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ORIGINAL ARTICLE

Reference frame stability and nonlinear distortion in minimum-constrained network adjustment

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Abstract The aim of this paper is to investigate the influence of the minimum constraints (MCs) on the reference frame parameters in a free-net solution. The non-estimable part of these parameters (which is not reduced by the available data) is analysed in terms of its stability under a numerical perturbation of the constrained datum functionals. In practice, such a perturbation can be ascribed either to hidden errors in the known coordinates/velocities that participate in the MCs or to a simple change of their a priori values due to a datum switch on a different fiducial dataset. In addition, a perturbation of this type may cause a nonlinear variation to the estimable part of the reference frame parameters, since it theoretically affects the adjusted observations that are implied by the network's nonlinear observational model. The aforementioned effects have an impact on the quality of a terrestrial reference frame (TRF) that is established via a minimum-constrained adjustment, and our study shows that they are both controlled through a characteristic matrix which is inherently linked to the MC system (the so-called TRF stability matrix). The structure of this matrix depends on the network's spatial configuration and the 'geometry' of the datum constraints, while its main role is the filtering of any MC-related errors into the least-squares adjustment results. A number of examples with different types of geodetic networks are also presented to demonstrate the theoretical findings of our study.

Keywords Minimum constraints · Free-net adjustment · Reference frame stability · Non-linearity · S transformation

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1 Introduction

The establishment of terrestrial reference frames (TRFs) is a fundamental task in geodesy, closely related to the zero-order design or datum choice problem of network optimization theory (Grafarend 1974; Teunissen 1985; Dermanis 1998). Due to the inherent datum deficiency in all types of geodetic measurements, a set of external conditions is always required to obtain a unique and well-defined TRF realization from a geodetic network adjustment. The use of minimum constraints signifies an optimal choice of such conditions in the sense that they provide the required information for the datum definition without interfering with the network's estimable characteristics (e.g. Grafarend and Schaffrin 1974). As a result, a minimum-constrained network is theoretically free of any geometrical distortion that could originate from the external datum conditions, while its estimable TRF parameters (if any) are determined solely from the available measurements without being affected by the user's minimum constraints. The latter affect only the non-estimable part of the reference frame parameters which is not reduced (defined) by the data, yet they influence the quality of the entire coordinate-based representation of the adjusted network (e.g. the covariance matrix of the estimated positions and their external reliability level).

Starting from the early works of Meissl (1969), Blaha (1971a) and Baarda (1973), several aspects of the minimum-constrained network adjustment and the datum choice problem have been investigated in the geodetic literature, focusing on topics like free networks and the role of inner constraints (Wolf 1973; Perelmuter 1979; Papo and Perelmuter 1981; Blaha 1982a,b; Papo 1986; Dermanis 1994), S-transformation (van Mierlo 1980; Strang van Hees 1982; Teunissen 1985), estimability analysis and invariance properties in network adjustment (Grafarend and Schaffrin 1974,

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1976; Grafarend and Livieratos 1978; Delikaraoglou 1985), zero-order network optimization (Grafarend 1974; Dermanis 1985; Schaffrin 1985; Teunissen 1985), hypothesis testing of non-estimable functions in network adjustment models (Koch 1985; Xu 1995) and nonlinear aspects in network datum definition (Xu 1997; Dermanis 1998). A comprehensive review on the use of minimum constraints for the least-squares inversion of rank-deficient geodetic models and their fundamental role within the ITRF methodology can be found in Sillard and Boucher (2001); see also Altamimi et al. (2002a,b), Altamimi and Dermanis (2009).

A realized TRF through a network adjustment is subject to quality limitations originating from the type of minimum constraints that are used for the datum definition. This is a well-known fact to geodesists which is theoretically justified by the dependency of the covariance matrix of the estimated positions with regard to the selected minimum constraints. The propagated data noise on the realized TRF depends strongly on the chosen datum conditions, a fact that has been the foundation of the zero-order network optimization and the formulation of the so-called inner constraints for geodetic network adjustment problems (e.g. Blaha 1971a, 1982a). An equally important issue, which however has not been systematically investigated in the geodetic literature, is the *frame stability* that can be achieved from a network adjustment with a given set of minimum constraints. This represents a crucial aspect for the overall quality of a TRF realization and the objective comparison of different datum definition strategies. The basic question associated with this issue is not concerned with the propagated effect of the measurement errors into the adjustment results, but rather with the identification and quantification of the criteria under which a set of minimum constraints can provide a more stable TRF than another set of minimum constraints for the same network. A recent study by Coulot et al. (2010) tackled the above problem in the context of an optimal search for global reference sub-networks that guarantee better orientation stability of the weekly SLR solutions with respect to ITRF2005; see also Heinkelmann et al. (2007) for a relevant study on the comparison of global VLBI solutions under different datum choices for their TRF realization.

The main objective of this paper is to present a general framework for analysing the frame stability in minimum-constrained networks. For this purpose, the influence of minimum constraints $\mathbf{H}\,(\mathbf{x}-\mathbf{x}_o)=\mathbf{c}$ on the realized TRF is studied via a perturbation analysis for the network solution under a variation $\mathbf{dc}$ of the constrained 'datum functionals'. In practice, such a variation can be attributed either to existing errors in the coordinates/velocities of the reference stations that participate in the datum conditions, or to a change of their a priori values due to a datum-switch into a different fiducial dataset (note that a well-designed geodetic network should be fairly robust against such datum disturbances). Our analysis will show that a fundamental matrix always exists which characterizes the frame stability of any set of minimum constraints in a given network and it can be used as a criterion matrix for an objective analysis of different datum definition strategies.

An important aspect that is also treated in our study is the geometrical distortion on a minimum-constrained network due to the aforementioned variation of the constrained datum functionals. This is an indirect nonlinear effect that remains hidden within the linearized framework of least-squares adjustment in rank-deficient nonlinear models, yet it theoretically exists and it can affect the estimable characteristics of a $\mathbf{dc}$-perturbed network solution. From a geodetic perspective, such an effect corresponds to a nonlinear propagation of datum-related errors into the adjusted observations of a minimum-constrained network, and it may cause a degradation of the actual accuracy level that is implied by their formal covariance matrix. In the present paper it is shown that the significance of these 'higher-order' errors is controlled, to a certain extent, by the frame-stability matrix of the underlying network.

The structure of the paper is organized as follows: in Sect. 2 a brief overview of the free-network solution concept and the role of minimum constraints is given, along with a discussion on the frame instability and the geometrical distortion that may occur in a minimum-constrained network; in Sect. 3 a number of important algebraic formulae for the free-net adjustment and the S-transformation are reviewed—some of these formulae are not usually found in the classic geodetic literature so their mathematical proofs are also given in a related appendix; Sect. 4 is fully devoted to the perturbation analysis for free-net solutions and the role of the TRF stability matrix (the latter being a newly introduced concept in this paper); a number of practical examples with different types of geodetic networks are given in Sect. 5; a stochastic perspective for the frame stability in free-net solutions is presented in Sect. 6 and a concluding summary is finally provided in Sect. 7.

2 Free networks and minimum constraints

2.1 General background

A $m \times m$ singular system of the so-called *normal equations* (NEQs)

$$\mathbf{N}(\mathbf{x}-\mathbf{x}_o)=\mathbf{u} \tag{1}$$

provides the fundamental setting for network adjustment problems and the establishment of spatial reference frames from terrestrial and/or space geodetic data. Typically, the above system is deduced from the linearized least-squares

(LS) inversion of a coordinate-based nonlinear parametric model ($\mathbf{y} = \mathbf{f}(\mathbf{x}) + \mathbf{v}$) that describes a noisy set of geodetic measurements in a local, regional or global network. Its analytic form depends on the rank-deficient Jacobian matrix $\mathbf{A} = \mathbf{f}_{\mathbf{x}}(\mathbf{x}_o)$ of the network observables, according to the well-known relationships (e.g. Blaha 1971a)

$$\mathbf{N} = \mathbf{A}^T\mathbf{P}\mathbf{A} \text{ and } \mathbf{u} = \mathbf{A}^T\mathbf{P}(\mathbf{y} - \mathbf{f}(\mathbf{x}_o)) \tag{2}$$

where $\mathbf{x}_o$ is an initial approximation of the model parameters, $\mathbf{P}$ is the data weight matrix and $\mathbf{x}$ is the unknown vector originating either from a static (coordinates only) or a dynamic (coordinates and velocities) modelling of the network stations with respect to an Earth-fixed reference system.

Any solution of Eq. (1) corresponds to what is commonly known as a *free-network (free-net) solution* (Sillard and Boucher 2001) and it holds a key role for the optimal analysis of a geodetic network on the basis of datum-deficient noisy measurements. Such solutions are theoretically equivalent to each other in the sense that they produce the same linearly adjusted observables $\hat{\mathbf{y}}$ and thus maintaining the same information about the network's estimable characteristics that are embedded in the given measurements (e.g. Grafarend and Schaffrin 1976). Their basic characteristic is that they provide an unequivocal least-squares fit to the data vector $\mathbf{y}$, and they offer the standard framework for the realization of TRFs that directly reflect the data quality without being distorted by external datum-related biases.

The differences between free-net solutions are rigorously described through a linear transformation that depends on the frame parameters which are not reduced (defined) by the geodetic observables in the underlying network. This transformation is known in the geodetic literature as *S-transformation* (Baarda 1973; van Mierlo 1980) and it provides a fundamental tool for the analysis of network adjustment problems.

2.2 Minimum constraints

The determination of a single solution of Eq. (1) requires a set of external conditions to specify a coordinate frame with respect to which the adjusted positions of the network stations shall be computed. A free-net solution is always associated with a set of k independent linear equations

$$\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c} \tag{3}$$

whose number is equal to the rank defect of the normal matrix ($k = m$-rank$\mathbf{N}$). These equations constitute the so-called *minimum constraints* (MCs), and they are theoretically satisfied by one, and only one, solution of the rank-deficient system $\mathbf{N}(\mathbf{x}-\mathbf{x}_o) = \mathbf{u}$ (Koch 1999). The terms $\mathbf{H}$ and $\mathbf{c}$ characterize completely a free-net solution and they provide the necessary information for its numerical computation either through a constrained LS estimator from the data vector $\mathbf{y}$, or through a S-transformation based on another solution of the same NEQ system.

From a theoretical perspective, the $k \times m$ matrix $\mathbf{H}$ needs to be of full-row rank and it has to satisfy the algebraic condition (Blaha 1971a)

$$\text{rank}\begin{bmatrix}\mathbf{N}\\ \mathbf{H}\end{bmatrix} = \text{rank}\mathbf{N} + \text{rank}\mathbf{H} = (m-k) + k = m \tag{4}$$

whereas the k-dimensional vector $\mathbf{c}$ is free to take any values within the column space (also called the range) of the matrix $\mathbf{H}$. The previous condition ensures the inversion of the extended NEQ system

$$\left(\mathbf{N} + \mathbf{H}^T\mathbf{W}\mathbf{H}\right)(\mathbf{x} - \mathbf{x}_o) = \mathbf{u} + \mathbf{H}^T\mathbf{W}\mathbf{c} \tag{5}$$

which contains the (minimum) required information for the datum definition in terms of a 'pseudo-observation' vector $\mathbf{c}$ that is associated with a 'design' matrix $\mathbf{H}$ and a symmetric positive definite 'weight' matrix $\mathbf{W}$. Given the condition in Eq. (4), the extended NEQs have a unique solution that satisfies both the original NEQ system (1) and the MC system (3), and it is independent of the weight matrix $\mathbf{W}$ (a fact that is sometimes overlooked in the geodetic literature)[1]; see also Sect. 3 and the related proofs given in the Appendix.

The theoretical freedom in the numerical selection of the auxiliary vector $\mathbf{c}$ could result in free-net solutions that are mathematically correct (in the sense that they satisfy both the singular NEQs and the imposed MCs) yet geodetically problematic due to the unreasonable magnitude of the estimated positions and/or the significant distortion in the geometrical characteristics of the linearly adjusted network; for some practical examples, see Xu (1997). In fact, even a MC vector $\mathbf{c}$ with arbitrarily small entries may have a 'distorting influence' on the free-net solution from Eq. (5) if a slight perturbation in its values is significantly amplified by the matrix $\left(\mathbf{N} + \mathbf{H}^T\mathbf{W}\mathbf{H}\right)^{-1}\mathbf{H}^T\mathbf{W}$. Note that this effect does not imply an ill-conditioned form for the constrained normal matrix $\mathbf{N} + \mathbf{H}^T\mathbf{W}\mathbf{H}$, yet it points to a frame-related instability for the adjusted network with respect to the adopted MCs.

Remark on terminology. The terms *free-net solution* and *minimum constraints* occasionally appear with different meanings in the geodetic literature. For example, a free-net solution is sometimes referred to as the solution of a (nearly) singular NEQ system without explicitly introducing any datum conditions, whereas the notion of MCs in several

[1] The independence of the solution of Eq. (5) from the weight matrix $\mathbf{W}$ does not hold if the system $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}$ contains more equations than the network datum defect ($k > m$-rank$\mathbf{N}$). In this case the solution of Eq. (5) will not generally satisfy Eqs. (1) and (3); such 'over-constrained' adjustment schemes are not treated in this paper.

papers is often identified with the implementation of inner constraints on the non-estimable TRF parameters. Herein, we adhere to the meaning of these terms as described in the previous paragraphs, without hopefully causing any confusion to readers who are used to a different denotation.

2.3 Algebraic versus geodetic admissibility of minimum constraints

The admissibility of the minimum constraints $\mathbf{H}(\mathbf{x}-\mathbf{x}_o) = \mathbf{c}$, as stated in most geodetic textbooks and related papers, relies on the fundamental condition (4) or on some of its algebraic-equivalent forms that can be found in the relevant literature (Blaha 1971a, 1982a; Schaffrin 1985; Koch 1999; Sillard and Boucher 2001). This (algebraic) admissibility depends on the structure of the matrices $\mathbf{H}$ and $\mathbf{N}$, and it is generally fulfilled by the datum definition strategies that are employed in geodetic practice (e.g. fixing a minimum number of station positions or applying inner constraints on the non-estimable frame parameters over some or all of the network stations). A point of concern is the existence of degenerate network configurations that may cause a problematic adjustment for certain options of datum constraints due to the remaining rank deficiency in the extended NEQ system. Such special cases of singular MCs were investigated by Blaha (1971b); Tsimis (1973) and later by Papo (1987), but it was Veis (1960) who first pointed out the possibility of singularities in the LS adjustment of satellite geodetic networks due to a geometrical faulty structure of the datum constraints; see also Delikaraoglou (1985) and the references given therein.

An algebraically admissible set of minimum constraints guarantees the inversion of the augmented normal matrix $\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H}$, yet it is not sufficient to secure a geodetically meaningful solution for the original NEQs. In order to better understand the meaning of this peculiar statement, it is helpful to clarify the role of the system $\mathbf{H}(\mathbf{x}-\mathbf{x}_o) = \mathbf{c}$ in the context of free-net adjustment. The primary aim of this system is not the designation of any arbitrary coordinate frame, but the establishment of a coordinate frame in the neighbourhood of an existing frame that is realized by the approximate positions of the network stations. The need to refer a free-net solution into a TRF which is close to the one implied by the initial vector $\mathbf{x}_o$ stems from the linearization that is implicitly associated with the formation of the NEQ system in Eq. (1). An attempt to overcome this restriction was presented some years ago from Xu (1997) by assimilating into the singular NEQs the non-estimable frame parameters of the network adjustment problem. Despite the theoretical interest of his over-parameterization approach, a free-net adjustment is (still) based on the logic of a ‘linearized datum implementation’ whose practical significance is revealed from the example mentioned in the following remark:

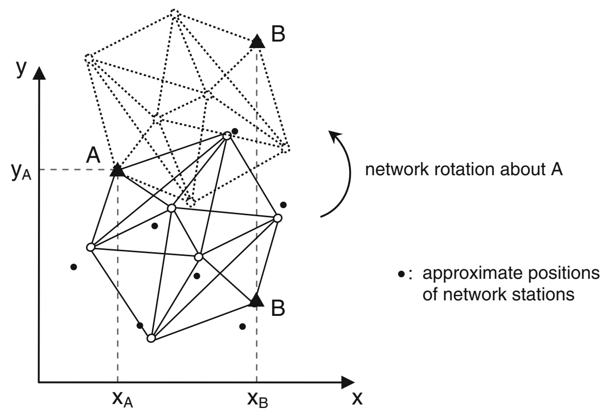

Fig. 1 The minimum constraints $x_A = const.$, $y_A = const.$ and $x_B = const.$ for the adjustment of a horizontal trilateration network do not theoretically yield a unique datum definition, since they cannot distinguish between the two symmetrical solutions that are shown in the above figure. However, a unique adjusted solution is practically obtained through these constraints, which is the one that lies closer to the approximate coordinates of the network stations

Remark The fixation of a minimum number of station coordinates to some a priori values does not (always) represent a valid datum definition scheme under a nonlinear treatment of the rank-deficient network. Nevertheless, such an option is valid for the linearized minimum-constrained adjustment as it leads to a unique datum specification relative to the frame of the initial (approximate) station coordinates; a straightforward example is depicted in Fig. 1.

The constant vector of the MC system controls the closeness between the TRFs of the adjusted coordinates and the approximate coordinates on the basis of a minimum number of ‘datum functionals’ $\mathbf{c}$. It is critical, though, that this term does not cause any detectable disturbance on the true (nonlinear) geometrical characteristics $\hat{\mathbf{y}} = \mathbf{f}(\hat{\mathbf{x}})$ of the adjusted network and the TRF parameters that are already reduced by the available data. Moreover, a small variation of the elements of $\mathbf{c}$ (e.g. due to coordinate/velocity errors at the reference stations that participate in the datum constraints) should not spawn an instability in the TRF of the adjusted network, neither interfere with its estimable characteristics. These are the main aspects behind a geodetically meaningful free-net solution which cannot be guaranteed by an admissible MC matrix $\mathbf{H}$, as they are directly influenced by the MC vector $\mathbf{c}$ and the sensitivity of the constrained NEQs with respect to its disturbance. From a geodetic perspective, the MC vector cannot take *any* values within the range of the MC matrix, a fact that creates a convoluted dependence among the basic components of the free-net adjustment problem.

Let us give a didactic example concerning the minimum-constrained adjustment of a horizontal trilateration network based on the fixation of three coordinates over its stations, namely x_A, y_A and x_B (see Fig. 2). In this case, the terms $\mathbf{H}$

and $\mathbf{c}$ have the general form:

$$\mathbf{H} = \begin{bmatrix} 1\,0\,0\,0\,\ldots\,0 \\ 0\,1\,0\,0\,\ldots\,0 \\ 0\,0\,1\,0\,\ldots\,0 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} \tilde{x}_\mathrm{A} - x_\mathrm{A}^\mathrm{o} \\ \tilde{y}_\mathrm{A} - y_\mathrm{A}^\mathrm{o} \\ \tilde{x}_\mathrm{B} - x_\mathrm{B}^\mathrm{o} \end{bmatrix} \quad (6)$$

where x_A^o, y_A^o, x_B^o are the approximate values of the datum-specifying coordinates, and $\tilde{x}_\mathrm{A}$, $\tilde{y}_\mathrm{A}$, $\tilde{x}_\mathrm{B}$ denote their fixed values which jointly define the TRF origin and orientation of the horizontal network. In the absence of any geometrical configuration defect, the matrix $\mathbf{H}$ will fulfill the condition (4) and it will impose a valid datum definition for the linearly adjusted network. However, if the value $\tilde{x}_\mathrm{B}$ (or, more precisely, the value of the difference $\tilde{x}_\mathrm{B} - \tilde{x}_\mathrm{A}$) exceeds a certain threshold then the free-net solution will be deformed, thus affecting the network scale that is implicitly defined through the distance measurements (see Fig. 2). The initial configuration of the network stations ($\mathbf{x}_\mathrm{o}$) plays a key role in quantifying the threshold for the reference coordinates' variation that could cause such a problematic solution. In the particular example it is evident that, as the point B lies closer to the x axis, the adjusted network could be effectively distorted under smaller disturbances of the MC vector. Note that even if the difference $\tilde{x}_\mathrm{B} - \tilde{x}_\mathrm{A}$ does not exceed a critical limit, the orientation of the free-net solution becomes increasingly unstable in this case (see Fig. 2).

The influence of the MC vector on a free-net solution that is obtained from a set of minimum constraints $\mathbf{H}(\mathbf{x}-\mathbf{x}_\mathrm{o}) = \mathbf{c}$ has not been systematically treated in the geodetic literature. In previous studies the role of this term was investigated in the context of its deterministic or stochastic interpretation and the resulting implications on the statistical properties of the optimal estimate $\hat{\mathbf{x}}$ (e.g. Blaha 1982a). Our main focus in this paper, on the other hand, is directed towards its frame-disturbance effect and the algebraic framework that is required to analyse such an effect in practice.

Concluding this section, we need to make a final comment in view of the nonlinear character of LS network adjustment. Since a free-net solution is practically determined through an iterative adjustment scheme, any set of $k = m - rank(\mathbf{N})$ datum conditions $\mathbf{H}(\mathbf{x} - \mathbf{x}_\mathrm{o}) = \mathbf{c}$ yielding a convergent LS estimate should always lead to the same geometrical form $\hat{\mathbf{y}} = \mathbf{f}(\hat{\mathbf{x}})$ for the adjusted network (note, however, that a rigorous convergence analysis for the linearized LS solution in rank-deficient nonlinear models does not currently exist in the geodetic literature). The crucial point to be emphasized here is that a convergent free-net solution does not necessarily realize a stable TRF over the network stations, and its geometrical characteristics may be affected under a small perturbation $\mathbf{dc}$ of the constrained datum functionals. These important issues are schematically described in Fig. 2 for the simple case of a horizontal network, and they will be analysed under a more general setting in Sect. 4.

3 Mathematical background

A number of important algebraic formulae that are relevant to the free-net adjustment problem are reviewed in this section. Our presentation gives only an overview of the required material for the TRF stability analysis in the next sections, without focusing on mathematical details but rather outlining the essential theoretical tools for the purpose of this paper.

3.1 Basic relationships

The general solution of a singular NEQ system $\mathbf{N}(\mathbf{x}-\mathbf{x}_\mathrm{o}) = \mathbf{u}$ can be expressed by the formula

$$\hat{\mathbf{x}} = \mathbf{x}_\mathrm{o} + \mathbf{N}^{-}\mathbf{u} + (\mathbf{I} - \mathbf{N}^{-}\mathbf{N})\mathbf{z} \quad (7)$$

where $\mathbf{N}^{-}$ is a generalized inverse of the normal matrix $\mathbf{N}$ and $\mathbf{z}$ is an arbitrary vector. The above expression is valid in view of the fundamental property $\mathbf{N}\mathbf{N}^{-}\mathbf{A}^\mathrm{T} = \mathbf{A}^\mathrm{T}$ that applies when $\mathbf{N} = \mathbf{A}^\mathrm{T}\mathbf{P}\mathbf{A}$ (Koch 1999, p. 51).

The primary link of Eq. (7) with the formulation of the free-net adjustment problem is rooted in the basic formula:

$$\mathbf{N}^{-} = (\mathbf{N} + \mathbf{H}^\mathrm{T}\mathbf{W}\mathbf{H})^{-1} \quad (8)$$

which gives the generalized inverse of a symmetric semi-positive definite matrix $\mathbf{N}$ in terms of a full-row rank matrix $\mathbf{H}$ that satisfies the algebraic condition (4) and an arbitrary symmetric positive definite matrix $\mathbf{W}$ (see Appendix).

When the NEQs generalized inverse originates from Eq. (8) then the condition $\mathbf{H}\mathbf{N}^{-}\mathbf{A}^\mathrm{T} = \mathbf{0}$ is always fulfilled (see also Appendix). Consequently, by multiplying both sides of Eq. (7) with the matrix $\mathbf{H}$, we deduce that the general NEQs' solution complies with a system of independent linear equations

$$\mathbf{H}(\hat{\mathbf{x}} - \mathbf{x}_\mathrm{o}) = \mathbf{H}\mathbf{z} \quad (9)$$

which corresponds to the required MCs for the datum definition in a free-net solution. Note that the MC vector is now identified as $\mathbf{c} = \mathbf{H}\mathbf{z}$, a fact that reveals an important issue which was mentioned in our previous discussions: the minimum constraints should form a consistent linear system and thus their constant vector must belong to the range of the MC matrix. Fortunately, the way that the MC vector is numerically constructed in the geodetic practice conforms to such a mathematical requirement, as it will be explained later in the paper.

The weight matrix $\mathbf{W}$ that is used in the computation of $\mathbf{N}^{-}$ does not affect the NEQ solution in Eq. (7), and it does not interfere with the validity of the minimum constraints in Eq. (9); a proof is provided in the Appendix. A free-net solution remains therefore independent of the weighting with which

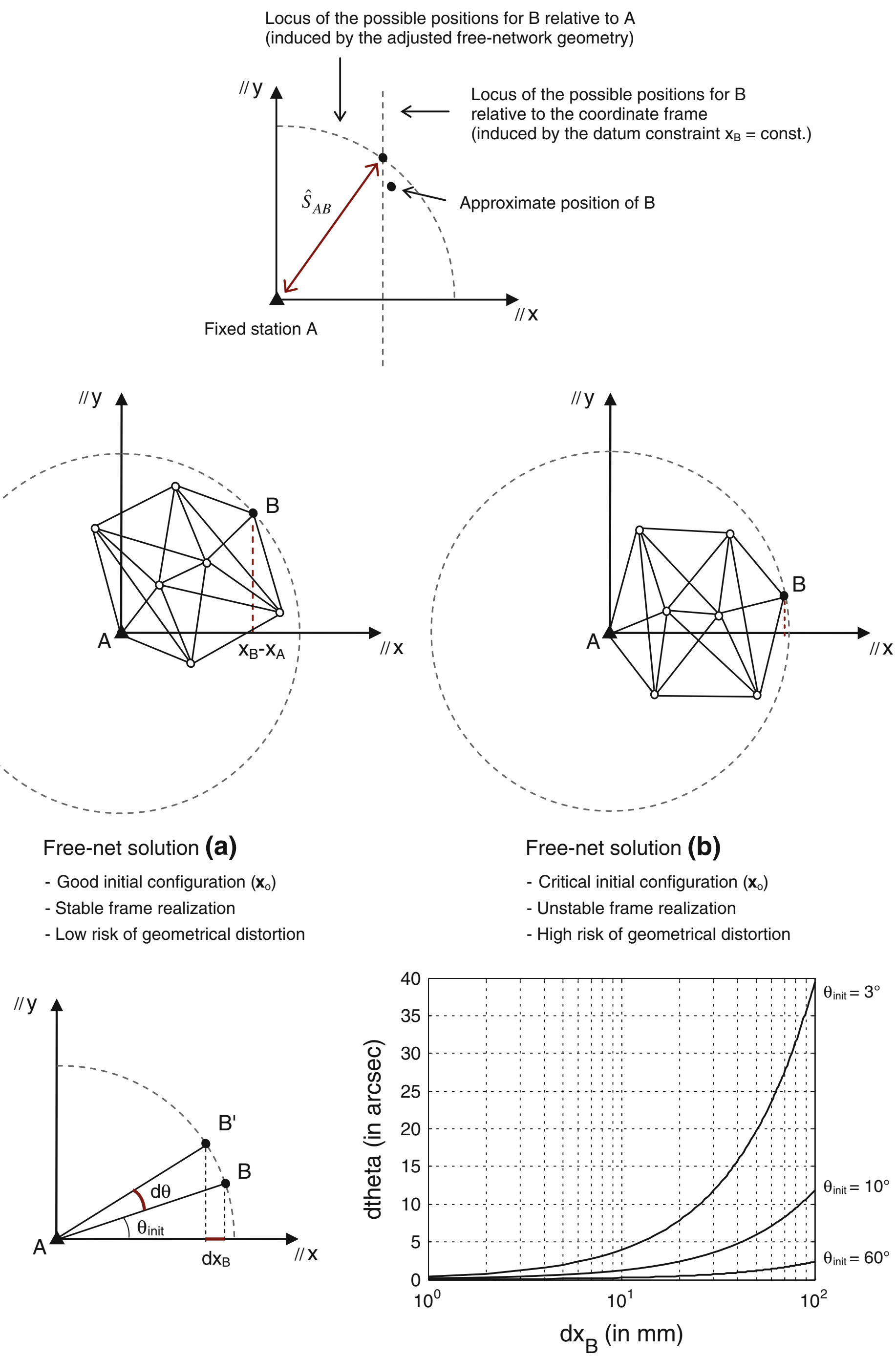

Fig. 2 A schematic description for the distorting effect that the minimum constraints may cause on the free-net solution of a horizontal trilateration network. The datum conditions refer to the fixation of the x and y coordinates of point A and the x coordinate of point B. If the difference of the fixed x-coordinates exceeds a critical value then a distortion will occur in the geometrical form $\hat{\mathbf{y}} = \mathbf{f}(\hat{\mathbf{x}})$ of the linearly adjusted network (in such case a convergent LS solution cannot be achieved through an iterative adjustment scheme). Note that the critical value corresponds to the adjusted geometrical distance between points A and B. The free-net solution (**b**) is more vulnerable than the free-net solution (**a**) due to the weak configuration of the network stations with respect to the coordinate reference frame. The lower two graphs depict the orientation disturbance of the minimum-constrained solution under a small change in the fixed x-coordinate of B (the numerical graph is based on an adjusted geometrical distance between A and B of 10 km)

the MCs are implemented into the LS adjustment algorithm; however, its statistical accuracy assessment may account for the prior uncertainty of the MC vector (see also Sect. 6).

For any $m \times m$ NEQ system with rank defect $k = m$-rank$\mathbf{N}$ there exists a class of $k \times m$ full-row rank matrices $\mathbf{E}$ with the fundamental property (Blaha 1971a; Koch 1999)

$$\mathbf{A}\mathbf{E}^{\mathrm{T}} = \mathbf{0} \text{ and } \text{ thus } \mathbf{N}\mathbf{E}^{\mathrm{T}} = \mathbf{0} \tag{10}$$

These matrices are identified in this paper as *type-E matrices* and they hold a crucial role in network adjustment problems. Any type-E matrix is an algebraically admissible MC matrix that generates the so-called *inner constraints* $\mathbf{E}(\mathbf{x}-\mathbf{x}_{\mathrm{o}}) = \mathbf{c}$ with well-known optimal statistical properties for the corresponding solution of the NEQ system. For more details, see Blaha (1982a), van Mierlo (1980), Papo and Perelmuter (1981).

In the context of our present study, the following equations are of particular importance (their proof is given in the Appendix):

$$(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{N} = \mathbf{I}-\mathbf{E}^{\mathrm{T}}(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{H} \tag{11}$$

$$(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H} = \mathbf{E}^{\mathrm{T}}(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{H} \tag{12}$$

where $\mathbf{H}$ corresponds to any MC matrix that can be associated with the original NEQ system. The above expressions define the fundamental projector matrices which are used in the formulation of the S-transformation, as described in the next section.

3.2 S-transformation

The *S-transformation* is a key tool that relates different free-net solutions of the same singular NEQ system (e.g. Koch 1999, p. 192). In its simplest form, it can be expressed by the following formula:

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}' + \mathbf{E}^{\mathrm{T}}\boldsymbol{\theta} \tag{13}$$

where $\mathbf{E}$ is a full-row rank matrix that satisfies the fundamental property (10). The vector $\boldsymbol{\theta}$ reflects the degrees of freedom in the inversion of the rank-deficient normal matrix $\mathbf{N}$, and it quantifies the difference between free-net solutions through k 'datum transformation parameters'.

Since there are infinitely many type-E matrices, the transformation parameters $\boldsymbol{\theta}$ are not uniquely defined and they depend on the choice of $\mathbf{E}$ that appears in Eq. (13). Their values can be determined in a straightforward way by multiplying both sides of the previous equation with an *arbitrary MC matrix* and then solving for $\boldsymbol{\theta}$, in which case we get

$$\boldsymbol{\theta} = (\tilde{\mathbf{H}}\mathbf{E}^{\mathrm{T}})^{-1}\tilde{\mathbf{H}}(\hat{\mathbf{x}}-\hat{\mathbf{x}}') \tag{14}$$

The above result is invariant with respect to the MC matrix $\tilde{\mathbf{H}}$ and/or the possible use of a positive definite weight matrix $\tilde{\mathbf{P}}$ (i.e. $\boldsymbol{\theta} = (\tilde{\mathbf{H}}\tilde{\mathbf{P}}\mathbf{E}^{\mathrm{T}})^{-1}\tilde{\mathbf{H}}\tilde{\mathbf{P}}(\hat{\mathbf{x}}-\hat{\mathbf{x}}')$) provided that both vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$ correspond to distinct solutions of the same NEQ system.

Based on Eq. (14), the forward S-transformation may also be expressed by the equivalent model

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}' + \mathbf{E}^{\mathrm{T}}(\tilde{\mathbf{H}}\mathbf{E}^{\mathrm{T}})^{-1}\tilde{\mathbf{H}}(\hat{\mathbf{x}}-\hat{\mathbf{x}}') \tag{15}$$

where $\tilde{\mathbf{H}}$ denotes again an arbitrary MC matrix. The above formula admits a straightforward geometrical interpretation within the parameter space R^m of the free-net adjustment problem, in view of the projection property $\left(\mathbf{P}_{\tilde{H}}^2 = \mathbf{P}_{\tilde{H}}\right)$ of the matrix $\mathbf{P}_{\tilde{H}} = \mathbf{E}^{\mathrm{T}}\left(\tilde{\mathbf{H}}\mathbf{E}^{\mathrm{T}}\right)^{-1}\tilde{\mathbf{H}}$.

If the S-transformed vector $\hat{\mathbf{x}}$ needs to satisfy a particular set of minimum constraints $\mathbf{H}(\mathbf{x}-\mathbf{x}_{\mathrm{o}}) = \mathbf{c}$, then Eq. (15) takes the following form:

$$\hat{\mathbf{x}} = (\mathbf{I}-\mathbf{E}^{\mathrm{T}}(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{H})\hat{\mathbf{x}}' + \mathbf{E}^{\mathrm{T}}(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}(\mathbf{c}+\mathbf{H}\mathbf{x}_{\mathrm{o}}) \tag{16}$$

The last equation provides the fundamental basis for analysing the effect of the MC vector $\mathbf{c}$ (and its disturbance) on the geodetic admissibility of a free-net solution that is compliant with a given MC matrix $\mathbf{H}$.

3.3 'Helmertization' of S-transformation

For every singular NEQ system there exists a type-E matrix which (i) is independent of the data weighting and (ii) depends only on the datum defect and the spatial configuration of the network stations ($\mathbf{x}_{\mathrm{o}}$). The corresponding S-transformation parameters $\boldsymbol{\theta}$ admit a straightforward interpretation and they reflect (the differences of) the non-estimable TRF parameters in $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$ due to the different datum conditions that were used in each solution.

The aforementioned matrix $\mathbf{E}$ is formally known as the *inner-constraint matrix* and it stems from the linearized Helmert transformation that describes a differential similarity between 'nearby' Cartesian coordinate frames over an N-point network (Blaha 1971a, p.23)

$$\mathbf{x}_{\mathrm{TRF}} = \mathbf{x}'_{\mathrm{TRF}'} + \mathbf{G}^{\mathrm{T}}\mathbf{q} \tag{17}$$

where the Helmert transformation matrix $\mathbf{G}$ is

$$\mathbf{G} = \begin{array}{c} t_x \\ t_y \\ t_z \\ \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \delta s \end{array} \begin{bmatrix} 1 & 0 & 0 & \dots & \dots & 1 & 0 & 0 \\ 0 & 1 & 0 & \dots & \dots & 0 & 1 & 0 \\ 0 & 0 & 1 & \dots & \dots & 0 & 0 & 1 \\ 0 & z'_1 & -y'_1 & \dots & \dots & 0 & z'_{\mathrm{N}} & -y'_{\mathrm{N}} \\ -z'_1 & 0 & x'_1 & \dots & \dots & -z'_{\mathrm{N}} & 0 & x'_{\mathrm{N}} \\ y'_1 & -x'_1 & 0 & \dots & \dots & y'_{\mathrm{N}} & -x'_{\mathrm{N}} & 0 \\ x'_1 & y'_1 & z'_1 & \dots & \dots & x'_{\mathrm{N}} & y'_{\mathrm{N}} & z'_{\mathrm{N}} \end{bmatrix} \tag{18}$$

and the vector $\mathbf{q}$ contains the seven parameters of the linearized similarity transformation, namely three translations (t_x, t_y, t_z), three small rotation angles ($\varepsilon_x, \varepsilon_y, \varepsilon_z$) and one differential scale factor (δs); see Leick and van Gelder (1975).

The inner-constraint matrix $\mathbf{E}$ consists of the particular rows of $\mathbf{G}$ that correspond to the non-estimable TRF parameters of the observed network. In case of dynamic networks, where both coordinates and velocities need to be jointly estimated from a combined adjustment of time-dependent data, the matrix $\mathbf{G}$ (and also $\mathbf{E}$) should be expanded to include additional rows for the rates of the TRF parameters according to the time-varying similarity transformation model; see Sillard and Boucher (2001); Altamimi et al. (2002a,b); Soler and Marshall (2003).

Note that the inner-constraint matrix $\mathbf{E}$ should be formed by the approximate coordinates of the network stations, upon which the design matrix $\mathbf{A}$ and the normal matrix $\mathbf{N}$ were also computed—otherwise Eq. (10) will not theoretically hold true. Hence, the S-transformation parameters between two free-net solutions $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$ are not strictly identical to their Helmert transformation parameters, since the latter correspond to a linearization of the similarity transformation model with respect to either $\hat{\mathbf{x}}$ or $\hat{\mathbf{x}}'$ (and not $\mathbf{x}_o$). For practical purposes, though, this difference is negligibly small and it does need to be further considered.

The Helmert matrix $\mathbf{G}$ and the parameter vector $\mathbf{q}$ of the linearized similarity transformation model, for a given network, can be generally decomposed as

$$\mathbf{G} = \begin{bmatrix} \mathbf{E} \\ \bar{\mathbf{E}} \end{bmatrix}, \mathbf{q} = \begin{bmatrix} \boldsymbol{\theta} \\ \bar{\boldsymbol{\theta}} \end{bmatrix} \tag{19}$$

where $\mathbf{E}$ is the inner-constraint matrix that refers to the non-estimable TRF parameters $\boldsymbol{\theta}$ of the underlying network, and $\bar{\mathbf{E}}$ is the complement matrix corresponding to the estimable TRF parameters $\bar{\boldsymbol{\theta}}$ that are inherently defined through the network observations. The above partition will be later used in the discussion on the nonlinear distortion of minimum-constrained networks (see Sect. 4.2).

4 MC-perturbation analysis of free networks

The behaviour of a free-net solution under a perturbation of its associated MCs reflects (an important part of) the TRF quality that can be achieved through a geodetic network adjustment. Our aim in this section is to study the above effect and to expose any problems related to the geodetic admissibility of a set of MCs for a given network.

4.1 Effect on the network's non-estimable characteristics

Let $\hat{\mathbf{x}}$ be a free-net solution of a singular NEQ system that is compliant with a particular set of minimum constraints, namely $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}$. The disturbance of such a solution due to a variation of the constant vector $\mathbf{c}$ is given from the expression:

$$\mathbf{d}\hat{\mathbf{x}} = \mathbf{E}^{\mathrm{T}} \left(\mathbf{H}\mathbf{E}^{\mathrm{T}}\right)^{-1} \mathbf{dc} \tag{20}$$

which is obtained by differentiating the S-transformation formula in Eq. (16). Alternatively, the previous equation may be derived through the differentiation of the free-net solution from the extended NEQs in Eq. (5) taking also into account the fundamental relationship in Eq. (12).

The induced disturbance of the non-estimable TRF parameters in the adjusted network is

$$\mathbf{d}\boldsymbol{\theta} = (\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{dc} \tag{21}$$

and it essentially corresponds to the S-transformation parameters between the initial solution (based on $\mathbf{H}$ and $\mathbf{c}$) and the disturbed solution (based on $\mathbf{H}$ and $\mathbf{c}$+$\mathbf{dc}$). Note that the auxiliary matrix $\mathbf{W}$ does not influence any of the previous terms due to the algebraic insensitivity of free-net adjustment with respect to the MCs weight matrix.

The last equation is important for the analysis of free networks as it dictates the influence of the MCs to each of the non-estimable frame parameters. The inverse of the square matrix $\mathbf{H}\mathbf{E}^{\mathrm{T}}$ controls the datum sensitivity in the network solution and it has a key role for the frame stability in the presence of a perturbation (error) in the selected minimum constraints. This matrix is not necessarily diagonal, a fact that signifies that each constraint may affect more than one, or even all, of the non-estimable TRF components in the adjusted network (we will return to this issue in later sections).

The geodetic admissibility of the minimum constraints $\mathbf{H}(\mathbf{x} - \mathbf{x}_o) = \mathbf{c}$ is influenced by the form of the matrix $(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}$. Depending on the numerical structure of this matrix, an 'unstable' free-net solution could emerge through Eq. (5) or (16) in the sense that a small error in the datum conditions may corrupt significantly not only the non-estimable frame parameters but also the estimable information that is contained in the linearly adjusted observations (see next section). The occurrence of this unfavourable effect depends on the spatial configuration of the underlying network in tandem with the type of its datum deficiency and the 'geometry' of the selected MCs, all of which are reflected into the *TRF stability matrix* $(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}$.

As an example, let us recall the simple case of the horizontal trilateration network given in Fig. 2. The minimum constraints in this example refer to the fixation of the three coordinates x_A, y_A, x_B, and they lead to the following matrix expressions:

$$\mathbf{H}\mathbf{E}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & y_{\mathrm{A}}^{\mathrm{o}} \\ 0 & 1 & -x_{\mathrm{A}}^{\mathrm{o}} \\ 1 & 0 & y_{\mathrm{B}}^{\mathrm{o}} \end{bmatrix} \tag{22}$$

and

$$(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1} = \frac{1}{y_{\mathrm{A}}^{\mathrm{o}} - y_{\mathrm{B}}^{\mathrm{o}}} \begin{bmatrix} -y_{\mathrm{B}}^{\mathrm{o}} & 0 & y_{\mathrm{A}}^{\mathrm{o}} \\ x_{\mathrm{A}}^{\mathrm{o}} & y_{\mathrm{A}}^{\mathrm{o}} - y_{\mathrm{B}}^{\mathrm{o}} & -x_{\mathrm{A}}^{\mathrm{o}} \\ 1 & 0 & -1 \end{bmatrix} \tag{23}$$

If the azimuth between the datum points A and B is close to $\pm 90° \left(y_{\mathrm{A}}^{\mathrm{o}} \approx y_{\mathrm{B}}^{\mathrm{o}}\right)$, then a LS adjustment with respect to an 'unstable' datum will take place from which both the origin and the orientation of the realized TRF will be weakly defined. Note that similar problems may also arise in other types of static or dynamic 2D/3D networks whose datum definition is associated with a 'problematic' matrix $(\mathbf{HE}^{\mathrm{T}})^{-1}$.

4.2 Effect on the network's estimable characteristics

The estimable characteristics of a free network are rendered into two basic components: the vector of the adjusted observations and the TRF parameters that are inherently reduced through the available measurements. Theoretically, both of these components remain invariant after a numerical perturbation ($\mathbf{dc}$) of the MCs or, more generally, under any S-transformation applied to the estimated vector $\hat{\mathbf{x}}$. This property is valid within the linearized framework of LS inversion in rank-deficient nonlinear models, yet it does not provide an exact assessment of the distortionless behaviour in any MC solution. Some general aspects about the potential distortion of the estimable characteristics of free networks will be now outlined.

4.2.1 Adjusted observations

The vector of the linearly adjusted observations from a LS network adjustment is given by the formula

$$\begin{aligned}\hat{\mathbf{y}} &= \mathbf{f}(\mathbf{x}_{\mathrm{o}}) + \mathbf{A}(\hat{\mathbf{x}} - \mathbf{x}_{\mathrm{o}}) \\ &= \mathbf{f}(\mathbf{x}_{\mathrm{o}}) + \mathbf{f}_{\mathbf{x}}(\mathbf{x}_{\mathrm{o}})(\hat{\mathbf{x}} - \mathbf{x}_{\mathrm{o}})\end{aligned} \tag{24}$$

Considering an iterative implementation of the network's adjustment algorithm (where the approximate vector $\mathbf{x}_{\mathrm{o}}$ is replaced at each step by the previously estimated position vector $\hat{\mathbf{x}}$ until a satisfactory convergence is achieved), the above estimate after sufficient iterations is practically compatible with the original nonlinear observational model, that is $\hat{\mathbf{y}} \simeq \mathbf{f}(\hat{\mathbf{x}})$.

The coordinate-based disturbance of the adjusted observations is expressed as

$$\mathbf{d}\hat{\mathbf{y}} = \mathbf{A}\mathbf{d}\hat{\mathbf{x}} \tag{25}$$

which, in view of Eq. (20), becomes

$$\mathbf{d}\hat{\mathbf{y}} = \mathbf{A}\mathbf{E}^{\mathrm{T}}(\mathbf{HE}^{\mathrm{T}})^{-1}\mathbf{dc} = \mathbf{0} \tag{26}$$

thus confirming the invariance of the adjusted observations under a MC disturbance within the free-net adjustment.

The previous property holds only to a first-order approximation of the observational model since it neglects the contribution of its nonlinear terms to the variation of the network observables. Based on a second-order approximation, for example, we would have that

$$\begin{aligned}\hat{\mathbf{y}}_{(\mathbf{H},\mathbf{c}+\mathbf{dc})} &= \mathbf{f}(\hat{\mathbf{x}} + \mathbf{d}\hat{\mathbf{x}}) \\ &\simeq \mathbf{f}(\hat{\mathbf{x}}) + \mathbf{A}\mathbf{d}\hat{\mathbf{x}} + \xi \\ &= \hat{\mathbf{y}}_{(\mathbf{H},\mathbf{c})} + \underbrace{\mathbf{A}\mathbf{d}\hat{\mathbf{x}}}_{\mathbf{0}} + \xi\end{aligned} \tag{27}$$

where ξ denotes the second-order term in the Taylor series expansion of the adjusted observables for the perturbed free-net solution.

The disturbance of the adjusted observables, up to a second order, is

$$\mathbf{d}\hat{\mathbf{y}} = \hat{\mathbf{y}}_{(\mathbf{H},\mathbf{c}+\mathbf{dc})} - \hat{\mathbf{y}}_{(\mathbf{H},\mathbf{c})} = \xi \tag{28}$$

where each element of ξ is given by the quadratic expression

$$\xi_i = \frac{1}{2}\mathbf{d}\hat{\mathbf{x}}^{\mathrm{T}}\mathbf{Q}_i\mathbf{d}\hat{\mathbf{x}} \tag{29}$$

and $\mathbf{Q}_i$ is the Hessian matrix of the respective observable. Taking into account Eq. (20), the previous equation takes the form

$$\xi_i = \frac{1}{2}\mathbf{dc}^{\mathrm{T}}(\mathbf{EH}^{\mathrm{T}})^{-1}\mathbf{EQ}_i\mathbf{E}^{\mathrm{T}}(\mathbf{HE}^{\mathrm{T}})^{-1}\mathbf{dc} \tag{30}$$

or equivalently

$$\xi_i = \frac{1}{2}\mathbf{d}\boldsymbol{\theta}^{\mathrm{T}}\mathbf{EQ}_i\mathbf{E}^{\mathrm{T}}\mathbf{d}\boldsymbol{\theta} \tag{31}$$

Therefore, a MC disturbance causes a change in the adjusted observables which are implied by the original nonlinear model and thus affects, in principle, the network's estimable characteristics. The last equation is particularly important as it relates the nonlinear variation in each adjusted observable to the perturbation of the non-estimable frame parameters.

4.2.2 The meaning of ξ_i

The terms ξ_i represent an important nonlinear element that has been neglected up to now in geodetic network analysis. Their values correspond to the geometrical effect of 'transformed linearization errors' between TRFs with respect to which a free-net solution can be determined. In our case, the corresponding frames arise from the MC systems $\mathbf{H}(\mathbf{x} - \mathbf{x}_{\mathrm{o}}) = \mathbf{c}$ and $\mathbf{H}(\mathbf{x} - \mathbf{x}_{\mathrm{o}}) = \mathbf{c} + \mathbf{dc}$, which do not necessarily share the same behaviour regarding the influence of linearization errors in the free-net adjustment. From Eq. (30) we can conclude that the TRF stability matrix plays a role in controlling whether a MC disturbance is able to trigger significant linearization errors into the geometrical characteristics of the adjusted network.

4.2.3 Estimable TRF parameters

Following the notation given at the end of Sect. 3.3, let us model the difference between the initial ($\hat{\mathbf{x}}$) and the perturbed

$(\hat{\mathbf{x}} + \mathbf{d}\hat{\mathbf{x}})$ MC solutions in terms of a (full) similarity transformation

$$\begin{aligned} \mathbf{d}\hat{\mathbf{x}} &= \mathbf{G}^{\mathrm{T}}\mathbf{dq} \\ &= \mathbf{E}^{\mathrm{T}}\mathbf{d}\boldsymbol{\theta} + \bar{\mathbf{E}}^{\mathrm{T}}\mathbf{d}\bar{\boldsymbol{\theta}} \end{aligned} \tag{32}$$

where $\mathbf{d}\boldsymbol{\theta}$ and $\mathbf{d}\bar{\boldsymbol{\theta}}$ are the changes of the non-estimable and the estimable TRF parameters, respectively. Based on a simple LS adjustment of the above model and taking into account that the vector $\mathbf{d}\hat{\mathbf{x}}$ is given by the perturbation formula (20), we obtain the result

$$\mathbf{d}\bar{\boldsymbol{\theta}} = \mathbf{0} \text{ and } \mathbf{d}\boldsymbol{\theta} = (\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{dc} \tag{33}$$

as it should be expected due to the theoretical invariance of any estimable quantity under a MC perturbation within a free-net solution.

However, the aforementioned invariance is an apparent theoretical element that exists only within the linearized framework of the differential similarity transformation. A simple approach to quantify a likely variation of the estimable TRF characteristics in a $\mathbf{dc}$-perturbed free network is to perform a stepwise LS estimation of the similarity transformation parameters from certain types of nonlinear datum functionals (the latter being respectively computed from the vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}} + \mathbf{d}\hat{\mathbf{x}}$). For example, a TRF scale-change factor may be directly estimated from chord differences over an independent set of network baselines, whereas TRF rotation parameters can be obtained from the differences of appropriately formed directional angles among the network stations; for more details on such stepwise schemes for transformation parameter estimation, see Leick and van Gelder (1975) and Han and van Gelder (2006). This approach has been actually implemented in a numerical example that is presented in Sect. 5.

4.3 A note on the TRF stability matrix

In the preceding sections we exposed the role of the matrix $(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}$ in free-net adjustment problems. For any set of minimum constraints $\mathbf{H}(\mathbf{x} - \mathbf{x}_{\mathrm{o}}) = \mathbf{c}$ in a rank-deficient NEQ system, the aforementioned matrix dictates (1) the stability of the non-estimable frame parameters and (2) the second-order nonlinear distortion of the geometrical characteristics in the linearly adjusted network, under the presence of a perturbation (error) in the MC vector $\mathbf{c}$.

In every network adjustment there exists an algebraic form for the MCs such that their TRF stability matrix always becomes a unit matrix. Indeed, if we multiply the original system $\mathbf{H}(\mathbf{x} - \mathbf{x}_{\mathrm{o}}) = \mathbf{c}$ with the matrix $(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}$, then an equivalent set of minimum constraints is obtained

$$(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{H}(\mathbf{x} - \mathbf{x}_{\mathrm{o}}) = (\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{c} \tag{34a}$$

or, in a more compact form

$$\mathbf{B}(\mathbf{x} - \mathbf{x}_{\mathrm{o}}) = \mathbf{c}_{\theta} \tag{34b}$$

where $\mathbf{B} = (\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{H}$ and $\mathbf{c}_{\theta} = (\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{c}$. Obviously, the TRF stability matrix of the above MCs is

$$(\mathbf{B}\mathbf{E}^{\mathrm{T}})^{-1} = ((\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1} = \mathbf{I} \tag{35}$$

whereas the perturbations of the non-estimable quantities in the adjusted network are now given by the simplified expressions

$$\mathbf{d}\hat{\mathbf{x}} = \mathbf{E}^{\mathrm{T}}\mathbf{dc}_{\theta} \tag{36}$$

$$\mathbf{d}\boldsymbol{\theta} = \mathbf{dc}_{\theta} \tag{37}$$

Hence, it seems that the MCs in Eq. (34) are more suitable for the implementation of a free-net adjustment (compared with the 'short' MCs given in (3)), as they ensure uniform stability and zero aliasing on the frame parameters under an error in the constrained datum functionals, see Eq. (21) versus Eq. (37). This is, however, only a pseudo-regularization characteristic of the constrained LS adjustment since the effect of the original TRF stability matrix remains hidden within the vector $\mathbf{c}_{\theta}$ (and its possible variation) as indicated by the relationship

$$\mathbf{dc}_{\theta} = (\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{dc} \tag{38}$$

From a geodetic perspective, the importance of the matrix $(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}$ is due to the fact that a type-$\mathbf{dc}$ disturbance of a free network is more relevant than a generic type-$\mathbf{dc}_{\theta}$ disturbance. A brief explanation of this vital fact is provided in the rest of this section.

Despite the equivalency of the MC systems $\mathbf{H}(\mathbf{x} - \mathbf{x}_{\mathrm{o}}) = \mathbf{c}$ and $\mathbf{B}(\mathbf{x} - \mathbf{x}_{\mathrm{o}}) = \mathbf{c}_{\theta}$ (i.e. both of them give theoretically the same free-net solution), their constrained elements are fundamentally different and they depend on some external TRF information ($\mathbf{x}^{\mathrm{ext}}$) according to the hierarchical scheme:

$$\mathbf{x}^{\mathrm{ext}} \rightarrow \boxed{\mathbf{c} = \mathbf{H}(\mathbf{x}^{\mathrm{ext}} - \mathbf{x}_{\mathrm{o}})} \rightarrow \boxed{\mathbf{c}_{\theta} = (\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{H}(\mathbf{x}^{\mathrm{ext}} - \mathbf{x}_{\mathrm{o}})} \tag{39}$$

The vector $\mathbf{c}$ contains the reduced values for a minimum number of datum functionals, such as the coordinates at individual points, the azimuth of a specific baseline, or other more complicated types like the position/velocity of the network's centroid or the magnitude of the network's angular momentum over some or all of its stations. These values are determined within a linear approximation from $\mathbf{x}^{\mathrm{ext}}$ and $\mathbf{x}_{\mathrm{o}}$—the MC matrix $\mathbf{H}$ contains the partial derivatives of the constrained datum functionals with respect to the network station positions. On the other hand, the vector $\mathbf{c}_{\theta}$ represents the (non-estimable) frame transformation parameters between $\mathbf{x}^{\mathrm{ext}}$ and $\mathbf{x}_{\mathrm{o}}$ that are inferred from the differences of the preceding datum functionals of each frame.

Both of the previous MC systems force the free-net solution $\hat{\mathbf{x}}$ to be computed in the same frame as $\mathbf{x}^{\text{ext}}$ in the following sense:

$$\begin{aligned}\boldsymbol{\theta}_{\hat{\mathbf{x}},\mathbf{x}^{\text{ext}}} &= (\mathbf{HE}^{\mathrm{T}})^{-1}\mathbf{H}(\hat{\mathbf{x}} - \mathbf{x}^{\text{ext}}) \\ &= (\mathbf{HE}^{\mathrm{T}})^{-1}\mathbf{H}(\hat{\mathbf{x}} - \mathbf{x}_{\text{o}}) - (\mathbf{HE}^{\mathrm{T}})^{-1}\underbrace{\mathbf{H}(\mathbf{x}^{\text{ext}} - \mathbf{x}_{\text{o}})}_{\mathbf{c}} \\ &= (\mathbf{HE}^{\mathrm{T}})^{-1}(\mathbf{H}(\hat{\mathbf{x}} - \mathbf{x}_{\text{o}}) - \mathbf{c}) \\ &= \mathbf{B}(\hat{\mathbf{x}} - \mathbf{x}_{\text{o}}) - \mathbf{c}_{\theta} \\ &= \mathbf{0}\end{aligned} \tag{40}$$

that is, the (non-estimable) frame transformation parameters between $\hat{\mathbf{x}}$ and $\mathbf{x}^{\text{ext}}$ vanish when their determination is based on the datum functionals which are specified by the MC matrix $\mathbf{H}$.

Any hidden errors in the external TRF information imply a disturbance to the constrained elements in Eq. (39), thus causing a variation to the adjusted network as discussed in Sects. 4.1 and 4.2. Therefore, the matrix $(\mathbf{HE}^{\mathrm{T}})^{-1}$ is the network's 'filter' against any $\mathbf{x}^{\text{ext}}$-related datum errors (which are given by $\mathbf{dc} = \mathbf{Hdx}^{\text{ext}}$) and it controls their propagated effect into the various components of the free-net solution.

Remark The choice $\mathbf{c} = \mathbf{0}$ is the one that has been mostly treated in the geodetic literature, since the approximate vector $\mathbf{x}_{\text{o}}$ is often formed on the basis of $\mathbf{x}^{\text{ext}}$ (although this is not always the case and, certainly, it is not a requirement for the implementation of a free-net adjustment). Such a case represents only a special homogeneous form for the minimum constraints, which does not affect the rationale of the MC-perturbation analysis that was presented in the previous sections.

5 Examples

5.1 Horizontal network

The adjustment of a horizontal network with measured distances was previously evoked as a simple example to explain some of the concepts related to the present study. Here, we present a few numerical tests for this example that reveal the effect of the TRF stability matrix in free-net solutions. The test network and its observed baselines are shown in Fig. 3, the approximate coordinates of the network stations are listed in Table 1, and the distance measurements are given in Table 2.

Five different types of MCs are tested in this network, namely

(1) fixing the x and y coordinates of point A and the x coordinate of point B,

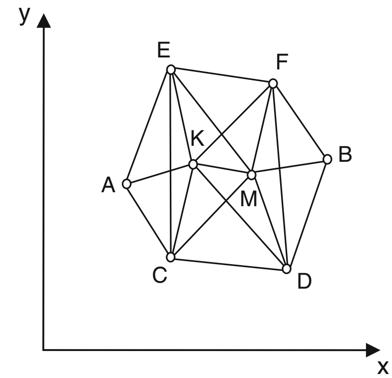

Fig. 3 A horizontal trilateration test network

Table 1 The approximate coordinates of the horizontal test network

Point	x	y
A	1,024.436	1,345.886
B	15,968.266	1,438.569
C	5,322.097	−4,507.417
D	11,343.332	−3,665.593
E	4,989.587	7,231.325
F	10,205.645	6,155.168
K	5,830.092	2,287.682
M	9,817.173	1,983.554

Units in meters

(2) fixing the x and y coordinates of point A and the x coordinate of point E,

(3) fixing the x and y coordinates of point A and the azimuth of the baseline A–F,

(4) using inner constraints for the TRF origin and orientation over the points A, B, M,

(5) using inner constraints for the TRF origin and orientation over all network stations.

For each case we computed the TRF stability matrix $(\mathbf{HE}^{\mathrm{T}})^{-1}$, its trace, and its condition number (i.e. the ratio between its maximum and minimum eigenvalue), as well as the condition number of the constrained normal matrix $\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{H}$ assuming that the data weight matrix and the MC weight matrix are both equal to a unit matrix. All the results are summarized in Table 3, from which the following conclusions can be drawn:

- the elements of the TRF stability matrix in case 1 have significantly larger magnitudes than the other cases, suggesting that the particular option leads to an unstable free-net solution;
- the datum instability in case 1 is not reflected to the condition number of the TRF stability matrix, but rather to its trace and the large magnitudes of its off-diagonal ele-

Table 2 The values of the observed baselines in the horizontal test network

S_{AC} = 7,261.601	S_{BM} = 6,175.190	S_{DK} = 8,114.402	S_{EF} = 5,325.915
S_{AE} = 7,096.529	S_{CK} = 6,814.065	S_{DM} = 5,851.674	S_{FK} = 5,839.768
S_{AK} = 4,897.078	S_{CE} = 11,743.450	S_{DF}= 9,886.433	S_{FM} = 4,189.663
S_{BD} = 6,887.845	S_{CM} = 7,895.462	S_{EK} = 5,014.600	S_{KM} = 3,998.668
S_{BF} = 7,446.749	S_{CD} = 6,079.788	S_{EM} = 7,130.546	

Units in meters

Table 3 The TRF stability matrix and its algebraic characteristics for different MCs

Type of MCs	$(\mathbf{HE}^{\mathrm{T}})^{-1}$	Trace and condition number of $(\mathbf{HE}^{\mathrm{T}})^{-1}$		Condition number of $\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{H}$
Case 1 (three fixed coordinates)	$\begin{bmatrix} 15.52 & 0.00 & -14.52 \\ -11.05 & 1.00 & 11.05 \\ -0.01 & 0.00 & 0.01 \end{bmatrix}$	16.53	5.86×10^4	53.68
Case 2 (three fixed coordinates)	$\begin{bmatrix} 1.23 & 0.00 & -0.23 \\ -0.17 & 1.00 & 0.17 \\ -0.00 & 0.00 & 0.00 \end{bmatrix}$	2.23	9.59×10^3	64.44
Case 3 (one fixed station and a fixed azimuth)	$\begin{bmatrix} 1.00 & 0.00 & -0.09 \\ 0.00 & 1.00 & 0.07 \\ 0.00 & 0.00 & 0.00 \end{bmatrix}$	2.00	1.51×10^4	1.02×10^2
Case 4 (partial inner constraints)	$\begin{bmatrix} 0.36 & -0.13 & -0.00 \\ -0.13 & 1.04 & 0.00 \\ -0.00 & 0.00 & 0.00 \end{bmatrix}$	1.40	3.83×10^8	2.04×10^9
Case 5 (full inner constraints)	$\begin{bmatrix} 0.13 & -0.05 & -0.00 \\ -0.05 & 0.37 & 0.00 \\ -0.00 & 0.00 & 0.00 \end{bmatrix}$	0.50	3.03×10^8	2.44×10^9

ments. Also, it does not affect the inversion of the constrained normal matrix $\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{H}$, which actually exhibits its most stable form in this case (compared with the other MCs choices);

- the TRF stability matrix in case 5 shows the best behaviour (i.e. it is closer to a diagonal matrix and it has the smallest trace) among all datum definition schemes, a fact that reveals the optimality of the full inner constraints ($\mathbf{H}=\mathbf{E}$) towards the realization of a stable reference frame through a minimum-constrained network adjustment.

Based on Eq. (21), we determined the variations of the non-estimable TRF parameters under a perturbation in the a priori coordinates of point A. The nonlinear effect on the network's scale was also estimated from the baseline-length variations that were induced by the corresponding disturbance vector $\mathbf{d}\hat{\mathbf{x}}$ in Eq. (20). The results from these experiments are plotted in Figs. 4 and 5. The first figure clearly depicts the datum instability in the free-net solution from case 1: a change of a few cm in the a priori value x_A causes a TRF shift at the meter level and a TRF rotation up to 300 arcsec, whereas the corresponding effects for the other MCs cases are smaller by at least an order of magnitude (up to three orders in the case of inner constraints). A notable nonlinear change in the TRF scale is also seen in case 1, which implies a potential distortion in the geometrical characteristics of the linearly adjusted network under a perturbation of the a priori value x_A. This scale change amounts to a few ppm when the latter varies by >15 cm, and it corresponds to cm-level baseline distortions for the given network size. For the other MCs cases the nonlinear scale disturbance due to the x_A change is practically negligible. On the other hand, a change of the a priori value y_A does not cause any notable frame instability in any of the tested MCs types (see Fig. 5).

Note that a small change of the a priori value x_B would generate a significant frame instability in case 1, much similar to the one shown in Fig. 4. This can be easily inferred by looking at the columns of the corresponding TRF stability matrix $(\mathbf{HE}^{\mathrm{T}})^{-1}$ in Table 3, which represent the network's 'frame filter' against the perturbations in each of the reference coordinates (x_A, y_A, x_B).

5.2 Regional GNSS networks

The datum deficiency in this type of networks refers to the freedom of the TRF origin and its temporal evolution (the latter applies for non-static GNSS network adjustment problems). A free-net solution is obtained either by constraining

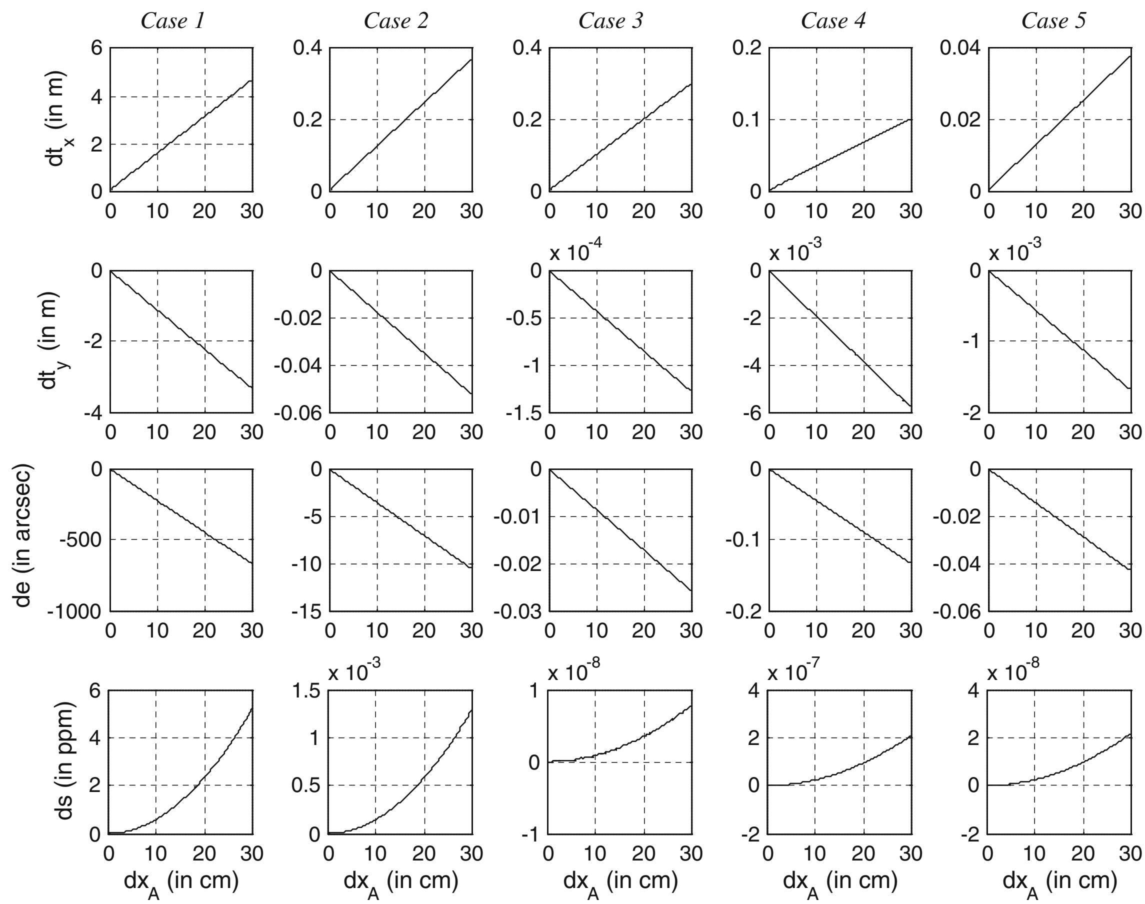

Fig. 4 Variations of the non-estimable TRF parameters (origin shifts dt_x, dt_y and orientation disturbance de) and the nonlinear effect on the TRF scale (ds) under a perturbation of the a priori coordinate x_A. The latter participates in all tested cases of MCs for the LS adjustment of the horizontal network. Note the large TRF instability associated with case 1 compared to the other MCs cases

a single reference station, or preferably by enforcing a no-net-translation (NNT) condition to the coordinates/velocities of a subset of network stations. As a result, the TRF stability matrix for regional GNSS networks has a diagonal form (the proof is straightforward and it is omitted):

$$(\mathbf{HE}^{\mathrm{T}})^{-1} = \frac{1}{\bar{m}}\mathbf{I} \tag{41}$$

where $\bar{m}$ indicates the number of reference stations participating in the NNT conditions, and $\mathbf{I}$ is the unit matrix. Note that the case $\bar{m} = 1$ corresponds to a free-net solution with a single fixed station.

Taking into account Eq. (39), a perturbation in the MCs of a regional GNSS network can be generally expressed as

$$\mathbf{dc} = \mathbf{Hdx}^{\mathrm{ext}} = \begin{bmatrix} \mathbf{dx}_1 + \cdots + \mathbf{dx}_{\bar{m}} \\ \mathbf{d\dot{x}}_1 + \cdots + \mathbf{d\dot{x}}_{\bar{m}} \end{bmatrix} \tag{42}$$

where $\mathbf{dx}_i$ and $\mathbf{d\dot{x}}_i$ denote the variation in the a priori coordinates and velocities of each reference station. Based on Eq. (21), the induced change on the frame origin of the minimum-constrained network will be

$$\mathbf{dt} = \frac{1}{\bar{m}}\left(\mathbf{dx}_1 + \cdots + \mathbf{dx}_{\bar{m}}\right) \tag{43}$$

and (in case of dynamic solutions)

$$\mathbf{d\dot{t}} = \frac{1}{\bar{m}}\left(\mathbf{d\dot{x}}_1 + \cdots + \mathbf{d\dot{x}}_{\bar{m}}\right) \tag{44}$$

where the vectors $\mathbf{dt}$ and $\mathbf{d\dot{t}}$ contain the TRF translation disturbances and their rates. Evidently, the effect of an error at a reference station will be attenuated by $1/\bar{m}$, a fact that reveals the advantage of the NNT conditions over a single-point datum realization.

The present example exposes an important limitation of the TRF stability matrix: if only a translation datum defect exists in the free-net adjustment problem, then the matrix $(\mathbf{HE}^{\mathrm{T}})^{-1}$ is 'blind' to the spatial distribution and the geographical coverage of the network stations. Hence, the disturbance terms $\mathbf{dt}$ and $\mathbf{d\dot{t}}$ will not include the instability of the TRF origin due to the non-global extent of a regional GNSS network. On the other hand, if an orientation datum defect is additionally present in the free-net adjustment problem, then

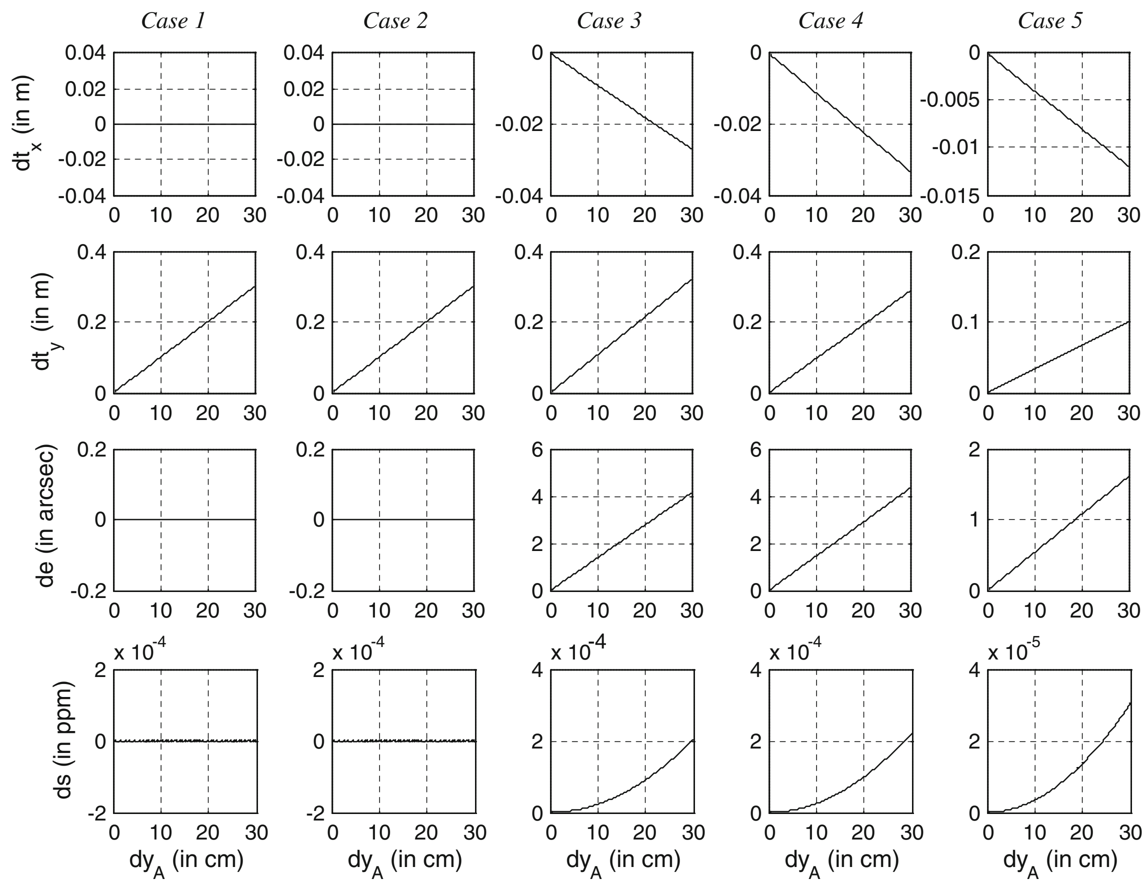

Fig. 5 Variations of the non-estimable TRF parameters (origin shifts dt_x, dt_y and orientation disturbance de) and the nonlinear effect on the TRF scale (ds) under a perturbation of the a priori coordinate y_A. The latter participates in all tested cases of MCs for the LS adjustment of the horizontal network. Note that no significant TRF instability occurs in any of the MCs cases

the TRF stability matrix not only will depend on the geometrical configuration of the network stations, but it will also unveil the error aliasing between the MCs and the realized frame of the adjusted network (e.g. whether an error in the z coordinate of a reference station will significantly affect the TRF origin along the x and y axes); see next example in Sect. 5.3.

The diagonal structure of the TRF stability matrix in Eq. (41) allows us to obtain an explicit formula for the nonlinear distortion of a MC-perturbed regional GNSS network. In the case of a static solution, for example, the quadratic term from Eq. (30) is now reduced to the form

$$\xi_{(i)} = \frac{1}{2\bar{m}^2} \sum_{k,l=1}^{\bar{m}} \mathbf{dx}_k^{\mathrm{T}} \mathbf{M}_{(i)} \mathbf{dx}_l \tag{45}$$

where the 3×3 matrix $\mathbf{M}_{(i)}$ is related to the Hessian $\mathbf{Q}_{(i)}$ of each network observable according to the relation $\mathbf{M}_{(i)} = \mathbf{E}\mathbf{Q}_{(i)}\mathbf{E}^{\mathrm{T}}$ (see Sect. 4.2). The above equation may be also used for computing the distortion in the adjusted baseline lengths due to small coordinate errors $\{\mathbf{dx}_k\}_{k=1,\ldots,\bar{m}}$ at the reference stations of the GNSS network datum definition. In such case the auxiliary matrix $\mathbf{M}_{(i)}$ takes the following form:

$$\mathbf{M}_{(i)} = \begin{bmatrix} \frac{\partial^2 S_{pq}}{\partial x_p^2} + \frac{\partial^2 S_{pq}}{\partial x_q^2} + 2\frac{\partial^2 S_{pq}}{\partial x_p \partial x_q} & \frac{\partial^2 S_{pq}}{\partial x_p \partial y_q} + \frac{\partial^2 S_{pq}}{\partial x_q \partial y_p} + \frac{\partial^2 S_{pq}}{\partial x_p \partial y_p} + \frac{\partial^2 S_{pq}}{\partial x_q \partial y_q} & \frac{\partial^2 S_{pq}}{\partial x_p \partial z_q} + \frac{\partial^2 S_{pq}}{\partial x_q \partial z_p} + \frac{\partial^2 S_{pq}}{\partial x_p \partial z_p} + \frac{\partial^2 S_{pq}}{\partial x_q \partial z_q} \\ & \frac{\partial^2 S_{pq}}{\partial y_p^2} + \frac{\partial^2 S_{pq}}{\partial y_q^2} + 2\frac{\partial^2 S_{pq}}{\partial y_p \partial y_q} & \frac{\partial^2 S_{pq}}{\partial y_p \partial z_q} + \frac{\partial^2 S_{pq}}{\partial y_q \partial z_p} + \frac{\partial^2 S_{pq}}{\partial y_p \partial z_p} + \frac{\partial^2 S_{pq}}{\partial y_q \partial z_q} \\ symm. & & \frac{\partial^2 S_{pq}}{\partial z_p^2} + \frac{\partial^2 S_{pq}}{\partial z_q^2} + 2\frac{\partial^2 S_{pq}}{\partial z_p \partial z_q} \end{bmatrix} \tag{46}$$

where S_{pq} is the spatial distance between two arbitrary network stations p and q. Using the analytical expressions for the second-order derivatives that appear in Eq. (46), it can be verified that the above matrix is always equal to zero, thus leading to the important conclusion: the nonlinear geometrical distortion due to a datum perturbation within the minimum-constrained network adjustment vanishes in the case of any geodetic network whose datum deficiency concerns only the translational part of its coordinate reference frame.

5.3 VLBI networks

A TRF realization from global VLBI solutions is based on a set of 12 MCs: 3+3 NNT conditions on station coordinates and velocities to specify the TRF origin and its temporal evolution, and 3+3 NNR conditions on station coordinates and velocities to specify the TRF orientation and its temporal evolution. These conditions are applied over a number of reference stations that are selected on the basis of several 'objective' criteria, such as their observational history, their spatial and temporal coverage and the existence of tectonic events (e.g. earthquakes) or other episodic station motions (Heinkelmann et al. 2007). Based on the concept of the TRF stability matrix, we present herein a simple comparison of different datum definition strategies that have been followed by various analysis centers (ACs) of the International VLBI Service for Geodesy and Astrometry (Schlüter and Behrend 2007) in the computation of their global TRF solutions.

The test network consists of 154 VLBI stations included in the gsfc2007a solution (see IVS 2011) that was computed by the Goddard Space Flight Center (NASA/GSFC) using 35 reference stations with a priori positions and velocities in ITRF2000 (t = 1997.0). Three alternative scenarios are also considered for the TRF realization in this network, which all rely on a subset of the 35 reference stations and they have been used by other IVS/ACs for their own TRF/EOPs solutions (see Table 4).

For each case in Table 4 we have computed the respective TRF stability matrix $(\mathbf{HE}^{\mathrm{T}})^{-1}$ using the MC matrix $\mathbf{H}$ that is associated with each selection of reference stations. The station coordinates that were used for the computations were taken from the a priori values in the SINEX file of the gsfc2007a solution (IVS 2011). In all cases, the TRF stability matrix has a block-diagonal symmetric structure, as follows:

$$(\mathbf{HE}^{\mathrm{T}})^{-1} = \begin{bmatrix} \underset{3\times3}{\mathbf{U}} & \underset{3\times3}{\mathbf{F}} & \underset{3\times3}{\mathbf{0}} & \underset{3\times3}{\mathbf{0}} \\ \underset{3\times3}{\mathbf{F}^{\mathrm{T}}} & \underset{3\times3}{\mathbf{V}} & \underset{3\times3}{\mathbf{0}} & \underset{3\times3}{\mathbf{0}} \\ \underset{3\times3}{\mathbf{0}} & \underset{3\times3}{\mathbf{0}} & \underset{3\times3}{\tilde{\mathbf{U}}} & \underset{3\times3}{\tilde{\mathbf{F}}} \\ \underset{3\times3}{\mathbf{0}} & \underset{3\times3}{\mathbf{0}} & \underset{3\times3}{\tilde{\mathbf{F}}^{\mathrm{T}}} & \underset{3\times3}{\tilde{\mathbf{V}}} \end{bmatrix} \tag{47}$$

where the 3×3 sub-matrices indicate the 'resistance' of the TRF parameters to a perturbation in the adopted minimum constraints. For instance, $\mathbf{U}$ and $\mathbf{F}$ reflect the sensitivity of the TRF origin with respect to a perturbation in the coordinate-based NNT and NNR conditions, whereas $\mathbf{V}$ and $\tilde{\mathbf{V}}$ reflect the sensitivity of the TRF orientation and its temporal evolution under a perturbation in the coordinate- and velocity-based NNR conditions, respectively. Note that due to the form of the inner-constraint matrix $\mathbf{E}$ for time-dependent networks (e.g. Sillard and Boucher 2001), we also have $\mathbf{U} = \tilde{\mathbf{U}}$, $\mathbf{V} = \tilde{\mathbf{V}}$ and $\mathbf{F} = \tilde{\mathbf{F}}$.

The above sub-matrices for each choice of reference stations are displayed in Fig. 6. The 35-station and 26-station options yield similar behaviour in terms of their TRF stability, with their $\mathbf{U}$ and $\mathbf{V}$ components having a stronger diagonal structure (and also their $\mathbf{F}$ component being significantly closer to the zero matrix) compared with the other reference station subsets. These aspects are important for the quality of the realized TRF, as they ensure a small aliasing level among the datum constraints and the non-estimable frame parameters (e.g. an error in the z coordinate of a reference station will not seriously affect the TRF origin along the x and y axes). Note that in Fig. 6 we have also included the hypothetical case of all 154 network stations participating in the NNT/NNR conditions, which theoretically gives the optimal TRF stability for the VLBI network.

Our results reveal that the least preferable option for the reference frame realization is the 11-stations subset. This is also supported by the numerical comparison of trace$\mathbf{U}$, trace$\mathbf{V}$ and sum$\mathbf{F}$ (the latter denotes the sum of all matrix elements), as these terms quantify, in some sense, the TRF stability that is achieved in each case. It is interesting that the 8-stations subset performs clearly better than the 11-stations subset, largely due to the participation of the reference stations HARTRAO and HOBART26 which are both located in the southern hemisphere (see Table 4).

The previous comparisons expose only the TRF stability of the global VLBI network with respect to different configurations of the selected reference stations, and they do not consider the influence of other factors that may additionally affect the quality of the reference frame realization (e.g. existence of stations with constrained velocities and/or discontinuous positions, choice of the defining sources for the celestial reference frame, etc.).

6 Reference frame stability of MCs from a stochastic perspective

Thus far the role of the matrix $(\mathbf{HE}^{\mathrm{T}})^{-1}$ has been considered from a deterministic perspective in view of 'ill-conditioned' datum definition schemes in free networks. Such schemes arise from MCs with a problematic TRF stability matrix and they can lead to a mathematically correct but geodetically improper (frame unstable) free-net solution. Note that the frame-related instability does not interfere with the inversion of the constrained normal matrix $\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{WH}$ per se, but it affects only the behaviour of the matrix operator

$$(\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{WH})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{W} = \mathbf{E}^{\mathrm{T}}(\mathbf{HE}^{\mathrm{T}})^{-1} \tag{48}$$

Table 4 Different sets of reference stations that have been used by various IVS/ACs for the TRF realization in global VLBI solutions

Reference stations	Case 1 (MAO, 8 stations) *mao2003a*	Case 2 (IAA, 11 stations) *iaaa2007a*	Case 3 (BKG, 26 stations) *bkg2007a*	Case 4 (GSFC, 35 stations) *gsfc2007a*
ALGOPARK	×	×	×	×
BR-VLBA			×	×
DSS45			×	×
FD-VLBA			×	×
FORTLEZA	×	×	×	×
HARTRAO	×		×	×
HATCREEK				×
HAYSTACK				×
HN-VLBA				×
HOBART26	×		×	×
KASHIM34			×	×
KASHIMA			×	×
KAUAI			×	×
KOKEE	×	×	×	×
KP-VLBA				×
LA-VLBA		×	×	×
MATERA		×	×	×
MK-VLBA		×	×	×
NL-VLBA			×	×
NOTO		×	×	×
NRAO20			×	×
NRAO85_3			×	×
NYALES20		×	×	×
ONSALA60	×	×	×	×
OV-VLBA				×
OVRO_130				×
PIETOWN				×
RICHMOND			×	×
SANTIA12			×	×
SC-VLBA			×	×
SESHAN25			×	×
TSUKUB32				×
VNDNBERG				×
WESTFORD	×	×	×	×
WETTZELL	×	×	×	×

which acts on the constrained datum functionals **c** within the LS adjustment algorithm (the proof of (48) can be found in the Appendix).

We shall now adopt a statistical view of Eqs. (20) and (21) so that we may evaluate the reference frame stability in a free-net solution through an appropriate covariance (CV) matrix $\mathbf{\Sigma}_\theta$. For this purpose, the perturbation vector **dc** in Eq. (21) is identified as a zero-mean random error which causes a corresponding zero-mean random error $\mathbf{d\theta}$ in the non-estimable TRF parameters of the adjusted network. By applying the error covariance propagation to Eq. (21), we get the formula

$$\mathbf{\Sigma}_\theta = (\mathbf{HE}^{\mathrm{T}})^{-1}\mathbf{\Sigma}_c(\mathbf{EH}^{\mathrm{T}})^{-1} \tag{49}$$

where $\mathbf{\Sigma}_\theta$ and $\mathbf{\Sigma}_c$ denote the error CV matrices of the (non-estimable) TRF parameters and the constrained datum functionals, respectively. The latter is determined from the general equation

$$\mathbf{\Sigma}_c = \mathbf{H\Sigma}_x^{\text{prior}}\mathbf{H}^{\mathrm{T}} \tag{50}$$

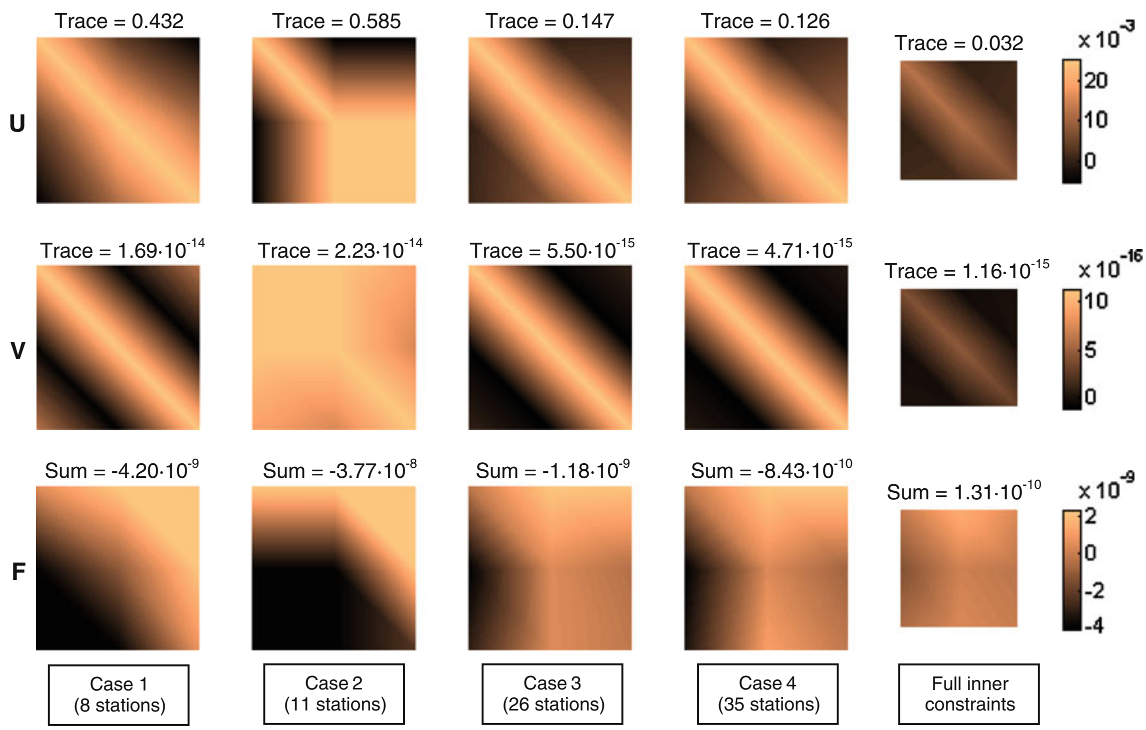

Fig. 6 Image representations of the 3×3 sub-matrices **U**, **V** and **F** of the TRF stability matrix in the VLBI test network. Each *column of the above plots* corresponds to a certain selection of reference stations for the TRF realization in the VLBI test network. For comparison purposes, we include also the case where all network stations participate in the NNT/NNR datum conditions

which is obtained from Eq. (39) on the basis of a CV matrix $\mathbf{\Sigma}_x^{\text{prior}}$ that specifies the accuracy of the external reference frame with respect to which the free-net solution is aligned. This matrix does not need to contain prior statistical information for all network stations but only for those involved in the underlying MCs, and it can be generally expressed as

$$\mathbf{\Sigma}_x^{\text{prior}} = \begin{bmatrix} \mathbf{\Sigma}_{x_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

in accordance with an equivalent partition of the parameter vector $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^{\text{T}} & \mathbf{x}_2^{\text{T}} \end{bmatrix}^{\text{T}}$, where $\mathbf{x}_1$ refers to the network stations with a priori given positions in the external reference frame and $\mathbf{x}_2$ refers to the remaining (new) network stations. In practice, this matrix originates either from the result of a previous adjustment, or by an empirical selection for the accuracy level of the available reference stations.

Taking into account Eqs. (49) and (50), we finally have the result

$$\mathbf{\Sigma}_\theta = (\mathbf{HE}^{\text{T}})^{-1}\mathbf{H}\mathbf{\Sigma}_x^{\text{prior}}\mathbf{H}^{\text{T}}(\mathbf{EH}^{\text{T}})^{-1} \tag{51}$$

which specifies, in a statistical sense, the TRF stability as a function of the adopted minimum constraints and the joint uncertainty (including the possible correlations) of the reference stations. It should be emphasized that $\mathbf{\Sigma}_\theta$ is different from the covariance matrix representing the so-called *reference system effect* (RSE) in LS network adjustment (see Eqs. (18) and (19) in Sillard and Boucher 2001) and it will not reflect the total TRF accuracy of a free-net solution. The above matrix does not include the error contribution of the available measurements (as the RSE covariance matrix does), but it evaluates the frame stability that can be achieved by different choices of MCs in a given network.

The uncertainty of the estimated station positions due to the 'TRF-stability effect' is obtained by applying the error CV propagation to Eq. (20), thus yielding

$$\begin{aligned}\bar{\mathbf{\Sigma}}_{\hat{x}} &= \mathbf{E}^{\text{T}}\mathbf{\Sigma}_\theta\mathbf{E} \\ &= \mathbf{E}^{\text{T}}(\mathbf{HE}^{\text{T}})^{-1}\mathbf{\Sigma}_c(\mathbf{EH}^{\text{T}})^{-1}\mathbf{E} \\ &= \mathbf{E}^{\text{T}}(\mathbf{HE}^{\text{T}})^{-1}\mathbf{H}\mathbf{\Sigma}_x^{\text{prior}}\mathbf{H}^{\text{T}}(\mathbf{EH}^{\text{T}})^{-1}\mathbf{E} \\ &= (\mathbf{I}-\mathbf{N}^{-}\mathbf{N})\mathbf{\Sigma}_x^{\text{prior}}(\mathbf{I}-\mathbf{N}^{-}\mathbf{N})^{\text{T}}\end{aligned} \tag{52}$$

where the last equality stems from (11) and (12). The matrix $\bar{\mathbf{\Sigma}}_{\hat{x}}$ represents the contribution of the external frame's noise to the total accuracy of a free-net solution, and it is related to the following CV decomposition:

$$\mathbf{\Sigma}_{\hat{x}} = \mathbf{N}^{-}\mathbf{N}\mathbf{N}^{-} + \bar{\mathbf{\Sigma}}_{\hat{x}} \tag{53}$$

which is obtained when a full error propagation is applied to the generalized inversion formula in Eq. (7) under a stochastic interpretation for the auxiliary vector $\mathbf{z}$ (i.e. $\mathbf{\Sigma}_z \rightarrow \mathbf{\Sigma}_x^{\text{prior}}$).

The previous CV components, that is $\mathbf{N}^{-}\mathbf{N}\mathbf{N}^{-}$ and $\bar{\mathbf{\Sigma}}_{\hat{x}}$, correspond to $m \times m$ singular matrices with rank defect equal to m-rank$\mathbf{N}$ and rank$\mathbf{N}$, respectively, and they are both affected by the MC matrix $\mathbf{H}$. A well-known result in free-net adjustment theory dictates that the inner constraints yield the best accuracy for the estimated positions in the sense that they minimize the trace of the first CV matrix in (53); see, e.g. Blaha (1982a). Indeed, in this case (i.e. H = E) the reflexive generalized inverse $\mathbf{N}^{-}\mathbf{N}\mathbf{N}^{-}$ becomes equal to the pseudo-inverse $\mathbf{N}^{+}$ which is known to have the smallest trace among all the symmetric reflexive generalized inverses of the NEQ matrix (Koch 1999, p. 62).

On the other hand, the trace minimization of the matrix $\bar{\mathbf{\Sigma}}_{\hat{x}}$ represents a zero-order network optimization task which has not been unveiled in the geodetic literature (at least to the author's knowledge). The corresponding optimal $\mathbf{N}^{-}$ and its associated MC matrix $\mathbf{H}$ will provide the most stable alignment of the free-net solution with an external frame that is characterized by an a priori CV matrix $\mathbf{\Sigma}_{x}^{\text{prior}}$. The solution of such a problem does not always lead to the classic inner constraints ($\mathbf{H} = \mathbf{E}$) and its detailed treatment lies beyond the scope of the present paper.

7 Conclusions

The influence of the MCs on the reference frame stability in a free-net solution has been investigated in this paper. Our study considered the distortion effect due to a perturbation $\mathbf{dc}$ of the constrained datum functionals $\mathbf{c} = \mathbf{H}\left(\mathbf{x}^{\text{ext}} - \mathbf{x}_{\text{o}}\right)$ by analysing its propagation on the non-estimable and the estimable components of the adjusted network. The main findings along with some brief final remarks can be summarized as follows:

- The matrix $(\mathbf{HE}^{\mathrm{T}})^{-1}$ plays a crucial role for the reference frame stability in a free-net solution, and it controls the impact of each MC to the (non-estimable) frame parameters of the adjusted network. This is a fundamental matrix which can be associated with alternative datum implementation strategies within the same physical network, but it could be also used for comparing the TRF stability from different network configurations with varying physical locations of their terrestrial stations.
- A 'problematic' TRF stability matrix tends to amplify any external perturbation of the MC vector and causes the aliasing of the positioning errors of the reference stations into different frame parameters. Theoretically, such an aliasing effect does not occur if a translation datum defect is only present in the network, in which case the matrix $(\mathbf{HE}^{\mathrm{T}})^{-1}$ admits a simple diagonal form (as discussed in Sect. 5.2).
- Any errors that are present in the a priori positions of the reference stations cause a nonlinear distortion on the estimable characteristics of a free-net solution and thus they affect the internal geometry of the minimum-constrained network (except, again, for the case of a translation-only datum deficient network). The linearized LS framework is 'blind' to this type of distortion whose practical significance in geodetic applications remains though to be investigated. A second-order modelling scheme of this effect has been presented in the paper, and it depends on the TRF stability matrix of the adopted MCs for the free-net solution (see Sect. 4.2).

Based on the last of the previous findings, it is concluded that the datum choice problem may interfere with the determination of the geometrical form of a minimum-constrained network under a nonlinear observation model—the notion of a truly distortionless free-net solution requires therefore additional theoretical steps to be rigorously defined.

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Appendix

Proofs of some relations in the theory of minimum constraints

Let $\mathbf{N} = \mathbf{A}^{\mathrm{T}}\mathbf{PA}$ be a $m \times m$ singular matrix with rank defect $k = m$-rank$\mathbf{N}$ and $\mathbf{H}$ any $k \times m$ full-row rank matrix that satisfies the algebraic condition in Eq. (4). The matrices $\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{WH}$ and $\mathbf{HE}^{\mathrm{T}}$ are then invertible, where $\mathbf{W}$ denotes a symmetric positive definite matrix and $\mathbf{E}$ corresponds to a type-E matrix that satisfies $\mathbf{AE}^{\mathrm{T}} = \mathbf{0}$ and $\mathbf{NE}^{\mathrm{T}} = \mathbf{0}$.

Starting with the following equation

$$(\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{WH})\mathbf{E}^{\mathrm{T}} = \mathbf{NE}^{\mathrm{T}} + \mathbf{H}^{\mathrm{T}}\mathbf{WHE}^{\mathrm{T}} = \mathbf{H}^{\mathrm{T}}\mathbf{WHE}^{\mathrm{T}} \quad \text{(A1)}$$

which is equivalent to

$$(\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{WH})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{WHE}^{\mathrm{T}} = \mathbf{E}^{\mathrm{T}} \quad \text{(A2)}$$

we finally get

$$(\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{WH})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{W} = \mathbf{E}^{\mathrm{T}}(\mathbf{HE}^{\mathrm{T}})^{-1} \quad \text{(A3)}$$

By multiplying (from the right) both sides of the last equation with $\mathbf{H}$, we obtain

$$(\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{WH})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{WH} = \mathbf{E}^{\mathrm{T}}(\mathbf{HE}^{\mathrm{T}})^{-1}\mathbf{H} \quad \text{(A4)}$$

The above equation can be also expressed as

$$(\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{WH})^{-1}(\mathbf{N} + \mathbf{H}^{\mathrm{T}}\mathbf{WH} - \mathbf{N}) = \mathbf{E}^{\mathrm{T}}(\mathbf{HE}^{\mathrm{T}})^{-1}\mathbf{H} \quad \text{(A5)}$$

which easily leads to

$$(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{N}=\mathbf{I}-\mathbf{E}^{\mathrm{T}}(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{H} \tag{A6}$$

Note that (A4) and (A6) are identical to Eqs. (11) and (12) given in Sect. 3.1. If we multiply (from the left) both sides of the last equation with $\mathbf{N}$, we obtain

$$\mathbf{N}(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{N}=\mathbf{N}-\mathbf{N}\mathbf{E}^{\mathrm{T}}(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}\mathbf{H}=\mathbf{N} \tag{A7}$$

which verifies that the matrix $(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}$ is indeed a generalized inverse of $\mathbf{N}$.

Also, if we multiply (from the left) both sides of (A3) with $\mathbf{A}$, we get the relationship

$$\mathbf{A}(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{W}=\mathbf{A}\mathbf{E}^{\mathrm{T}}(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}=\mathbf{0} \tag{A8}$$

and since the matrix $\mathbf{W}$ is invertible, we have

$$\mathbf{A}(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}=\mathbf{0} \tag{A9}$$

or equivalently

$$\mathbf{H}(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{A}^{\mathrm{T}}=\mathbf{0} \tag{A10}$$

Since $(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}$ is a generalized inverse of $\mathbf{N}=\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}$ the following relationship holds (Koch 1999, p. 51):

$$\mathbf{N}(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{A}^{\mathrm{T}}=\mathbf{A}^{\mathrm{T}} \tag{A11}$$

If we multiply (from the left) both sides of (A10) with $\mathbf{H}^{\mathrm{T}}$, we also have

$$\mathbf{H}^{\mathrm{T}}\mathbf{H}(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{A}^{\mathrm{T}}=\mathbf{0} \tag{A12}$$

and by adding together the last two equations, we get

$$(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{H})(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{A}^{\mathrm{T}}=\mathbf{A}^{\mathrm{T}} \tag{A13}$$

or equivalently

$$(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{A}^{\mathrm{T}}=(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1}\mathbf{A}^{\mathrm{T}} \tag{A14}$$

The last equation in conjunction with (A3) verify that the free-net solution $\hat{\mathbf{x}}$ obtained from the constrained NEQ system in Eq. (5) will be independent of the MCs weight matrix $\mathbf{W}$.

Furthermore, if we multiply (from the left) both sides of (A3) with the matrix $\mathbf{H}$, we have that

$$\mathbf{H}(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{W}=\mathbf{H}\mathbf{E}^{\mathrm{T}}(\mathbf{H}\mathbf{E}^{\mathrm{T}})^{-1}=\mathbf{I} \tag{A15}$$

and therefore

$$\mathbf{H}(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}=\mathbf{W}^{-1} \tag{A16}$$

Taking into account (A10) and (A16), it is easily verified that the free-net solution $\hat{\mathbf{x}}$ from the augmented NEQ system in Eq. (5) satisfies the MC system, i.e.

$$\begin{aligned}\mathbf{H}(\hat{\mathbf{x}}-\mathbf{x}_{\mathrm{o}})&=\mathbf{H}(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}(\mathbf{u}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{c})\\&=\mathbf{H}(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{u}+\mathbf{H}(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{c}\\&=\mathbf{0}+\mathbf{W}^{-1}\mathbf{W}\mathbf{c}=\mathbf{c}\end{aligned} \tag{A17}$$

and, using (A9) and (A11), it is deduced that the same solution satisfies also the original singular NEQ system, i.e.

$$\begin{aligned}\mathbf{N}(\hat{\mathbf{x}}-\mathbf{x}_{\mathrm{o}})&=\mathbf{N}(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}(\mathbf{u}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{c})\\&=\mathbf{N}(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{u}+\mathbf{N}(\mathbf{N}+\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{H})^{-1}\mathbf{H}^{\mathrm{T}}\mathbf{W}\mathbf{c}\\&=\mathbf{u}+\mathbf{0}=\mathbf{u}\end{aligned} \tag{A18}$$

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