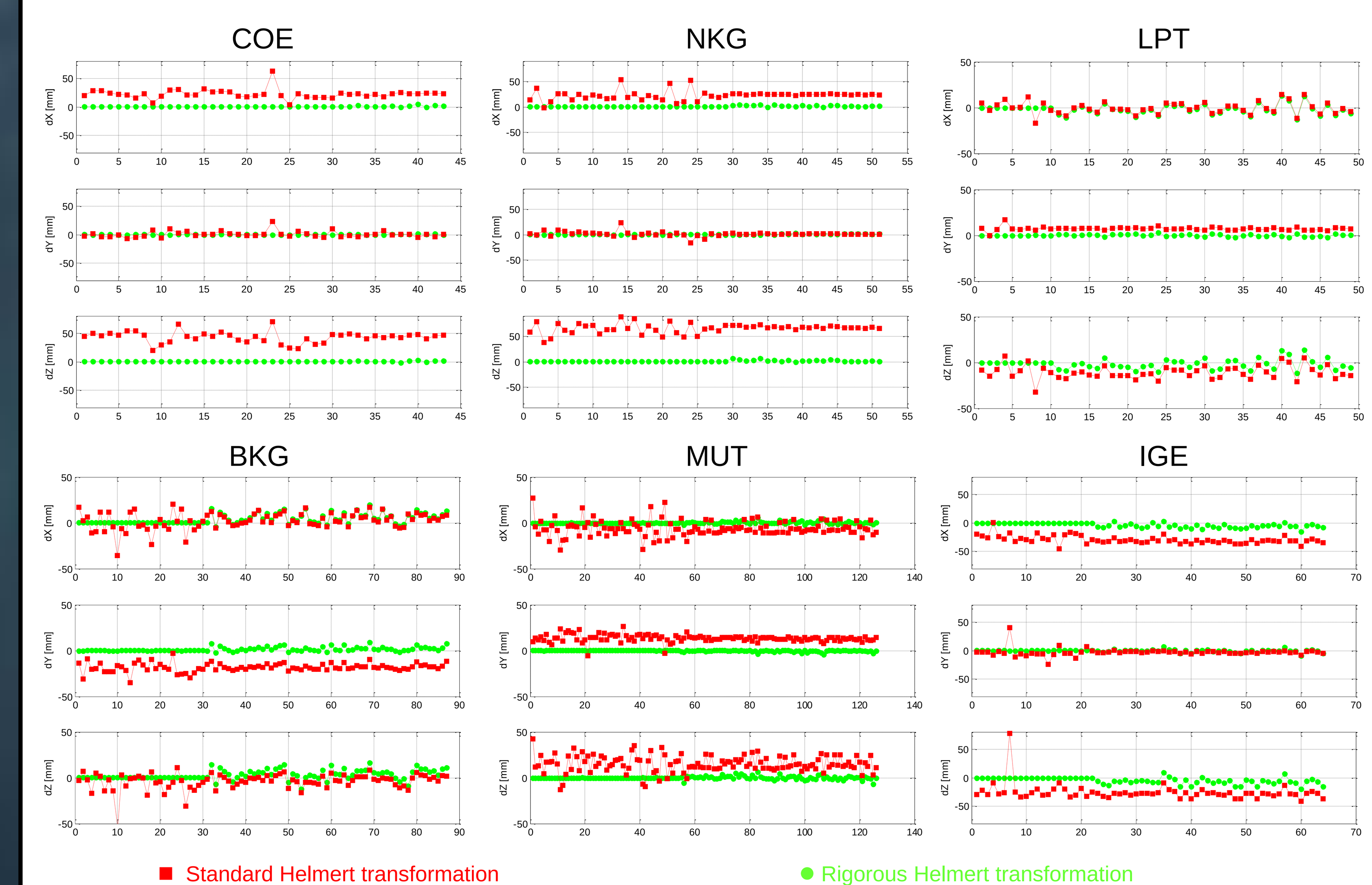
Both approaches can be used for the alignment of a regional network to a global reference frame. **The assessment of their differences will be based on their agreement with the result of the constrained adjustment of the regional network from its original data directly to the global frame** (SEE NEXT).

4. Numerical experiments

For our experiments we use weekly SNX files (*GPS week 1788*) from different LACs of the EUREF Permanent Network. The numerical tests involve the comparison of the ITRF08 coordinates from the constrained adjustment of each LAC's sub-network against the ITRF08 coordinates that are determined via the Helmert transformation approach on the same sub-networks – see following flowchart.



DIFFERENCES BETWEEN THE ITRF08-CONSTRAINED WEEKLY SOLUTIONS & THE ITRF08 HELMERT-TRANSFORMED COORDINATES FOR EPN/LAC SUBNETS



Transformation parameters between ITRF08 and EPN/LACs free weekly solutions

	T_x (cm)	T_y (cm)	T_z (cm)	ε_x (mas)	ε_y (mas)	ε_z (mas)	δs (ppb)
COE	-1.04	-0.12	-3.30	0.001	-0.219	-0.052	0.71
NKG	-1.92	1.53	-3.67	-0.701	-0.318	0.229	-0.61
LPT	6.24	-6.84	-0.18	1.504	2.481	-1.200	2.22
BKG	-7.23	-2.22	4.32	1.089	-2.148	-0.330	-0.81
MUT	-0.66	0.74	1.94	-0.875	-0.848	0.418	-0.34
IGE	7.30	6.19	3.66	-1.614	0.939	1.079	1.49

5. Conclusions

This paper demonstrates a rigorous implementation of the Helmert transformation approach for aligning regional networks to global reference frames. In contrast to the standard approach (which is known to suffer by significant apparent biases in the transformed coordinates, Altamimi 2003), our revised scheme improves significantly the consistency with the constrained solution of regional networks directly to the global frame. The most general case of the problem with inter-frame coordinate correlations will be treated in a forthcoming paper.