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Transforming ellipsoidal heights and geoid undulations between different geodetic reference frames

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Abstract Transforming height information that refers to an ellipsoidal Earth reference model, such as the geometric heights determined from GPS measurements or the geoid undulations obtained by a gravimetric geoid solution, from one geodetic reference frame (GRF) to another is an important task whose proper implementation is crucial for many geodetic, surveying and mapping applications. This paper presents the required methodology to deal with the above problem when we are given the Helmert transformation parameters that link the underlying Cartesian coordinate systems to which an Earth reference ellipsoid is attached. The main emphasis is on the effect of GRF spatial scale differences in coordinate transformations involving reference ellipsoids, for the particular case of heights. Since every three-dimensional Cartesian coordinate system 'gauges' an attached ellipsoid according to its own accessible scale, there will exist a supplementary contribution from the scale variation between the involved GRFs on the relative size of their attached reference ellipsoids. Neglecting such a scale-induced indirect effect corrupts the values for the curvilinear geodetic coordinates obtained from a similarity transformation model, and meter-level apparent offsets can be introduced in the transformed heights. The paper explains the above issues in detail and presents the necessary mathematical framework for their treatment.

Keywords Ellipsoidal height · Geoid undulation · Reference ellipsoid · Geodetic reference frame (GRF) · Scale · Similarity transformation

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1 Introduction

The current advancements in the field of global and regional geodetic positioning have greatly enhanced the potential of conventional Earth-fixed reference systems by offering access to 3D spatial datums of very high accuracy, integrity and consistency (Altamimi et al. 2002; Craymer 2006; Ray et al. 2004; Snay et al. 2002). In addition, the continuous developments in the acquisition, modeling and processing of Global Positioning System (GPS) data have provided geodesists not only with highly reliable and precise external control to evaluate global and regional models for the Earth's gravity field (e.g. Denker 1998; Featherstone et al. 2001; Kiamehr and Sjoberg 2005), but they have also allowed the synergetic use of GPS with the continuously improved gravimetric geoid solutions for the definition and realization of modern satellite-based vertical reference systems (e.g. Kearsley et al. 1993; Denker et al. 2000; Nahavandchi and Soltanpour 2006).

To take full advantage of the above developments, careful treatment of several reference frame issues is required to ensure a coherent spatial framework for data analysis and to avoid datum-related biases and artifacts in the results (Pavlis 1998). Considering the recurring need in many Earth science applications to convert spatial information for vertical geodetic positions to a common reference system, the objective of the present paper is to investigate the transformation of *ellipsoid-dependent heights* from one geodetic reference frame (GRF) to another, when we are given the Helmert transformation parameters that link the underlying Cartesian coordinate systems to which a conventional reference ellipsoid is attached (see Fig. 1). Such a task is of crucial importance in several geodetic applications, including

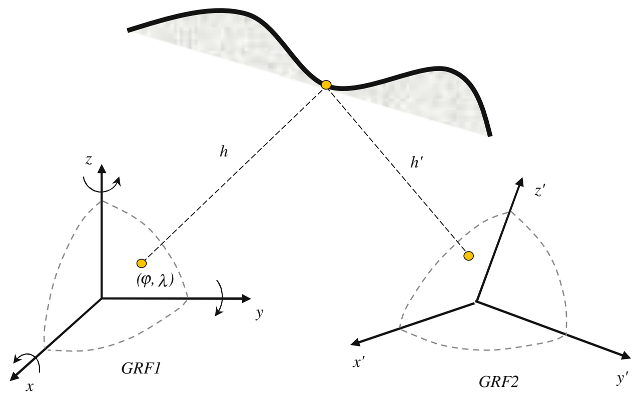

Fig. 1 Ellipsoidal height transformation between different geodetic reference frames

– the consistent combination of ellipsoidal, orthometric and geoid heights in the context of GPS-based leveling;
– the external validation of gravimetric geoid models using geometrically derived undulations with respect to a common datum;
– the update of existing geoid models to comply with current definitions and realizations of global geodetic reference systems;
– the reduction of sea surface heights obtained from satellite altimetry data to a preferred geodetic reference frame for mean sea level monitoring; and
– the reduction of ellipsoidal heights obtained at different epochs to a common reference frame for vertical crustal deformation studies.

An explicit treatment is given for the GRF transformation of geometric (ellipsoidal) heights obtained from GPS measurements, and geoid undulations determined either directly from a gravimetric geoid model or indirectly through GPS and leveled orthometric heights. A similar approach to the one presented herein can be followed for the GRF transformation of other ellipsoid-dependent height quantities, such as the sea surface height obtained from satellite altimetry data or the height anomaly computed from a spherical harmonic model for the Earth's gravitational potential.

Using the standard framework of Euclidean similarity transformation, a critical issue that affects the transformed GPS or geoid heights from one GRF to another (e.g. WGS84 → ITRF2000) is whether or not the adopted reference ellipsoid should be 'adapted' to each reference frame's scale. Regardless of the physical meaning that can be associated with the existence of spatial scale differences between modern GRFs (e.g., variation in their inherent values for fundamental Earth parameters such as GM, observation technique's biases, undetected instrument-related systematic errors, other modeling errors), the following dilemma can be stated: should an adopted reference ellipsoid retain the value of its semi-major axis in every GRF it is attached to, or should every Cartesian coordinate system 'gauge' an attached reference ellipsoid according to its own accessible scale that is inherent in the x, y, z values of the control stations used for the particular GRF realization?

The above predicament requires some clarification and elaboration, since the use of a reference ellipsoid entails a more or less conventional choice and, apparently, it should not be dictated by the spatial scale difference of the specific GRFs that will utilize such a conventional model. However, if the same reference ellipsoid (*thought of as a conventional geometric Earth model with invariant physical dimensions*) needs to be utilized in different, in terms of their accessible spatial scale, GRFs, then a distinct value for its semi-major axis should be assigned in each case.

Specifically, in order to maintain the invariance of the reference ellipsoid's physical surface in every 3D Cartesian coordinate system associated with it, we should 're-define' the value of its semi-major axis by $a' = (1+\delta s)a$, where δs is the differential scaling factor between two reference frames GRF and GRF′, a is the value of the reference ellipsoid's semi-major axis when attached to GRF, and a' is the value of the semi-major axis of the *same* reference ellipsoid when attached to GRF′. Neglecting such a scale-induced indirect effect corrupts the resulting values for the curvilinear geodetic coordinates obtained from a similarity transformation and *meter-level apparent offsets* can be introduced in the transformed ellipsoidal heights.

Based on a similar approach followed by Soler and van Gelder (1987), this paper presents an extended similarity-type model for the datum transformation of ellipsoid-dependent heights, which offers a proper de-coupling of the height variation originating from (i) the GRF scale differences and (ii) the actual change of the physical size of the reference ellipsoid. Various options that may appear in practice for the conventional selection of the reference ellipsoid attached to the 'new' GRF (i.e., the geodetic frame with respect to which the transformed heights are defined) are discussed, and their corresponding effects on the transformed heights are evaluated as a function of the GRFs' scale difference δs.

2 Similarity transformation for ellipsoid-dependent heights

2.1 GPS heights

Let us consider the well-known Euclidean similarity transformation model, which is used to convert Cartesian coordinates between two geodetic reference frames that differ in terms of three translation parameters (t_x, t_y, t_z), three orientation parameters $(\varepsilon_x, \varepsilon_y, \varepsilon_z)$ and a factor of uniform spatial scale change (δs)

$$\begin{bmatrix} x'-x \\ y'-y \\ z'-z \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} + \begin{bmatrix} \delta s & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & \delta s & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & \delta s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (1)$$

Equation (1) corresponds to a first-order approximation of the rigorous Helmert conformal transformation (Leick and van Gelder 1975; Soler 1998)

$$\mathbf{x}'_{\mathrm{GRF2}} = \mathbf{t} + (1+\delta s)\mathbf{R}(\varepsilon_x,\varepsilon_y,\varepsilon_z)\mathbf{x}_{\mathrm{GRF1}} \quad (2)$$

where $\mathbf{R}$ is an orthogonal matrix that performs three successive rotations around the axes of GRF1 so that they become parallel to the corresponding axes of GRF2, $\mathbf{t} = [t_x t_y t_z]^{\mathrm{T}}$ is the Cartesian coordinate vector of the origin of GRF1 with respect to GRF2, δs is a differential unitless factor expressing the scale difference between the two frames, $\mathbf{x}_{\mathrm{GRF1}} = [xyz]^{\mathrm{T}}$ is the Cartesian coordinate vector of an arbitrary point with respect to GRF1, and $\mathbf{x}'_{\mathrm{GRF2}} = [x'y'z']^T$ is the Cartesian coordinate vector of the same point with respect to GRF2; see Fig. 1.

The use of the approximate model in Eq. (1) instead of the rigorous expression in Eq. (2) has a negligible effect on the transformed coordinates and it is justified for most geodetic applications where the rotation angles do not exceed a few arc seconds and the differential scale factor is of the order of 10^{-5} or less (e.g., Harvey 1995, pp. 101–102). In contrast to the non-standard convention followed by the International Earth Rotation and Reference Systems Service (IERS) for terrestrial GRF transformations (McCarthy and Petit 2004, Chap. 4), Eq. (1) is consistent with the usual hypothesis that counter-clockwise rotations around the GRF1 frame axes are positive, whereas clockwise rotations around the same frame axes are considered negative (Soler 1998).

To derive a one-step formula for the similarity transformation of ellipsoidal heights determined from GPS, we need to combine Eq. (1) with the relationship between Cartesian and curvilinear geodetic coordinates

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (\mathrm{N}+h)\cos\varphi\cos\lambda \\ (\mathrm{N}+h)\cos\varphi\sin\lambda \\ (\mathrm{N}(1-e^2)+h)\sin\varphi \end{bmatrix} \quad (3)$$

where N (to be distinguished from the symbol N that denotes the geoid height) is the prime vertical radius of curvature

$$\mathrm{N} = \frac{a}{W} = \frac{a}{\sqrt{1-e^2\sin^2\varphi}}. \quad (4)$$

The quantities a and e^2 correspond to the length of the semi-major axis and the squared eccentricity of the reference ellipsoid that is used for the definition of the geodetic coordinates φ, λ, h that appear in Eq. (3).

If we differentiate Eq. (3) with respect to the variation in the curvilinear geodetic coordinates, we get

$$\begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{bmatrix} = \mathbf{J} \begin{bmatrix} \mathrm{d}\varphi \\ \mathrm{d}\lambda \\ \mathrm{d}h \end{bmatrix} \quad (5)$$

where the Jacobian matrix $\mathbf{J}$ has the form (Soler 1976)

$$\mathbf{J} = \begin{bmatrix} -(\mathrm{M}+h)\sin\varphi\cos\lambda & -(\mathrm{N}+h)\cos\varphi\sin\lambda & \cos\varphi\cos\lambda \\ -(\mathrm{M}+h)\sin\varphi\sin\lambda & (\mathrm{N}+h)\cos\varphi\cos\lambda & \cos\varphi\sin\lambda \\ (\mathrm{M}+h)\cos\varphi & 0 & \sin\varphi \end{bmatrix} \quad (6)$$

and $M = a(1-e^2)/W^3$ is the meridian radius of curvature. Note that the auxiliary term W denotes the latitude-dependent unitless quantity $\sqrt{1-e^2\sin^2\varphi}$.

Substituting the left-hand side in Eq. (5) with the Cartesian coordinate transformation from Eq. (1), and then solving for dh, we obtain the following expression that corresponds to the direct (linearized) *similarity transformation model for ellipsoidal heights*

$$h'-h = \delta h(t_x) + \delta h(t_y) + \delta h(t_z) + \delta h(\varepsilon_x) + \delta h(\varepsilon_y) + \delta h(\delta s) \quad (7)$$

where the individual transformation terms are given by

$$\delta h(t_x) = t_x\cos\varphi\cos\lambda \quad (8)$$

$$\delta h(t_y) = t_y\cos\varphi\sin\lambda \quad (9)$$

$$\delta h(t_z) = t_z\sin\varphi \quad (10)$$

$$\delta h(\varepsilon_x) = -\varepsilon_x \mathrm{N}e^2\sin\varphi\cos\varphi\sin\lambda \quad (11)$$

$$\delta h(\varepsilon_y) = \varepsilon_y \mathrm{N}e^2\sin\varphi\cos\varphi\cos\lambda \quad (12)$$

$$\delta h(\delta s) = (aW+h)\delta s \quad (13)$$

Notice that, due to the rotational symmetry of the reference ellipsoid with respect to the z axis, the rotation angle ε_z does not affect the change of the ellipsoidal height from GRF1 to GRF2. More details for the above differential transformation procedure can be found in Molodensky et al. (1962), Soler (1976), Soler and van Gelder (1987) and Rapp (1993).

2.2 Geoid heights

Equations (7) to (13) perform, in one step, the similarity transformation of ellipsoidal heights from one geodetic reference frame (h with respect to GRF1) to another geodetic reference frame (h' with respect to GRF2) at any point in space with known curvilinear coordinates φ, λ and h; see Fig. 1.

If we assume that the point whose ellipsoidal height being transformed is located on the geoid, then Eq. (7) is reduced to the direct (linearized) *similarity transformation model for geoid heights*

$$N'-N = \delta N(t_x) + \delta N(t_y) + \delta N(t_z) + \delta N(\varepsilon_x) + \delta N(\varepsilon_y) + \delta N(\delta s) \quad (14)$$

where

$$\delta N(t_x) = t_x \cos\varphi \cos\lambda \tag{15}$$

$$\delta N(t_y) = t_y \cos\varphi \sin\lambda \tag{16}$$

$$\delta N(t_z) = t_z \sin\varphi \tag{17}$$

$$\delta \mathrm{N}(\varepsilon_x) = -\varepsilon_x \mathrm{N}e^2 \sin\varphi \cos\varphi \sin\lambda \tag{18}$$

$$\delta \mathrm{N}(\varepsilon_y) = \varepsilon_y \mathrm{N}e^2 \sin\varphi \cos\varphi \cos\lambda \tag{19}$$

Since the evaluation point in this case is located on the geoid surface, its ellipsoidal height will be identical to the geoid undulation with respect to the same GRF ($h = N$). Thus, the scale-dependent term from Eq. (13) should take the form

$$\delta N(\delta s) = (aW + N)\delta s \tag{20}$$

where N is the geoid height with respect to the initial reference frame.

In principle, the curvilinear geodetic coordinates φ and λ that enter in the above transformation formulae refer to the horizontal position of the evaluation point (located on the geoid) with respect to GRF1. In practice, φ and λ correspond to the horizontal geodetic coordinates of a terrestrial point located on the Earth's surface, whose geoid undulation is available with respect to GRF1 and needs to be transformed to a new reference frame GRF2.

Remark 1 Note that Eqs. (7) and (14) perform only a geometric-type datum transformation for ellipsoidal and geoid heights respectively, without considering additional variations in the values of other fundamental Earth parameters (e.g. a, e^2, W_o, J_2, etc.) for the reference system associated with the new frame GRF2.

Remark 2 Similar formulae like Eqs. (7) or (14) can be used for the transformation of other ellipsoid-dependent height quantities that are commonly employed in geodetic applications, such as the height anomaly determined by a spherical harmonic model for the Earth's gravitational field or the sea surface height obtained from satellite altimetry measurements.

2.3 Apparent change of the reference ellipsoid

Consider the case where the underlying GRFs have the same origin and orientation (i.e. $t_x = t_y = t_z = \varepsilon_x = \varepsilon_y = \varepsilon_z = 0$) and they differ only in terms of a spatial scaling factor. Under these constraints, the similarity transformation model for ellipsoidal heights in Eq. (7) yields

$$h' - h = \delta h(\delta s) = (aW + h)\delta s \tag{21}$$

or equivalently

$$h' = (1 + \delta s)h + aW\delta s \tag{22}$$

Likewise, the similarity transformation model for geoid heights in Eq. (14) takes the simplified form

$$N' - N = \delta N(\delta s) = (\mathrm{a}W + N)\delta s \tag{23}$$

or equivalently

$$N' = (1 + \delta s)N + aW\delta s \tag{24}$$

The additive latitude-dependent term $aW\delta s$ in Eqs. (22) and (24) corresponds to the effect of an 'apparent' change in the length of the semi-major axis of the reference ellipsoid due to the scale difference between the involved GRFs. Its magnitude can be quite significant, reaching more than 6 m when $\delta s = 10^{-6}$ (1 ppm) and dropping to about 7 mm for $\delta s = 10^{-9}$ (1 ppb). The direct consequences of this apparent offset in the transformed height values were studied in Soler and van Gelder (1987) to explain the detected z-shift between Doppler-based and SLR geodetic reference systems, as well as other types of inconsistencies that occurred from the results of curvilinear geodetic coordinate transformations in various studies that were performed in the 1960s, 1970s and 1980s.

If the reference ellipsoid retains its physical size in both GRFs, then the transformed ellipsoidal height (when $t_x = t_y = t_z = \varepsilon_x = \varepsilon_y = \varepsilon_z = 0$) should be given only by a simple re-scaling

$$h' = (1 + \delta s)h \tag{25}$$

since the same physical length (i.e., the distance between a fixed point in space and its orthogonal projection onto a single reference ellipsoid) needs to be quantified with respect to two coinciding GRFs that differ only by a uniform scale factor δs. The same argument applies also for the case of geoid height transformation.

In order to counter-balance the effect of the apparent variation term $aW\delta s$ in Eqs. (22) and (24), and also to properly account for an actual change in the physical dimensions of the reference ellipsoid, the six-parameter similarity transformation model for ellipsoidal and geoid heights needs to be further extended as described in Sect. 3.

2.4 Rigorous non-linear approach

The most rigorous approach for performing the similarity-type transformation of ellipsoidal or geoid heights between different datums requires, in principle, a three-step procedure according to the non-linear scheme depicted in Fig. 2.

The first step is executed through the curvilinear-to-Cartesian coordinate conversion from Eq. (3), while the second intermediate step employs the standard Helmert transformation model from Eq. (1) or Eq. (2). The last step requires the inversion of the curvilinear-to-Cartesian transformation formula, which is usually implemented through an iterative or

$$(\varphi,\lambda,h)_{GRF1} \xrightarrow{step\,1} (x,y,z)_{GRF1} \xrightarrow{step\,2} (x',y',z')_{GRF2} \xrightarrow{step\,3} (\varphi',\lambda',h')_{GRF2}$$

Fig. 2 The most rigorous approach for transforming the ellipsoidal height at a known point (φ, λ, h) between different geodetic reference frames

closed-form algorithm (e.g., Pollard 2002; Fukushima 2006; Zhang et al. 2005).

For the particular case of geoid height transformation, the procedure in Fig. 2 should be applied using the initially known value of the geoid height N in place of h, so that the computed Cartesian coordinates from step 1 refer to the spatial position of the evaluation point on the geoid with respect to GRF1. From the two subsequent steps, the Cartesian and curvilinear geodetic coordinates of the same point with respect to the new reference frame are obtained. Obviously, the ellipsoidal height h' that is computed at the final step will correspond to the transformed geoid undulation N' with respect to GRF2.

The implementation of step 1 and step 3 entails the conventional adoption of a specific reference ellipsoid that should be attached to the GRF1 and GRF2 frames, in order to define the triplets of curvilinear geodetic coordinates φ, λ, h (or N) and φ', λ', h'(or N'), respectively. It should be noted that the presence of the term $aW\delta s$ in Eqs. (22) and (24), which corresponds to an apparent variation of the size of the reference ellipsoid as explained in the previous section, is consistent with the use of the *same numerical value* for the semi-major axis of the reference ellipsoids in steps 1 and 3.

The difference in the transformed heights obtained from the rigorous non-linear scheme of Fig. 2 and the direct one-step transformation formulae given in Eqs. (7) or (14) is due to the linearization errors of the curvilinear-to-Cartesian conversion formula, which was inherently employed in a differential form for the development of Eqs. (7) and (14); see Sect. 2.1. The effect of these linearization errors on the transformed heights is at the cm-level, which for certain applications (e.g., geodynamic studies) may pose an unacceptable source of error.

As an example, let us consider the problem of transforming the ellipsoidal height at a point with known geodetic coordinates $\varphi = 50^\circ.0034$, $\lambda = 11^\circ.0028$ and $h = 547.19$ m from the German national coordinate system (DHDN) to the European Terrestrial Reference Frame 1989 (ETRF89). The Helmert transformation parameters are $t_x = 582.00$ m, $t_y = 105.00$ m, $t_z = 414.00$ m, $\varepsilon_x = -1''.040$, $\varepsilon_y = -0''.350$, $\varepsilon_z = 3''.080$ and $\delta s = 8.30$ ppm (Ihde and Lindstrot 1995). The semi-major axis of the reference ellipsoid associated with the DHDN geodetic coordinates is set to $a = 6378137.000$ m and its flattening $f = 0.00335281068118$ (GRS80 values). In this case, the transformed ellipsoidal height obtained by applying the direct transformation formula in Eq. (7) is $h' = 1297.253$ m, whereas the corresponding value determined through the rigorous non-linear methodology of Fig. 2 is $h' = 1297.256$ m.

3 Extended similarity transformation for ellipsoid-dependent heights

3.1 Considering the effect of the reference ellipsoid change

The length of the semi-major axis (a) and the flattening (f) shall be adopted as the two fundamental parameters that uniquely define the geometrical size and shape of a reference Earth ellipsoid.

In order to account for a likely change in the physical dimensions of the reference ellipsoid in height transformation problems, we should again perform a differentiation of the curvilinear-to-Cartesian coordinate conversion formula in Eq. (3) as follows

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \mathbf{J}_1 \begin{bmatrix} d\varphi \\ d\lambda \\ dh \end{bmatrix} + \mathbf{J}_2 \begin{bmatrix} da \\ df \end{bmatrix} \tag{26}$$

where the first Jacobian matrix $\mathbf{J}_1$ is identical to the matrix $\mathbf{J}$ given in Eq. (6), while the analytical form of the second Jacobian matrix $\mathbf{J}_2$ is (Soler 1976)

$$\mathbf{J}_2 = \begin{bmatrix} \dfrac{\cos\varphi\cos\lambda}{W} & \dfrac{a(1-f)\sin^2\varphi\cos\varphi\cos\lambda}{W^3} \\ \dfrac{\cos\varphi\sin\lambda}{W} & \dfrac{a(1-f)\sin^2\varphi\cos\varphi\sin\lambda}{W^3} \\ \dfrac{(1-e^2)\sin\varphi}{W} & (\mathrm{M}\sin^2\varphi - 2\mathrm{N})(1-f)\sin\varphi \end{bmatrix} \tag{27}$$

The differential quantities da and df express the variation of the geometrical size and shape of the reference ellipsoid due to a change in the length of its semi-major axis and/or the value of its flattening.

Setting the left-hand side in Eq. (26) equal to zero, and then solving for dh, we obtain the ellipsoidal height variation associated with the change of the reference ellipsoid which, in conjunction with Eq. (7), leads to the direct (linearized) *extended similarity transformation model for ellipsoidal heights*

$$h' - h = \delta h(t_x) + \delta h(t_y) + \delta h(t_z) + \delta h(\varepsilon_x) + \delta h(\varepsilon_y) + \delta h(\delta s) + \delta h(\delta a) + \delta h(\delta f) \tag{28}$$

where the additional terms $\delta h(\delta a)$ and $\delta h(\delta f)$ are given by (Soler 1976; Rapp 1993)

$$\delta h(\delta a) = -W\delta a \tag{29}$$

$$\delta h(\delta f) = \frac{a(1-f)}{W}\sin^2\varphi\,\delta f \tag{30}$$

The quantities $\delta a = a' - a$ and $\delta f = f' - f$ correspond to the difference in the numerical values for the semi-major axis and the flattening of the reference ellipsoid, as these are used in the respective reference frames, GRF1 and GRF2. The translation, rotation, and scale dependent terms in Eq. (28) have already been defined in Sect. 2.

Assuming that the evaluation point is located on the geoid, the corresponding direct (linearized) *extended similarity transformation model for geoid heights* is obtained

$$N' - N = \delta N(t_x) + \delta N(t_y) + \delta N(t_z) + \delta N(\varepsilon_x) + \delta N(\varepsilon_y) + \delta N(\delta s) + \delta N(\delta a) + \delta N(\delta f) \tag{31}$$

where the terms $\delta N(\delta a)$ and $\delta N(\delta f)$ are exactly the same as in the case of ellipsoidal height transformation

$$\delta N(\delta a) = \delta h(\delta a) \tag{32}$$

$$\delta N(\delta f) = \delta h(\delta f) \tag{33}$$

while the translation, rotation, and scale dependent terms in Eq. (31) have been also defined in Sect. 2.

Remark 3 With the exclusion of the rotation- and scale-dependent terms, Eqs. (28) and (31) correspond to the standard *Molodensky differential transformation formulae* (Molodensky et al. 1962) which have often been used for transforming ellipsoidal heights between different geodetic datums (National Imagery and Mapping Agency 1996, pp. 7.3–7.4) and for determining the Earth's mean equatorial radius and center of mass through the joint analysis of geometrically derived and gravimetric geoid heights (e.g. Anderle 1974; Rapp and Rummel 1976; Grappo 1980); see also Badekas (1969); Hotine (1969); Heiskanen and Moritz (1967) and Rapp (1993).

3.2 Geoid fitting

A simplified version of the geoid height transformation given in Eq. (31), namely

$$N' - N = \delta N(t_x) + \delta N(t_y) + \delta N(t_z) + N_o \tag{34}$$

is frequently employed for studies related to the evaluation and the accuracy assessment of geoid models, particularly in view of their potential use in GPS-based leveling projects.

In such cases, the values N and N' correspond to the known geoid heights obtained from different data sources or Earth gravity models (e.g., N from a gravimetric geoid model and N' from GPS and leveling data or from another geoid model) and they are available over a network of terrestrial control stations. Following a standard least-squares adjustment, Eq. (34) has been often applied for the comparative validation of global and/or regional geoid models, with the datum-shift terms t_x, t_y, t_z and the geoid-shift term N_o being treated as unknown fitting or correction parameters (Rapp and Rummel 1976).

The estimated value of N_o that is determined by the aforementioned approach corresponds to the mean offset between the geoid models under comparison, and it gives a collective indication of the various constant biases that are inherent in them. The parameter N_o will absorb any uncorrected effects originating from the orientation, scale and reference ellipsoid discrepancies between the underlying datums, as well as the zero-order effect coming from the gravity potential and mass differences between the equipotential surfaces realized by the values N and N', respectively. As a result, the conclusions drawn from this type of evaluation scheme are likely to be obscured by the fusion of several distinct sources into the N_o estimate, which could lead to a deceptive interpretation about the actual quality of the geoid model(s) under study.

A relatively large value of N_o may arise merely from the spatial scale difference between the GRFs associated with the geoid heights N and N'. The scale-dependent term $\delta N(\delta s) = (aW + N)\delta s$, which will be absorbed within the estimated value of N_o, can reach more than 6 m for datum scale differences of the order of 10^{-6}. On the other hand, a small value of N_o could be the result of reciprocal cancellations for a number of individually significant error sources that may exist in each data set.

If we have a clear knowledge about the particular GRFs with respect to which the original geoid heights N and N' are defined, then the transformation model of Eq. (31) should be initially applied in a forward manner to reduce the two data sets to a common reference system. Subsequently, the comparison and the modeling of the differences between the datum-corrected geoid heights can provide a more realistic assessment and quantification about the remaining biases, or other systematic effects, that may still exist in them.

3.3 What should we use for δa?

An issue that needs to be clarified in the context of the similarity-type transformation for ellipsoid-dependent heights is the proper evaluation of the terms $\delta h(\delta a)$ and $\delta N(\delta a)$, which give the ellipsoidal and geoid height variation due to the change of the semi-major axis for the reference ellipsoids adopted by the frames GRF1 and GRF2. Note that if we simply set $\delta a = 0$ (which corresponds to the case where both datums use the same value for the semi-major axis of their ellipsoids), then the influence of the additive term $aW\delta s$ for the apparent variation of the reference ellipsoid will fully remain in the transformed height values; see Eqs. (28) and (31).

If not corrected, the presence of such a 'non-physical' effect in the transformed vertical positions can cause problems for certain geodetic applications (e.g., vertical crustal deformation from GPS data, or mean sea level monitoring by satellite altimetry), since it will appear as a hidden, almost constant, offset in the transformed heights which, however, will not correspond to any real vertical movement at the evaluation points.

In general, the length of the semi-major axis of the reference ellipsoid attached to GRF2 can be expressed as

$$a' = (1 + \delta s)a + \delta\bar{a} \tag{35}$$

where a is the length of the semi-major axis of the reference ellipsoid attached to GRF1, δs is the scale change factor between the two frames, and $\delta\bar{a}$ corresponds to the actual change of the *physical length* of the semi-major axis of the GRF2 ellipsoid with respect to the *physical length* of the semi-major axis of the GRF1 ellipsoid.

In this way, we have the relationship

$$\delta a = a' - a = a\delta s + \delta\bar{a} \tag{36}$$

which shows the influence of the differential scale factor δs on the total variation δa of the semi-major axis for the reference ellipsoid adopted by the GRF2 frame; see also Soler and van Gelder (1987). Hence, the transformation terms $\delta h(\delta a)$ and $\delta N(\delta a)$ can be decomposed as

$$\delta h(\delta a) = \delta N(\delta a) = -Wa\delta s - W\delta\bar{a} \tag{37}$$

Based on the above, let us consider again the case where $t_x = t_y = t_z = \varepsilon_x = \varepsilon_y = \varepsilon_z = 0$, and additionally $\delta f = 0$. Under these constraints, the extended similarity transformation model for ellipsoidal heights in Eq. (28) yields

$$h' - h = \delta h(\delta s) + \delta h(\delta a) \tag{38}$$

or, using Eqs. (13) and (37), in the equivalent form

$$h' = (1 + \delta s)h - W\delta\bar{a} \tag{39}$$

In contrast to the expression obtained by the simple (non-extended) similarity transformation model in Eq. (22), the above result complies with geometrical intuition that dictates that the transformed ellipsoidal height should be determined by a simple re-scaling *if* the underlying GRFs have the same origin and orientation *and* also use the same reference ellipsoid in terms of physical dimensions ($\delta f = 0, \delta\bar{a} = 0$!). A similar analysis can also be carried out for the case of geoid height transformation $N \rightarrow N'$.

After having explained the proper evaluation of the term $\delta a = a' - a$, we can now give the final form of the direct similarity-type transformation for ellipsoid-dependent heights, as shown in Table 1.

Table 1 Conversion of ellipsoid-dependent heights between different geodetic reference frames according to the Helmert similarity transformation model and considering a change of the physical size of the reference ellipsoid

Transformation of ellipsoid-dependent heights from GRF1 to GRF2		
$h' = h + \delta h(t_x) + \delta h(t_y) + \delta h(t_z) + \delta h(\varepsilon_x) + \delta h(\varepsilon_y) + \delta h(\delta s) + \delta h(\delta a) + \delta h(\delta f)$		
GRF trtanslation terms	$\delta h(t_x) = t_x \cos\varphi \cos\lambda$	where t_x, t_y, t_z are the coordinates of the origin of GRF1 with respect to GRF2
	$\delta h(t_y) = t_y \cos\varphi \sin\lambda$	
	$\delta h(t_z) = t_z \sin\varphi$	
GRF orientation terms	$\delta h(\varepsilon_x) = -\varepsilon_x \mathrm{N}e^2 \sin\varphi \cos\varphi \sin\lambda$	where $\varepsilon_x, \varepsilon_y$ are the rotation angles around the x and y axes of GRF1 (anti-clockwise rotations are assumed positive)
	$\delta h(\varepsilon_y) = \varepsilon_y \mathrm{N}e^2 \sin\varphi \cos\varphi \cos\lambda$	
GRF scaling term	$\delta h(\delta s) = (aW + h)\delta s$	where δs is the differential scale factor between GRF1 and GRF2
Reference ellipsoid variation terms	$\delta h(\delta a) = -Wa\delta s - W\delta\bar{a} = -W\delta a$	where δa and $\delta\bar{a}$ are the changes in the *numerical value* and the *physical length*, respectively, of the semi-major axis of the reference ellipsoid, and δf is the change of its flattening
	$\delta h(\delta f) = \frac{a(1-f)}{W} \sin^2\varphi\delta f$	

Notes:
- The values of a, e^2 and f refer to the reference ellipsoid adopted by the first reference frame (GRF1)
- The quantities φ, λ, h refer to the position of the evaluation point with respect to the first reference frame (GRF1)
- The auxiliary term $W = (1 - e^2 \sin^2\varphi)^{1/2}$ and the prime vertical radius of curvature $\mathrm{N} = a/W$ are evaluated using the aforementioned quantities
- For the transformation of geoid heights, h should be replaced by N in all above formulae

Remark 4 The option of employing a different reference ellipsoid in the context of the rigorous non-linear datum transformation procedure for ellipsoid-dependent heights (see Sect. 2.4), can be handled by using appropriate values for its defining geometrical parameters (a, f) during the implementation of step 1 and step 3 (see Fig. 2). In any case, the value of the semi-major axis of the reference ellipsoid that will be used in step 3 can be related to the value of the semi-major axis of the reference ellipsoid adopted in step 1 through the general expression of Eq. (35). Therefore, using $a' = (1 + \delta s)a$ in step 3 of the non-linear transformation procedure corresponds to retaining the physical size of the reference ellipsoid in both datums, since in this case we have that $\delta\bar{a} = 0$. On the other hand, if we choose to employ the convention $a' = a$ in step 3, the physical size of the reference ellipsoid is different in the new datum since from Eq. (35) it holds that $\delta\bar{a} = -a\delta s$.

3.4 Joint variation of ellipsoidal and geoid heights

If we consider a point P that is located on the Earth's surface then its ellipsoidal height variation $\delta h_P = h'_P - h_P$, due to a change of the underlying GRF, will not be necessarily equal to its corresponding geoid height variation $\delta N_P = N'_P - N_P$. The discrepancy between the values δh_P and δN_P is caused solely by the scale-dependent variation terms $\delta h(\delta s)$ and $\delta N(\delta s)$, which are generally different at a point located outside the geoid, i.e.

$$\delta h(\delta s) = (aW + h_P)\delta s \neq (aW + N_P)\delta s = \delta N(\delta s) \quad (40)$$

However, the rest of the height variation terms, that depend on the datum translation parameters (t_x, t_y, t_z), the datum orientation parameters $(\varepsilon_x, \varepsilon_y, \varepsilon_z)$ and the datum ellipsoid parameters $(\delta a, \delta f)$, are *always* the same regardless of the particular quantity (ellipsoidal or geoid height) being transformed; see Table 1.

The fact that $\delta h_P \neq \delta N_P$, in the case where the GRF1 $\rightarrow$ GRF2 transformation involves a non-zero scaling factor δs, does not contradict the well-known relationship among ellipsoidal, orthometric and geoid heights (e.g., Heiskanen and Moritz 1967)

$$h_P = H_P + N_P \quad (41)$$

A common misconception that often arises from Eq. (41) is to claim that the ellipsoidal height variation is *always* equivalent to the geoid height variation under a change of the geodetic reference frame in which the ellipsoid-dependent quantities h and N have been determined.

The primary reasoning in this case is summarized as follows: since the point P remains fixed with respect to the Earth's surface and the orthometric height depends only on the geoid-crust vertical separation, then changing solely the geodetic reference system implies that $\mathrm{d}H_P = 0$ and thus $\mathrm{d}h_P = \mathrm{d}N_P$.

However, Eq. (41) presumes that all three height types refer to a common spatial scale. If we change the reference scale in which the GRF-dependent heights h and N are expressed to, then we need also to rescale accordingly the orthometric height H in order for Eq. (41) to remain valid. In this sense, the transformation model given in Table 1 is fully consistent with Eq. (41), since

$$\begin{aligned}\delta h_P - \delta N_P &= \delta h(\delta s) - \delta N(\delta s)\\ &= (aW + h_P)\delta s - (aW + N_P)\delta s\\ &= (h_P - N_P)\delta s\\ &= H_P\delta s\\ &= \delta H_P\end{aligned} \quad (42)$$

where $\delta H_P = H_P\delta s$ is the required correction that needs to be applied to the orthometric height so that Eq. (41) retains its consistency.

4 Implementation options

Several cases can be identified for the practical implementation of the similarity-type transformation model that was discussed in the previous section. Assuming that the Helmert transformation parameters $(t_x, t_y, t_z, \varepsilon_x, \varepsilon_y, \varepsilon_z, \delta s)$ between the underlying frames are known, these cases relate to the evaluation of the terms $\delta h(\delta a)$ or $\delta N(\delta a)$, and they essentially correspond to choosing how to treat the physical size of the reference ellipsoid with respect to the involved GRFs.

Note that, unlike the quantities $\delta h(\delta a)$ and $\delta N(\delta a)$, the terms $\delta h(\delta f)$ and $\delta N(\delta f)$ are insensitive to any GRF scale difference. That is because the ellipsoid flattening, $f = (a - b)/a$, is a unitless geometrical measure that does not depend on the spatial scale of the particular datum.

Equation (35) provides the basic model upon which each of the following implementation options can be distinguished. The value of a associated with the initial reference system (GRF1) and the scaling factor δs are considered known quantities, in every case. In many practical applications, the value of a' associated with the target reference system (GRF2) is also fixed, particularly for studies where the transformed heights need to be combined with other types of geodetic data that are already available with respect to the GRF2 datum and a conventionally given reference ellipsoid (a', f').

In the latter case, the transformed heights obtained through the similarity-type model of Eq. (28) or Eq. (31) will contain the effect of the inherent change $\delta\bar{a} = a' - (1 + \delta s)a$ of the *physical dimensions* of the GRF2 ellipsoid with respect to the GRF1 ellipsoid. Such an effect can create significant apparent biases in the vertical position of the evaluation points, especially in the common case where $a' = a$, and it will affect any other auxiliary geodetic quantity that may be subsequently computed from these transformed heights (e.g. normal gravity, gravity anomaly, gravity disturbance, etc.).

Based on Eq. (35), the following three cases can be distinguished. The critical point for each case is the combined treatment of the terms $\delta h(\delta s)$ and $\delta h(\delta a)$, which give the variation due to the change of the semi-major axis of the reference ellipsoid and the spatial scale difference in the underlying reference frames.

Case 1: Keeping the same physical size of the reference ellipsoid in both datums

One alternative is to select $\delta\bar{a} = 0$, which implies that the physical length of the semi-major axis of the reference ellipsoid is invariant within the underlying GRFs. According to Sect. 3, the combined effect of the variation terms $\delta h(\delta s)$ and $\delta h(\delta a)$ is

$$\begin{aligned}\delta h(\delta s, \delta a) &= \delta h(\delta s) + \delta h(\delta a)\\ &= (aW + h)\delta s - W(a\delta s + \delta\bar{a})\\ &= h\delta s\end{aligned} \tag{43}$$

whereas, for the case of geoid heights, we have that

$$\begin{aligned}\delta N(\delta s, \delta a) &= \delta N(\delta s) + \delta N(\delta a)\\ &= (aW + N)\delta s - W(a\delta s + \delta\bar{a})\\ &= N\delta s\end{aligned} \tag{44}$$

which corresponds to a negligible correction for most practical purposes (<1 cm even for $\delta s = 10$ ppm).

In this case, all numerical calculations involving the semi-major axis of the reference ellipsoid with respect to the GRF2 datum (e.g. conversion of Cartesian coordinates to curvilinear coordinates and vice versa, computation of normal gravity values, etc.) should be compatible with the new re-scaled value

$$a' = (1 + \delta s)a \tag{45}$$

and not the initial value a which is used for similar calculations with respect to the GRF1 datum; see also Soler and van Gelder (1987).

The implementation of this particular option is useful if we want to ensure that various ellipsoid-dependent geodetic quantities that are observed or computed in one datum, they will keep referring to the same spatial reference surface when they are transferred to a new datum through a similarity-type transformation model.

Case 2: Using the same numerical value for the semi-major axis of the reference ellipsoid in both datums (a′ = a)

The second alternative is to set a-priori $\delta a = 0$, which implies that the same numerical value for the semi-major axis of the reference ellipsoid is adopted and used by both reference systems, GRF1 and GRF2. This is a rather common option in practice, since most geodetic datums today make use of the physical and geometric parameter values associated with GRS80.

In this case, the combined effect of the terms $\delta h(\delta s)$ and $\delta h(\delta a)$ is

$$\begin{aligned}\delta h(\delta s, \delta a) &= \delta h(\delta s) + \delta h(\delta a)\\ &= (aW + h)\delta s - W\delta a\\ &= h\delta s + aW\delta s\end{aligned} \tag{46}$$

whereas, for the case of geoid heights, we have

$$\begin{aligned}\delta N(\delta s, \delta a) &= \delta N(\delta s) + \delta N(\delta a)\\ &= (aW + N)\delta s - W\delta a\\ &= N\delta s + aW\delta s.\end{aligned} \tag{47}$$

The magnitude of the additional correction term $aW\delta s$ that appears in Eqs. (46) and (47) is quite significant, since it can reach more than 6 m even for δs = 1 ppm; see Fig. 3. Its actual value is practically independent of the particular ellipsoid (a, f) adopted by the initial datum GRF1, and it appears as an almost constant offset in all transformed heights since it has a very weak dependence on latitude variations (i.e. it varies by less than ±1 cm, over the ellipsoidal surface).

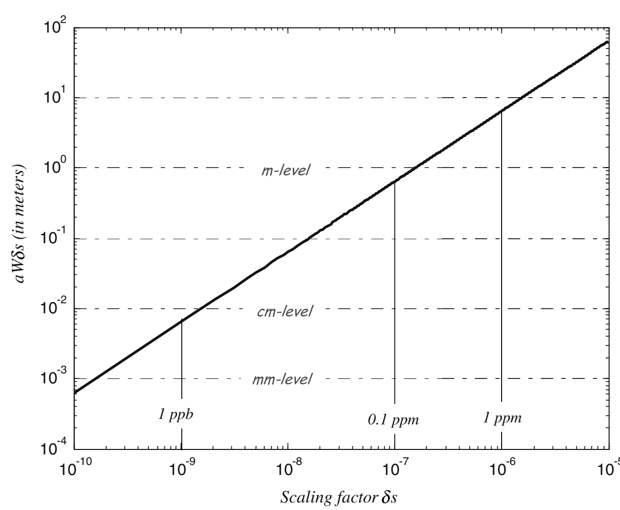

Fig. 3 The effect of the apparent variation term $aW\delta s$ on the transformed ellipsoid-dependent heights, as a function of the scaling factor involved in the GRF transformation. The particular graph shows the influence at a point with $\varphi = 45°$ for a reference ellipsoid with $a = 6378137$ m and $e^2 = 0.006694380$

Adopting the same numerical value for the semi-major axis of the reference ellipsoid that is attached in both datums carries an inherent change in its physical dimensions, if the scaling factor $\delta s \neq 0$. Indeed, by setting $\delta a = 0$ in Eq. (36), we get

$$\delta\bar{a} = -a\delta s \tag{48}$$

which gives the required change in the physical length of the semi-major axis of the reference ellipsoid attached to GRF2 in order for the numerical values a and a' to be equal. This, in turn, may cause significant changes in the transformed heights values, as shown in Fig. 3, depending on the value of the scaling factor δs.

Case 3: Using an arbitrary, conventionally fixed, value for the semi-major axis of the reference ellipsoid in the second datum ($\mathbf{a}' \neq \mathbf{a}$)

The last option can be considered as a generalization of Case 2. Now, a given conventional numerical value a' for the semi-major axis of the GRF2 reference ellipsoid is used, which is generally different from the conventional value a that was adopted for the GRF1 ellipsoid. This new value is intended for use in all related calculations involving positional information with respect to the GRF2 datum (e.g., conversion of Cartesian coordinates to curvilinear coordinates and vice versa, computation of normal gravity values, etc.), and it may have been obtained by a revision of the geometrical parameters associated with the best geodetic Earth model (e.g., Grafarend and Ardalan 2000).

In this case, the actual change in the physical length of the semi-major axes of the GRF1 and GRF2 ellipsoids should be determined by Eq. (35) as

$$\delta\bar{a} = -a\delta s + (a' - a) \tag{49}$$

Taking into account the similarity-type transformation model from Table 1, the combined effect of the terms $\delta h(\delta s)$ and $\delta h(\delta a)$ is

$$\begin{aligned}\delta h(\delta s, \delta a) &= \delta h(\delta s) + \delta h(\delta a)\\ &= (aW + h)\delta s - W\delta a\\ &= h\delta s + aW\delta s - W(a' - a)\end{aligned} \tag{50}$$

whereas, for the case of geoid heights, we have

$$\begin{aligned}\delta N(\delta s, \delta a) &= \delta N(\delta s) + \delta N(\delta a)\\ &= (aW + N)\delta s - W\delta a\\ &= N\delta s + aW\delta s - W(a' - a)\end{aligned} \tag{51}$$

Depending on the magnitude of the difference $a' - a$ and the accuracy standards for the transformed heights, the last correction term may be significant and it should be taken into account. The other two correction terms shown in Eqs. (50) and (51) have already been explained under Case 1 and Case 2, respectively.

Numerical example. Consider the problem of transforming the EGM96 geoid height from the WGS84(G873) frame to the ITRF94 frame. The horizontal geodetic coordinates of the evaluation point, with respect to the WGS84(G873) frame, are given $\varphi = 50°.0000, \lambda = 11°.0000$ and its EGM96/WGS84(G873) geoid height is $N = 47.193$ m. The values of the Helmert transformation parameters from WGS84(G873) to ITRF94 (at epoch $t = 1997.0$) are $t_x = 9.6$ cm, $t_y = 6.0$ cm, $t_z = 4.4$ cm, $\varepsilon_x = -2.2$ mas, $\varepsilon_y = -0.1$ mas, $\varepsilon_z = 1.1$ mas and $\delta s = -14.3$ ppb (Malys et al. 1997). The reference ellipsoid associated with the EGM96 geoid and the WGS84(G873) geodetic position is defined in terms of $a = 6378137.00$ m and $f = 0.00335281066475$ (National Imagery and Mapping Agency 1996).

By applying each of the previous cases for the geoid height conversion from WGS84(G873) to ITRF94, we obtain:

Case 1 ($\delta\bar{a} = 0, \delta f = 0$— retaining the physical size of the WGS84 reference ellipsoid)
$\delta N = N' - N = 0.102$ m
$N' = 47.295$ m

Case 2 ($\delta a = 0, \delta f = 0$— retaining the numerical values for the semi-major axis and the flattening of the WGS84 reference ellipsoid)
$\delta N = N' - N = 0.011$ m
$N' = 47.204$ m

Case 3 ($a' = 6378136.602$ m, $f' = 0.00335281969240$— using the zero-tide reference ellipsoid of the World Geodetic Datum 2000; see Grafarend and Ardalan 1999)
$\delta N = N' - N = 0.442$ m
$N' = 47.635$ m

The above results verify the important role of the a-priori conventional specification of the reference ellipsoid attached to the new GRF, with respect to which the transformed height shall refer to. Although the EGM96/ITRF94 geoid height obtained from Case 2 can be considered identical, within ± 1 cm, to the original EGM96/WGS84(G873) geoid height, the implementation of Case 1 increases the EGM96/ITRF94 geoid height by almost 9 cm. Furthermore, in Case 3, the transformed geoid undulation exhibits an additional increase of about 30 cm with respect to the corresponding value that is referenced to the physical body of the WGS84 ellipsoid. Note that for the computation of the transformed geoid height in Case 3, the flattening change term $\delta N(\delta f)$ has been taken into account.

Remark 5 It should be noted that none of the three aforementioned options should be treated as more 'correct' or 'false' in the sense of physical reality. The problem of selecting a value for the semi-major axis of a reference ellipsoid is always an issue of conventional choice. In the context of similarity-type transformation for geodetic quantities, it is up to the user to ensure that his conventional choice will not create inconsistencies or other apparent biases with existing data sets, when transferring geodetic data from one datum to another.

5 Discussion and conclusions

Maintaining a conventional, yet geometrically and physically invariant, Earth reference model is a fundamental issue in geodesy, particularly in view of the increasing need to monitor global change parameters, such as mean sea level or landmass subsidence. However, when a geodetic reference system is used in practice via an accessible group of stations with known spatial positions, the adopted reference ellipsoid that is required to define several important geodetic quantities does not refer to an 'ideal' scale unit, but rather to the best spatial scale that geodesists are able to reproduce by means of their current data, measurement techniques and combination procedures (Soler and van Gelder 1987).

As a result, any geodetic datum 'detects' an attached reference ellipsoid, as well as every length-type quantity that depends on it (e.g., ellipsoidal height, geoid height, height anomaly, sea surface height), according to its own accessible spatial scale that is inherent in the Cartesian coordinate values of its realization points. In cases of classical terrestrial geodetic datums, the spatial scale itself is defined through the conventional adoption of a reference ellipsoid in which the horizontal positions of its realization points are given.

Taking into account the above considerations, we have investigated the problem of ellipsoidal height ($h \rightarrow h'$) and geoid height ($N \rightarrow N'$) conversion between different GRFs by providing a general similarity-type transformation that incorporates the contribution of GRF scale variation to the relative size of the reference ellipsoids adopted by each datum. Several options that can be followed for the conventional selection of the semi-major axis of the reference ellipsoid in the target GRF have been analyzed, and their practical implications were highlighted.

Considering the growing trend for the definition and realization of modern GPS-based vertical reference systems, as well as the requirement of accurate vertical control for the evaluation of Earth gravity field models, the extended height transformation model presented herein provides an adequate tool for the consistent combination of GPS and/or geoid heights obtained from different GRFs with different scale realizations. Its usage in 'inverse' datum studies, for the purpose of estimating unknown datum transformation parameters using heterogeneous height information, will be explored in future studies.

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