Development of a Semi-Automated Approach for Regional Corrector Surface Modeling in GPS-Levelling

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Overview

- Introduction to problem
- Background to previous work
- Choosing the ‘best’ model …
- Assessing model performance
- Testing parameter significance
- Description of data
  - Switzerland
  - Canada
- Discussion of results
- Conclusions
**Introduction** (1/4)

**Standard practice:** Use of a corrector surface to model the datum discrepancies and systematic effects when combining GPS, geoid and orthometric heights.

**Theory:** \( h_i - H_i - N_i = 0 \)  \( \rightarrow \)  \( N_{i}^{GPS/levelling} = N_i \)

**Practice:** \( h_i - H_i - N_i = l_i \)  \( \rightarrow \)  \( N_{i}^{GPS/levelling} \neq N_i \)

**Model:** \( l_i = h_i - H_i - N_i = a_i^T x + v_i \)

- residuals
- parametric ‘corrector’ model/surface
Introduction (2/4)

• Profound reasoning for choosing a specific model is missing

• Spatial modelling and analysis of the adjusted residual values over a network of GPS/levelling benchmarks are useful for a variety of applications:
  – External accuracy evaluation of spherical harmonic models of the Earth’s gravity field and regional gravimetric geoid solutions
  – Refinement of regional geoid solutions by eliminating long wavelength errors through ties to GPS/levelling benchmarks
  – Check and improve the accuracy of vertical datums through combining geoid, GPS and levelling data
Development of corrector surface models to be used with GPS and gravimetric geoid models for GPS-Levelling

\[ H_p = h_p - N_p - \mathbf{a}_p^T \hat{\mathbf{x}} \]

- corrector surface
- known height data
- orthometric height at new point

**Data**

- GPS: \( h_i, \Delta h_{ij} \)
- Orthometric heights: \( H_i, \Delta H_{ij} \)
- Geoid model: \( N_i, \Delta N_{ij} \)

Prediction surface → aim is to derive a surface from data which is to be applied to new data
**Objective:** To eliminate some of the arbitrariness in both **choosing the model type** and **assessing its performance**

**General Pointwise Case:**

\[ h_i - H_i - N_i = a_i^T x + v_i \]

where,
- \( x \) … vector of unknown parameters
- \( a_i \) … vector of known coefficients (depend on horizontal coords)
- \( v_i \) … residuals
Data
\( h_i, H_i, N_i \)
\( \Delta h_{ij}, \Delta H_{ij}, \Delta N_{ij} \)

Corrector Models
\( h_i - H_i - N_i - a_i^T x = 0 \)

Least-squares adjustment
\( \hat{x}_k \)

Statistics of adjusted residuals
\( \hat{v}_i = h_i - H_i - N_i - a_i^T \hat{x} \)

Final model selection

- network geometry
- pre-corrected
- pre-adjusted
- over constrained adjustment

- polynomial (order)?
- base functions (trig.)?
- “physical” meaning of terms
Corrector Surface Model Selection

Corrector Models

\[ h_i - H_i - N_i - a_i^T x = 0 \]

- Selection of analytical model suffers from a degree of arbitrariness (*Why?*)
  - type of model (i.e. polynomial)
  - type of base functions (i.e. trigonometric)
  - number of coefficients

- Need statistical tools to
  - assess choices made
  - compare different models

- Factors for model selection/analysis may vary if
  - nested models
  - orthogonal vs. non-orthogonal models

*No straightforward answer, data dependent (geometry)*
Assessing the Goodness of Fit

Statistics of adjusted residuals
\[ \hat{v}_i = h_i - H_i - N_i - a_i^T \hat{x} \]

Coefficient of determination
\[ R^2 = 1 - \frac{\sum_{i=1}^{n} (\ell_i - \hat{v}_i)^2}{\sum_{i=1}^{n} (\ell_i - \bar{\ell}_i)^2} \]  
\[ \ell_i = h_i - H_i - N_i \]
\[ n \quad \ldots \# \text{ of observations} \]

Adjusted coefficient of determination
\[ \overline{R}^2 = 1 - \frac{\left[ \sum_{i=1}^{n} (\ell_i - \hat{v}_i)^2 \right] / (n-m)}{\left[ \sum_{i=1}^{n} (\ell_i - \bar{\ell}_i)^2 \right] / (n-1)} \]  
\[ m \quad \ldots \# \text{ of parameters} \]
Cross-Validation

- Use a subset of all points to compute the model parameters \( \hat{x} \)
- **Predict** the residual values at a new point and compare the predicted value with the ‘known’ height value
  \[
  \Delta \hat{v}_p = h_p - H_p - N_p - a_p^T \hat{x}
  \]
- Repeat for each point and compute the average rms, \( \sum_{i=1}^{n} \sqrt{\mu_i^2 + \sigma_i^2} \)
Reasons for reducing the number of model parameters

• Simplicity, computational efficiency

• Over-parameterization (i.e. high-degree trend models)
  \[\rightarrow\] unrealistic extrema in data voids where control points are missing

• Unnecessary terms may bias other parameters in model
  \[\rightarrow\] hinders capability to assess model performance

Need for automated selection process
Stepwise Procedures

Backward Elimination Procedure
• Start with highest order model
• Eliminate less-significant terms one-by-one (or several at once)
• *Criteria* for determining parameter deletion
  – Partial F-test
  – Level of significance, $\alpha$
  – *Problem*: correlation between parameters

Forward Selection Procedure
• Start with simple model
• Add parameter with the highest coefficient of determination (or partial F-value)

Stepwise Procedure
• Combination of backward elimination and forward selection procedures
• Starts with no parameters and selects parameters one-by-one (or several)
• After inclusion, examine every parameter for significance (partial F-test)
Testing Parameter Significance

- Statistical tests are more powerful in pointing out inappropriate models rather than establishing model validity

- Test if a set of parameters in the model is significant or not:

\[
x = \begin{bmatrix} x(I) \\ x_I \end{bmatrix}
\]

\(I\) ... set of parameters tested

\(I\) ... remaining parameters (complement)

**hypothesis**

\[H_0 : x_I = 0 \quad \text{vs} \quad H_a : x_I \neq 0\]

**test statistic**

\[\tilde{F} = \frac{\hat{x}_I Q^{-1} \hat{x}_I}{k \hat{\sigma}^2}
\]

\(k \quad \text{……. number of ‘tested’ terms}\)

\(Q_{\hat{x}_I} \quad \text{…… submatrix of } Q = N^{-1}\)

**criteria**

\[\tilde{F} \leq F_{\alpha, k, f}\]

\(H_0 \text{ accepted} \checkmark\)
Testing Parameter Significance

- Test statistic (regardless of form) is a function of observations

\[ \tilde{F} = \frac{\hat{\chi}_I Q^{-1} \hat{\chi}_I}{k\hat{\sigma}^2} \]

\[ \hat{F} = \frac{[\sum (\ell - \hat{\nu})^2_{\text{partial}} - \sum (\ell - \hat{\nu})^2_{\text{full}}]}{[\sum (\ell - \hat{\nu})^2_{\text{full}}]/(n-m)} \]

- No need to repeat combined least-squares adjustment (first case)

Problems
- No unique answer (depends on initial selection, \( \alpha \))
- High parameter correlation may skew results
- Highly correlated parameters should be deleted (detection)
**Stepwise Procedure**

1. Enter parameter
2. Perform regression
3. Re-compute partial F-values for each model parameter
4. Check: \( \tilde{F} \leq F_{out}^\alpha \)
   - Delete parameter
5. Compute partial F-values, choose the highest one
   - Check: \( \tilde{F} \geq F_{in}^\alpha \)
     - Yes: Terminate
     - No: Backward elimination
6. Start
   - Select regression model
   - Least-squares adjustment
Description of Data

- 111 stations in Switzerland
- 343 km × 212 km region
- Form ‘residuals’:
  \[ \ell_i = h_i - H_i - N_i \]

Statistics of residuals before fit

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</thead>
<tbody>
<tr>
<td>min</td>
<td>-4.9 cm</td>
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<tr>
<td>max</td>
<td>19 cm</td>
</tr>
<tr>
<td>mean</td>
<td>1.1 cm</td>
</tr>
<tr>
<td>std</td>
<td>3.8 cm</td>
</tr>
<tr>
<td>rms</td>
<td>3.9 cm</td>
</tr>
</tbody>
</table>
Description of Data

- 63 stations in Southern British Columbia & Alberta
- 495 km × 334 km region
- Form ‘residuals’:
  \[ \ell_i = h_i - H_i - N_i \]

Stats of residuals before fit

<table>
<thead>
<tr>
<th>min</th>
<th>-17.1 cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>max</td>
<td>25.2 cm</td>
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<tr>
<td>mean</td>
<td>4.5 cm</td>
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<tr>
<td>std</td>
<td>8.1 cm</td>
</tr>
<tr>
<td>rms</td>
<td>9.3 cm</td>
</tr>
</tbody>
</table>

GPS on Benchmarks (and residuals)
Analytical Models

Nested bilinear polynomial series

\[ 1 \, d\phi \, d\lambda \, d\phi d\lambda \, d\phi^2 \, d\lambda^2 \, d\phi^2 d\lambda \, d\phi d\lambda^2 \, d\phi^3 \, d\lambda^3 \, d\phi^2 d\lambda^2 \, d\phi^3 d\lambda \, d\phi d\lambda^3 \, d\phi^4 \, d\lambda^4 \]

Classic trigonometric-based polynomial fits

\[ 1 \, \cos\phi \cos\lambda \, \cos\phi \sin\lambda \, \sin\phi \]

\[ 1 \, \cos\phi \cos\lambda \, \cos\phi \sin\lambda \, \sin\phi \, \sin^2\phi \]

Differential similarity transformation

\[
\begin{array}{cccc}
\cos\phi \cos\lambda & \cos\phi \sin\lambda & \sin\phi & \frac{\sin\phi \cos\phi \sin\lambda}{W} \\
\cos\phi \sin\lambda & \sin\phi & \frac{\sin\phi \cos\phi \cos\lambda}{W} & \frac{1 - f^2 \sin^2\phi}{W} \\
\sin\phi & \sin\phi & \frac{\sin^2\phi}{W}
\end{array}
\]

where, \( W = \sqrt{1 - e^2 \sin^2\phi} \)
Analytical Models (polynomials)

4th order

3rd order

2nd order

1st order
Other Analytical Models

Classic 4-parameter

Classic 5-parameter

7-parameter differential similarity

Notes
- all values shown in m
- GPS BMs in Switzerland used
- Full models shown (no parameters omitted)
Example - Coefficient of Determination

$0 \leq R^2 \leq 1$

- A 1\textsuperscript{st} order polynomial
- B Classic 4-parameter
- C Classic 5-parameter
- D 2\textsuperscript{nd} order polynomial
- E Differential Similarity
- F 3rd order polynomial
- G 4\textsuperscript{th} order polynomial

Model performance
Conclusions

Residuals after fit
→ 4th order polynomial

Prediction (external test)
→ Any model except 4th order polynomial

Not enough of a difference between models to justify statistical parameter significance testing
→ use lowest order model
Results - Southern BC/AB

Differential Similarity Fit (7-parameters)

\[
\begin{align*}
\cos \varphi \cos \lambda & \quad \cos \varphi \sin \lambda & \quad \sin \varphi & \quad \frac{\sin \varphi \cos \varphi \sin \lambda}{W} & \quad \frac{\sin \varphi \cos \varphi \cos \lambda}{W} & \quad \frac{1 - f^2 \sin^2 \varphi}{W} & \quad \sin^2 \varphi \\
\end{align*}
\]

Selection criteria

\[
\begin{array}{|c|c|}
\hline
R^2 & 0.4805 \\
\hline
\bar{R}^2 & 0.2311 \\
\hline
\sqrt{v^T \hat{v}} & 53 \text{ cm} \\
\hline
\text{condition number} & 1.52 \times 10^{12} \\
\hline
\text{rms after fit} & 6.7 \text{ cm} \\
\hline
\text{rms (prediction)} & 7.9 \text{ cm} \\
\hline
\end{array}
\]

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Results - Switzerland

**Classic 4-parameter fit**

\[ l \cos \varphi \cos \lambda \cos \varphi \sin \lambda \sin \varphi \]

### Selection criteria

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R^2 )</td>
<td>0.5668</td>
</tr>
<tr>
<td>( \bar{R}^2 )</td>
<td>0.5181</td>
</tr>
<tr>
<td>( \sqrt{\hat{\nu}^T \hat{\nu}} )</td>
<td>24.5 cm</td>
</tr>
<tr>
<td>Condition number</td>
<td>2.77 \times 10^7</td>
</tr>
<tr>
<td>rms after fit</td>
<td>2.4 cm</td>
</tr>
<tr>
<td>rms (prediction)</td>
<td>2.4 cm</td>
</tr>
</tbody>
</table>
Conclusions

- **Semi-automated procedure** for **comparing corrector surface models** and **assessing model performance** was presented.

- **Semi**
  - no unique straightforward solution
  - some user intervention required

- In most cases, the best test is cross-validation (prediction)
  - independent ‘external’ test
  - depends on quality of data

- When model parameters are highly correlated (as is the case with polynomial regression), statistical testing may not be conclusive.

- Use orthogonal polynomials to eliminate problems with high correlation between parameters (i.e. Fourier Series).

- Procedure should include a **combination** of empirical **and** statistical testing.