Estimation of Variance Components Through a Combined Adjustment of GPS, Geoid and Levelling Data

G. Fotopoulos, C. Kotsakis, and M.G. Sideris
Department of Geomatics Engineering, University of Calgary
2500 University Drive N.W., Calgary, Alberta, Canada, T2N 1N4, Email: gfotopou@ucalgary.ca

Abstract. The method of GPS-levelling for obtaining orthometric heights is not a new concept. In fact, many studies have proven its usefulness and the question of whether GPS-levelling can provide a viable alternative to traditional techniques is no longer an issue. An important question, however, that has yet to be satisfactorily solved is, *'What accuracy level can be achieved using this approach?'* Over the past decade, numerous advances have been made which have placed us in a position where we can begin to address the issue with more confidence, namely (i) improved mathematical models/techniques for dealing with GPS and geoid data, (ii) increased data availability for gravimetric geoid models, and (iii) improved data processing capabilities. In this paper a statistical approach for estimating the variance components of heterogeneous groups of observations is used in the combined adjustment of GPS, geoid and levelling data. Specifically, the iterative minimum norm quadratic unbiased estimation algorithm is employed to determine the individual variance components for each of the three height types. The challenges encountered when implementing this well-known algorithm in practice with real data are discussed. The analysis provides some indication into the practicality and effectiveness of estimating variance components in mixed vertical networks. Notably, the estimation of realistic variance components provides us with important insight regarding the GPS-levelling problem in addition to other uses of combined GPS, geoid and levelling data, such as assessing the accuracy of a gravimetric geoid model.

1 Introduction

The reliable combined adjustment of GPS (h), orthometric (H) and geoid (N) height data depends on two main factors, namely (i) the suitability of a parametric model for the systematic effects and biases and (ii) the correctness of the stochastic model for the observational noise. The former has been a topic of interest for researchers and practitioners alike in light of its benefits for GPS-based determination of orthometric heights in a local vertical datum. Although the use of a parametric model designed to absorb the systematic effects and datum inconsistencies inherent among the GPS, geoid and levelling height data is an interesting topic on its own, it will not be dwelled on extensively herein. Instead, the issue of stochastic modelling for observations in the mixed adjustment of heterogeneous height data will be the main focus of this paper. It should, however, be emphasized that both aforementioned factors are interrelated and need to be addressed in order to rigorously and reliably combine the different types of height data. Of course, both the modelling of systematic effects and random errors presupposes the absence of gross errors/blunders in the observational data. Furthermore, improper stochastic modelling may lead to certain deviations in the results, the very effect we try to model through parameterization of the height misclosures ($h - H - N$).

The motivation for addressing this issue is quite clear given the vast applications requiring optimally combined height data, including, but not limited to:

- modernize regional vertical datums
- unify national regional datums between neighbouring countries
- transform between various height data sets (incl. vertical reference systems)
- refine existing gravimetric geoid models

The chosen approach presented herein for testing/improving the stochastic model is the well known statistical tool of variance component estimation (VCE). Many different algorithms for VCE have been studied with regards to geodetic data analysis and a comprehensive literature review is given in Grafarend (1985). Recent geodetic

applications include the statistical analysis of VLBI data, GPS observations, deformation monitoring schemes and simulated gravity field models from upcoming satellite missions such as GOCE. Despite these advances, the implementation of VCE for combined data types, more specifically GPS, orthometric and geoid heights, has not been suitably addressed in the geodetic literature. A properly applied VCE technique coupled with appropriate stochastic modelling will allow for the optimal combination of these heterogeneous data leading to the inclusive goal of more realistic accuracy measures for height-related applications.

2 Problem Description

Given a network of points with co-located GPS, orthometric and geoid height data, a combined adjustment can be performed. We start with the general linear model (see Kotsakis and Sideris, 1999 for details),

$$\mathbf{l} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{v} \tag{1a}$$

$$E\{\mathbf{v}\} = \mathbf{0} \tag{1b}$$

$$E\{\mathbf{v}\mathbf{v}^{\mathrm{T}}\} = \mathbf{C}_{\mathbf{v}} = \sigma^2 \mathbf{Q}_{\mathbf{v}} \tag{1c}$$

where the vector of observations $\mathbf{l}$ is composed of the height 'misclosure' at each GPS/levelling benchmark as follows

$$l_i = h_i - H_i - N_i \tag{2}$$

$E\{\cdot\}$ is the mathematical expectation operator, $\mathbf{A}$ is the design matrix which depends on the parametric model type (see Sect. 5.4), $\mathbf{B}$ is the block-structured matrix $\mathbf{B} = [\mathbf{I} \quad -\mathbf{I} \quad -\mathbf{I}]$, where each $\mathbf{I}$ is an $m \times m$ unit matrix (m is the number of observation equations), $\mathbf{x}$ is a vector containing the unknown parameters corresponding to the selected parametric model and $\mathbf{v}$ is a vector of random errors, with zero mean (Eq. 1b), described by the following formula

$$\mathbf{v} = \left[\mathbf{v}_{\mathrm{h}}^{\mathrm{T}} \quad \mathbf{v}_{\mathrm{H}}^{\mathrm{T}} \quad \mathbf{v}_{\mathrm{N}}^{\mathrm{T}}\right]^{\mathrm{T}} \tag{3}$$

where $\mathbf{v}_{(\cdot)}$ is an $m \times 1$ vector of random errors for each of the h, H, N data types. The corresponding CV matrix is described by Eq. (1c), where the positive-definite cofactor matrix $\mathbf{Q}_{\mathbf{v}}$ is scaled by the variance factor σ^2. For the case of heterogeneous disjunctive observations, a block-diagonal CV model is used

$$\mathbf{C}_{\mathbf{v}} = \begin{bmatrix} \mathbf{C}_{\mathbf{h}} & 0 & 0 \\ 0 & \mathbf{C}_{\mathbf{H}} & 0 \\ 0 & 0 & \mathbf{C}_{\mathbf{N}} \end{bmatrix} \tag{4}$$

where $\mathbf{C}_{(\cdot)}$ is the CV matrix for each of the height types. An additive covariance matrix model for the observations is given by:

$$\mathbf{C}_{\mathbf{l}} = \mathbf{B}\mathbf{C}_{\mathbf{v}}\mathbf{B}^{\mathrm{T}} = \sigma_h^2 \mathbf{Q}_{\mathbf{h}} + \sigma_H^2 \mathbf{Q}_{\mathbf{H}} + \sigma_N^2 \mathbf{Q}_{\mathbf{N}} \tag{5}$$

where $\mathbf{Q}_{\mathbf{h}}$, $\mathbf{Q}_{\mathbf{H}}$, and $\mathbf{Q}_{\mathbf{N}}$ are known positive-definite cofactor matrices for the GPS, orthometric and geoid height data respectively. The unknown variance components are σ_h^2, σ_H^2, σ_N^2.

The problem, therefore, is to solve for the unknown parameters of the parametric model, $\mathbf{x}$, **and** the individual variance components for each of the height data types, σ_h^2, σ_H^2, σ_N^2.

3 Variance Component Estimation

There are a number of methods available to perform VCE within the context of least-squares adjustment [Crocetto et al., 2000]. In this paper the minimum norm quadratic unbiased estimation (MINQUE) procedure is followed [Rao, 1971]. It has been shown that many VCE approaches (i.e., best quadratic unbiased estimation, MINQUE), under the assumption of normally distributed observations, give seemingly numerically equivalent results. However, distinguishable characteristics in the formulation of the problem for each algorithm exist. Ultimately, the selection of the appropriate technique should rely on the desired estimator properties, such as translation invariance, unbiasedness, computational efficiency, etc. The rationale detailing our preference for this particular approach is not included due to space restrictions, however it can be stated that this approach works well for the posed problem and is relatively easy to implement in practice.

The selected VCE algorithm is briefly described below. Due to space restrictions, only the basic

formulas are given without a detailed account as to their derivation. The problem is to solve the following system

$$\mathbf{S}\,\hat{\boldsymbol{\theta}} = \mathbf{q} \tag{6}$$

where $\hat{\boldsymbol{\theta}}$ is a vector containing the unknown variance components (also indicated by $\hat{\boldsymbol{\sigma}}^2$). The composition of the matrix $\mathbf{S}$ is denoted by

$$\mathbf{S} = \begin{bmatrix} s_{hh} & s_{hH} & s_{hN} \\ s_{Hh} & s_{HH} & s_{HN} \\ s_{Nh} & s_{NH} & s_{NN} \end{bmatrix} \tag{7}$$

Each element s_{ij} in the matrix is computed from

$$s_{ij} = tr\left(\mathbf{R}\mathbf{Q}_{\mathrm{i}}\mathbf{R}\mathbf{Q}_{\mathrm{j}}\right), \quad \mathrm{i,j} = h, H, N \tag{8}$$

where $tr(\cdot)$ is the trace operator, $\mathbf{Q}_{(\cdot)}$ is a positive-definite cofactor matrix for each group of observations, $\mathbf{R}$ is the matrix defined by

$$\mathbf{R} = \mathbf{C}_{\mathbf{l}}^{-1}[\mathbf{I} - \mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{C}_{\mathbf{l}}^{-1}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{C}_{\mathbf{l}}^{-1}] \tag{9}$$

where $\mathbf{A}$ is the same design matrix as in Eq. (1a) and $\mathbf{C}_{\mathbf{l}}$ is the CV matrix of the observations. The vector $\mathbf{q}$ contains the quadratic forms

$$\mathbf{q} = \{q_{\mathrm{i}}\}, \quad q_{\mathrm{i}} = \hat{\mathbf{v}}_{\mathrm{i}}^{\mathrm{T}}\mathbf{Q}_{\mathrm{i}}^{-1}\hat{\mathbf{v}}_{\mathrm{i}} \tag{10}$$

where $\hat{\mathbf{v}}_{\mathrm{i}} = \mathbf{Q}_{\mathrm{i}}\mathbf{R}\,\mathbf{l}$ are the separated residuals for each group of observations and is easily derived from the mixed model equation formulation; see Kotsakis and Sideris (1999) for more details.

It is evident from the expression for the $\mathbf{R}$ matrix (Eq. 9) that an iterative procedure must be employed since the unknown variance components are embedded in $\mathbf{C}_{\mathbf{l}}$ (Eq. 5) that is used to compute $\mathbf{R}$. Therefore, initial values for the unknowns must be provided along with an appropriate convergence criterion. The iterative application of this algorithm is known as iterative minimum norm quadratic unbiased estimation (I-MINQUE). The general iterative scheme is shown in Figure 1 where each $\hat{\theta}_i^{\alpha}$ value represents a ratio computed at each iteration α and the final estimated variance components are obtained from the product of all ratios after n iterations. More information regarding the estimator properties can be found in Rao (1971); Rao and Kleffe (1988).

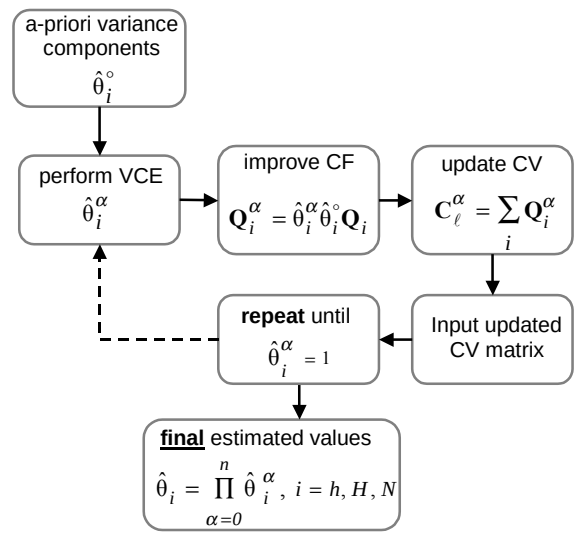

Fig. 1. Iterative variance component estimation scheme

Notes. The problem, as described in the previous section, outlines a rare characteristic for geodetic data, that of *balanced* data. Normally, when heterogeneous types of geodetic data are used in a mixed adjustment, the number of observations per each group of data is not the same. In this case, the problem is pre-designed such that all three height groups are available for each network benchmark. Thus, we can estimate the variance components from balanced data - a less demanding task, in general, than dealing with unbalanced data. An additional advantage offered by the design of this particular problem is a relatively low computational load. Only three variance components are sought. The largest matrix inversion will be on the order of the number of observations, m, which in the absolute case is equivalent to the number of network benchmarks. Thus, unlike many other VCE-related applications, where the main obstacle encountered is the high computational load, the problem here, lies in the absence of independently derived and reliable variance estimates for each height type.

4 Description of Data

The Swiss national network, consisting of 111 GPS/levelling benchmarks distributed throughout an ~343 km × 212 km region, was used for the numerical tests. At each point the observations, as given by Eq. (2), were computed (statistics: mean ~ 1.1 cm, min. –4.9 cm, max. 19 cm, RMS ~ 4 cm). Fully populated initial CV matrices were obtained for *h*, *H* and *N*. The original CV matrix for the ellipsoidal heights $\mathbf{Q}_h$ was extracted from the results of a commercial post-processing software package of GPS data. Typical for GPS, the output CV matrix was overly optimistic, a direct result of neglecting (even partially) correlations (temporal, spatial and physical) between GPS phases. In practice, this situation is sometimes rectified by *arbitrarily* scaling the CV matrix by some factor. The validity of this all too common practice is addressed in sect. 5.1. $\mathbf{Q}_H$ comes directly from the rigorous national adjustment of all first and second order levelling measurements. As expected, the correlation between nearby neighbouring stations is very high. Finally, $\mathbf{Q}_N$ at the GPS benchmarks was obtained by straightforward application of error propagation to the least-squares collocation geoid solution [see Marti, 2002 for more details]. Table 1 summarizes some characteristics of the 'a-priori' covariance matrices for comparison purposes.

Table 1. Initial covariance matrix characteristics

	GPS	Levelling	Geoid
condition number	146.2	2.23×10^7	4.50×10^5
average σ (cm)	0.79	0.75	1.93

The comparatively high condition number corresponding to the levelling data matrix is of particular interest. This high value may lead to numerical instability problems when the matrix needs to be inverted. Therefore, a simple ridge regression was applied to alleviate any numerical problems. Other values listed in the table include the average standard deviation (σ) computed from $\sqrt{tr(\mathbf{Q}_i)/m}$, which gives an indication of the overly optimistic a-priori levels, especially evident for the ellipsoidal heights.

Note that, where possible, the GPS/levelling observations were excluded from the computation of the gravimetric geoid to ensure independence among the cofactor matrices satisfying the assumption of disjunctive observation groups, i.e. the cross-covariance matrices for $i, j = h, H, N$ are given by:

$$E\left\{\mathbf{v}_i \mathbf{v}_j^T\right\} = \mathbf{0}, \quad for\ i \neq j \tag{11}$$

5 Numerical Examples – Case Studies

In this section, the results of four different empirical tests are described, whereby some of the key issues related to the implementation of the I-MINQUE method are analyzed in detail, demonstrating the use of VCE for practical height-related applications.

5.1 Case Study I: A-priori CV matrices

The purpose of this case study is to test the effect of different a-priori CV matrices on the final estimated variance components. The I-MINQUE scheme does not provide any guarantee of the *correctness* of the final estimated values. Therefore, mere convergence cannot be taken as a positive re-enforcer. To overcome this uncertainty, one can compute the CV matrix for $\hat{\boldsymbol{\theta}}$ using the following formula [Crocetto et al., 2000]:

$$\mathbf{C}_{\hat{\boldsymbol{\theta}}} = 2\mathbf{S}^{-1} \tag{12}$$

From this equation, the standard deviations corresponding to each of the estimated variance components can be extracted and, depending on the relative magnitudes, inferences can be made regarding the 'goodness' of the estimated values. Another method, though beyond the scope of this paper, is the computation of the associated confidence intervals for each of the variance components.

In this study, a more empirical approach is followed whereby a number of different a-priori variance values for each height type are tested to see if they all yield the same solution. Interestingly, the results showed that the initial a-priori values for a single group of observations (e.g., *N*) do not have a noticeable effect on the remaining groups, even though theoretically co-dependence is present. It was also discovered that through proper VCE, the arbitrariness of certain pre-selected factors (such as

scaling $\mathbf{Q}_h$) can be alleviated. For instance, in the Swiss network, a factor of 10^2 was empirically determined as a scale for $\mathbf{Q}_h$. By re-scaling the original CV matrix with different a-priori factors, this independently-derived value was verified as an appropriate scale for $\mathbf{Q}_h$. This is an important realization as it ascribes some statistical reasoning towards an otherwise 'arbitrary' scaling practice and leads to a better understanding of the true measures of error.

5.2 Case Study II: Non-negative variances

One of the major pitfalls of the described VCE technique is that no provision has been made to ensure non-negative variance values (i.e. $\hat{\boldsymbol{\theta}} \in \Re_+^3$). In general, negative outcomes of variance components can be attributed to an insufficient number of observations compared to unknown parameters and/or an incorrect stochastic model. Thus, a negative variance outcome yields important information regarding the problem set up, information that is lost if the estimator is constrained to give only positive outcomes. Nonetheless, alternate algorithms exist which are constrained to provide positive variance factors. One such algorithm, known as iterative almost unbiased estimation (IAUE), can be implemented through the following formula [Rao and Kleffe, 1988]

$$\hat{\boldsymbol{\theta}}^{\mathrm{i}}_{new} = \frac{\hat{\boldsymbol{\theta}}^{\mathrm{i}}_{old} \mathbf{l}^{\mathrm{T}} \mathbf{R} \mathbf{Q}_{\mathrm{i}} \mathbf{R}^{\mathrm{T}} \mathbf{l}}{tr(\mathbf{R}\,\mathbf{Q}_{\mathrm{i}})} \qquad (13)$$

where $i = h, H, N$, $\hat{\boldsymbol{\theta}}^{\mathrm{i}}_{new}$ and $\hat{\boldsymbol{\theta}}^{\mathrm{i}}_{old}$ represent the current and previous iteratively-derived variance estimates respectively, and all other terms have been previously described. This algorithm was tested with the real data and proved to give almost identical results to those obtained using the I-MINQUE method. An added benefit of this method is that it is computationally simpler and converges approx-imately 50% faster than the other approach. Thus, in cases where computational efficiency is an issue, IAUE offers a viable alternative to I-MINQUE.

To determine the effect of data redundancy on the estimated variance components, using the iterated MINQUE approach, a test was performed whereby observations were eliminated (one-by-one) and the final values of the estimated components were noted. For the case of the Swiss test network, it was found that at least 49 observations are required for convergence and positive-valued final components, which corresponds to 44% of the available GPS-on-benchmarks in this region.

5.3 Case Study III: Effect of correlations

In practice, the availability of fully populated variance-covariance matrices for all groups of height data at the same GPS benchmarks is a luxury. To test the effect of correlations between observations of the same type on the estimated variance components, numerical experiments were conducted with fully populated and diagonal covariance matrices. The results are given in Table 2 (note that the estimated variance components are unitless scale factors). It is evident that due to the correlated nature of the observations, a diagonal covariance matrix is further from the 'true' CV matrix and therefore requires more iterations to obtain the final estimated values. Also, by neglecting the off-diagonal elements, we obtain overly optimistic CV matrices compared to the fully populated case. Results will vary depending on the degree of correlation, however it is clear that unrealistically 'good' results are obtained when correlations are ignored, as expected.

Table 2. Effect of correlations on estimated variances

Covariance Matrices	$\hat{\sigma}_h^2$	$\hat{\sigma}_H^2$	$\hat{\sigma}_N^2$	iterations
Full	2.83	5.06	1.02	99
Diagonal	0.71	3.63	1.07	152

5.3 Case Study IV: Parametric model type

As mentioned in the introduction, the optimal combination of h, H and N requires the incorporation of a parametric model to deal with the systematic errors inherent in the data and the fact that each height type refers to a different reference surface. The parametric model, or corrector surface

as it is commonly termed, can be incorporated as follows (Kotsakis and Sideris, 1999):

$$\mathbf{h} - \mathbf{H} - \mathbf{N} = \mathbf{A}\mathbf{x} + \mathbf{v} \tag{14}$$

where the parametric term **Ax** describes the corrector surface. VCE procedures pre-suppose that *no biases* or *systematic effects* are present in the vector **v**. Any unmodelled effects may propagate into the estimated variances and give unreliable results. This case study was designed to determine the role of the parametric model type, if any, on the final estimated variance components. Five models were selected, namely the classic 4-parameter transformation model, as shown in the following formula

$$p_C(\varphi,\lambda) = x_\circ + x_1 \cos\varphi \cos\lambda + x_2 \cos\varphi \sin\lambda + x_4 \sin\varphi \tag{15}$$

and four simple nested polynomial regression models from first to fourth order as given by

$$p_{nm}(\varphi,\lambda) = \sum_{m=0}^{M}\sum_{n=0}^{N} x_{nm}(\varphi-\varphi_\circ)^n(\lambda-\lambda_\circ)^m \tag{16}$$

where φ,λ are the horizontal coordinates and $\varphi_\circ,\lambda_\circ$ represent the midpoint of the network. The estimated variance components (unitless) corresponding to each parametric model type are shown in Table 3.

Table 3. Estimated variance components

Parametric Model Type	$\hat{\sigma}_h^2$	$\hat{\sigma}_H^2$	$\hat{\sigma}_N^2$
Eq. 15	2.82	5.06	1.01
Eq. 16, 1st order	2.83	4.80	1.01
2nd order	2.94	4.50	1.05
3rd order	3.08	3.96	0.97
4th order	divergence, negative estimates		

Predictably, a solution was not achievable in all cases leading to divergence and/or negative estimates. This may indicate that an inadequate model was used for the systematic effects resulting in 'residual' biases that corrupt the performance of the VCE method. Another possibility is the presence of numerical instabilities caused by over-parameterization, which occurred when a full 4th order model was used (Eq. 16, 4th order). In any case, these first results are revealing as they hint towards a means for identifying the effectiveness of the selected corrector surface. The latter comment is stated with prudence as further studies are currently underway to verify this deduction.

6 Conclusions and Future Work

The implementation of VCE in the optimal combined adjustment of heterogeneous height data, namely GPS, orthometric and geoid heights was presented. The I-MINQUE procedure was described and used to test the supplied CV matrices for each height type. Through various numerical case studies with real data, a number of key issues involved in most height-related applications were studied in detail. In the future, studies will be conducted in order to apply this approach for assessing the accuracy of global gravity field models (i.e. EGM96 and those obtained from GRACE/GOCE).

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