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ORIGINAL ARTICLE

Estimation of the zero-height geopotential level W_o^{LVD} in a local vertical datum from inversion of co-located GPS, leveling and geoid heights: a case study in the Hellenic islands

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Abstract The estimation of the zero-height geopotential level of a local vertical datum (LVD) is a key task towards the connection of isolated physical height frames and their unification into a common vertical reference system. Such an estimate resolves, in principle, the 'ambiguity' of a traditional crust-fixed LVD by linking it with a particular equipotential surface of Earth's gravity field under the presence of an external geopotential model. The aim of this paper is to study the estimation scheme that can be followed for solving the aforementioned problem based on the joint inversion of co-located GPS and leveling heights in conjunction with a fixed Earth gravity field model. Several case studies with real data are also presented that provide, for the first time, precise estimates of the LVD offsets for a number of Hellenic islands across the Aegean and Ionian Sea.

Keywords Local vertical datum · Zero-height level · Vertical datum unification · Geopotential · Hellenic islands

1 Introduction

The primary component of any vertical reference system for physically meaningful heights is an equipotential surface of Earth's gravity field, which represents what is commonly called *a vertical datum*. Regardless of the particular type of physical heights within a vertical reference system, the underlying vertical datum defines an unequivocal *zero-height level* relative to which terrestrial vertical positions can be obtained by geodetic leveling techniques. It should be kept in mind, though, that a geometrical interpretation of such vertical positions may not be always feasible (i.e. dynamic heights) or it may be associated not with the vertical datum per se but with other auxiliary non-equipotential reference surfaces of rather abstract structure (i.e. normal heights). In fact, only the use of orthometric heights theoretically permits a simple geometrical relationship between physical vertical positions and their inherent vertical datum (Heiskanen and Moritz 1967, ch. 4). Nevertheless, the role of a vertical datum is equally important for all physical height types that quantify absolute vertical positions in terms of (scaled) geopotential differences with respect to a conventional zero-height level.

In geodetic theory the rigorous definition of a vertical datum adheres to the fundamental equation

$$W(x, y, z) = W_o$$

which specifies a single equipotential surface of Earth's gravity field in terms of a reference geopotential value. The choice of a particular value W_o is, more or less, arbitrary and it relies on a conventional postulation under certain geodetic and/or oceanographic criteria, most of which are related to the well-known notion of the *Gauss-Listing geoid* (Heck and Rummel 1990; Hipkin 2003; Heck 2004; Burša et al. 2007). However, the traditional methodology that has been followed in practice for the realization of most vertical datums neither incorporates an a priori value for W_o nor does it entail a gravity field model $W(\cdot)$ for their spatial representation. Instead of a virtual realization scheme as suggested from the previous definition, the establishment of a vertical datum has been commonly based on a more tangible approach by constraining one, or more, terrestrial stations to be situated at known vertical distance from an unknown reference equipotential surface. From a practical viewpoint,

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the adoption of such *origin points* with fixed height, yet poorly known or unknown gravity potential, has been sufficient for setting up local vertical datums (LVDs) and physical height frames (PHFs) over a regional or even continental scale via terrestrial leveling techniques. In most cases, the established ‘crust-fixed’ LVDs refer to the long-term average of the local mean sea level (MSL) that is observed at one, or more, tide-gauge stations which are vertically tied to their corresponding origin points. Evidently, the lack of a reference geopotential value for a LVD does not prohibit the precise determination of vertical positions through relative measurements from existing benchmarks, neither the connection with other vertical datums using direct or indirect leveling ties.

Despite the absence of the fundamental parameter W_o^{LVD} from the primary realization of most vertical datums, the knowledge of an optimal estimate $\hat{W}_o^{\mathrm{LVD}}$ provides a useful tool for vertical positioning problems and other related applications. Theoretically, such an estimate allows the transformation of LVD heights to any other vertical reference system that is specified by a known geopotential value with respect to a conventional gravity field model. The connection of isolated PHFs and their unification into a common vertical datum, as well as the harmonization of terrestrial gravity databases from different vertical datum zones, are well-known geodetic problems whose treatment relies on the implementation of such a height transformation, and thus on the knowledge of the reference geopotential value for the corresponding LVDs (or, at least, of their relative geopotential offsets).

Essentially, an estimated value $\hat{W}_o^{\mathrm{LVD}}$ provides an external identification for a LVD with respect to an available gravity field representation, in a similar sense that a set of translation/rotation transformation parameters specify a 3D regional spatial reference frame relative to a global geocentric reference frame. This identification should be understood in terms of an equipotential surface which originates from the existing gravity field representation and it optimally approximates the zero-height level of the LVD at a number of known benchmarks.

The estimation accuracy of $\hat{W}_o^{\mathrm{LVD}}$ holds an important role for the practical significance of the above identification scheme of an local vertical datum. If an errorless geopotential model is supposedly available, then the error variance of $\hat{W}_o^{\mathrm{LVD}}$ will reflect the *LVD stability* over the leveling benchmarks that were used in the estimation procedure. In reality, though, the error variance of $\hat{W}_o^{\mathrm{LVD}}$ yields a quality measure that characterizes the consistency between the LVD heights and the auxiliary geopotential model due to their inherent errors, and it signifies the minimum expected uncertainty in absolute vertical positioning (within the particular vertical datum) from the combined use of an Earth gravity field model with spatial positioning techniques (e.g. GPS). We will return to these accuracy-related issues later in the paper.

An important problem that may occur in PHFs associated with regional LVDs is the existence of systematic distortions in their vertical coordinates. In the presence of such distortions, one may question the rationale of LVD identification in terms of a single geopotential estimate for its zero-height level. For example, there is a notable difficulty in the interpretation of $\hat{W}_o^{\mathrm{LVD}}$ if the LVD was originally established by an over-constrained adjustment of a leveling network to several tide-gauge stations. The fixed origin points, in such cases, realize a zero-height level that is not theoretically matched to a single equipotential surface due to the distorting effect of sea surface topography. Nevertheless, the key role of $\hat{W}_o^{\mathrm{LVD}}$ is not necessarily to identify the ‘true’ equipotential surface of a LVD, but rather to specify an equipotential surface that originates from a current gravity field model and it optimally fits the known vertical positions of terrestrial leveling benchmarks.

The estimation of the reference geopotential value of an existing vertical datum has been an active area of geodetic research for many years. Various solution strategies have been developed that rely on the synergy between a gravity field model with other terrestrial and space geodetic data. Their corresponding algorithms comprise well-known formulae and linearized modeling procedures from physical geodesy, which can be classified into two main approaches depending on the available data types and the treatment of Earth’s geopotential signal. The latter may either be assumed fully known beforehand, or it can be estimated through an integrated approach using local gravity anomaly data that have been referenced to their own vertical datum(s).

More specifically, the first approach exploits the availability of high-quality geopotential (or geoid/quasi-geoid) models along with precise GPS measurements at a number of leveling benchmarks, which can jointly resolve the reference gravity potential of a LVD, or equivalently its offset from a conventional level $W(\cdot) = W_o$, with a statistical accuracy better than 1 $\mathrm{m}^2/\mathrm{s}^2$ or within a few cm in terms of terrestrial spatial scale. Numerous such studies exist in the geodetic literature for different geographical regions, including the Baltic Sea countries (Grafarend and Ardalan 1997; Pan and Sjöberg 1998; Ardalan et al. 2002), New Zealand (Amos and Featherstone 2009; Tenzer et al. 2011), Iran (Ardalan and Safari 2005), Pakistan (Sadiq et al. 2009), South-East Asia countries (Kasenda and Kearsley 2003), the North American Great Lakes (Jekeli and Dumrongchai 2003), and a multitude of other regional and national vertical datums (Rapp 1994; Burša et al. 1999, 2001, 2004). The rationale of this approach is primarily suitable for single-LVD analysis, yet its simultaneous implementation over different geographical zones is also useful for unifying multiple LVDs with respect to a common reference surface that is specified either by a conventional W_o value (Burša et al. 2004) or through an external W_o-free geoid model (Rapp 1994).

The second approach is based on an extended formulation of the geodetic boundary value problem (GBVP) using 'biased' gravity anomaly data from multiple LVD zones which should theoretically cover the entire globe. The fundamental unknowns are the geopotential offsets among the involved LVDs, while their optimal estimates can be jointly obtained through a modified least-squares (LS) adjustment using an external constraint that specifies a 'global' zero-height level. A formal analysis of this approach is given in Rummel and Teunissen (1988) whose work was essentially a more rigorous continuation of earlier studies performed by Colombo (1980) and Hajela (1983); for more details, see also Heck and Rummel (1990). Additional contributions and important theoretical extensions have been made by many authors, including Xu and Rummel (1991), Rapp and Balasubramania (1992), Balasubramania (1994), Sansò and Usai (1995), Lehmann (2000), Sansò and Venuti (2002), Sacerdote and Sansò (2004) and Ardalan et al. (2010) among others. For some examples on the simplified implementation of this approach using real height data in Fennoscandia, see Pan and Sjöberg (1998) and Nahavandchi and Sjöberg (1998), whereas several numerical investigations about its quality performance based on global simulation tests can be found in Xu and Rummel (1991), Xu (1992) and van Onselen (1997).

The focus of this study lies on the first approach, namely the estimation of W_o^{LVD} under a fixed model of Earth's gravity field that is directly employed for analyzing heterogeneous height data over a regional network of LVD benchmarks. The underlying methodology does not require the knowledge of gravity anomaly data from different LVD zones, and it is thus suitable for geographically isolated regions with limited gravity coverage. Although many aspects of the joint inversion of co-located geometric and physical heights for W_o^{LVD} estimation have been already discussed in the geodetic literature (Jekeli 2000; Burša et al. 2001), some key issues are elucidated herein in more detail; see Sect. 2. A number of case studies with real data are also presented (Sect. 3) which reveal, for the first time, precise estimates of the zero-height geopotential level in several Hellenic islands across the Aegean and Ionian Sea. Our results are based on recent GPS measurements over a network of 483 control points that are located within the selected islands (and they are directly tied to each island's LVD) and the use of the EGM2008 geopotential model (Pavlis et al. 2008). The corresponding vertical datums were established by the Hellenic Military Geographic Service through the fixed MSL at a single tide-gauge station in each island (Antonopoulos 1999), and our results suggest that they all lie consistently lower from the conventional global geoidal surface ($W_o = 62636856.00\,\mathrm{m^2/s^2}$, IERS 2010) by an amount ranging from a few cm up to several dm.

2 Determination of $\hat{W}_o^{\mathrm{LVD}}$

2.1 General aspects

The estimation of the zero-height level of an existing vertical datum relies on the knowledge of the spatial position for a number of leveling benchmarks (or other control points that are tied to the particular vertical datum) and a detailed representation of Earth's gravity field over the test area. The latter is usually based on a high-resolution spherical harmonic model which may be augmented by additional local gravity data, whereas the 3D spatial coordinates of the LVD benchmarks are obtained by space geodetic techniques with respect to a geocentric reference frame. Note that to ensure a bias-free estimate of W_o^{LVD}, the permanent tide effects and other temporal height variations should be properly taken into account and treated consistently in all data sets used within the estimation procedure (Ekman 1989; Mäkinen and Ihde 2009).

The simplest approach for the recovery of W_o^{LVD} is applicable when the vertical coordinates of the leveling benchmarks are given as geopotential numbers with respect to the underlying vertical datum. In this case, an estimate of the fundamental LVD parameter can be computed from the equation

$$\hat{W}_o^{\mathrm{LVD}} = W(P) + c(P), \tag{1}$$

where $c(P)$ is the known geopotential number at a leveling benchmark P, and $W(P)$ is the gravity potential computed at the same point through an external model. Obviously, if P coincides with the origin point of the vertical datum, then the estimation of W_o^{LVD} is simply a matter of evaluating Earth's gravity potential $W(\cdot)$ at that particular point.

If more than one LVD benchmarks are used, then the previous approach leads to an averaging procedure which yields an improved estimate in terms of the sample mean

$$\hat{W}_o^{\mathrm{LVD}} = \frac{1}{K}\sum_{i=1}^{K}[W(P_i) + c(P_i)]. \tag{2}$$

In case that error co-variances are available for the geopotential numbers and/or the gravity potential values, a more rigorous averaging formula can be derived via a weighted LS adjustment of the basic observation equation: $y_i = W(P_i) + c(P_i) = W_o^{\mathrm{LVD}} + e_i$.

The zero-degree term of Earth's gravity potential must be properly considered for the implementation of the above procedure. Specifically, if the normal and disturbing geopotential components are used for synthesizing $W(P_i)$ at each benchmark, then the term $(GM - GM')/r(P_i)$ needs to be included in the determination of the total gravity potential value. The quantities GM and GM' denote the geocentric gravitational constant of the actual Earth and its reference ellipsoid, while $r(P_i)$ is the geocentric radial distance of the corresponding benchmark. Theoretically, a centrifugal zero-degree term is also required if the rotational velocity of the reference ellipsoid is inconsistent with the up-to-date value of Earth's rotational velocity; however, the effect of such a residual term is negligible and it does not need to be further considered in this paper.

Remark 1 The uncertainty of Earth's geocentric gravitational constant inflicts a relative error in the order of 10^{-9} to the estimate of the vertical datum parameter W_o^{LVD}. The dominant contribution stems from the uncertainty of the zero-degree term of Earth's gravity potential, which affects the accuracy of the result by Eq. (2) as follows:

$$\sigma^2_{\hat{W}_o^{\mathrm{LVD}}} = \sigma^2_{GM}\left[\frac{1}{K}\sum_{i=1}^{K}\frac{1}{r(P_i)}\right]^2 \approx \frac{\sigma^2_{GM}}{R_o^2}. \tag{3}$$

Based on some representative values of the *GM* uncertainty and the mean radius of the Earth, e.g. $\sigma_{GM} = 8\times 10^5\,\mathrm{m}^3/\mathrm{s}^2$ and $R_o = 6363672.6\,\mathrm{m}$ (Petit and Luzum 2010), the last equation yields an accuracy level of $\sigma_{\hat{W}_o^{\mathrm{LVD}}} \approx 0.13\,\mathrm{m}^2/\mathrm{s}^2$ which corresponds to a spatial (vertical) uncertainty of more than 1 cm for the zero-height surface of the vertical datum. *This signifies the absolute accuracy limitations in vertical positioning with respect to any equipotential reference surface that is specified by a given geopotential value.*

In practice, the geopotential numbers $c(P_i)$ may not always be available since the LVD vertical coordinates are often provided in terms of normal or orthometric heights. In the first case, the corresponding geopotential numbers are easily re-computable using the normal height and the geodetic latitude for each leveling benchmark, along with the defining parameters of the associated normal gravity field. Hence, the general estimator from Eq. (2) can still be applied in the equivalent form

$$\hat{W}_o^{\mathrm{LVD}} = \frac{1}{K}\sum_{i=1}^{K}\left[W(P_i) + \bar{\gamma}_i H^*(P_i)\right], \tag{4}$$

where $H^*(P_i)$ is the known normal height and $\bar{\gamma}_i$ denotes the average normal gravity along the normal plumbline between the telluroid and the reference ellipsoid. The value of $\bar{\gamma}_i$ is computed from a truncated (usually up to second-order term) latitude-dependent power series of $H^*(P_i)$ that incorporates the fundamental parameters of the normal gravity field (Heiskanen and Moritz 1967, p. 170).

Conversely, if orthometric heights are provided in the LVD, the estimation of W_o^{LVD} can be performed with the modified formula

$$\hat{W}_o^{\mathrm{LVD}} = \frac{1}{K}\sum_{i=1}^{K}\left[W(P_i) + \bar{g}_i H(P_i)\right] \tag{5}$$

where $H(P_i)$ is the known orthometric height (usually given by its Helmert-type approximation) and $\bar{g}_i$ denotes the average gravity along the physical plumbline between the Earth's surface and the LVD reference surface. The computation of $\bar{g}_i$ is commonly based on the knowledge of surface gravity $g(P_i)$ at each benchmark and the use of a conventional model for the crust density and terrain roughness of the topographic masses over the test area. In the case of Helmert orthometric heights, for example, the term $\bar{g}_i$ should be calculated from the simplified Poincare-Prey gravity reduction (Heiskanen and Moritz 1967, pp. 163–167). For other approximation schemes related to orthometric height computation, see Kingdon et al. (2005), Tenzer et al. (2005), and Santos et al. (2006). Alternatively, if a Bouguer gravity anomaly model is available for the test area, the known orthometric heights could be first transformed to their equivalent normal heights (Heiskanen and Moritz 1967, pp. 327–328) and then used within Eq. (4) for estimating the LVD datum parameter, as suggested in Jekeli (2000, p. 18).

The previous framework summarizes what might be called the conventional approach for W_o^{LVD} estimation, where the gravity potential needs to be evaluated only at control points on the Earth's surface. An alternative estimation scheme based on the spheroidal free-air potential reduction at the LVD's zero-height level can be found in Grafarend and Ardalan (1997), Ardalan et al. (2002) and Ardalan and Safari (2005).

2.2 Estimation accuracy of the LVD datum parameter

The estimation accuracy of $\hat{W}_o^{\mathrm{LVD}}$ is mainly affected by two error sources: (a) the uncertainty of Earth's gravity field model $W(\cdot)$, and (b) the uncertainty of the LVD vertical coordinates. It needs to be emphasized that these error sources must be clearly distinguished and (if possible) their effects should be separately quantified, as they have different physical significance for the final result and its estimation accuracy.

The propagated random errors from the LVD vertical coordinates into $\hat{W}_o^{\mathrm{LVD}}$ reflect the *inner accuracy or spatial stability* of the vertical over the network of control points $\{P_i\}$. Hence, in the hypothetical case of an errorless geopotential model, the error variance of $\hat{W}_o^{\mathrm{LVD}}$ represents the uncertainty of the LVD zero-height level with respect to an equipotential surface that optimally fits a set of known benchmarks; see Fig. 1.

The random errors in the adopted geopotential model cause an increase on the error variance of $\hat{W}_o^{\mathrm{LVD}}$ that reflects the additional uncertainty with which the reference surface $W(\cdot) = \hat{W}_o^{\mathrm{LVD}}$ is tied to the real Earth's gravity field. A characteristic example is the contribution of the *GM* uncertainty which dictates that the recovered reference equipotential surface of a LVD cannot be presently known to an

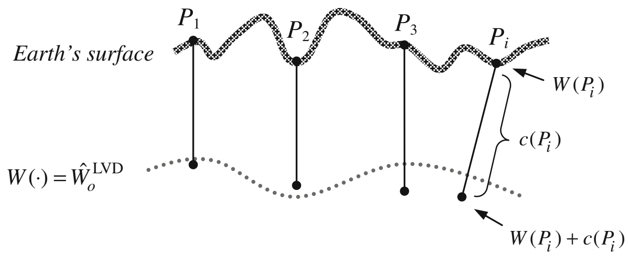

Fig. 1 Assuming an error-free geopotential model $W(\cdot)$, the error variance of the estimated parameter $\hat{W}_o^{\mathrm{LVD}}$ reflects the inner accuracy of the local vertical datum over its particular realization of leveling benchmarks with known vertical coordinates $c(P_i)$

absolute accuracy better than 1–2 cm, in a spatial (vertical) sense (see Remark 1).

The presence of systematic errors in the geopotential model, especially at spatial wavelengths of larger size than the extent of the test area, may lead to a biased LVD parameter in the sense shown in Fig. 2. Depending on their magnitude and spatial behavior, such systematic effects provide a crucial error source for the result of Eq. (2) (similarly for (4) or (5)) whose treatment remains largely an open problem in vertical datum studies (see also Sect. 2.5).

Existing systematic errors in the LVD vertical coordinates obviously affect the estimated value $\hat{W}_o^{\mathrm{LVD}}$, yet they do not diminish its importance as a 'datum identification parameter' for the given LVD realization (see Fig. 1). Note that any consideration about the bias of $\hat{W}_o^{\mathrm{LVD}}$ from a true value W_o^{LVD} relies on the presumption that such a true value exists and it is recoverable, within the data noise limits, from a set of vertical coordinates and external geopotential information. If, for example, the underlying physical heights originate from an overconstrained LVD adjustment to more than one tide-gauge stations, it is rather ineffectual or even meaningless to be concerned with the inherent bias $\hat{W}_o^{\mathrm{LVD}} - W_o^{\mathrm{LVD}}$.

The uncertainty in the 3D geometrical coordinates of the test points, especially in their vertical component, may create a notable error contribution to the geopotential values $W(P_i)$ that can consecutively affect the estimation accuracy of the LVD datum parameter. For example, a geocentric radial error of 3 cm in the spatial location of each benchmark inflicts an error of about 0.3 m^2/s^2 on their computed gravity potential values. Depending on the nature (random or systematic) of this error, its propagated effect on $\hat{W}_o^{\mathrm{LVD}}$ will be either statistically reduced by a factor of $\sqrt{K}$ (where K is the number of test points) or it will inflict a constant bias of equal magnitude. In most cases, however, such spatial positioning errors are practically masked under the uncertainty imposed by the noise level of the external geopotential model itself.

2.3 Use of an external geoid model

Often in LVD studies, instead of computing the gravity potential at leveling benchmarks with known 3D spatial positions, an external geoid or quasi-geoid model is alternatively used for the recovery of the unknown parameter W_o^{LVD}. The estimation procedure relies on the well-known relationship between geometric and physical heights over the Earth's surface, and it is theoretically equivalent to the approach that was outlined in Sect. 2.1, as long as the geoid/quasi-geoid model and the geopotential signal $W(\cdot)$ originate from the same gravity field representation. The advantage of the geoid-based approach is that it does not require additional surface gravity information at the control points when orthometric heights need to be analyzed for the determination of their zero-height level (i.e. compared to Eq. (5)).

Here, we study the case where orthometric heights are employed along with a gravimetric geoid and GPS heights, which forms the basic framework for the case studies that will be later presented in this paper. The main aspects of the following methodology are also applicable, with minor modifications, for LVD studies based on Molodensky's normal heights and quasi-geoid models (e.g. Burša et al. 2004).

In the absence of any random or systematic errors, the orthometric (H_i), ellipsoidal (h_i) and geoid (N_i) heights at any terrestrial benchmark should fulfill the theoretical constraint (expressed in linearized form):

$$h_i - H_i - N_i = \frac{W_o - W_o^{\mathrm{LVD}}}{\gamma_i}, \qquad (6)$$

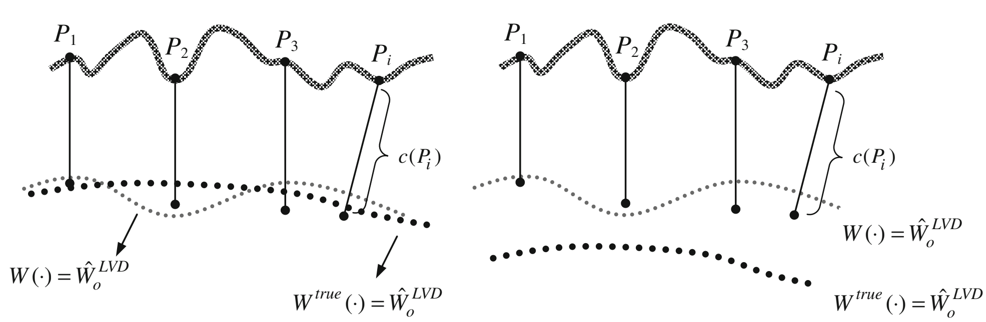

Fig. 2 Schematic illustration of the 'biased' and 'unbiased' character of the estimated LVD zero-height level due to the presence of systematic errors in the external geopotential model

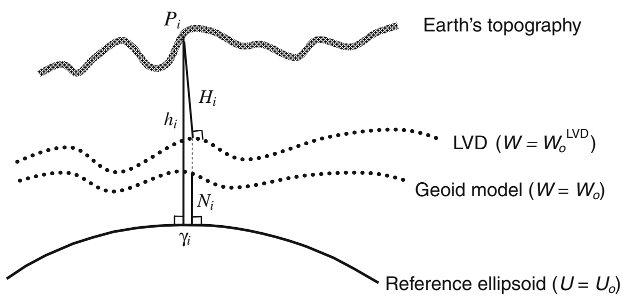

Fig. 3 The relationship among ellipsoidal, orthometric and geoid heights, and their corresponding reference surfaces

where W_o^{LVD} and W_o denote the gravity potential on the LVD reference surface and the equipotential surface that is realized by the geoid model, respectively. The value γ_i corresponds to the normal gravity on the reference ellipsoid, and it is computed through Somigliana's formula at the known geodetic latitude of the particular benchmark (see Fig. 3).

The following remarks should be stated regarding the validity of Eq. (6).

- The deflection of the vertical at the LVD benchmark is ignored, so that all height types are treated as geometrically straight distances along the same normal direction. The approximation error caused in Eq. (6) is below the mm-level which is considered negligible for most geodetic applications.
- A more rigorous formulation of Eq. (6) should employ the gravity g_i on the geoidal surface, instead of the normal gravity γ_i on the reference ellipsoid. Assuming that the gravity anomaly $\Delta g_i = g_i - \gamma_i$ does not exceed a maximum of 500 mGal, the approximation error in Eq. (6) is typically below the mm-level and thus also negligible for most geodetic applications.
- The geoid height contains the contribution of a zero-degree term which accounts for the mass difference between the actual Earth (GM) and its reference ellipsoid (GM'), and it also specifies the particular equipotential surface that is realized by the geoid model (see Smith 1998). This zero-degree term entails the a priori choice of a reference geopotential value W_o which explicitly appears in Eq. (6).
- The value W_o should be numerically close to W_o^{LVD} in order to minimize the linearization error of Eq. (6). In practice, W_o is selected either as an optimal approximation to the global mean sea level from satellite altimetry data (e.g. Burša et al. 2002) or as a postulated parameter of a global vertical reference system (GVRS) that needs to be connected with the LVD via a geoid model. In any case, the linearization error of Eq. (6) is practically negligible (< 1 mm) for any 'reasonable' choice of W_o.
- If the geoid model does not incorporate a zero-degree term, the theoretical constraint in Eq. (6) should take the form (based on a spherical approximation):

$$h_i - H_i - \tilde{N}_i = \frac{1}{\gamma_i}\left(\frac{GM - GM'}{R_o} + U_o - W_o^{\text{LVD}}\right),$$

where $\tilde{N}_i$ is the geoid height without the contribution of a zero-degree term, U_o is the normal gravity potential on the reference ellipsoid, and R_o denotes the mean Earth radius. In principle, the equipotential surface realized by the geoid model will now correspond to the geopotential value:

$$W_o = U_o + \frac{GM - GM'}{R_o}.$$

- The geometric height and the geoid height should refer to a common geodetic reference system with respect to a single Earth reference ellipsoid. Datum inconsistencies between these height types need to be accounted for and either eliminated beforehand through a suitable transformation, or included as additional parameterized corrections within the general model of Eq. (6). A compendium of height transformation formulae between different geodetic reference systems can be found in Kotsakis (2008).
- Effects due to the permanent tidal deformation of the Earth's crust and its gravity field, as well as other temporal height variations, need to be consistently modeled beforehand in all data types within Eq. (6).

Using Eq. (6) as an 'observation equation' over a network of GPS/leveling stations and also assuming the same height noise level at all points, the following LS estimate is obtained for the LVD datum parameter:

$$\begin{aligned}\hat{W}_o^{\text{LVD}} &= W_o - \frac{\sum_{i=1}^{K} \frac{1}{\gamma_i}(h_i - H_i - N_i)}{\sum_{i=1}^{K} \frac{1}{\gamma_i^2}} \\ &= W_o - \frac{\sum_{i=1}^{K} \frac{1}{\gamma_i^2}\gamma_i(h_i - H_i - N_i)}{\sum_{i=1}^{K} \frac{1}{\gamma_i^2}}. \end{aligned} \tag{7}$$

The error variance of the above estimate is given from the equation

$$\sigma^2_{\hat{W}_o^{\text{LVD}}} = \frac{\sigma^2}{\sum_{i=1}^{K} \frac{1}{\gamma_i^2}}, \tag{8}$$

where σ^2 represents the total height data accuracy at each LVD benchmark, i.e. $\sigma^2 = \sigma_h^2 + \sigma_H^2 + \sigma_N^2$.

The previous result has the form of a weighted averaging of the 'geopotentialized' residuals $\gamma_i(h_i - H_i - N_i)$ whose non-zero trend is theoretically caused by the offset between the equipotential surfaces $W(\cdot) = W_o^{\mathrm{LVD}}$ and $W(\cdot) = W_o$. A more rigorous implementation of this averaging scheme requires the use of geoidal gravity values (instead of normal gravity values) in order to properly account for the non-parallelism of the involved equipotential surfaces of Earth's gravity field. Nevertheless, in practice, the last two equations can be safely reduced to the simplified expressions:

$$\hat{W}_o^{\mathrm{LVD}} = W_o - \gamma_{\mathrm{ave}} \frac{\sum_{i=1}^{K} (h_i - H_i - N_i)}{K} \tag{9}$$

and

$$\sigma^2_{\hat{W}_o^{\mathrm{LVD}}} = \gamma^2_{\mathrm{ave}} \frac{\sigma^2}{K}, \tag{10}$$

where γ_{ave} corresponds to the average normal gravity on the reference ellipsoid over the test area, and K denotes the number of the GPS/leveling benchmarks. The difference in the results that are computed by Eqs. (7) and (9) is not larger than $10^{-3}\,\mathrm{m}^2/\mathrm{s}^2$, even for test networks extending from the equator up to the poles.

The statistical accuracy of $\hat{W}_o^{\mathrm{LVD}}$ from the above procedure depends on the noise level and the amount of the heterogeneous height data. As an example, for a total noise level of $\sigma = 11\,\mathrm{cm}$ (which is roughly composed by $\sigma_h = 3\,\mathrm{cm}$, $\sigma_H = 2\,\mathrm{cm}$ and $\sigma_N = 10\,\mathrm{cm}$), the LVD datum parameter can be estimated with a formal accuracy of about $\pm 0.5\,\mathrm{m}^2/\mathrm{s}^2$ using only four control points ($K = 4$). This corresponds to a relative error of 10^{-8} which conforms with the statistical uncertainty of several $\hat{W}_o^{\mathrm{LVD}}$ estimates that have been determined for various LVDs around the world (Burša et al. 2001, 2004).

Note that the previous formulae neglect the effect of heteroscedastic and/or geographically correlated errors in each height dataset, and they may lead to a suboptimal result and a deceptive evaluation of its true estimation accuracy. The error variance from Eq. (8) or (10) should thus be considered as a statistical measure representing the consistency of the heterogeneous height data in the gravity potential domain, and not necessarily as the actual accuracy level of $\hat{W}_o^{\mathrm{LVD}}$.

2.4 A note on the 'global' parameter W_o

A seemingly simple, yet important, aspect within the previous methodology is the influence of the global value W_o on the estimation accuracy of (a) the parameter $\hat{W}_o^{\mathrm{LVD}}$ and (b) the geopotential offset $W_o - \hat{W}_o^{\mathrm{LVD}}$. The latter is a key quantity for implementing the height transformation between the underlying LVD and a unified vertical datum defined by W_o, as well as for obtaining LVD vertical positions directly from space geodetic measurements and a geoid model representing a particular equipotential surface $W(\cdot) = W_o$ of Earth's gravity field.

Based on the geoid height decomposition in terms of its zero- and higher-degree terms (Smith 1998), we have:

$$N_i = N_i^o + \tilde{N}_i = \frac{1}{\gamma_i}\left(\frac{GM - GM'}{R_o} + U_o - W_o\right) + \tilde{N}_i, \tag{11}$$

where the higher-degree part $\tilde{N}_i$ is independent of W_o. Substituting the previous expression into the general estimator of Eq. (7), we get the equation:

$$\hat{W}_o^{\mathrm{LVD}} = U_o + \frac{GM - GM'}{R_o} - \frac{\sum_{i=1}^{K} \frac{1}{\gamma_i}(h_i - H_i - \tilde{N}_i)}{\sum_{i=1}^{K} \frac{1}{\gamma_i^2}} \tag{12}$$

or equivalently,

$$W_o - \hat{W}_o^{\mathrm{LVD}} = (W_o - U_o) - \frac{GM - GM'}{R_o} + \frac{\sum_{i=1}^{K} \frac{1}{\gamma_i}(h_i - H_i - \tilde{N}_i)}{\sum_{i=1}^{K} \frac{1}{\gamma_i^2}}. \tag{13}$$

From the last two equations, the following conclusions can be drawn:

- The choice of the global value W_o does not influence the actual estimate $\hat{W}_o^{\mathrm{LVD}}$ and its formal accuracy. However, apart from the random errors in the heterogeneous height data, a realistic assessment of the error variance of $\hat{W}_o^{\mathrm{LVD}}$ should take into account the uncertainty of Earth's geocentric gravitational constant, as already stated in Remark 1.
- Theoretically, the estimation accuracy of the difference $W_o - \hat{W}_o^{\mathrm{LVD}}$ is affected by the inherent uncertainty of W_o. However, if the latter corresponds to a GVRS conventional parameter, we may plausibly enforce the condition $\sigma_{W_o} = 0$ and thus the accuracy of $W_o - \hat{W}_o^{\mathrm{LVD}}$ becomes formally equivalent with the accuracy of the estimated parameter $\hat{W}_o^{\mathrm{LVD}}$.
- Taking into account an a priori error variance $\sigma_{W_o} \neq 0$ is meaningful if we need to evaluate the offset between the LVD zero-height level and a global equipotential surface approximating the mean sea level in terms of an estimated value W_o (as obtained, for example, from the experimental analysis of satellite altimetry data).

Curiously enough, the estimation accuracy of $\hat{W}_o^{\mathrm{LVD}}$ that has been reported in various geodetic studies appears to be worse than the corresponding accuracy of the difference $W_o - \hat{W}_o^{\mathrm{LVD}}$ (e.g. Tables 1 and 2 in Burša et al. 2004). However, according to Eqs. (12) and (13), we must have that

$$\sigma^2_{W_o - \hat{W}_o^{\mathrm{LVD}}} = \sigma^2_{\hat{W}_o^{\mathrm{LVD}}} + \sigma^2_{W_o} \tag{14}$$

since the estimate of the LVD zero-height level from GPS/leveling data is independent of the global reference value W_o; see also Jekeli (2000, pp. 15–19).

2.5 Treatment of systematic data errors

In the presence of systematic effects and spatially correlated errors in the height data, the LS estimator from Eq. (7) yields a biased result due to improper data modeling. In such cases the original values $h_i - H_i - N_i$ do not follow a typical trend of a constant offset, but they reveal strong spatial tilts or even a more complex oscillatory pattern over the GPS/leveling benchmarks. Several types of systematic errors usually contribute to this problem, including geometrical distortions in the leveling height data (e.g. overconstrained LVD to several tide-gauge stations, poor modeling of the local topography in orthometric height approximation, etc.), long and medium wavelength errors in the geoid model, datum inconsistencies between the ellipsoidal heights and the geoid heights, unmodeled time-dependent variations and inconsistent treatment of the permanent tide effect among the different height types.

The removal of systematic effects from the height data is a requisite task for the determination of the LVD datum parameter and it can be performed either beforehand through appropriate corrections and spatial de-trending of the raw height residuals, or concurrently with the estimation of W_o^{LVD} using an extended observation equation

$$h_i - H_i - N_i = \frac{W_o - W_o^{\mathrm{LVD}}}{\gamma_i} + \mathbf{a}_i^{\mathrm{T}}\mathbf{x} + v_i, \tag{15}$$

where the additional term absorbs the systematic errors through a set of nuisance parameters $\mathbf{x}$ and $\mathbf{a}_i$ is a vector of known coefficients depending on the spatial position of each benchmark, while v_i is a stochastic term with the remaining random errors in the height data. Some examples of parametric models that have been used for the description of systematic effects in a mixed set of geometric, orthometric and geoid heights can be found in Fotopoulos (2003) and the references given therein.

A key issue for the effective inversion of Eq. (15) is the separability between the term $W_o - W_o^{\mathrm{LVD}}/\gamma_i$ and the nuisance parameters of the 'bias corrector' model. If we aim to get a realistic estimate of the LVD zero-height level, then the vertical-offset term should not be strongly correlated with the adopted parametric model, otherwise problematic results may arise from the height data adjustment. Therefore, $\mathbf{a}_i^{\mathrm{T}}\mathbf{x}$ should not contain any components that remain constant or almost constant over the test network, which in turn implies that any long-wavelength data errors will be necessarily absorbed by the LVD datum parameter.

The inseparability between the geopotential offset $W_o - W_o^{\mathrm{LVD}}/\gamma_i$ and a bias within the height data is a crucial problem in all related techniques that have been used for the practical determination of LVD datum parameters. A simple modification of the error variance of $\hat{W}_o^{\mathrm{LVD}}$ may be applied in practice to account for the propagated effect of an unknown height bias into the LS adjustment result. For example, if the height errors are assumed to contain an uncorrelated random part (data noise) and a perfectly correlated part (data bias), then the accuracy of the simple LS estimator (9) should take the extended form:

$$\sigma^2_{\hat{W}_o^{\mathrm{LVD}}} = \gamma^2_{\mathrm{ave}} \left(\frac{\sigma^2_{\mathrm{noise}}}{K} + \sigma^2_{\mathrm{bias}} \right), \tag{16}$$

where σ_{noise} is the combined noise level of the orthometric, geoid and GPS heights, while σ_{bias} represents (in statistical terms) the total bias in the available data. A meaningful contributor to the latter is the geoid commission error over the spatial wavelengths that overly exceed the test area size.

The geographical coverage of the test network is an important factor regarding the influence of geoid systematic errors on the estimated value $\hat{W}_o^{\mathrm{LVD}}$. In fact, a part of these errors may start to average out through the LS adjustment of Eq. (6) or (15), as the number of GPS/leveling benchmarks increases over a larger area. This fact was pointed out in Burša et al. (2007) where a sufficiently large network of GPS/leveling sites with a spatial extent of about 2,000 km was suggested in order to keep the effect of EGM96 geoid errors below $0.1\ \mathrm{m^2/s^2}$. On the other hand, in areas with limited geographical coverage such as the ones in our following investigation over the Hellenic islands, the long-wavelength geoid errors will inevitably contaminate the LS estimate $\hat{W}_o^{\mathrm{LVD}}$ and they cannot be filtered out on the basis of the local height data. Nevertheless, their degrading effect can be accounted for through an a posteriori 'correction' on the error variance of the zero-height level, like the one shown in Eq. (16).

A few final comments need to be given about the auxiliary parameterization of height systematic errors within Eq. (15). Specifically, a 'dangerous' model that should not be used for LVD-related studies is the well-known 4-parameter model

$$\mathbf{a}_i^{\mathrm{T}}\mathbf{x} = x_o + x_1 \cos\varphi_i \cos\lambda_i + x_2 \cos\varphi_i \sin\lambda_i + x_3 \sin\varphi_i \tag{17}$$

which is often employed in gravimetric geoid evaluation with GPS/leveling data. Even if we omit its constant term x_o for the reasons that were previously explained, the LS inversion of Eq. (15) using the reduced form of the above model will lead to a non-realistic estimate for the offset $W_o - W_o^{\mathrm{LVD}}$. The reason is that the combined effect of the three remaining terms (which depend on x_1, x_2 and x_3) represents a spatial translation between the LVD reference surface and the equipotential surface of the geoid model—the last three terms in the four-parameter model $\mathbf{a}_i^{\mathrm{T}}\mathbf{x}$ describe a geodetic datum shift on the geoid height (Heiskanen and Moritz 1967, pp. 213). As a result, a strong correlation will occur with the term $W_o - W_o^{\text{LVD}}/\gamma_i$ which corresponds to a strictly vertical shift between the same surfaces; for more details and some numerical examples, see Kotsakis and Katsambalos (2010).

On the other hand, a single-parameter scaling model $\mathbf{a}_i^{\text{T}}\mathbf{x} = \delta s\, H_i$ yields a simple, yet effective, way to describe a certain class of systematic data errors in W_o^{LVD} estimation problems. The raw residuals $h_i - H_i - N_i$ are often topographically correlated, a fact that can be perceived as an apparent scale difference between the LVD orthometric heights and the GPS/geoid-based orthometric heights (i.e. the parameter δs can be interpreted as a scale factor between alternative realizations of orthometric heights). Note that the successful use of this model requires a significant height variability in the test network, so that the offset $W_o - W_o^{\text{LVD}}$ and the scale parameter δs can be sufficiently separated through the LS inversion of Eq. (15). Some results from the practical implementation of this model are presented in Sect. 4.

3 Case studies in Hellenic islands

3.1 General remarks

The fundamental parameter W_o^{LVD} has been estimated in a number of Hellenic islands across the Aegean and Ionian Sea. The corresponding vertical datums were independently established, within the period 1963–1986, by the Hellenic Military Geographic Service (HMGS) in cooperation with the Hydrographic Service of the Hellenic Military Navy, based on the local MSL that was fixed at a single tide-gauge station in each island (Takos 1989). The exact time period of sea level observations that were used for the MSL/LVD definition is generally unknown, with the exception of Crete where the local leveling network is known to be constrained at the tide-gauge station located in Heraklion using the observed MSL time series over the years 1955–1978 (Antonopoulos 1999). Note that to date, to the authors' knowledge, there has not been any attempt towards the connection of LVDs in the Hellenic islands and/or their unification with the official vertical datum of the Hellenic mainland which is currently defined at the Pireaus' tide-gauge station from sea level recordings over the period 1933–1978 (Mylona-Kotrogianni 1989; Takos 1989).

A sample of 16 Hellenic islands have been selected for our study (see Fig. 4). In each of them, a sufficient number of geodetic control points are available with known 3D spatial positions (ITRF2000) and also known Helmert-type orthometric heights relative to the LVD of the respective island (see Table 1). All control points correspond to pillar markers of the national Hellenic geodetic network, and they are tied to the local PHF of each island through leveling surveys from nearby LVD benchmarks. A brief description of the original height data and the geoid undulation computations that were performed with the EGM2008 model is provided in the following sections.

3.2 Ellipsoidal heights

A nation-wide GPS campaign was undertaken in 2007 in Greece to establish a dense national network with accurately known 3D spatial coordinates in an ITRF-type coordinate system. These activities have been initiated by the Ministry for the Environment, Planning and Public Works and the financial support of the EU and the Hellenic State, and they were part of the HEPOS project (Hellenic Positioning System) that led to the launch of a GNSS-based positioning service for cadastral, mapping and other surveying applications in Greece; for more details, consult the webpage http://www.hepos.gr and the technical references given therein.

The aforementioned GPS campaign involved more than 2450 control stations of the Hellenic geodetic network, part of which are the 483 test points that are listed in Table 1. The original scope of the campaign was to provide a sufficient number of well-distributed stations for the determination of a coordinate transformation model between the official Hellenic Geodetic Reference System of 1987 and other ITRF/ETRF-type frames. The fieldwork was performed within a 6-month period (March to September 2007) using 12 dual-frequency Trimble 5700/5800 GPS receivers with Zephyr or R8 internal antennas. In all cases, a 15-s sampling rate and an 15° elevation cut-off angle were used for the data collection. Thirty-three of the re-surveyed control points were selected as base reference stations with 24-h continuous GPS observations, while the rest of the re-surveyed points were treated as rover stations with observation periods ranging between 3 and 6 h. Note that the maximum baseline length that was observed through the above procedure did not exceed 35 km.

After the processing of the GPS carrier phase observations using EUREF/EPN ties and IGS precise orbits, the geocentric Cartesian coordinates of all stations, including the 483 test points over the 16 Hellenic islands, were determined in ITRF2000 (epoch: 2007.236) and their corresponding geometric heights were derived relative to the GRS80 ellipsoid. The formal accuracy of the geometric heights ranged between 2 and 5 cm, while the horizontal positioning accuracy with respect to ITRF2000 was marginally better by 1–2 cm (1σ level).

The original geometric heights derived from the GPS data refer, in principle, to a tide-free system in terms of the treatment of the permanent tide effect (Poutanen et al. 1996). For our following LVD analysis (Sect. 4), their values have been transformed to the zero-tide system based on the formula

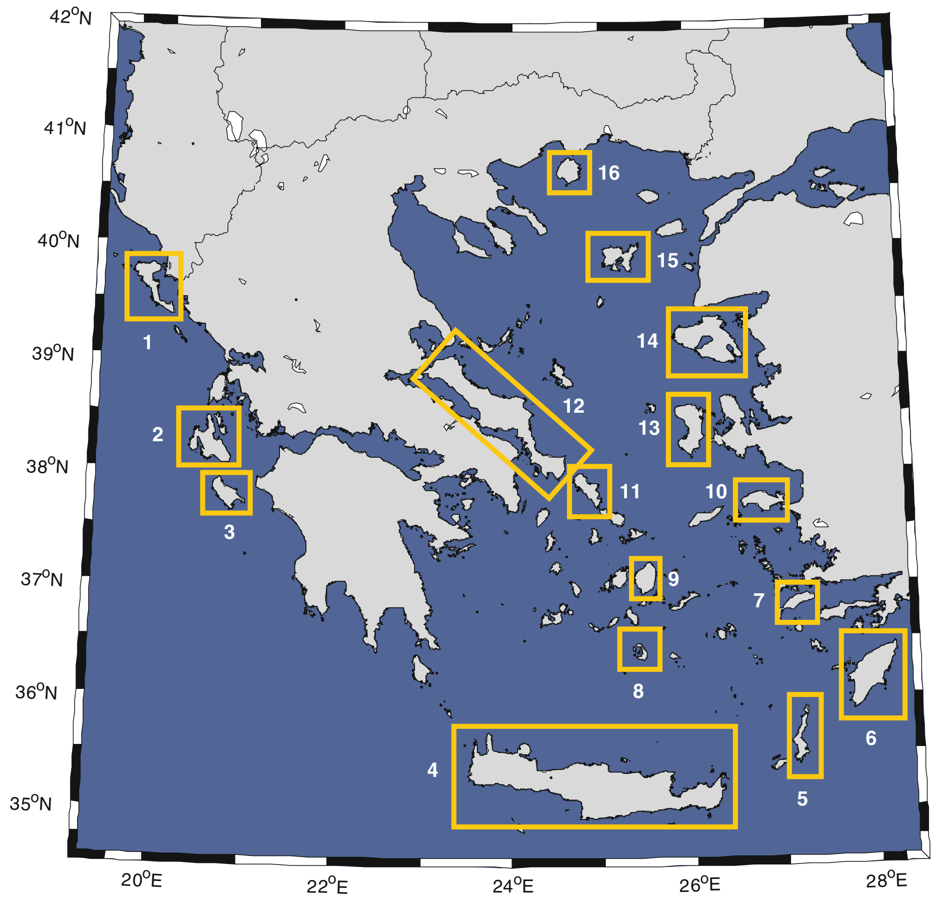

Fig. 4 The 16 Hellenic islands used in our study

Table 1 The 16 Hellenic islands of our study and the available GPS/leveling benchmarks in each of them

Code no.	Island	No. of test points
1	*Corfu*	13
2	*Kefallonia*	19
3	*Zakinthos*	11
4	*Crete*	126 (1)
5	*Karpathos*	18
6	*Rodos*	36
7	*Kos*	13
8	*Santorini*	8
9	*Naxos*	16
10	*Samos*	14 (1)
11	*Andros*	12
12	*Evoia*	81 (1)
13	*Chios*	24
14	*Lesvos*	39
15	*Limnos*	18
16	*Thasos*	8

The parenthetical values correspond to the number of 'problematic' points that were rejected from our analysis due to identified blunders in their original height values

(Ekman 1989; Mäkinen and Ihde 2009):

$$h^{ZT} = h^{FT} + \ell_2(0.099 - 0.296 \sin^2\varphi) \text{ (m)}, \tag{18}$$

where ℓ_2 is the conventional (second-degree) *Love number* that is approximately equal to 0.62 (Ekman 1989). The terms h^{FT} and h^{ZT} correspond to the geometric height of the tide-free and zero($\equiv$mean)-tide crust relative to a geometrically fixed (conventional) reference ellipsoid. The above correction is always negative over our test areas, reaching up to -1.6 cm at the most northern control points.

3.3 Helmert-type orthometric heights

The 483 test points within the selected islands are tied to the corresponding LVDs through local surveys from nearby benchmarks, which were performed by HMGS using spirit and/or precise trigonometric leveling (Takos 1989). The formal precision for the resulting leveled heights, as stated by HMGS, is approximately 1–2 cm, yet their true accuracy level is largely unknown. It should be pointed out that the determination of Helmert orthometric heights in the primary leveling network of each island and at the control points of the Hellenic geodetic network was not based on observed surface gravity information, but mostly on interpolated gravity values obtained from free-air gravity anomaly maps (Mylona-Kotrogianni 1989; Antonopoulos 1999).

Since no tidal corrections were originally applied to the field measurements and the results of the leveling work performed by HMGS in Greece (Antonopoulos 1999), the available orthometric heights refer, in principle, to a mean-tide system. For our subsequent analysis in Sect. 4, these values have been transformed to the zero-tide system using the formula (Ekman 1989; Mäkinen and Ihde 2009):

$$H^{\mathrm{ZT}} = H^{\mathrm{MT}} + (0.099 - 0.296 \sin^2\varphi) \text{ (in m)}. \qquad (19)$$

The term H^{MT} corresponds to the orthometric height of the zero (≡mean)-tide crust with respect to the mean-tide geoid, while H^{ZT} denotes the orthometric height of the zero (≡mean)-tide crust with respect to the zero-tide geoid. The above correction is always negative over our test areas, reaching up to −2.5 cm at the most northern control points.

Note that the orthometric heights that are used in our study represent official values which were determined over different time periods in each island. Since these time periods are generally unknown, it is not possible to assign a specific epoch to the orthometric height data. Also, as no re-leveling field campaign or any other kind of height monitoring effort has been performed in these islands, it is not possible to infer the time variability of the orthometric height data. As a result, the ignored vertical displacements at the test points will cause the estimated values $\hat{W}_o^{\mathrm{LVD}}$ to reflect the status of the LVDs at the time of their establishment, and not necessarily at the current epoch.

3.4 EGM2008-based geoid heights

Geoid heights were determined at all 483 test points from the EGM2008 global geopotential model (Pavlis et al. 2008) and the normal gravity field parameters of the GRS80 reference ellipsoid (Moritz 1992). The numerical computation was based on the general formula (Rapp 1997)

$$N = \zeta + \frac{\Delta g^{\mathrm{FA}} - 0.1119\,(\mathrm{mGal/m}) \times H}{\bar{\gamma}} H + N_o, \qquad (20)$$

where ζ and Δg^{FA} denote the height anomaly and free-air gravity anomaly signals that are obtained from their respective spherical harmonic expansion using the EGM2008 fully normalized potential coefficients (from $n = 2$ up to $n_{\max} = 2{,}190$) and the GRS80 normal gravity field parameters. Their computation has been performed in the zero-tide system using the harmonic_synth_v02 software program that is freely provided by the NGA/EGM development team (Holmes and Pavlis 2006). Note that the term H stands for the known orthometric height at each computation point, whereas $\bar{\gamma}$ symbolizes the mean normal gravity along the normal plumbline between the telluroid and the reference ellipsoid. The latter is computed by an approximate formula (Heiskanen and Moritz 1967, Eq. 4–42) based on a truncated latitude-dependent power series of the geodetic, instead of normal, height; for more details see Rapp (1997) and Flury and Rummel (2009).

The additive term N_o represents the contribution of the zero-degree harmonic to the EGM2008 geoid height with respect to the GRS80 ellipsoid. It has been separately computed by the equation (Heiskanen and Moritz 1967)

$$N_o = \frac{GM - GM'}{R\gamma} - \frac{W_o - U_o}{\gamma}, \qquad (21)$$

where the parameters GM' and U_o correspond to the Somigliana-Pizzeti normal gravity field generated by the GRS80 ellipsoid ($GM' = 398600.50 \times 10^9\,\mathrm{m^3/s^2}$ and $U_o = 62636860.85\,\mathrm{m^2/s^2}$). The Earth's geocentric gravitational constant and the geoidal gravity potential were set equal to the values $GM = 398600.4415 \times 10^9\,\mathrm{m^3/s^2}$ and $W_o = 62636856.00\,\mathrm{m^2/s^2}$, respectively. The value $R = 6371008.771$ m induced by GRS80 was adopted for the mean Earth radius, while the normal gravity γ on the reference ellipsoid was computed at each point from Somigliana's formula. Based on these choices, the zero-degree term in Eq. (21) yields an almost constant negative value ($N_o \approx -0.443$ m) throughout our test regions, which has been included in the determination of the final geoid heights at the 483 GPS/leveling benchmarks.

4 Results and discussion

The zero-height level in each LVD has been estimated under different scenarios, depending on the parametric model that was used for the joint LS inversion of the geometric, orthometric and geoid heights. Specifically, the general observation equation (15) has been implemented with the following options for its 'bias corrector' term.

Null model: $\mathbf{a}_i^{\mathrm{T}}\mathbf{x} = 0$

No systematic errors or other biases are modeled within the height data adjustment. In this case, the LS estimate $\hat{W}_o^{\mathrm{LVD}}$ corresponds to the result of the simple formula in Eq. (7).

Model 1: $\mathbf{a}_i^{\mathrm{T}}\mathbf{x} = \delta s H_i$

The systematic differences between the GPS/EGM2008 and the LVD-based orthometric heights are modeled in terms of a scaling factor. The use of this particular model is justified by the presence of a significant topographic correlation in the

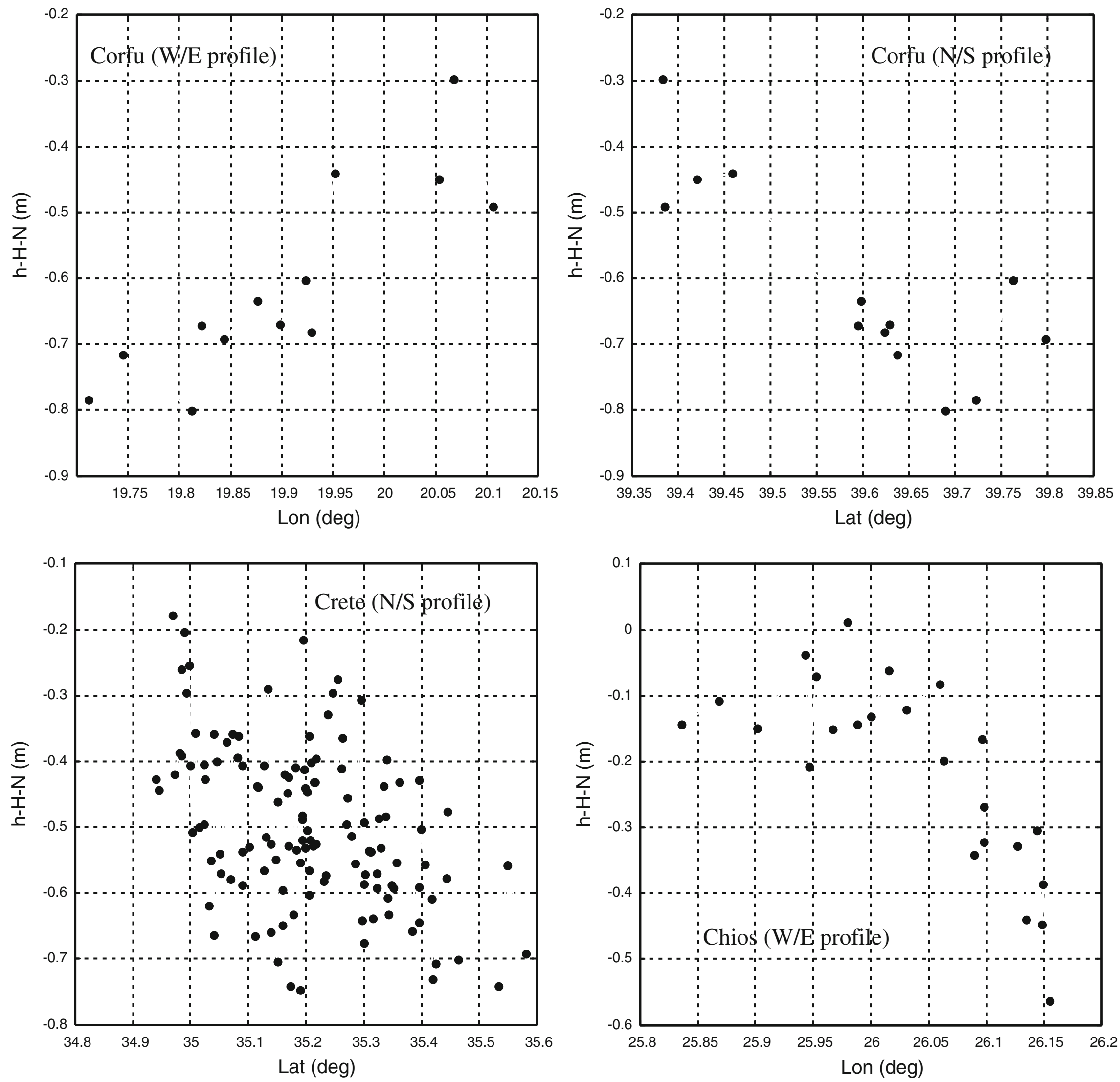

Fig. 5 Examples of systematic spatial tilts in the original height residuals $h-H-N^{\mathrm{EGM2008}}$ for some of the tested Hellenic islands

original height residuals $h_i - H_i - N_i$, at least in some of the tested islands (Kos, Chios, Kefallonia).

Model 2: $\mathbf{a}_i^{\mathrm{T}}\mathbf{x} = x_1(\varphi_i - \varphi_o) + x_2(\lambda_i - \lambda_o)\cos\varphi_i$

The systematic differences between the GPS/EGM2008 and the LVD-based orthometric heights are described by a two-parameter model representing a spatial tilt between their corresponding reference surfaces. The overall tilt consists of a N/S component (parameter x_1) and a W/E component (parameter x_2) with respect to the centroid of the test network. Some examples of the existing systematic tilts in the original height residuals are shown in Fig. 5.

Combined model: $\mathbf{a}_i^{\mathrm{T}}\mathbf{x} = x_1(\varphi_i - \varphi_o) + x_2(\lambda_i - \lambda_o)\cos\varphi_i + \delta s H_i$

A combination of the previous parameterization schemes is employed to describe the systematic differences between the GPS/EGM2008 and the LVD-based orthometric heights. Note that, in some cases, the combined model improves the consistency among the geometric, orthometric and geoid heights over the Hellenic islands by several 3 cm compared to the performance of models 1 or 2 (see Table 2).

In all cases, the LS inversion of Eq. (15) is performed with a unit weight matrix, thus assuming a constant (and uncorrelated) noise level in the known heights at all data points within each island. The auxiliary parametric model $\mathbf{a}_i^{\mathrm{T}}\mathbf{x}$ absorbs, in principle, some of the ignored part of the correlated data errors. The total noise level of the height data is estimated by the a posteriori variance factor of each adjustment test, and it is subsequently employed for the uncertainty assessment of

Table 2 Root mean square (rms) values of the adjusted height residuals from the LS estimation of $\hat{W}_o^{\mathrm{LVD}}$

Island	RMS of adjusted height residuals (cm)			
	Null model	*Model 1*	*Model 2*	*Combined model*
Corfu	14.8	14.6	6.6	5.2
Kefallonia	9.9	8.8	9.8	8.8
Zakinthos	6.5	6.3	5.5	5.3
Crete	12.1	12.1	10.7	10.7
Karpathos	9.8	9.8	6.7	5.4
Rodos	15.8	14.5	11.0	10.6
Kos	9.3	6.5	8.0	6.3
Santorini	5.1	4.5	2.3	2.3
Naxos	8.7	8.7	8.2	8.2
Samos	9.5	9.5	9.2	9.2
Andros	8.9	8.6	8.1	7.5
Evoia	10.0	10.0	9.6	9.6
Chios	14.7	13.3	9.1	6.1
Lesvos	8.4	7.9	6.9	6.7
Limnos	9.4	9.4	8.3	7.6
Thasos	6.3	6.2	5.8	5.7

the recovered LVD parameters. The estimated values $\hat{W}_o^{\mathrm{LVD}}$ along with their standard errors (1σ) from each modeling scheme in every island are listed in Table 3. It should be noted that our error estimates indicate only an approximate statistical accuracy and not necessarily the true accuracy level of the final results, since no variance correction has been applied to account for the EGM2008 long-wavelength errors over each island's area (see Eq. (16)).

The largest relative differences among the 16 LVDs occur between the islands of Corfu and Rodos ($\delta\hat{W}_o^{\mathrm{LVD}} = 6.42\,\mathrm{m}^2/\mathrm{s}^2$ *combined model*) and also Samos and Rodos ($\delta\hat{W}_o^{\mathrm{LVD}} = 5.52\,\mathrm{m}^2/\mathrm{s}^2$ *combined model*). In general, the vertical offsets of the zero-height levels do not exceed a maximum of about 66 cm (Corfu-Rodos) and they typically range from a few centimeters (e.g. Andros-Evoia/5.1 cm, Crete-Naxos/1.4 cm) up to a few decimeters (e.g. Chios-Samos/28.4 cm, Corfu-Kefallonia/21.6 cm); for the detailed results see Fig. 6.

The discrepancy among the estimated values $\hat{W}_o^{\mathrm{LVD}}$ from the different error modeling schemes *within each LVD* does not exceed $\sim 1.3\,\mathrm{m}^2/\mathrm{s}^2$ and, in most cases, it remains below $0.5\,\mathrm{m}^2/\mathrm{s}^2$ (see Table 3). These numerical variations are caused by the different correlations between the offset term $W_o - W_o^{\mathrm{LVD}}$ and the nuisance parameters of each bias-corrector model $\mathbf{a}_i^{\mathrm{T}}\mathbf{x}$, in conjunction with the existing systematic effects in the height data of each island. Note that the LS solution from *model 2* remains identical with the one derived from the *null model*, since the adjusted spatial tilt is taken with respect to the centroid of the test network and thus it does not interfere with the mean offset of the LVD from the geoidal equipotential surface.

The geopotential values $\hat{W}_o^{\mathrm{LVD}}$ from the LS adjustment with the combined parametric model were selected as our final estimates for the zero-height level in each LVD. These values are listed in Table 4, along with the equivalent

Table 3 Estimated reference geopotential values for the LVDs in various Hellenic islands

Island	$\hat{W}_o^{\mathrm{LVD}}$ ($\mathrm{m}^2/\mathrm{s}^2$)			
	Null model	*Model 1*	*Model 2*	*Combined model*
Corfu	62636861.90 ± 0.40	62636861.47 ± 0.81	62636861.90 ± 0.20	62636862.74 ± 0.39
Kefallonia	60.12 ± 0.22	60.61 ± 0.31	60.12 ± 0.23	60.61 ± 0.34
Zakinthos	61.02 ± 0.19	60.79 ± 0.34	61.02 ± 0.18	61.29 ± 0.46
Crete	60.89 ± 0.11	61.00 ± 0.16	60.89 ± 0.09	60.95 ± 0.15
Karpathos	59.79 ± 0.23	59.81 ± 0.41	59.79 ± 0.17	60.42 ± 0.27
Rodos	56.68 ± 0.26	55.95 ± 0.38	56.68 ± 0.19	56.32 ± 0.30
Kos	59.84 ± 0.25	60.51 ± 0.27	59.84 ± 0.24	60.44 ± 0.32
Santorini	58.54 ± 0.18	59.03 ± 0.42	58.54 ± 0.09	58.46 ± 0.32
Naxos	60.78 ± 0.21	60.80 ± 0.33	60.78 ± 0.22	60.81 ± 0.36
Samos	61.88 ± 0.26	61.85 ± 0.37	61.88 ± 0.27	61.84 ± 0.39
Andros	61.22 ± 0.25	60.83 ± 0.52	61.22 ± 0.25	60.69 ± 0.52
Evoia	60.16 ± 0.11	60.22 ± 0.18	60.16 ± 0.11	60.19 ± 0.17
Chios	58.05 ± 0.29	58.94 ± 0.48	58.05 ± 0.19	59.06 ± 0.24
Lesvos	61.58 ± 0.13	61.92 ± 0.19	61.58 ± 0.11	61.77 ± 0.17
Limnos	60.64 ± 0.22	60.67 ± 0.35	60.64 ± 0.20	61.12 ± 0.36
Thasos	59.61 ± 0.22	59.67 ± 0.29	59.61 ± 0.24	59.65 ± 0.33

Three different modeling schemes have been used for the reduction of the systematic height errors within the least-squares adjustment procedure

Fig. 6 Mean vertical shifts of the LVDs in various Hellenic islands relative to the conventional equipotential surface defined by the IERS global reference value W_o (cm)

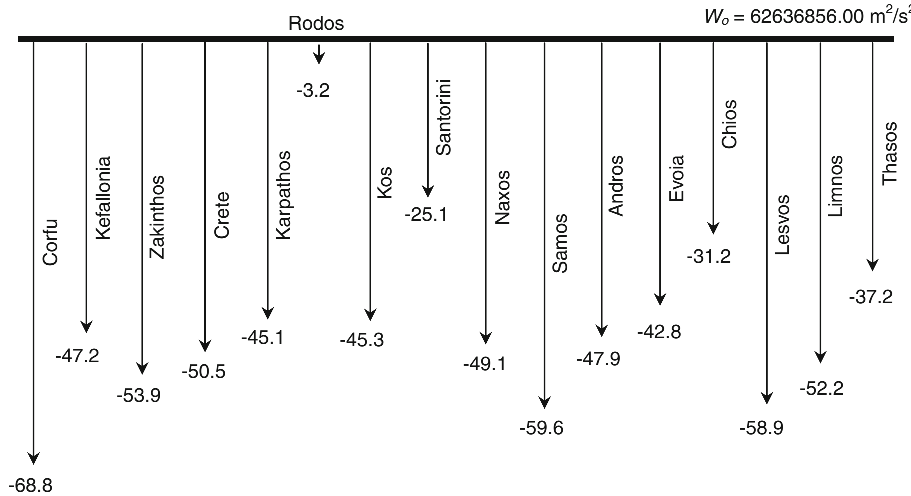

Table 4 Estimated reference geopotential values for the LVDs in various Hellenic islands (as obtained from the combined bias-corrector model), the equivalent geopotential offsets with respect to the IERS (2010) global value $W_o = 62636856.00\,\mathrm{m}^2/\mathrm{s}^2$ and their corresponding mean vertical shifts

Island	No. of GPS/lev BMs	W_o^{LVD} ($\mathrm{m}^2/\mathrm{s}^2$)	$W_o^{\mathrm{LVD}} - W_o$ ($\mathrm{m}^2/\mathrm{s}^2$)	$\delta H_o^{\mathrm{LVD}}$ (cm)
Corfu	13	62636862.74 ± 0.39	6.74 ± 0.39	-68.8 ± 3.9
Kefallonia	19	60.61 ± 0.34	4.62 ± 0.34	-47.2 ± 3.5
Zakinthos	11	61.29 ± 0.46	5.29 ± 0.46	-53.9 ± 4.7
Crete	125	60.95 ± 0.15	4.95 ± 0.15	-50.5 ± 1.5
Karpathos	18	60.42 ± 0.27	4.42 ± 0.27	-45.1 ± 2.8
Rodos	36	56.32 ± 0.30	0.32 ± 0.30	-3.2 ± 3.0
Kos	13	60.44 ± 0.32	4.44 ± 0.32	-45.3 ± 3.3
Santorini	8	58.46 ± 0.32	2.46 ± 0.32	-25.1 ± 3.2
Naxos	16	60.81 ± 0.36	4.81 ± 0.36	-49.1 ± 3.6
Samos	13	61.84 ± 0.39	5.84 ± 0.39	-59.6 ± 4.0
Andros	12	60.69 ± 0.52	4.69 ± 0.52	-47.9 ± 5.3
Evoia	80	60.19 ± 0.17	4.19 ± 0.17	-42.8 ± 1.8
Chios	24	59.06 ± 0.24	3.06 ± 0.24	-31.2 ± 2.5
Lesvos	39	61.77 ± 0.17	5.77 ± 0.17	-58.9 ± 1.8
Limnos	18	61.12 ± 0.36	5.12 ± 0.36	-52.2 ± 3.7
Thasos	8	59.65 ± 0.33	3.65 ± 0.33	-37.2 ± 3.3

offsets with respect to the IERS global conventional value W_o (Petit and Luzum 2010) and the mean vertical shifts between the equipotential surfaces $W^{\mathrm{EGM2008}}(\cdot) = W_o$ and $W^{\mathrm{EGM2008}}(\cdot) = \hat{W}_o^{\mathrm{LVD}}$; for a graphical representation of these results see Fig. 6. Note that the standard errors of $\delta W_o^{\mathrm{LVD}}$ and $\delta H_o^{\mathrm{LVD}}$ do not include the contribution of the inherent uncertainty of the global value W_o. The corresponding estimates of the systematic error parameters over the 16 Hellenic islands, namely the apparent scale difference and the spatial tilts between the GPS/EGM2008 and LVD-based orthometric heights, are also given in Table 5.

The choice of the aforementioned values is based on the rationale that the zero-height level $\hat{W}_o^{\mathrm{LVD}}$ should provide the optimal least-squares fit to the available orthometric heights of each island. From the results of Table 2, it is seen that the combined model satisfies such a requirement, although with a negligible improvement in its fitting performance over the other error modeling schemes in some areas (Naxos, Samos, Evoia, Thasos).

As a final remark, let us point out the discrepancy of $\hat{W}_o^{\mathrm{LVD}}$ for the island of Rodos compared to the zero-height levels of the other tested islands; see Fig. 6. Due to the lack of

Table 5 Estimated values of systematic error parameters from the LS height adjustment over the 16 Hellenic islands

Island	*Systematic error parameters*		
	δs	N/S tilt (cm/km)	W/E tilt (cm/km)
Corfu	4.77×10^{-4}	−0.2	1.6
Kefallonia	2.16×10^{-4}	0.0	0.1
Zakinthos	0.95×10^{-4}	−0.1	0.6
Crete	0.17×10^{-4}	−0.4	0.0
Karpathos	2.08×10^{-4}	−0.7	−0.2
Rodos	-1.77×10^{-4}	−0.7	1.3
Kos	4.49×10^{-4}	−0.2	0.0
Santorini	-0.34×10^{-4}	−0.3	1.9
Naxos	0.15×10^{-4}	−0.3	0.1
Samos	-0.14×10^{-4}	0.5	−0.1
Andros	-1.72×10^{-4}	−0.6	−0.3
Evoia	0.12×10^{-4}	0.2	0.2
Chios	3.84×10^{-4}	−0.4	−1.6
Lesvos	0.98×10^{-4}	−0.2	−0.3
Limnos	3.77×10^{-4}	−0.6	0.8
Thasos	0.25×10^{-4}	0.3	−0.2

The particular parameters refer to the apparent scale difference and spatial tilt between GPS/EGM2008 and LVD-based orthometric heights

any detailed technical reference on the LVD definition in the Hellenic islands by HMGS, it is rather difficult to provide a justified explanation for this particular result of our study. However, a similar result showing a significant offset for the recorded MSL at Rodos' tide-gauge station relative to other tide-gauge stations in Greece (with respect to the OSU91 model) was also reported in the study by Fenoglio-Marc (1996, p. 41).

5 Summary and conclusions

The realization of most vertical datums is based on a 'crust-fixed' approach using a physical-height constraint for at least a single terrestrial point, without incorporating either of the two fundamental constituents of the theoretical LVD definition, namely a geopotential model $W(\cdot)$ and a conventional geopotential value W_o^{LVD}. A representation of existing LVDs in the gravity field domain is nevertheless essential, since it allows the unification of different PHFs in the absence of any relative leveling measurements between them. Such a representation relies on a high-quality geopotential model (e.g. EGM2008) in conjunction with a zero-height level that needs to be recovered from a realization of the vertical datum at hand. The main aspects of the underlying estimation procedure were discussed in this paper, focusing on the inversion of co-located GPS, orthometric and geoid heights for the determination of the fundamental parameter W_o^{LVD}.

A critical open problem is the existence of hidden biases and other unknown systematic errors within the height data that are used in the LS adjustment algorithm. Although some of these errors can be taken into account through empirical modeling during the height data inversion, a part of them (especially at spatial wavelengths of larger size than the extent of the local test area) cannot be separated from the actual offset between the external geoid model and the LVD. Nevertheless, the estimated value $\hat{W}_o^{\mathrm{LVD}}$ retains, to a certain extent, its importance as a 'datum identification parameter' for the given LVD realization on the basis of a particular gravity field representation.

From the analysis of ellipsoidal, orthometric and EGM2008 geoid heights at 483 GPS/leveling benchmarks distributed over 16 Hellenic islands, a set of optimal estimates $\{\hat{W}_o^{\mathrm{LVD}}\}$ for their MSL-based vertical datums was derived in this study (see Table 4). These results correspond to the use of a three-parameter bias-corrector model within the LS height data adjustment, which effectively absorbs topography-correlated systematic differences and other tilting biases between GPS/EGM2008 and LVD-based orthometric heights. In our study, these effects were shown to cause vertical scale differences in the order of 10^{-5} to 10^{-4} and systematic tilts from −0.7 to 0.5 cm/km (north/south direction) and from −1.6 to 1.9 cm/km (west/east direction). The statistical accuracy of the estimated LVD offsets with respect to the conventional equipotential surface $W(\cdot) = W_o^{\mathrm{IERS}}$ is better than 4 cm, while the adjusted height residuals in the various islands retain an rms level that is well below 10 cm.

Our results indicate the existence of relative offsets among the LVDs of the tested islands ranging from a few cm up to several dm. These offsets reflect a mixture of the MSL variations throughout the Aegean and Ionian Sea, but they are also affected by the inconsistencies in the averaging procedure that was used by HMGS for the MSL determination at the tide-gauge reference station of each island. A part of these offsets should also be attributed to EGM2008 long-wavelength errors that cause an apparent bias (with a different magnitude in each island) into the computed geoid heights at the GPS/leveling benchmarks.

An external verification of our results requires, in principle, the determination of the gravity potential $W(\cdot)$ at the LVD origin point of each island, yet such a task is not possible to be performed at this point due to the lack of GPS measurements at the tide-gauge reference stations of the corresponding vertical datums. Although further investigations are still needed for a comprehensive LVD analysis over the Hellenic islands, our present study provides a starting point towards their unification into a common vertical datum for the area of Greece.

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