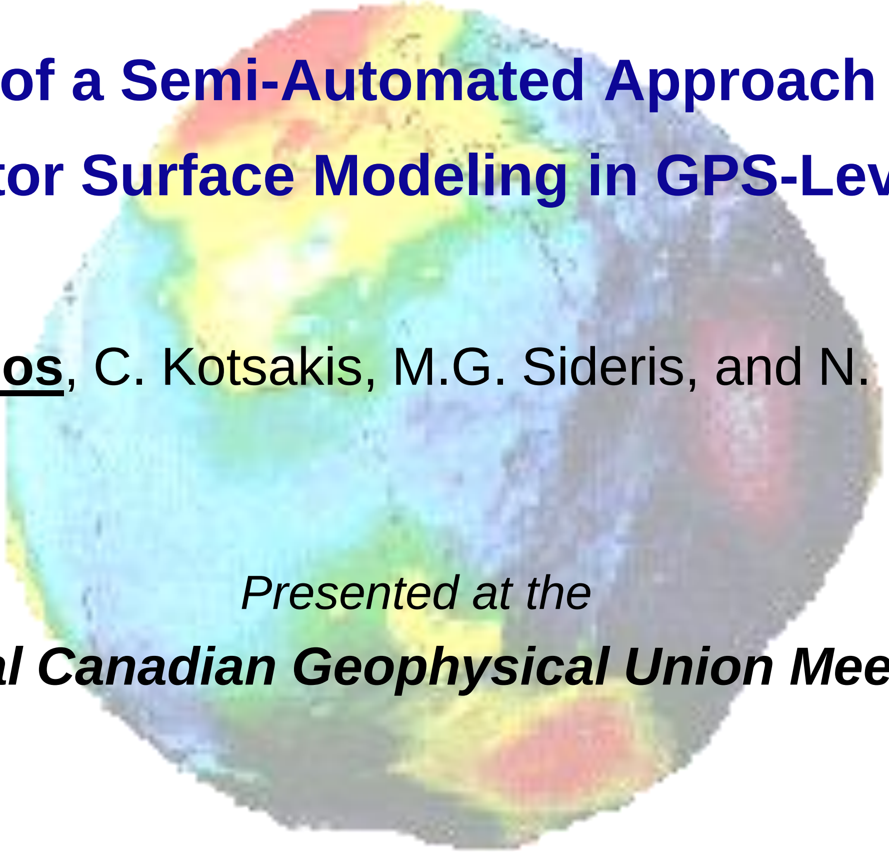

Development of a Semi-Automated Approach for Regional Corrector Surface Modeling in GPS-Levelling

G. Fotopoulos, C. Kotsakis, M.G. Sideris, and N. El-Sheimy

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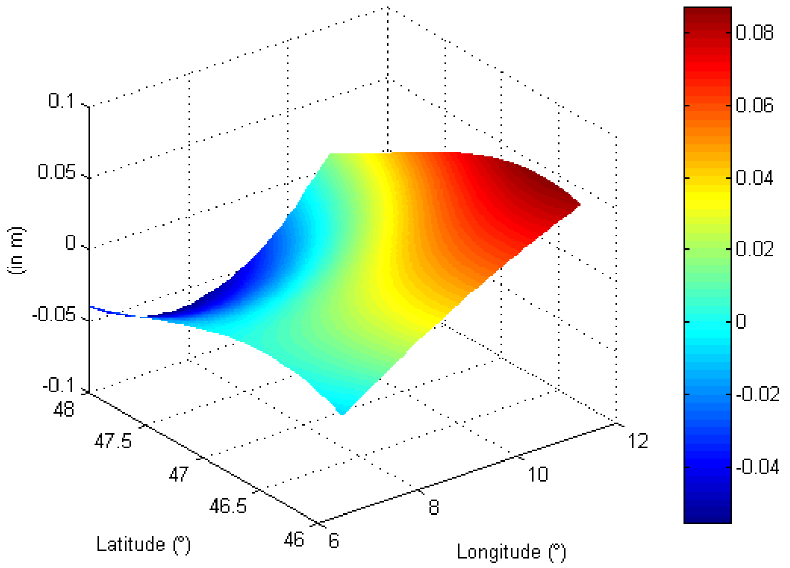
Overview

- Introduction to problem
- Background to previous work
- Choosing the ‘best’ model …
- Assessing model performance
- Testing parameter significance
- Description of data
  - Switzerland
  - Canada
- Discussion of results
- Conclusions



Introduction (1/4)

Standard practice: Use of a corrector surface to model the datum discrepancies and systematic effects when combining GPS, geoid and orthometric heights

Theory: $h_i - H_i - N_i = 0 \quad \rightarrow \quad N_i^{GPS\,/\,levelling} = N_i$

Practice: $h_i - H_i - N_i = l_i \quad \rightarrow \quad N_i^{GPS\,/\,levelling} \neq N_i$

Model: $l_i = h_i - H_i - N_i = \underbrace{\mathbf{a}_i^{\mathrm{T}}\mathbf{x}}_{\text{parametric 'corrector' model/surface}} + \underbrace{v_i}_{\text{residuals}}$

residuals

parametric *'corrector'*
model/surface

Introduction (2/4)

- Profound reasoning for choosing a specific model is missing
- Spatial modelling and analysis of the adjusted residual values over a network of GPS/levelling benchmarks are useful for a variety of applications:
  - External accuracy evaluation of spherical harmonic models of the Earth's gravity field and regional gravimetric geoid solutions
  - Refinement of regional geoid solutions by eliminating long wavelength errors through ties to GPS/levelling benchmarks
  - Check and improve the accuracy of vertical datums through combining geoid, GPS and levelling data

Introduction (3/4)

- Development of corrector surface models to be used with GPS and gravimetric geoid models for GPS-Levelling

$$\underset{\substack{\uparrow\\ \text{orthometric height at new point}}}{H_p} = \underbrace{h_p - N_p}_{\textit{known}\ \text{height data}} - \underset{\substack{\uparrow\\ \text{corrector surface}}}{\mathbf{a}_p^T}\hat{\mathbf{x}}$$

Data

GPS: $h_i, \Delta h_{ij}$ Orthometric heights: $H_i, \Delta H_{ij}$ Geoid model: $N_i, \Delta N_{ij}$

Prediction surface → aim is to derive a surface from data which is to be applied to new data

Introduction (4/4)

Objective: To eliminate some of the arbitrariness in both **choosing the model type** and **assessing its performance**

General Pointwise Case:

$$h_i - H_i - N_i = \mathbf{a}_i^{\mathrm{T}}\mathbf{x} + v_i$$

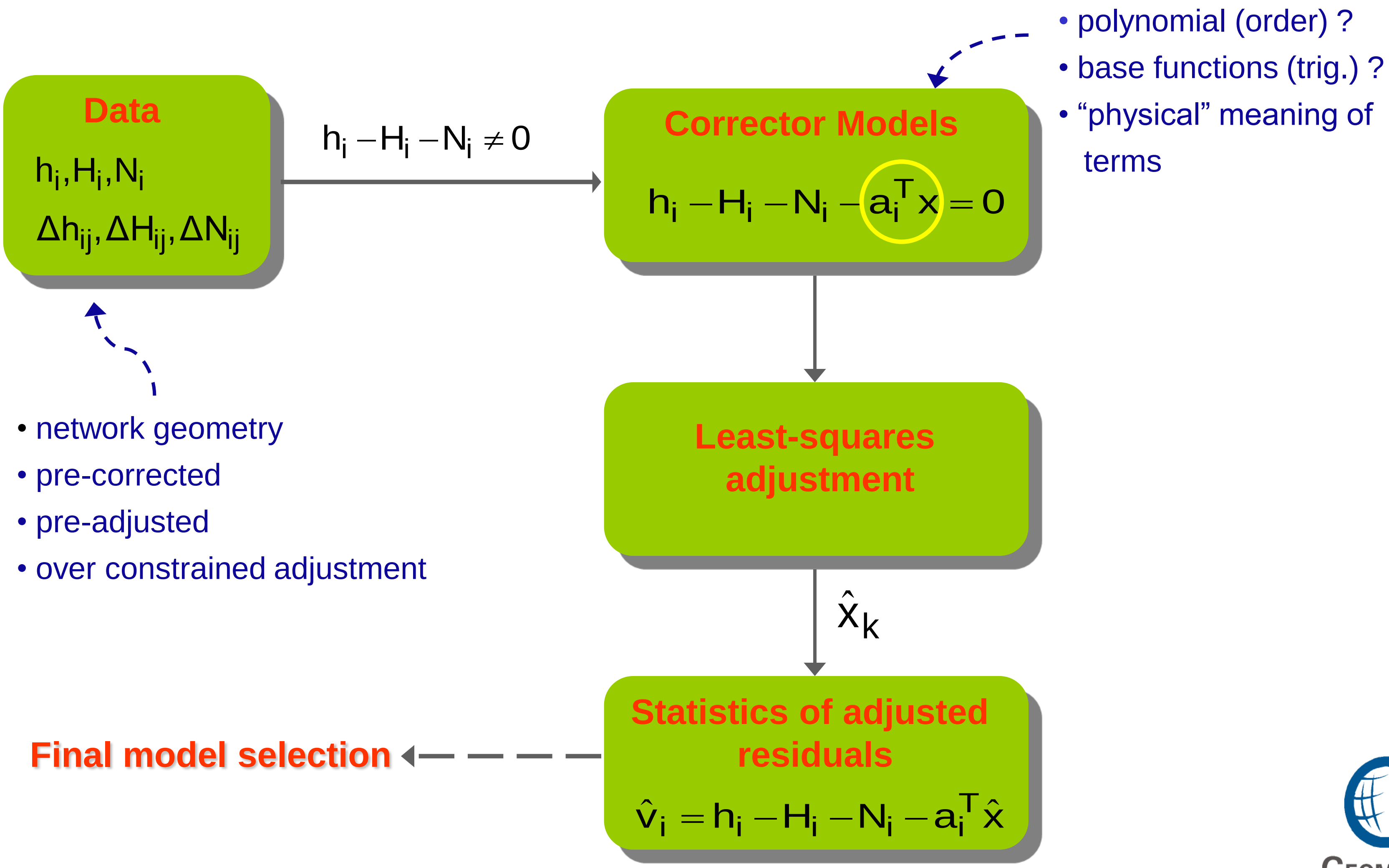
where,

$\mathbf{x}$ … vector of unknown parameters

$\mathbf{a}_i$ … vector of known coefficients (depend on horizontal coords)

v_i … residuals

Classic Empirical Approach



Data

h_i, H_i, N_i

$\Delta h_{ij}, \Delta H_{ij}, \Delta N_{ij}$

$h_i - H_i - N_i \neq 0$

Corrector Models

$h_i - H_i - N_i - a_i^T x = 0$

- polynomial (order) ?
- base functions (trig.) ?
- “physical” meaning of terms

Least-squares adjustment

$\hat{x}_k$

Statistics of adjusted residuals

$\hat{v}_i = h_i - H_i - N_i - a_i^T \hat{x}$

Final model selection

- network geometry
- pre-corrected
- pre-adjusted
- over constrained adjustment

Corrector Surface Model Selection

Corrector Models

$$h_i - H_i - N_i - a_i^T x = 0$$

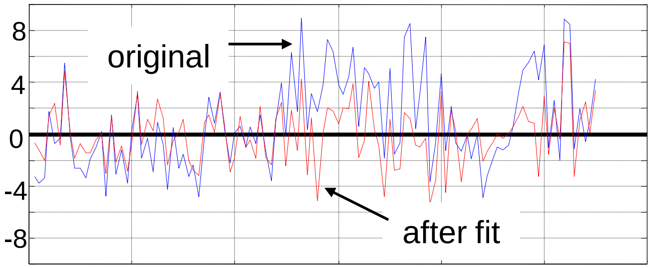
- Selection of analytical model suffers from a degree of arbitrariness *(Why?)*
  - type of model (i.e. polynomial)
  - type of base functions (i.e. trigonometric)
  - number of coefficients

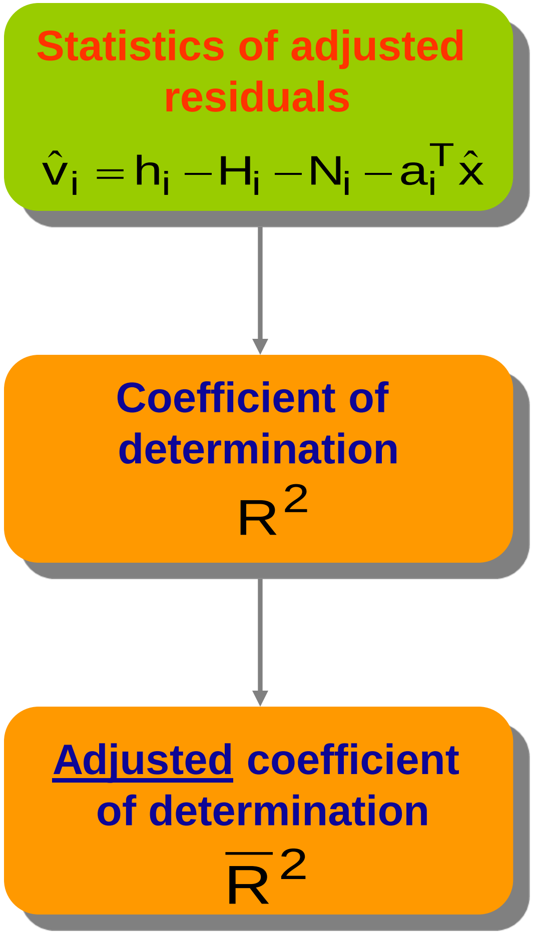
- Need statistical tools to
  - assess choices made
  - compare different models

- Factors for model selection/analysis may vary if
  - nested models
  - orthogonal vs. non-orthogonal models

No straightforward answer, data dependent (geometry)

Assessing the Goodness of Fit





Statistics of adjusted residuals

$$\hat{v}_i = h_i - H_i - N_i - a_i^T \hat{x}$$

Coefficient of determination

R^2

$$R^2 = 1 - \frac{\sum_{i=1}^{n} (\ell_i - \hat{v}_i)^2}{\sum_{i=1}^{n} (\ell_i - \bar{\ell}_i)^2}$$

$\ell_i = h_i - H_i - N_i$

n … # of observations

Adjusted coefficient of determination

$\bar{R}^2$

$$\bar{R}^2 = 1 - \frac{\left[\sum_{i=1}^{n} (\ell_i - \hat{v}_i)^2\right] / (n-m)}{\left[\sum_{i=1}^{n} (\ell_i - \bar{\ell}_i)^2\right] / (n-1)}$$

m … # of parameters

Additional Empirical Approach

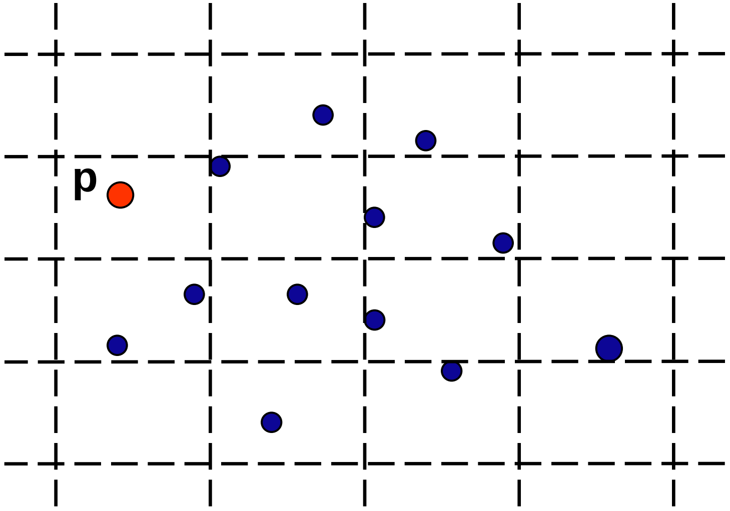
Cross-Validation

- Use a subset of all points to compute the model parameters $\hat{\mathbf{x}}$
- **Predict** the residual values at a new point and compare the predicted value with the 'known' height value

$$\Delta\hat{v}_p = h_p - H_p - N_p - a_p^T \hat{x}$$

- Repeat for each point and compute the average rms, $\sum_{i=1}^{n}\sqrt{\mu_i^2 + \sigma_i^2}$

Cross-validation (empirical approach)



Testing Parameter Significance

Reasons for reducing the number of model parameters

- Simplicity, computational efficiency
- Over-parameterization (i.e. high-degree trend models)
  → unrealistic extrema in data voids where control points are missing
- Unnecessary terms may bias other parameters in model
  → hinders capability to assess model performance

Parameter Significance → Need for automated selection process

Stepwise Procedures

Backward Elimination Procedure

- Start with highest order model
- Eliminate less-significant terms one-by-one (or several at once)
- *Criteria* for determining parameter deletion
  - Partial F-test
  - Level of significance, α
  - **Problem:** correlation between parameters

nested
models
only

Forward Selection Procedure

- Start with simple model
- Add parameter with the highest coefficient of determination (or partial F-value)

Stepwise Procedure

- Combination of backward elimination and forward selection procedures
- Starts with no parameters and selects parameters one-by-one (or several)
- After inclusion, examine every parameter for significance (partial F-test)

Testing Parameter Significance

- Statistical tests are more powerful in pointing out inappropriate models rather than establishing model validity
- Test if a set of parameters in the model is significant or not:

$$x = \begin{bmatrix} x_{(I)} \\ x_I \end{bmatrix}$$

I … set of parameters tested

(I) … remaining parameters (complement)

hypothesis $H_0 : x_I = 0 \quad \text{vs} \quad H_a : x_I \neq 0$

test statistic

$$\tilde{F} = \frac{\hat{x}_I \, Q_{\hat{x}_I}^{-1} \, \hat{x}_I}{k\hat{\sigma}^2}$$

k ……… number of ‘tested’ terms

$Q_{\hat{x}_I}$ …... submatrix of Q = N^{-1}

criteria $\tilde{F} \leq F_{k,f}^{\alpha}$ H_0 accepted ✓

Testing Parameter Significance

- Test statistic (regardless of form) is a function of observations

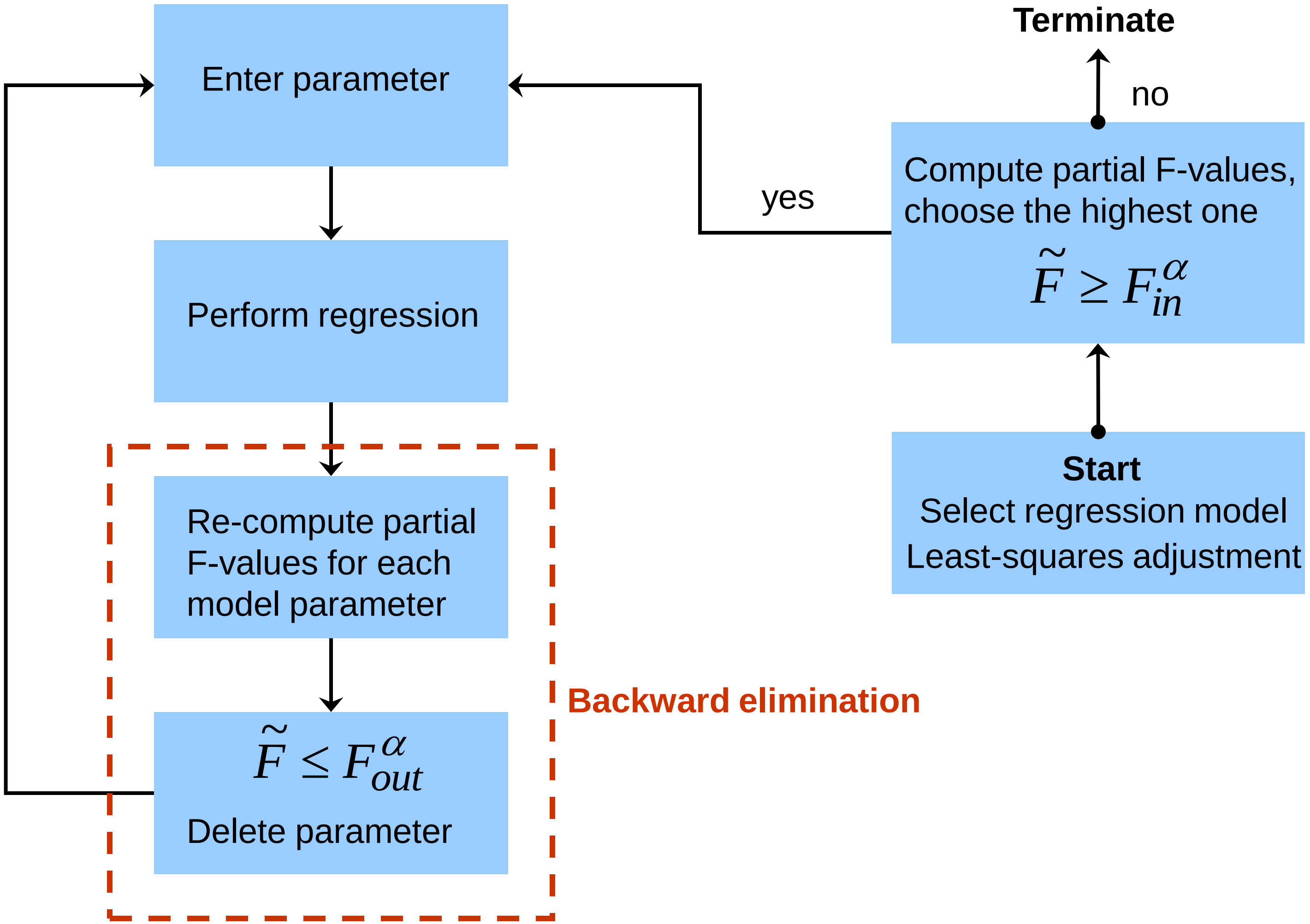
$$\tilde{F} = \frac{\hat{x}_I \, Q_{\hat{x}_I}^{-1} \, \hat{x}_I}{k\hat{\sigma}^2} \qquad \tilde{F} = \frac{\left[\sum(\ell - \hat{v})^2_{partial} - \sum(\ell - \hat{v})^2_{full}\right] \big/ k}{\left[\sum(\ell - \hat{v})^2_{full}\right] \big/ n - m}$$

- No need to repeat combined least-squares adjustment (first case)

Problems

- No unique answer (depends on initial selection, α)
- High parameter correlation may skew results
- Highly correlated parameters should be deleted (detection)

Stepwise Procedure



Start
Select regression model
Least-squares adjustment

Compute partial F-values, choose the highest one

$$\tilde{F} \geq F_{in}^{\alpha}$$

no

Terminate

yes

Enter parameter

Perform regression

Re-compute partial F-values for each model parameter

$$\tilde{F} \leq F_{out}^{\alpha}$$

Delete parameter

Backward elimination

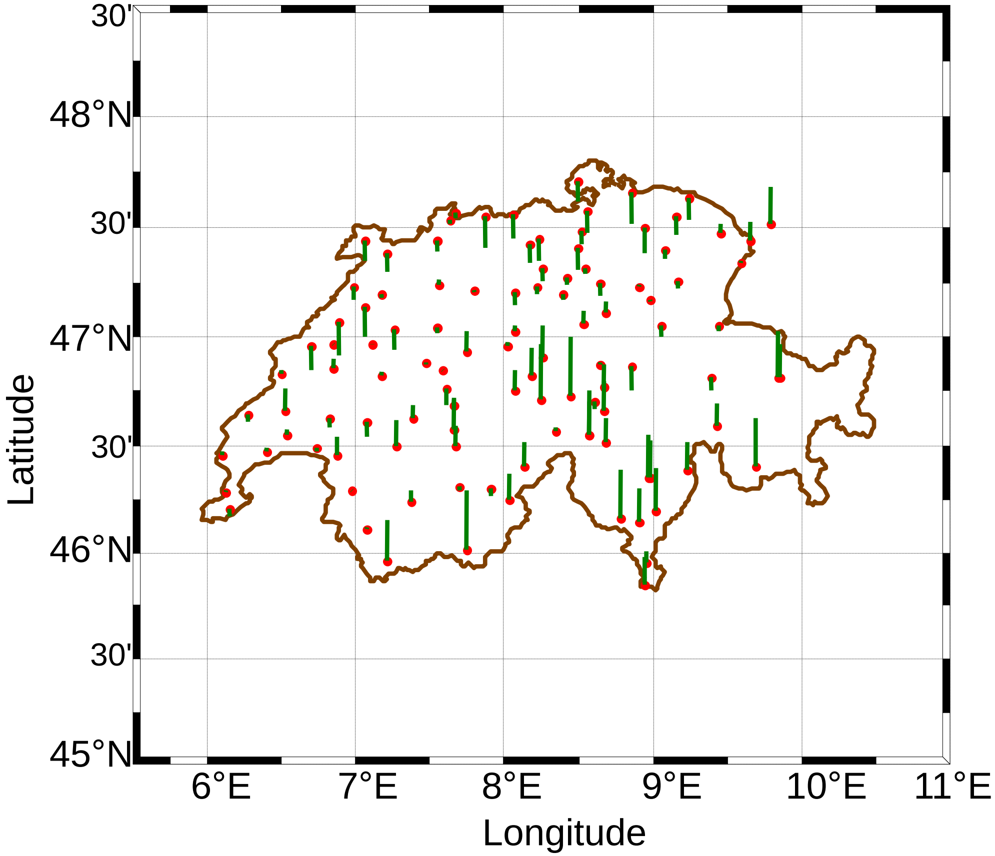
Description of Data

- 111 stations in **Switzerland**
- 343 km × 212 km region
- Form ‘residuals’:

$$\ell_i = h_i - H_i - N_i$$

Statistics of residuals before fit

min	-4.9 cm
max	19 cm
mean	1.1 cm
std	3.8 cm
rms	3.9 cm



GPS on Benchmarks (and residuals)

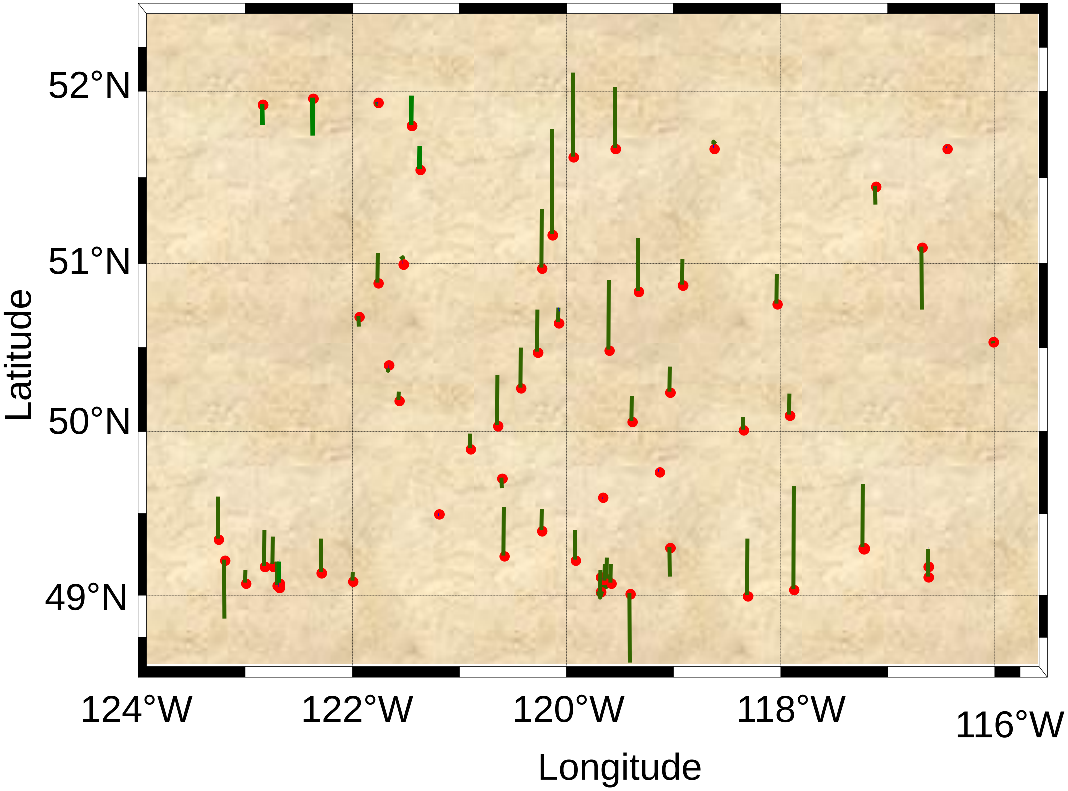
Description of Data

- 63 stations in **Southern British Columbia** & **Alberta**
- 495 km × 334 km region
- Form ‘residuals’:

$$\ell_i = h_i - H_i - N_i$$

Stats of residuals before fit

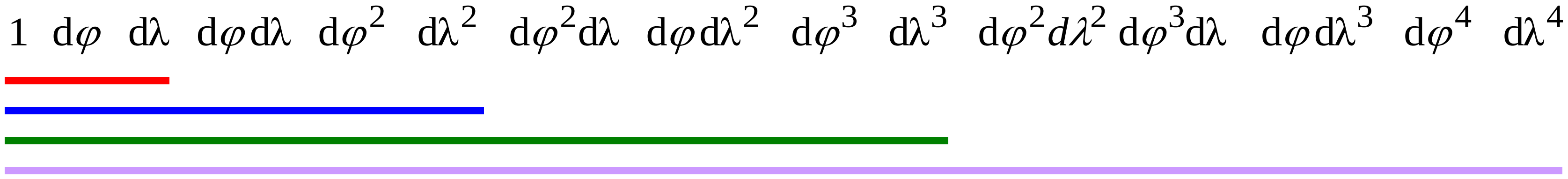
min	-17.1 cm
max	25.2 cm
mean	4.5 cm
std	8.1 cm
rms	9.3 cm



GPS on Benchmarks (and residuals)

Analytical Models

Nested bilinear polynomial series



$$1 \quad d\varphi \quad d\lambda \quad d\varphi d\lambda \quad d\varphi^2 \quad d\lambda^2 \quad d\varphi^2 d\lambda \quad d\varphi d\lambda^2 \quad d\varphi^3 \quad d\lambda^3 \quad d\varphi^2 d\lambda^2 \quad d\varphi^3 d\lambda \quad d\varphi d\lambda^3 \quad d\varphi^4 \quad d\lambda^4$$

Classic trigonometric-based polynomial fits

$$1 \quad \cos\varphi\cos\lambda \quad \cos\varphi\sin\lambda \quad \sin\varphi$$

$$1 \quad \cos\varphi\cos\lambda \quad \cos\varphi\sin\lambda \quad \sin\varphi \quad \sin^2\varphi$$

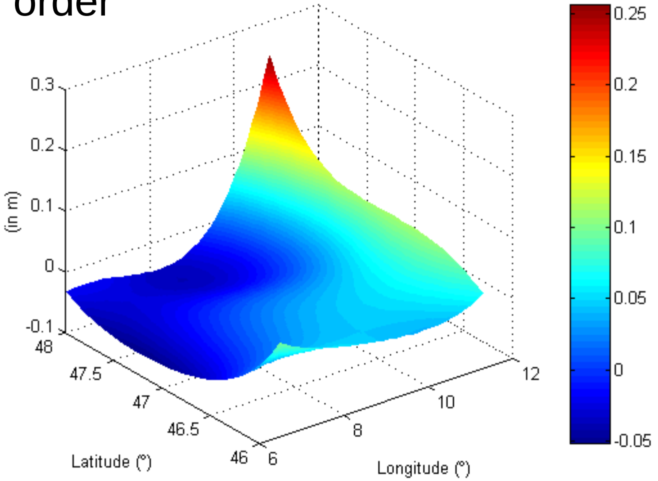
Differential similarity transformation

$$\cos\varphi\cos\lambda \quad \cos\varphi\sin\lambda \quad \sin\varphi \quad \frac{\sin\varphi\cos\varphi\sin\lambda}{W} \quad \frac{\sin\varphi\cos\varphi\cos\lambda}{W} \quad \frac{1-f^2\sin^2\varphi}{W} \quad \frac{\sin^2\varphi}{W}$$

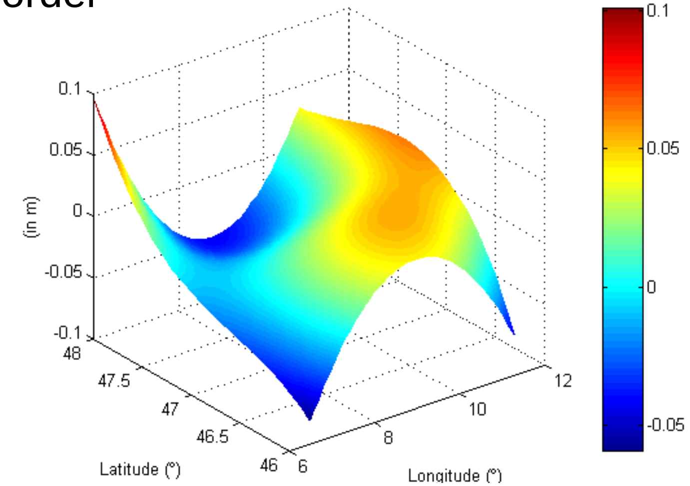
where, $W = \sqrt{1-e^2\sin^2\varphi}$

Analytical Models (polynomials)

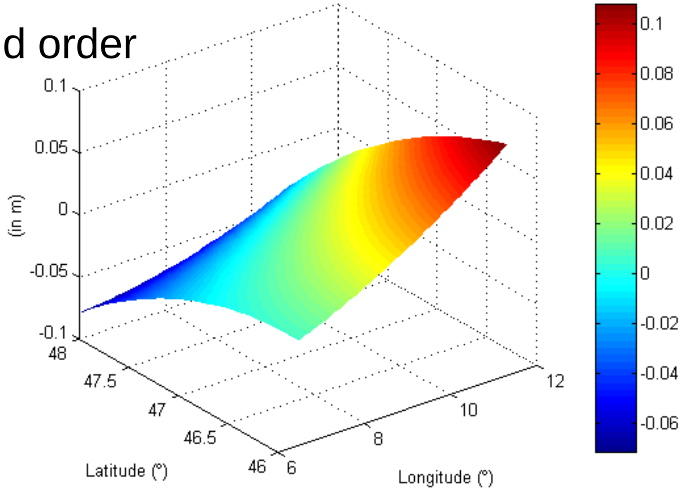
4th order



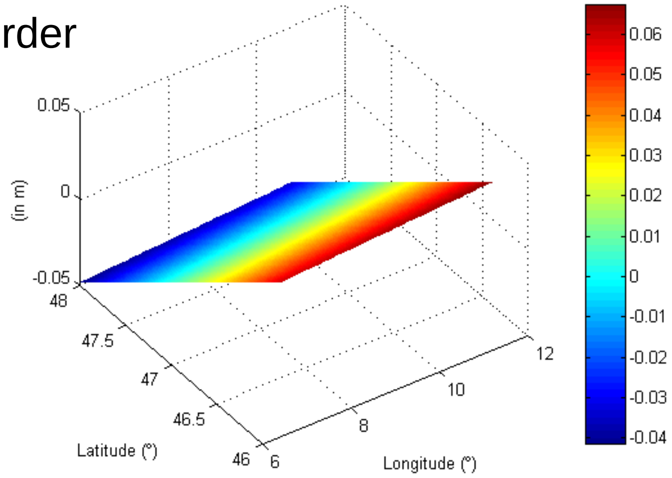
3rd order



2nd order

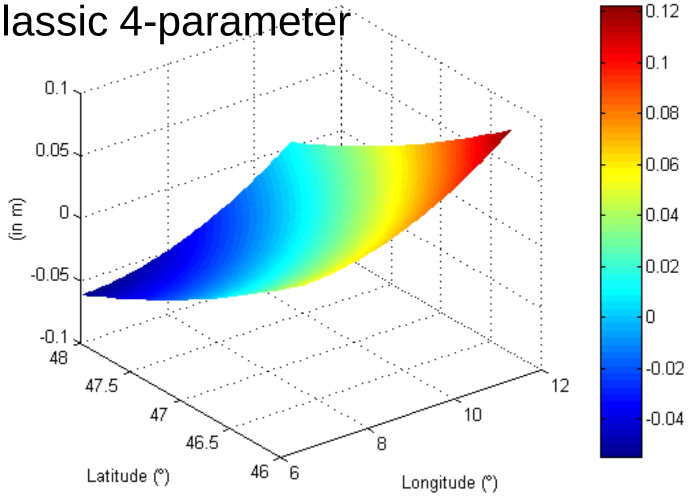


1st order

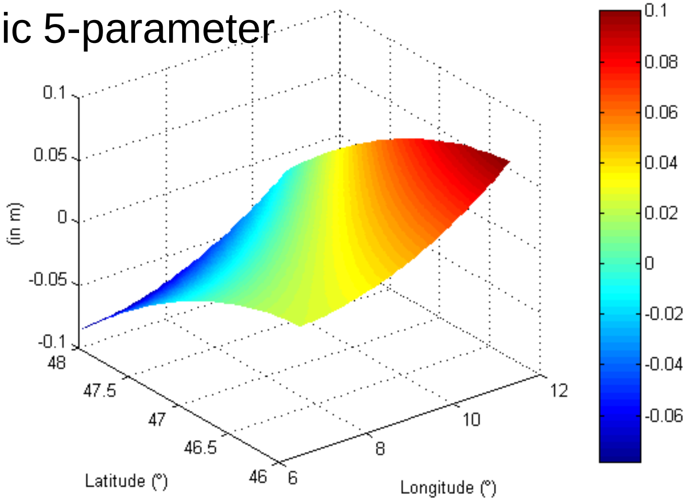


GEOMATICS ENGINEERING

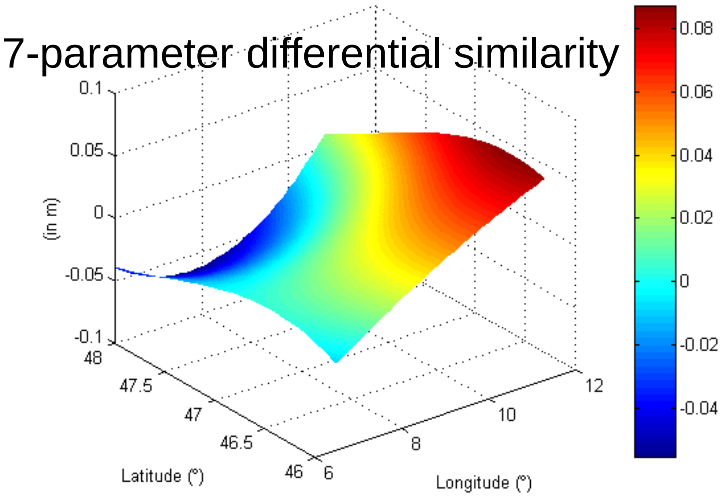
Other Analytical Models



Classic 4-parameter



Classic 5-parameter



7-parameter differential similarity

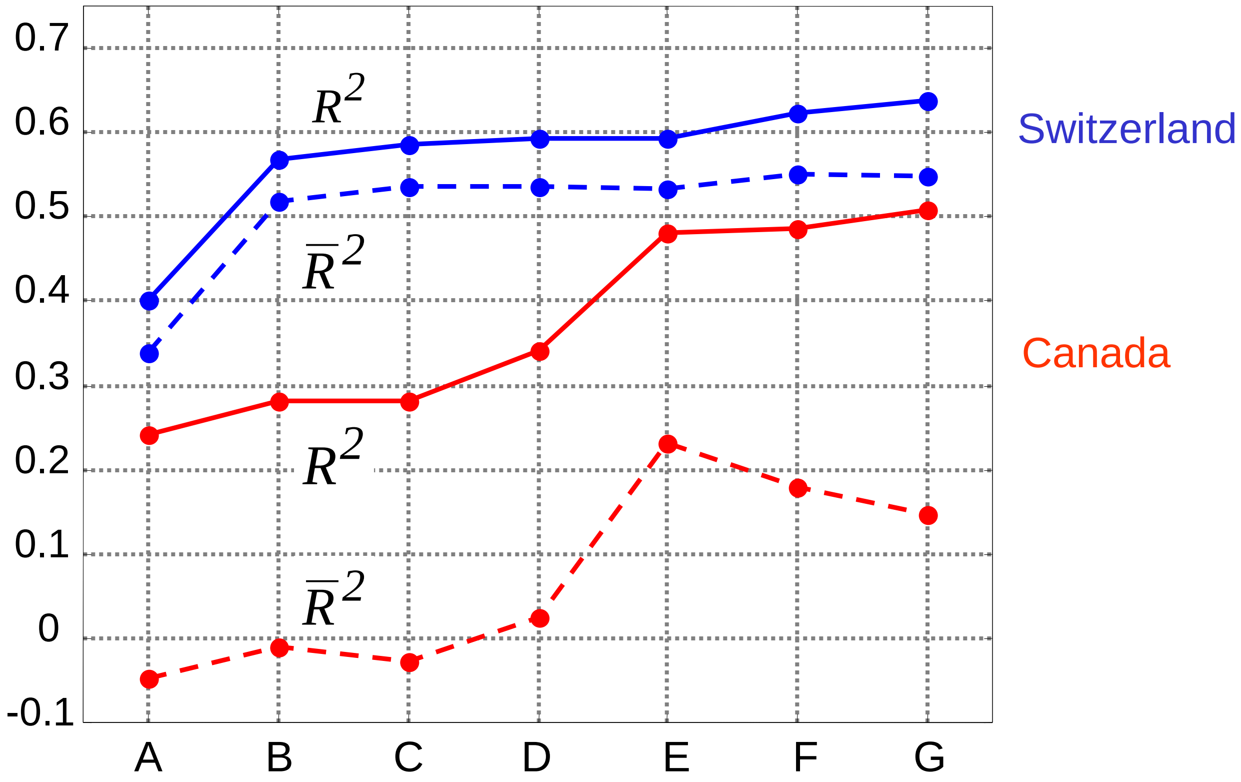
Notes

- all values shown in m
- GPS BMs in Switzerland used
- Full models shown (no parameters omitted)

Example - Coefficient of Determination

$$0 \le R^2 \le 1$$

model performance



A 1st order polynomial

B Classic 4-parameter

C Classic 5-parameter

D 2nd order polynomial

E Differential Similarity

F 3rd order polynomial

G 4th order polynomial

Empirical Testing

Conclusions

Residuals after fit

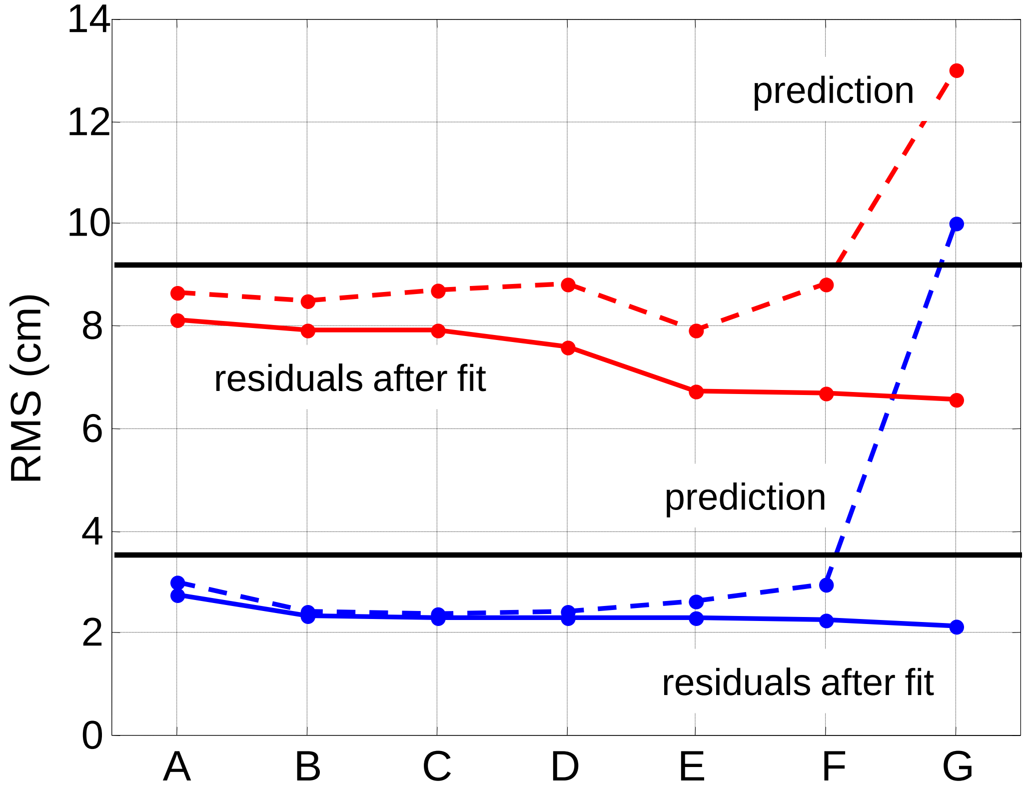
→ 4th order polynomial

Prediction (external test)

→ Any model except 4th order polynomial

Not enough of a difference between models to justify statistical parameter significance testing

→ use lowest order model



Switzerland Canada

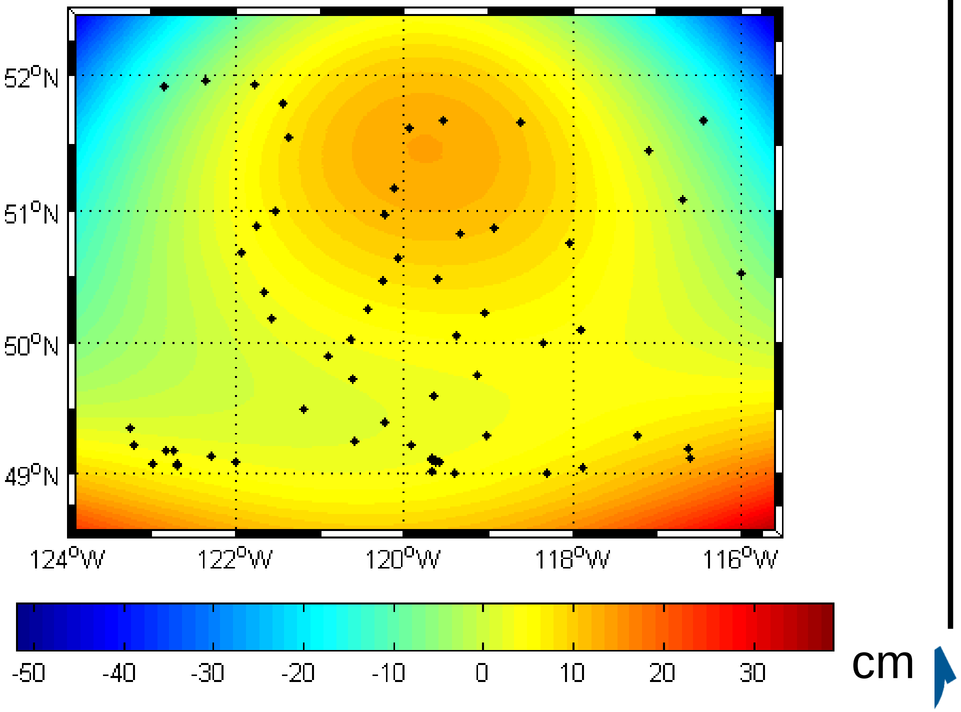
Results - Southern BC/AB

Differential Similarity Fit (7-parameters)

$$\cos\varphi\cos\lambda \quad \cos\varphi\sin\lambda \quad \sin\varphi \quad \frac{\sin\varphi\cos\varphi\sin\lambda}{W} \quad \frac{\sin\varphi\cos\varphi\cos\lambda}{W} \quad \frac{1-f^2\sin^2\varphi}{W} \quad \frac{\sin^2\varphi}{W}$$

Selection criteria

R^2	0.4805
$\overline{R}^2$	0.2311
$\sqrt{\hat{v}^T\hat{v}}$	53 cm
condition number	1.52×10^{12}
rms after fit	6.7 cm
rms (prediction)	7.9 cm



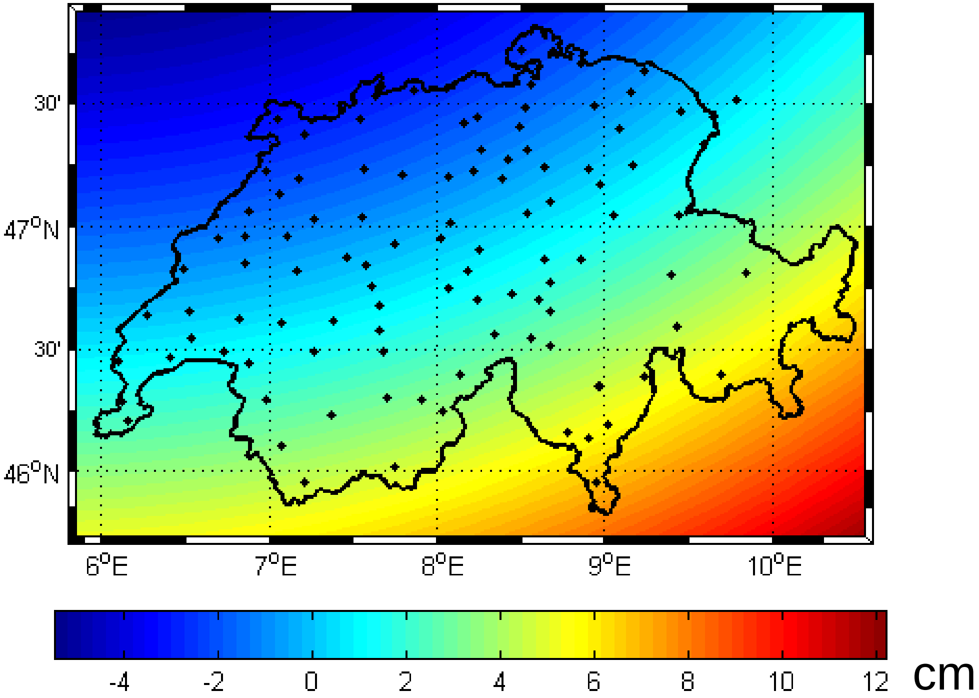
Results - Switzerland

Classic 4-parameter fit

$$1 \quad \cos\varphi\cos\lambda \quad \cos\varphi\sin\lambda \quad \sin\varphi$$

Selection criteria

R^2	0.5668
$\bar{R}^2$	0.5181
$\sqrt{\hat{v}^T\hat{v}}$	24.5 cm
condition number	2.77×10^7
rms after fit	2.4 cm
rms (prediction)	2.4 cm



Conclusions

- **Semi-automated procedure** for **comparing corrector surface models** and **assessing model performance** was presented
- ***Semi***
  - no unique straightforward solution
  - some user intervention required
- In most cases, the best test is cross-validation (prediction)
  - independent ‘external’ test
  - depends on quality of data
- When model parameters are highly correlated (as is the case with polynomial regression), statistical testing may not be conclusive
- Use orthogonal polynomials to eliminate problems with high correlation between parameters (i.e. Fourier Series)
- Procedure should include a **combination** of empirical **and** statistical testing