# On the problem of geoid height transformation between different geodetic reference frames

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**Abstract.** Transforming geoid heights between different geodetic reference frames (GRFs) is an essential task in gravity field modeling and its proper implementation is crucial for many applications involving the use of the geoid. In this paper an attempt is made to investigate the problem of geoid height transformation between different GRFs, without considering other variations in the auxiliary geophysical or geodynamical hypotheses that may be a-priori specified for the desired geoid type (e.g. variations in the values of $M$ and/or $W_o$, variations in terms of treatment of permanent tidal effects). The aim here is to present the required methodology to deal with the problem: "how should we transform geoid heights, referring to a fixed equipotential surface ($W = W_o$), from a given GRF to another GRF when we know the seven similarity transformation parameters linking the two frames?". Special emphasis is given on the effect of GRF scale variations in coordinate transformations involving reference ellipsoids, for the particular case of geoid heights. Since every Cartesian coordinate system "gauges" an attached ellipsoid according to its own particular scale, there will exist a small contribution from the scale variation between the involved GRFs on the relative size of their adopted reference ellipsoids. Neglecting such a scale-induced indirect effect corrupts the values for the curvilinear geodetic coordinates obtained from a similarity transformation model, and significant errors can be introduced in the transformed geoid heights. The paper explains the above issues in detail and presents the necessary mathematical framework for their solution.

**Keywords.** Geoid, similarity transformation, reference frame, reference ellipsoid, datum scale.

## 1 Introduction

High-precision studies in Earth gravity field modeling require a careful treatment of several reference frame issues in order to ensure a coherent framework for data analysis and to avoid datum-related biases and artifacts in the results. Transforming geoid heights, for example, between different geodetic reference frames is an essential and necessary component in gravity field modeling, and its proper implementation is crucial in many scientific applications involving the direct or indirect use of the geoid (e.g. consistent combination of ellipsoidal, orthometric and geoid height data for GPS-based leveling, external validation of gravimetric geoid models with GPS and leveling data, datum-consistent comparison between old and recent gravimetric, satellite-only or combined geoid models, update of existing geoid models to comply with current definitions and realizations of global geodetic reference systems, reduction of sea surface heights obtained from satellite altimetry data to a preferred geodetic reference frame, and proper utilization of geoid height information in datum transformation studies).

By definition, geoid heights refer to a specific geodetic reference system (GRS). Available geoid models (e.g. EGM96) or individually computed geoid height values (e.g. from GPS and leveling data) ought to be consistent with a particular realization of such a GRS, namely a geodetic reference frame (GRF). In this way, a gravimetric geoid determined through the generalized Stokes' formula (Heiskanen and Moritz 1967)

$$N = \frac{G\delta M}{R\gamma} - \frac{\delta W}{\gamma} + \frac{R}{4\pi\gamma}\iint_{\sigma} \Delta g\, S(\psi)\, d\sigma \qquad (1)$$

will be consistent with the geocentric GRF in which the spatial positions and the values of the gravity anomaly data $\Delta g$ refer to, whereas a geoid obtained through a spherical harmonic series expansion refers to the GRF that is realized by the positions of the satellite tracking stations which were estimated at the time of the model development (Pavlis 1998, Lemoine *et al.* 1998). On the other hand, geoid heights that are computed directly from GPS and leveling data, according to $N = h - H$, refer to the same GRF implied by the GPS coordinates of the evaluation point, while the use of the orthometric height in this case fixes (in principle) the particular

equipotential surface $W = W_o$ that we should treat as "the geoid".

Apart from the chosen spatial reference frame in which the relative position of the geoid with respect to a reference ellipsoid shall be expressed (at a particular epoch), there are additional aspects affecting the adopted GRS with respect to which a final geoid model can be delivered. Such aspects involve the treatment of the permanent tidal effects on the gravity field and the Earth's crust (zero-frequency geoid, non-tidal geoid, mean geoid), the consideration of other loading effects on the solid and liquid parts of the Earth, the adoption of specific numerical values for fundamental GRS parameters (or "constants") such as the Newton's gravitational constant $G$, the Earth's (including its atmosphere) mass $M$, the normal gravity potential value on the reference ellipsoid $U_o$, the gravity potential value on the geoid $W_o$, the semi-major axis and the flattening of the reference ellipsoid, the mean angular velocity of the Earth, etc.; see Groten (2004).

In this paper an attempt is made to highlight the essential points related to the problem of geoid height transformation between different GRFs, without considering other variations in the auxiliary geophysical or geodynamical hypotheses that may be a-priori specified for the desired geoid type (e.g. variations in the values of $M$ and/or $W_o$, variations in terms of treatment of permanent tidal effects, etc.). Our objective is to present the methodology and the required formulae to deal with the following problem: *"how should we transform geoid heights, referring to a fixed equipotential surface ($W = W_o$), from a given GRF to another GRF when we know the seven similarity transformation parameters linking the two frames?"*; see Fig. 1.

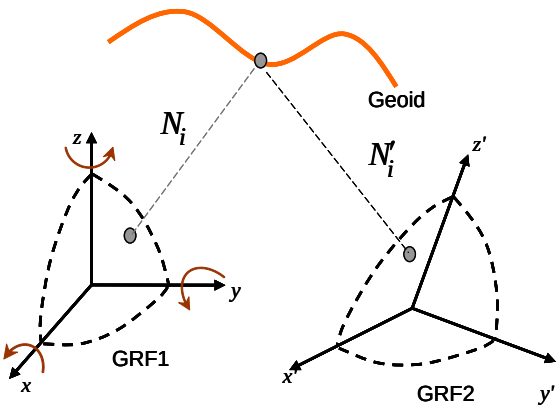


**Figure 1.** Geoid height transformation between different geodetic reference frames.

Special emphasis is given on the effect of GRF scale variations in coordinate transformations involving reference ellipsoids, which is an important issue that has not been sufficiently stressed in the geodetic literature; see Soler and van Gelder (1987). Since every Cartesian coordinate system "gauges" an attached reference ellipsoid according to its own particular scale, there will be a small contribution from the scale variation between the involved GRFs on the relative size of their adopted reference ellipsoids. For example, if the same ellipsoid (in terms of *physical dimensions*) is attached to two different GRFs, we should generally assign different values to its semi-major axis in each case *if* the GRFs are connected through a non-zero scale change factor. Neglecting such a scale-induced indirect effect corrupts the resulting values for the curvilinear geodetic coordinates obtained from a similarity transformation model, and significant errors can be introduced in the transformed geoid heights (Soler and van Gelder 1987).

To clarify these points, an extended similarity transformation model is presented which provides a proper "de-coupling" of the geoid height variation arising from (i) the GRF scale difference and (ii) the actual change of the *physical size* of the reference ellipsoid.

## 2 Similarity transformation model for geoid heights

Let us consider the well known Euclidean similarity transformation model which is used to convert Cartesian coordinates between two geodetic reference frames that generally differ in terms of three translation parameters ($t_x$, $t_y$, $t_z$), three orientation parameters ($\varepsilon_x$, $\varepsilon_y$, $\varepsilon_z$) and a factor of uniform spatial scale change ($\delta s$)

$$\begin{bmatrix} x'-x \\ y'-y \\ z'-z \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} + \begin{bmatrix} \delta s & \varepsilon_z & -\varepsilon_y \\ -\varepsilon_z & \delta s & \varepsilon_x \\ \varepsilon_y & -\varepsilon_x & \delta s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (2)$$

Note that the above model corresponds to a first-order linear approximation of the rigorous vector transformation formula

$$\mathbf{x}'_{\mathrm{GRF2}} = \mathbf{t} + (1+\delta s)\,\mathbf{R}(\varepsilon_x, \varepsilon_y, \varepsilon_z)\,\mathbf{x}_{\mathrm{GRF1}} \qquad (3)$$

with $\mathbf{R}$ being the total rotation matrix that performs three successive rotations around the axes of GRF1 so that they become parallel to the corresponding axes of GRF2, $\mathbf{t}$ is the Cartesian coordinate vector of the origin of GRF1 with respect to GRF2, and $\delta s$ is the scale difference factor between the two reference frames (see Fig. 1). The use of the approxi-

mate model (2) instead of the rigorous expression in (3) has a negligible effect on the transformed coordinates and it is justified for most geodetic applications where the rotation angles do not exceed a few arc seconds and the differential scale factor is of the order of $10^{-5}$ or less; for more details, see Hofmann-Wellenhof and Moritz (2005, ch. 5).

In order to derive the expression for the similarity transformation of geoid heights between the reference frames GRF1 and GRF2, we need also to consider the relationship between Cartesian and curvilinear geodetic coordinates

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (\mathsf{N}+h)\cos\varphi\cos\lambda \\ (\mathsf{N}+h)\cos\varphi\sin\lambda \\ (\mathsf{N}(1-e^2)+h)\sin\varphi \end{bmatrix} \tag{4}$$

where $\mathsf{N}$ (to be distinguished from the symbol $N$ that denotes the geoid height) is the prime vertical radius of curvature, given by the formula

$$\mathsf{N} = \frac{\mathrm{a}}{W} = \frac{\mathrm{a}}{\sqrt{1-e^2\sin^2\varphi}} \tag{5}$$

The quantities a and $e^2$ correspond to the length of the semi-major axis and the squared eccentricity of the adopted reference ellipsoid which is used for the definition of the curvilinear geodetic coordinates $\varphi$, $\lambda$ and $h$ in (4).

By differentiation of (4), we get

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \mathbf{J} \begin{bmatrix} d\varphi \\ d\lambda \\ dh \end{bmatrix} \tag{6}$$

where the Jacobian matrix **J** has the following form (Soler 1976)

$$\mathbf{J} = \begin{bmatrix} -(\mathsf{M}+h)\sin\varphi\cos\lambda & -(\mathsf{N}+h)\cos\varphi\sin\lambda & \cos\varphi\cos\lambda \\ -(\mathsf{M}+h)\sin\varphi\sin\lambda & (\mathsf{N}+h)\cos\varphi\cos\lambda & \cos\varphi\sin\lambda \\ (\mathsf{M}+h)\cos\varphi & 0 & \sin\varphi \end{bmatrix} \tag{7}$$

and $\mathsf{M} = \mathrm{a}\,(1-e^2)/W^3$ is the meridian radius of curvature.

Substituting the left hand-side in (6) according to the similarity transformation model given by (2), and then solving for $dh$, we obtain the following formula that corresponds to the *similarity transformation model for ellipsoidal heights* (Soler and van Gelder 1987)

$$\begin{aligned} h' - h &= \delta h(t_x) + \delta h(t_y) + \delta h(t_z) \\ &\quad + \delta h(\varepsilon_x) + \delta h(\varepsilon_y) + \delta h(\delta s) \end{aligned} \tag{8}$$

where

$$\delta h(t_x) = t_x \cos\varphi\cos\lambda \tag{9}$$

$$\delta h(t_y) = t_y \cos\varphi\sin\lambda \tag{10}$$

$$\delta h(t_z) = t_z \sin\varphi \tag{11}$$

$$\delta h(\varepsilon_x) = -\varepsilon_x \mathsf{N} e^2 \sin\varphi\cos\varphi\sin\lambda \tag{12}$$

$$\delta h(\varepsilon_y) = \varepsilon_y \mathsf{N} e^2 \sin\varphi\cos\varphi\cos\lambda \tag{13}$$

$$\delta h(\delta s) = (\mathrm{a}W + h)\,\delta s \tag{14}$$

Note that the rotation angle $\varepsilon_z$ does not affect the change of the ellipsoidal height from GRF1 to GRF2 due to the rotational symmetry of the reference ellipsoid.

The above formulae perform the linearized (i.e. neglecting terms of the order $e^4$ and higher) similarity transformation of ellipsoidal heights between two arbitrary GRFs at any point in space whose initial curvilinear coordinates (with respect to GRF1) are $\varphi$, $\lambda$ and $h$. If we assume, in particular, that the point whose ellipsoidal height being transformed is located on the geoid (see Fig. 1), then (8) is reduced to the *similarity transformation model for geoid heights*

$$\begin{aligned} N' - N &= \delta N(t_x) + \delta N(t_y) + \delta N(t_z) \\ &\quad + \delta N(\varepsilon_x) + \delta N(\varepsilon_y) + \delta N(\delta s) \end{aligned} \tag{15}$$

where $\delta N(t_x) = \delta h(t_x)$, $\delta N(t_y) = \delta h(t_y)$, $\delta N(t_z) = \delta h(t_z)$, $\delta N(\varepsilon_x) = \delta h(\varepsilon_x)$ and $\delta N(\varepsilon_y) = \delta h(\varepsilon_y)$.

It is important to mention that the ellipsoidal height of the evaluation point in this case is identical to the geoid height, and thus the scale-dependent term $\delta N(\delta s)$ should take the form

$$\delta N(\delta s) = (\mathrm{a}W + N)\,\delta s \tag{16}$$

**Remark 1.** In the special case where $t_x = t_y = t_z = 0$ and $\varepsilon_x = \varepsilon_y = \varepsilon_z = 0$, the previous geoid transformation model yields

$$N' - N = \delta N(\delta s) = (\mathrm{a}W + N)\,\delta s \tag{17}$$

or equivalently

$$N' = (1+\delta s)N + \mathrm{a}W\delta s \tag{18}$$

The interesting point in this particular case is the presence of the additional term 'a$W\delta s$' in (18), which can be understood as the effect of an "apparent" change in the physical dimensions of the reference ellipsoid due to the scale difference between the involved GRFs. The magnitude of this term can be quite significant, reaching more than 6 m when $\delta s = 10^{-6}$ (1 ppm) and dropping to about 1 cm for $\delta s = 10^{-9}$ (1 ppb).

If the reference ellipsoid remains the same in terms of its physical dimensions in both GRFs, then the geoid height change (when $t_x = t_y = t_z = 0$ and $\varepsilon_x = \varepsilon_y = \varepsilon_z = 0$) should be given *only* by a simple re-scaling

$$N' = (1+\delta s)N \tag{19}$$

since the same physical length (i.e. the distance between a point on the geoid and its orthogonal projection on the single reference ellipsoid) is "measured" with respect to two coinciding GRFs which differ only by a uniform scale factor $\delta s$.

In order to counter balance the effect of the term 'a$W\delta s$' in (18), and also to properly account for an actual change in the physical size of the reference ellipsoid, the similarity transformation model for geoid heights in (15) needs to be extended as described in the following section.

## 3 Considering the effect of the reference ellipsoid change

Let us adopt the length of the semi-major axis (a) and the flattening (f) as the two fundamental parameters that uniquely define the geometrical size of a reference ellipsoid.

In order to account for a possible change in the physical dimensions of the reference ellipsoid in geoid height transformation problems, we need first to differentiate the vector formula in (4) as follows

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \mathbf{J}_1 \begin{bmatrix} d\varphi \\ d\lambda \\ dh \end{bmatrix} + \mathbf{J}_2 \begin{bmatrix} d\mathrm{a} \\ d\mathrm{f} \end{bmatrix} \tag{20}$$

where the first Jacobian matrix $\mathbf{J}_1$ is identical to the matrix $\mathbf{J}$ given in (7), while the analytical form of the second Jacobian matrix $\mathbf{J}_2$ can be found in Soler (1976); see also Soler and van Gelder (1987). Setting the left-hand side in (20) equal to zero, and then solving for $dh$, we obtain the ellipsoidal height variation only from the change of the reference ellipsoid which, in conjunction with (8), leads to the following *extended similarity transformation model for ellipsoidal heights*

$$\begin{aligned} h' - h = {} & \delta h(t_x) + \delta h(t_y) + \delta h(t_z) \\ & + \delta h(\varepsilon_x) + \delta h(\varepsilon_y) + \delta h(\delta s) \\ & + \delta h(\delta\mathrm{a}) + \delta h(\delta\mathrm{f}) \end{aligned} \tag{21}$$

where the additional terms $\delta h(\delta\mathrm{a})$ and $\delta h(\delta\mathrm{f})$ are given by the equations (Soler and van Gelder 1987)

$$\delta h(\delta\mathrm{a}) = -W\delta\mathrm{a} \tag{22}$$

$$\delta h(\delta\mathrm{f}) = \frac{\mathrm{a}\,(1-\mathrm{f})}{W}\sin^2\varphi\,\delta\mathrm{f} \tag{23}$$

The quantities $\delta\mathrm{a} = \mathrm{a}' - \mathrm{a}$ and $\delta\mathrm{f} = \mathrm{f}' - \mathrm{f}$ correspond to the variation of the numerical values for the semi-major axis and the flattening of the reference ellipsoid, as these are used in the respective geodetic reference frames, GRF1 and GRF2.

With the exclusion of the terms $\delta h(\varepsilon_x)$, $\delta h(\varepsilon_y)$ and $\delta h(\delta s)$, the model in (21) is identical to the standard *Molodensky transformation formula* (Molodensky *et al.* 1962) which has often been used for transforming ellipsoidal heights between different geodetic datums (see, e.g., National Imagery and Mapping Agency 1996, pp. 7.3-7.4) and for determining the Earth's mean equatorial radius and center of mass through the joint analysis of geometrically derived and gravimetric geoid heights (see, e.g., Grappo 1980).

Taking into account (21), the corresponding *extended similarity transformation model for geoid heights* is obtained

$$\begin{aligned} N' - N = {} & \delta N(t_x) + \delta N(t_y) + \delta N(t_z) \\ & + \delta N(\varepsilon_x) + \delta N(\varepsilon_y) + \delta N(\delta s) \\ & + \delta N(\delta\mathrm{a}) + \delta N(\delta\mathrm{f}) \end{aligned} \tag{24}$$

where $\delta N(\delta\mathrm{a}) = \delta h(\delta\mathrm{a})$ and $\delta N(\delta\mathrm{f}) = \delta h(\delta\mathrm{f})$. The translation, rotational and scale-dependent terms in (24) have already been defined and explained in the previous section.

## 4 What should we use for $\delta$a ?

An important issue that remains to be clarified, in the context of geoid transformation, is the proper

usage of the term $\delta N(\delta \mathrm{a}) = -W\,\delta \mathrm{a}$ which gives the geoid height variation due to the difference $\delta \mathrm{a} = \mathrm{a}' - \mathrm{a}$ in the numerical values of the semi-major axis for the reference ellipsoids adopted by the frames GRF1 and GRF2.

In general, the length of the semi-major axis of the reference ellipsoid attached to GRF2 can be expressed as

$$\mathrm{a}' = (1+\delta s)\,\mathrm{a} + \delta\bar{\mathrm{a}} \tag{25}$$

where a is the length of the semi-major axis of the reference ellipsoid attached to GRF1, $\delta s$ is the scale change factor between the two frames, and $\delta\bar{\mathrm{a}}$ corresponds to the actual change of the *physical length* of the semi-major axis of the GRF2 ellipsoid with respect to the *physical length* of the semi-major axis of the GRF1 ellipsoid (see Fig. 2).

In this way, we have that

$$\delta \mathrm{a} = \mathrm{a}' - \mathrm{a} = \mathrm{a}\,\delta s + \delta\bar{\mathrm{a}} \tag{26}$$

and thus the geoid height variation term $\delta N(\delta \mathrm{a})$ becomes

$$\delta N(\delta \mathrm{a}) = -W\mathrm{a}\delta s - W\delta\bar{\mathrm{a}} \tag{27}$$

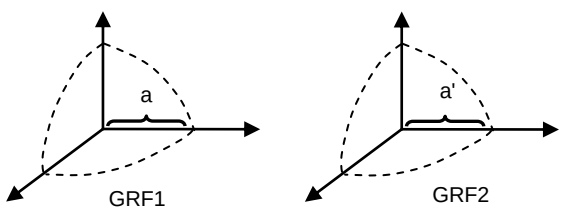


**Figure 2.** Each geodetic reference frame "gauges" the attached reference ellipsoid according to its own particular scale.

**Remark 2.** Let us consider again the special case where $t_x = t_y = t_z = 0$ and $\varepsilon_x = \varepsilon_y = \varepsilon_z = 0$, and additionally $\delta \mathrm{f} = 0$. Based on these assumptions, the extended similarity transformation model for geoid heights in (24) yields

$$N' - N = \delta N(\delta s) + \delta N(\delta \mathrm{a}) \tag{28}$$

Using (16) and (27), the last equation can be written in the equivalent form

$$N' = (1+\delta s)N - W\delta\bar{\mathrm{a}} \tag{29}$$

In contrast to the transformation formula obtained by the simple (non-extended) similarity transformation model in (18), the above result complies with geometrical intuition which dictates that the transformed geoid height should be determined by a simple re-scaling of the initial geoid height value *if* the underlying GRFs have the same origin and orientation *and* also use the same reference ellipsoid in terms of physical dimensions ($\delta \mathrm{f} = 0,\ \delta\bar{\mathrm{a}} = 0$). Note that the elimination of the "apparent" geoid variation term '$\mathrm{a}W\delta s$' that emerged in (18) has been inherently achieved by the inclusion of the term $\delta N(\delta \mathrm{a})$ as given in (27).

## 5 Summary – Open problems

When a GRS is used in practice via an established and accessible GRF, the adopted reference ellipsoid that is required to define and quantify several important geodetic quantities does not refer to an "ideal" scale unit (e.g. the light-based meter standard) but rather to the best scale which geodesists are able to reproduce by means of their current data, measurement techniques and optimal combination procedures (Soler and van Gelder 1987). Therefore, any GRF "detects" an attached reference ellipsoid, as well as every length-type quantity that depends on it (e.g. ellipsoidal height derived from known Cartesian coordinates with respect to a given ITRF), according to its own particular scale.

Taking into account the above considerations, we have investigated the problem of geoid height conversion between different GRFs by providing a general transformation model that incorporates the contribution of GRF scale variation on the relative size of the reference ellipsoids adopted by each datum. Specifically, if we know the seven similarity transformation parameters between two given GRFs, then the conversion of the geoid height from one GRF to another can be implemented through the formula

$$\begin{aligned} N' - N = {} & \delta N(t_x) + \delta N(t_y) + \delta N(t_z) \\ & + \delta N(\varepsilon_x) + \delta N(\varepsilon_y) \\ & + \delta N(\delta \mathrm{f}) + \delta N(\delta s, \delta \mathrm{a}) \end{aligned} \tag{30}$$

The critical point in the above model is the treatment of the last variation term, which contains the combined effect due to the GRF scale variation and the change of the semi-major axis of the reference ellipsoid. As explained in the previous sections, the combined term $\delta N(\delta s, \delta \mathrm{a}) = \delta N(\delta s) + \delta N(\delta \mathrm{a})$ can be expressed in the form

$$\delta N(\delta s, \delta \mathrm{a}) = (\mathrm{a}W + N)\,\delta s - W\delta \mathrm{a} \tag{31}$$

or, taking into account (26),

$$\delta N(\delta s, \delta \mathrm{a}) = (\mathrm{a}W + N)\,\delta s - W(\mathrm{a}\delta s + \delta \bar{\mathrm{a}}) \qquad (32)$$

where $\delta \bar{\mathrm{a}}$ is the change of the *physical length* of the semi-major axis of the reference ellipsoid.

In practice, there are two basic options for the implementation of the geoid height transformation model in (30). Both of these options relate to the evaluation of the term $\delta N(\delta s, \delta \mathrm{a})$ and they essentially correspond to choosing how to treat the physical size of the reference ellipsoid with respect to the underlying GRFs.

One alternative is to select $\delta \bar{\mathrm{a}} = 0$, which implies that the *physical length* of the semi-major axis of the reference ellipsoid is invariant within the underlying GRFs. In this case, we have

$$\delta N(\delta s, \delta \mathrm{a}) = N\delta s \qquad (33)$$

which is a negligible geoid correction for all purposes (i.e. less than 1 mm even for $\delta s$ = 10 ppm). Note, however, that all numerical calculations involving the semi-major axis of the reference ellipsoid *with respect to the GRF2 frame* (e.g. conversion of Cartesian coordinates to curvilinear coordinates and vice versa) should be made using the new value

$$\mathrm{a}' = (1 + \delta s)\,\mathrm{a} \qquad (34)$$

and not the initial value 'a' which is used for similar calculations with respect to the GRF1 frame; see also Soler and van Gelder (1987).

The other alternative for the evaluation of the term $\delta N(\delta s, \delta \mathrm{a})$ is to set $\delta \mathrm{a}$ = 0, which implies that the *same numerical value* for the semi-major axis of the reference ellipsoid is used in both frames GRF1 and GRF2. In this case, the geoid height variation term $\delta N(\delta s, \delta \mathrm{a})$ takes the form

$$\delta N(\delta s, \delta \mathrm{a}) = N\delta s + \mathrm{a}W\delta s \qquad (35)$$

As already mentioned, the magnitude of the above correction is quite significant and it must always be considered since the term 'a$W\delta s$' can reach more than 6 m for $\delta s$ = 1 ppm. Note that this alternative carries an *inherent change in the physical dimensions of the reference ellipsoid*, since from (26) we have that

$$\delta \mathrm{a} = 0 \;\Rightarrow\; \delta \bar{\mathrm{a}} = -\mathrm{a}\delta s \qquad (36)$$

In closing, let us add a final remark. In contrast to $\delta N(\delta \mathrm{a})$ given in (27), the term $\delta N(\delta \mathrm{f})$ which represents the geoid height variation due to the flattening change of the reference ellipsoid, is insensitive to a uniform GRF scale difference ($\delta s$) since the ellipsoid's flattening $\mathrm{f} = (\mathrm{a} - \mathrm{b})/\mathrm{a}$ does not depend on the scale unit of the underlying GRF.

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