Estimating crustal deformation parameters from geodetic data: Review of existing methodologies, open problems and new challenges

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Abstract

The study of crustal deformation using various types of geodetic data is a research topic whose practical importance needs hardly to be stressed, and its theoretical richness encompasses several scientific disciplines, including estimation theory, differential geometry, elasticity theory, geodynamics and physics. In this paper, an attempt is made to summarize the existing methodologies that are commonly applied in the geodetic practice for crustal deformation studies. Special emphasis is given on issues such as: (i) the definition and the estimability of frame-invariant quantities in time-dependent geodetic networks, (ii) the separation of rigid motion effects from actual body deformation changes, (iii) the problem of spatial and/or temporal interpolation of the crustal deformation field, and (iv) the separation of the total deformation field into a "horizontal" part and a "vertical" part. An assessment of the remaining open problems that exist within the currently used geodetic methodologies for crustal deformation analysis is also given, and finally a number of new challenges that are imposed by the availability of data types which are essentially continuous in space and/or time, is listed.

Keywords. Crustal deformation, estimation, interpolation, collocation, GPS, SAR

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1 Introduction

The assessment of reliable information concerning crustal motion is based on the fulfillment of criteria concerning a satisfactory signal-to-noise ratio and sufficient spatial and temporal resolution.

Resolution is directly related to the nature of the studied deformation, which may range from the slow and spatially smooth plate motion (e.g. Soler 1977) to the spatially more complex local deformation patterns (e.g. Dermanis et al. 1981, Kogan et al. 2000), or to temporally more abrupt landslide-related deformations (e.g. Prescott 1981).

Signal-to-noise ratio, after a long period of unsatisfactory performance which required long monitoring periods to detect persistent secular deformations, has finally met the criteria for the detection of deformations of smaller magnitude within shorter time periods. Since this growth in precision comes essentially from GPS observations, some quality assessment questions remain open, concerning the difference between realistic and nominal accuracy measure and the danger of interpreting as deformation other systematic influences on the relevant data (Davis et al. 1989).

The main breakthrough has taken place in the increase of temporal resolution to at least daily values, which exceeds far beyond the resolution required for the detection of steady crustal deformations. Apart from its role in detecting abrupt deformations, related e.g. to seismic events, the high temporal resolution provides a tool for hopefully resolving the quality assessment problem.

With respect to spatial resolution, the cost of maintaining a dense network of permanent GPS stations or of often repeated GPS surveys, turns our interest to different sources of densification data, such as laser scanning and SAR interferometry (Crippa et al. 2002, Hanssen 2005, Wright et al. 2004, Lohman and Simons 2005).

The advancement in observational performance, typically resolves some theoretical problems and on the other hand poses some new challenges, as more elaborate models and data analysis techniques become necessary.

A critical problem that remains beyond the reach of observational advancement is the obligatory restriction of the data on the two-dimensional physical surface of the earth while crustal deformation is by nature a three-dimensional physical process. Only a parallel separation of geophysical deformation models into a horizontal and vertical part may overcome this problem to a certain extend.

While spatial or temporal resolution remains beyond the point when it may be operationally considered as a continuous observational process, interpolation is the most crucial issue in data analysis methodologies. Such interpolation may be either direct or "masked" under a stochastic prediction model, where smooth deformation is modeled by a stochastic process (Dermanis 1976, 1988). In this case, correlation is a measure of similarity between nearby displacements which secure a smooth deformation field, or in fact a piecewise smooth field between discontinuities which are either spatial (faults) or temporal (seismic events).

A usual prerequisite for a successful interpolation using the stochastic tool of prediction, is trend removal, so that the reduced field can be indeed modeled by a zero-mean stochastic process. This trend removal is based partly on a redefinition of the dynamic behavior of the reference system. The original system must be replaced by an optimal one, which absorbs, as much as possible, from the apparent displacements reflecting an inappropriate reference system definition and not crustal deformation itself. One may argue that proper deformation parameters are invariant under changes of the reference system (Dermanis 1985, Xu et al. 2000), but this is true once continuous information is available. The interpolation that provides such continuous information may well be system dependent, thus leading to different values for the theoretically invariant deformation parameters.

We shall present here a whole armory of possible interpolation approaches for various possible data availability situations with the warning that the proper choice of data analysis methodology is a critical issue, which calls for a deep understanding of the "geophysical" part of the studied phenomena and necessitates a dialectic relation between geodesy and geophysics. A crucial part in this dialogue is that the final product resulting from the analysis of geodetic data, is accompanied by an, as realistic as possible, description of its overall quality and the assumptions involved in its derivation.

2 Deformation parameters in relation to the type of available data

Deformation parameters are one of the three elements involved in models of the "mechanics of continuous media", the other two being the acting forces and the parameters describing the behavior of the deforming material (Sokolnikoff 1956, Sansò 1982). As is the usual case in physics, the relevant models are differential equations describing the local aspects of the problem at each point of the deforming body. These are the well known "constitutional equations", also known with the less formal term "stress-strain relations". The geodetic input in this type of analysis is the deformation function, relating the coordinates of any material point at any epoch with its identifying coordinates, which may be its coordinates at a particular reference epoch (Lagrangean approach) or its present coordinates (Eulerian approach), or even its coordinates in a reference "undeformed" state that is not realized at any particular epoch.

If

$$\mathbf{x} = \boldsymbol{\psi}(P,t) = \boldsymbol{\psi}_t(P) \tag{1}$$

denotes the Cartesian coordinates of point P at epoch t, we may use the coordinates $\mathbf{x}_0 = \boldsymbol{\psi}(P,t_0) = \boldsymbol{\psi}_{t_0}(P)$ as independent parameters and use the point identifying relation $P = \boldsymbol{\psi}_{t_0}^{-1}(\mathbf{x}_0)$ in order to describe the coordinate at any point and epoch by a deformation function

$$\mathbf{x} = \mathbf{f}(\mathbf{x}_0,t) \equiv \boldsymbol{\psi}(\boldsymbol{\psi}_{t_0}^{-1}(\mathbf{x}_0),t) \tag{2}$$

In the constitutional relations enters not the deformation function itself but its local linear approximation by the corresponding "tangent mapping" described by the "deformation gradient" matrix

$$\mathbf{F}(\mathbf{x}_0,t) = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0}(\mathbf{x}_0,t) \tag{3}$$

Of course the particular value of the gradient matrix depends on the chosen instantaneous reference system at the epochs t_0 and t, and since t is an independent parameter, on the choice of the "dynamic" reference system $(O(t),\vec{\mathbf{e}}(t))$ which is a temporally smooth choice of reference systems for each particular epoch having $O(t)$ as the point of origin and $\vec{\mathbf{e}}(t) = [\vec{e}_1(t)\,\vec{e}_2(t)\,\vec{e}_3(t)]$ as the orthonormal vector basis.

A frame invariant description involves either numerical or physical invariants which are functions of the elements of the deformation gradient $\mathbf{F}$ (Boucher 1980, Sansò 1982, Dermanis 1981, 1985). The numerical invariants are eventually functions of the singular values λ_1, λ_2, λ_3 of $\mathbf{F}$ which are roots of the common eigenvalues of either the right Cauchy strain tensor matrix $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$ or the left Cauchy strain tensor matrix $\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2$. Here $\mathbf{U} = \mathbf{C}^{1/2}$ is the right stretch tensor matrix and $\mathbf{V} = \mathbf{B}^{1/2}$ is the left stretch tensor matrix. They are both symmetric matrices appearing in the polar decomposition of the deformation gradient $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$, where $\mathbf{R}$ is the orthogonal rotation matrix. All the above matrices represent in the chosen frame tensors bearing corresponding names.

More popular in classical deformation analysis is the strain tensor (Sansò 1982, Dermanis 1981)

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) \tag{4}$$

or even its "infinitesimal approximation"

$$\mathbf{E}_{\text{inf}} = \frac{1}{2}(\mathbf{J} + \mathbf{J}^T) \tag{5}$$

based on neglecting 2nd order terms in the displacement gradient $\mathbf{J} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}_0} = \frac{\partial(\mathbf{x} - \mathbf{x}_0)}{\partial \mathbf{x}_0} = \mathbf{F} - \mathbf{I}$. The situation becomes more transparent with the use of the singular value decomposition (SVD)

$$\mathbf{F} = \mathbf{Q}^T\mathbf{L}\mathbf{P} \tag{6}$$

for which it can be easily verified that $\mathbf{C} = \mathbf{P}^T\mathbf{L}^2\mathbf{P}$, $\mathbf{B} = \mathbf{Q}^T\mathbf{L}^2\mathbf{Q}$, $\mathbf{U} = \mathbf{P}^T\mathbf{L}\mathbf{P}$, $\mathbf{V} = \mathbf{Q}^T\mathbf{L}\mathbf{Q}$ and $\mathbf{R} = \mathbf{Q}^T\mathbf{P}$. From the diagonalizations $\mathbf{C} = \mathbf{P}^T\mathbf{L}^2\mathbf{P}$ and $\mathbf{B} = \mathbf{Q}^T\mathbf{L}^2\mathbf{Q}$ it follows that the diagonal elements $(\mathbf{L}^2)_{kk} = \lambda_k^2$ are the common eigenvalues of the symmetric matrices $\mathbf{C}$ and $\mathbf{B}$, $\mathbf{P}^T = [\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3]$ is the matrix with columns the eigenvectors $\mathbf{p}_k$ of $\mathbf{C}$ ($\mathbf{C}\mathbf{p}_k = \lambda_k^2\mathbf{p}_k$) and $\mathbf{Q}^T = [\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3]$ is the matrix with columns the eigenvectors $\mathbf{q}_k$ of $\mathbf{B}$ ($\mathbf{B}\mathbf{q}_k = \lambda_k^2\mathbf{q}_k$). For the corresponding physical interpretation we note that a change in the instantaneous reference system bases from $\vec{\mathbf{e}}(t_0)$ and $\vec{\mathbf{e}}(t)$ to $\vec{\mathbf{e}}'(t_0) = \vec{\mathbf{e}}(t_0)\mathbf{P}^T$ and $\vec{\mathbf{e}}'(t) = \vec{\mathbf{e}}(t)\mathbf{Q}^T$ is accompanied by coordinate transformations $\mathbf{x}'_0 = \mathbf{P}\mathbf{x}_0$ and $\mathbf{x}' = \mathbf{Q}\mathbf{x}$, so that the deformation gradient in the new systems becomes

$$\mathbf{F}' = \frac{\partial \mathbf{x}'}{\partial \mathbf{x}'_0} = \frac{\partial \mathbf{x}'}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} \frac{\partial \mathbf{x}_0}{\partial \mathbf{x}'_0} = \mathbf{Q}\mathbf{F}\mathbf{P}^T = \mathbf{L} \tag{7}$$

This means that, in the new systems, deformation does not change locally the directions of lines along the base vectors but only their lengths by a factor of λ_k. These directions are the principal directions which are physical invariants. The angular parameters defining the directions of the principal axes are not numerical invariants since they relate also to the axes of the reference system; see Dermanis (1981, 1985), Xu et al. (2000). The matrix $\mathbf{P}$ rotates the axes of the reference system at the reference epoch t_0 to the directions of the principal axes and $\mathbf{Q}$ does the same to the axes of the reference system at epoch t. The angular parameters defining the matrices $\mathbf{P}$ or $\mathbf{Q}$ are not numerical invariants. When the reference systems are changed their values are adjusted accordingly in order to define the same directions of the principal axes. Any other function of $\mathbf{F}$, which is a numerical invariant, is necessarily a function of the singular values of $\mathbf{F}$.

A different type of deformation analysis concerns the "rate of deformation" which is related to the time derivative of the deformation gradient $\dot{\mathbf{F}} = \frac{d}{dt}\mathbf{F}$, or the time derivatives $\dot{\varphi} = \frac{d}{dt}\varphi$ of numerical invariant functions $\varphi(\mathbf{F}) = \varphi(\lambda_1.\lambda_2, \lambda_3)$.

The available data are typically coordinates of discrete points at discrete epochs $\mathbf{x}_{i,t_k} = \mathbf{x}(\mathbf{x}_{0i}, t_k)$, for points P_i identified by their coordinates $\mathbf{x}_{0i}$ at the reference epoch t_0. In the case that both epochs t_0 and t_k are observation epochs, the calculation of the deformation gradient $\mathbf{F}$ at any point for the epoch t_k requires spatial interpolation which will provide spatial coordinates $\mathbf{x}_{t_k} = \mathbf{x}(\mathbf{x}_0, t_k)$ for "points" $\mathbf{x}_0$ other than the data points $\mathbf{x}_{0i}$. Temporal interpolation is required whenever deformation rates are required and the time derivative $\dot{\mathbf{F}}$ must be calculated. In addition it is required whenever observations at different points $\mathbf{x}_{0i}$ are performed not simultaneously (i.e. in a very short time period) but rather at different epochs t_j, in which case the observed coordinates must be reduced to a common epoch t_k.

The performance of the interpolation procedure is strongly related to the spatial and temporal resolution of the observations in relation to the spectral content of the relevant physical processes. In addition interpolation must be performed independently within separate regions and time intervals which are separated by discontinuities, i.e. faults and seismic events; for a finite element approach to the geodetic computation of two- and three-dimensional deformation parameters, see Dermanis and Grafarend (1992).

The establishment of permanent GPS networks has practically removed the need for temporal interpolation, since daily coordinate solutions provide a very satisfactory resolution for crustal deformations that evolves very slowly in time (Wernicke et al. 2004, Pietrantonio and Riguzzi 2004). However the spatial interpolation remains necessary, since densification of the observing networks raises considerably the relevant cost, while coordinates of nearby stations are affected by similar systematic errors which tend to be ultimately interpreted erroneously as additional contributions to deformation.

3 Definition of the reference frames as a tool of trend removal before interpolation.

Instead of spatially interpolating discrete coordinates $\mathbf{x}_i$ it is more convenient and effective to interpolate the corresponding displacements $\mathbf{u}_i = \mathbf{x}_i - \mathbf{x}_{0i}$ and to compute the deformation gradient $\mathbf{F} = \mathbf{I} + \mathbf{J}$ from the displacement gradient $\mathbf{J} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}_0}$. The available displacements refer to an externally defined reference system (e.g. ITRFxx) which may be sufficient for global studies of plate motions, but not so for studying local deformation. Note that, in order to avoid the impact of reference frame definition on the computed displacement vectors, any redundant a-priori constraints should be first removed from the individually adjusted coordinate sets, before applying any trend removal and deformation analysis procedure. The displacements with respect to such an external reference frame reflect at the same time the intrinsic deformation of the observing network as well as its total rigid motion (displacement and rotation) with respect to the adopted reference system.

Although it is possible to perform trend removal (absorbing mainly the "rigid-body motion" effects such as the Eulerian motion of tectonic plates) and subsequent spatial interpolation to any local displacement field that is expressed with respect to a common global frame, we shall here discuss a particular strategy that is capable of separating the actual deformation from the rigid-body motion component by introducing an intrinsic to the network reference system. This separation will on one hand provide estimates of the motion of the network as a whole and on the other hand provide intrinsic displacements which are smaller and thus more convenient for interpolation, in particular when this is carried out by statistical tools as prediction of functionals related to a zero-mean stochastic process. Of course trend removal can be achieved also by other means in order to provide a known mean displacement function. The definition of a network-intrinsic reference system simply removes a significant part of the trends which has nothing to do with local physical processes but rather has to do with the arbitrary use of a particular reference system serving a different purpose.

When the available data are discrete in time, the definition of the reference system in each epoch is based on constraints which are introduced for the coordinate differences $\Delta\mathbf{x}_i(t_k) = \mathbf{x}_i(t_k) - \mathbf{x}_i(t_{k-1})$. In the case of classical geodetic observations the constraints were incorporated into the network adjustment as "inner constraints" on the unknown parameters, by using the coordinates of the previous epoch as approximate coordinates for the linearization of the observation equations at the current epoch. Nowadays the system defining conditions may be introduced a posteriori by requiring that *(a)* the "barycenter" of the network is preserved

$$\mathbf{m}(t_k) \equiv \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i(t_k) = \mathbf{m}(t_{k-1}) \equiv \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_i(t_{k-1}) \tag{8}$$

and *(b)* that the vector of "relative angular momentum" of the network vanishes

$$\mathbf{h}_R \equiv \sum_{i=1}^{N} [\mathbf{x}_i(t_{k-1})\times]\,\mathbf{v}_i(t_{k-1}) = \tag{9}$$

$$= \sum_{i=1}^{N} [\mathbf{x}_i(t_{k-1})\times]\,\frac{\mathbf{x}_i(t_k) - \mathbf{x}_i(t_{k-1})}{t_k - t_{k-1}} = \mathbf{0} \tag{10}$$

These two conditions define respectively the position and orientation of the intrinsic to the network reference system, which is uniquely defined by the two equivalent conditions

$$\sum_{i=1}^{N} \Delta\mathbf{x}_i(t_k) = \mathbf{0} \tag{11}$$

and

$$\sum_{i=1}^{N} [\mathbf{x}_i(t_{k-1})\times]\,\Delta\mathbf{x}_i(t_k) = \mathbf{0} \tag{12}$$

If we remove the epoch t_k and set $\mathbf{x}_i(t_{k-1}) = \mathbf{x}_i^0$, we recognize 6 of the 7 inner constraints introduced by Meissl for the adjustment of so called "free networks" (Dermanis 2002). The 7th condition relates to the definition of scale in the network, by requiring that the quadratic mean distance of the network points from their barycenter remains constant

$$L(t_k) \equiv \sqrt{\frac{1}{N}\sum_{i=1}^{N} ||\mathbf{x}_i(t_k) - \mathbf{m}(t_k)||^2} = L(t_{k-1}) \tag{13}$$

which under the barycenter preservation condition and neglecting second order terms in the small displacements $\Delta\mathbf{x}_i(t_k)$ simplifies to

$$\sum_{i=1}^{N} \mathbf{x}_i(t_{k-1})^T \Delta\mathbf{x}_i(t_k) = 0 \tag{14}$$

However, even if the maintenance of a constant unit of length in the observations is questionable, the incorporation of the last condition into the analysis leads to the interpretation of a common increase or decrease of the network size as a whole as simply variation of the unit of scale. The opposite possibility has the danger of interpreting variations in the scale of measuring unit of length (in fact unit of time) as a change in the overall size of the network. In our opinion one should use both approaches and interpret scale changes as deformation only when their magnitude is well above the influence of unit instability in the system of reference clocks used in the observations.

Remark. The adoption of a network-intrinsic reference frame for the determination of local displacement vectors $\Delta\mathbf{x}_i(t_k)=\mathbf{x}_i(t_k)-\mathbf{x}_i(t_{k-1})$ between different epochs in a common spatial reference system will provide "snapshots" of the deformation field, as viewed from an observer who is continuously situated at the "center of mass" of the deforming network. Note that the center of mass of a terrestrial deformable network is physically varying in space (with respect to other stable points outside the network), due to the alteration of its physical shape and/ size caused by its underlying dynamical behavior. In this way, the continuous fixation of the network barycenter, though the use of an inner constraint adjustment framework via a common set of initial approximate coordinates at some reference epoch, still gives a "relative" picture of the actual deformation field, since it will mask the part that causes variations in the geometrical position of the network barycenter.

Since the definition of the reference system epoch by epoch using an analog of the Meissl constraints we may characterize the relevant procedure as a "Meissl ladder", which is a discrete approximation to the definition of the reference system under time-continuous data (Dermanis 2002).

In the case of time-continuous coordinates $\mathbf{x}_i(t)$ a transformation to an optimal frame is realized by a time dependent change of reference system

$$\tilde{\mathbf{x}}_i(t)=\mathbf{R}(t)\mathbf{x}_i(t)+\mathbf{c}(t) \tag{15}$$

which satisfies two optimality conditions. The preservation of the "barycenter" of the network

$$\tilde{\mathbf{m}}\equiv\frac{1}{N}\sum_{i=1}^{N}\tilde{\mathbf{x}}_i=\text{const.} \tag{16}$$

determines the translation parameters, which in view of $\tilde{\mathbf{m}}=\mathbf{Rm}+\mathbf{c}$, become $\mathbf{c}=\tilde{\mathbf{m}}-\mathbf{Rm}$ and the system transformation reduces to

$$\tilde{\mathbf{x}}_i=\mathbf{R}(\mathbf{x}_i-\mathbf{m})+\tilde{\mathbf{m}} \tag{17}$$

where $\mathbf{m}$ are the coordinates of the center of mass of the network and $\tilde{\mathbf{m}}$ an arbitrary constant vector, e.g. $\tilde{\mathbf{m}}=\mathbf{m}(t_0)$. The parameters of the rotation matrix $\mathbf{R}$ are determined by the "discrete Tisserand condition" that the vector of "relative angular momentum" of the network vanishes (Dermanis 2000, 2001)

$$\tilde{\mathbf{h}}_R\equiv\sum_{i=1}^{N}[(\tilde{\mathbf{x}}_i-\tilde{\mathbf{m}})\times]\frac{d(\tilde{\mathbf{x}}_i-\tilde{\mathbf{m}})}{dt}=\mathbf{0} \tag{18}$$

This is a discrete version of the condition for Tisserand axes for the earth, where the integral over the whole earth masses is replaced by a sum over the network points considered as point masses with the mass each. Introducing the relative rotation vector $\boldsymbol{\omega}$ as the axial vector of the antisymmetric matrix $[\boldsymbol{\omega}\times]=\frac{d}{dt}(\mathbf{R}^T)\mathbf{R}$, it can be shown that

$$\tilde{\mathbf{h}}_R=\mathbf{R}(-\mathbf{C}\boldsymbol{\omega}+\mathbf{h}_R)=\mathbf{0} \tag{19}$$

where

$$\mathbf{C}=-\sum_{i=1}^{N}[(\mathbf{x}_i-\mathbf{m})\times]^2 \tag{20}$$

is the inertia matrix and

$$\mathbf{h}_R=\sum_{i=1}^{N}[(\mathbf{x}_i-\mathbf{m})\times]\frac{d(\mathbf{x}_i-\mathbf{m})}{dt} \tag{21}$$

is the relative angular momentum vector of the network, with respect to original reference system. Thus

$$\boldsymbol{\omega}=\mathbf{C}^{-1}\mathbf{h}_R \tag{22}$$

and the parameters of the rotation matrix $\mathbf{R}$ are determined from the solution of the geometric Euler differential equations which are the axial part of the antisymmetric matrix relation $[\boldsymbol{\omega}\times]=\frac{d}{dt}(\mathbf{R}^T)\mathbf{R}$. To give a specific example consider the parameterization $\mathbf{R}=\mathbf{R}_m(\theta_m)\mathbf{R}_k(\theta_k)\mathbf{R}_n(\theta_n)$ by means of consecutive rotations around the n, k and m axis, in which case the angles $\boldsymbol{\theta}=[\theta_n\theta_k\theta_m]^T$ are determined by the solution of the differential equations

$$\frac{d\boldsymbol{\theta}}{dt}=\begin{bmatrix}\mathbf{i}_n^T\\ \mathbf{i}_k^T\mathbf{R}_n(\theta_n)\\ \mathbf{i}_m^T\mathbf{R}_k(\theta_k)\mathbf{R}_n(\theta_n)\end{bmatrix}\boldsymbol{\omega} \tag{23}$$

where $\mathbf{i}_p$ denotes the p[th] row of the identity matrix $\mathbf{I} = [\mathbf{i}_1 \mathbf{i}_2 \mathbf{i}_3]$. We may choose $\boldsymbol{\theta}(t_0) = \mathbf{0}$ which, in combination with $\tilde{\mathbf{m}} = \mathbf{m}(t_0)$, makes the new and the original frames identical at the reference epoch t_0. The resulting angles $\boldsymbol{\theta}(t)$ describe the rotation of the network as a whole while its translational motion $\mathbf{c}(t)$ is given by

$$\mathbf{c} = \tilde{\mathbf{m}} - \mathbf{R}(\boldsymbol{\theta})\mathbf{m} . \tag{24}$$

Further analysis will make use of the transformed coordinate functions $\tilde{\mathbf{x}}(t)$ and displacements $\tilde{\mathbf{u}}(t) = \tilde{\mathbf{x}}(t) - \tilde{\mathbf{x}}(t_0) = \tilde{\mathbf{x}}(t) - \tilde{\mathbf{x}}_0$, which we will further on denote by $\mathbf{x}(t)$ and $\mathbf{u}(t)$ for the sake of simplicity.

Let us also note that, apart from the trend removal related to linear (global) trends in the spatial displacement field, other seasonal (annual, semi-annual) station-dependent signals can also be removed from the available coordinate values, before applying any further interpolation procedure for local deformation analyses. Such information is nowadays often available and accessible through the analysis of GPS time series in various types of regional, continental and global networks.

4 Three-dimensional or two-dimensional deformation?

The physical process of crustal deformation is unquestionably three-dimensional (3D) and any deformation parameters entering the constitutional equations should naturally refer to the 3D case (Voosoghi 2000). On the other hand, there is a long tradition of separating vertical from horizontal information, leading to the study of a two-dimensional (2D) deformation of an abstract and somewhat unnatural entity, consisting of the projections of the material points on a horizontal plane. The fact is that available information is restricted on the 2D surface of the earth. In order to derive 3D deformation parameters, we need not only to spatially interpolate in the horizontal sense, but also to *extrapolate* in the vertical sense. Interpolation is always a less subject to errors process compared to extrapolation and it can be virtually overcome by further densification.

There have been attempts to derive 3D deformation from surface data by forming tetrahedrons and using a finite element approach where homogeneous deformation is assumed within each tetrahedron, which is an extension of the similar 2D approach where triangles are formed (Dermanis and Grafarend 1992, Voosoghi 2000). This approach may have some relevance when the central point of the tetrahedron is located higher than the external points, so that deforming earth masses are located inside the tetrahedron. In the opposite case one simply derives deformation for tetrahedrons consisting of "thin air". In any case, the low variation of surface altitude with respect to the usual mean distance between neighboring network stations leads to tetrahedrons with vertical dimension considerably smaller than the horizontal one and the homogeneity assumption hardly provides meaningful information in the vertical sense.

A proper unified approach which does not neglect, or treats separately, temporal height variation, is to study the deformation of the 2D earth surface as embedded in 3D space and to relate it to the 3D deformation of the crust. The extension of surface deformation to the only physically meaningful 3D crustal deformation is an improperly posed problem and can be solved only with of additional assumptions of geophysical nature. Geodesy can only provide boundary conditions on the surface of the earth for the elements of the deformation gradient of the actual crustal deformation.

The study of the deformation of the surface of the earth as embedded in 3D space is very complicated because the earth surface is not an Euclidean but rather a Riemannian manifold. In the literature for the mechanics of continuous media the use of deformation within Riemannian spaces can be found in the treatment of relativistic elasticity, which has its own particularities. We shall present the treatment of the surface deformation exploiting 3D coordinate information elsewhere. More details on the classical approach where 2D deformation is studied separately from height variation information can be found in Biagi and Dermanis (2005).

For the sake of generality, the analysis of the available data will be presented hereon, in a general way that it can incorporate both the 2D and the 3D case.

5 Interpolation methods: Interpretation and comparison

There are two basic approaches to interpolation, not completely independent from each other (Dermanis 1976, 1988; Dermanis and Rossikopoulos 1988). Analytic interpolation models the interpolating function by means of a number of unknown parameters, with most popular choice linear models of the form

$$f(P) = \sum_i a_i \; f_i(P) \tag{25}$$

where $f_i(P)$ are known base functions (e.g. algebraic polynomials, trigonometric polynomials, spline-type functions, etc.), and a_i are the unknown coefficients. For temporal interpolation the spatial variable P will be replaced by time t, while also simultaneous spatial-temporal interpolation is also possible. Discrete data are available of the form

$$b_k = f(P_k) + v_k = \sum_i a_i \, f_i(P_k) + v_k \tag{26}$$

polluted by the observation errors v_k. The objective is to determine parameter estimates $\hat{a}_i$ so that estimates of function values can be obtained at any point P by $\hat{f}(P) = \sum_i \hat{a}_i f_i(P) = \mathbf{f}_P^T \hat{\mathbf{a}}$.

There are two basic optimality criteria that can be also combined into a third hybrid approach. When the number of base functions is less than the number of data then the least-squares criterion $\mathbf{v}^T \mathbf{W} \mathbf{v} =$ min leads to the smoothing-interpolation solution

$$\hat{\mathbf{a}} = (\mathbf{G}^T \mathbf{W} \mathbf{G})^{-1} \mathbf{G}^T \mathbf{W} \mathbf{b} \tag{27}$$

where $\mathbf{G}$ is the matrix with elements $G_{ki} = f_i(P_k)$. When the number of base functions is more than the number of data, then a minimum norm criterion $\mathbf{a}^T \tilde{\mathbf{W}} \mathbf{a} = \min$ leads to the exact-interpolation solution

$$\hat{\mathbf{a}} = \tilde{\mathbf{W}}^{-1} \mathbf{G}^T (\mathbf{G} \tilde{\mathbf{W}}^{-1} \mathbf{G}^T)^{-1} \mathbf{b} \tag{28}$$

The interpolated function values in the last case are

$$\hat{f}(P) = \mathbf{f}_P^T \hat{\mathbf{a}} = \mathbf{f}_P^T \tilde{\mathbf{W}}^{-1} \mathbf{G}^T (\mathbf{G} \tilde{\mathbf{W}}^{-1} \mathbf{G}^T)^{-1} \mathbf{b} \tag{29}$$

The hybrid alternative refers to the criterion $\mathbf{v}^T \mathbf{W} \mathbf{v} + \mathbf{a}^T \tilde{\mathbf{W}} \mathbf{a} = \min$, with solution

$$\begin{aligned} \hat{\mathbf{a}} &= (\mathbf{G}^T \mathbf{W} \mathbf{G} + \tilde{\mathbf{W}})^{-1} \mathbf{G}^T \mathbf{W} \mathbf{b} \\ &= \tilde{\mathbf{W}}^{-1} \mathbf{G}^T (\mathbf{G} \tilde{\mathbf{W}}^{-1} \mathbf{G}^T + \mathbf{W}^{-1})^{-1} \mathbf{b} \end{aligned} \tag{30}$$

$$\mathbf{C}_{\hat{\mathbf{a}}} = (\mathbf{G}^T \mathbf{W} \mathbf{G} + \tilde{\mathbf{W}})^{-1} (\mathbf{G}^T \mathbf{W} \mathbf{G}) (\mathbf{G}^T \mathbf{W} \mathbf{G} + \tilde{\mathbf{W}})^{-1} \tag{31}$$

The interpolated values take in this case the form

$$\hat{f}(P) = \mathbf{f}_P^T \hat{\mathbf{a}} = \mathbf{f}_P^T \tilde{\mathbf{W}}^{-1} \mathbf{G}^T (\mathbf{G} \tilde{\mathbf{W}}^{-1} \mathbf{G}^T + \mathbf{W}^{-1})^{-1} \mathbf{b} =$$

$$= \mathbf{k}_P^T (\mathbf{K} + \mathbf{W}^{-1})^{-1} \mathbf{b} \tag{32}$$

It is possible to separate a factor α from the matrix $\tilde{\mathbf{W}} = \alpha \tilde{\mathbf{W}}_0$ in which case the criterion $\mathbf{v}^T \mathbf{W} \mathbf{v} + \alpha \mathbf{a}^T \tilde{\mathbf{W}}_0 \mathbf{a} = \min$ contains an additional regularization parameter α and the above solution is known as *Tikhonov regularization*.

At the antipodes of the deterministic analytical interpolation lies the interpolation by means of stochastic prediction, where the unknown function is assumed to be a zero-mean random field (stochastic process). The interpolating value $f(P)$ is a random variable which can be predicted from the sample values of the random variables $b_k = f(P_k) + v_k$ or $b_k = f(P_k)$. The minimum mean square prediction error leads to the well known (collocation) solution

$$\hat{f}(P) = \mathbf{c}_P^T \mathbf{C}_\mathbf{b}^{-1} \mathbf{b} = \mathbf{c}_P^T (\mathbf{C}_\mathbf{s} + \mathbf{C}_\mathbf{v})^{-1} \mathbf{b} \tag{33}$$

where $\mathbf{C}_\mathbf{b} = \mathbf{C}_\mathbf{s} + \mathbf{C}_\mathbf{v}$ is the covariance matrix of the observations, $\mathbf{C}_\mathbf{v}$ is the noise covariance matrix, $(\mathbf{C}_\mathbf{s})_{jk} = \sigma(f(P_j), f(P_k)) = C(P_j, P_k)$ and $(\mathbf{c}_P)_k = \sigma(f(P), f(P_k)) = C(P, P_k)$,
where $C(P, Q)$ is the covariance function of the random field.

The equations (32) and (33) have a striking similarity, which is by no means accidental. Assuming a diagonal matrix $\tilde{\mathbf{W}}$ for the sake of simplicity, the matrix appearing in (32) have elements

$$K_{jk} = \sum_i \frac{1}{\tilde{W}_{ii}} f_i(P_j) \, f_i(P_k) = k(P_j, P_k) \tag{34}$$

$$(\mathbf{k}_P)_k = \sum_i \frac{1}{\tilde{W}_{ii}} f_i(P) \, f_i(P_k) = k(P, P_k) \tag{35}$$

where $k(P, Q) = \sum_i \frac{1}{W_{ii}} f_i(P) \, f_i(Q)$ is a "reproducing kernel" in the set of all linear combinations of the base functions, whose number may even be infinite. It suffices to select $\frac{1}{\tilde{W}_{ii}} = \sigma^2(a_i)$, interpreting the coefficients as random variables (weight=reciprocal variance!), so that the reproducing kernel becomes in the light of covariance propagation identical with the covariance function

$$C(P,Q)=\sigma(f(P),\, f(Q))=\sum_i \frac{\partial f(P)}{\partial a_i}\frac{\partial f(Q)}{\partial a_i}\sigma^2(a_i)=$$

$$=\sum_i \sigma^2(a_i)\, f_i(P)\, f_i(Q)=k(P,Q) \tag{36}$$

Therefore, minimum-norm/least-squares (hybrid) becomes numerically equivalent to minimum mean square error stochastic prediction (Dermanis 1976).

Although the estimates in both cases are numerically the same, their corresponding error assessment differs. In the deterministic case of hybrid interpolation, only the influence of the measurement errors can be taken into account to obtain the covariance of interpolated values

$$\sigma(\hat f(P),\, \hat f(Q))=\mathbf{f}_P^T\mathbf{C}_{\hat{\mathbf{a}}}\mathbf{f}_Q=$$

$$=\mathbf{k}_P^T(\mathbf{K}+\mathbf{C}_\mathbf{v})^{-1}\mathbf{C}_\mathbf{v}(\mathbf{K}+\mathbf{C}_\mathbf{v})^{-1}\mathbf{k}_Q \tag{37}$$

In the case of stochastic prediction the covariance of the observed function values must be taken also into account. Assuming that the observation noise is uncorrelated with the random field, the covariance of the interpolated values is

$$\sigma(\hat f(P),\, \hat f(Q))=\mathbf{c}_P^T\mathbf{C}_\mathbf{b}^{-1}\mathbf{c}_Q=\mathbf{c}_P^T(\mathbf{C}_\mathbf{s}+\mathbf{C}_\mathbf{v})^{-1}\mathbf{c}_Q=$$

$$=\mathbf{c}_P^T(\mathbf{C}_\mathbf{s}+\mathbf{C}_\mathbf{v})^{-1}\mathbf{C}_\mathbf{v}(\mathbf{C}_\mathbf{s}+\mathbf{C}_\mathbf{v})^{-1}\mathbf{c}_Q+$$

$$+\mathbf{c}_P^T(\mathbf{C}_\mathbf{s}+\mathbf{C}_\mathbf{v})^{-1}\mathbf{C}_\mathbf{s}(\mathbf{C}_\mathbf{s}+\mathbf{C}_\mathbf{v})^{-1}\mathbf{c}_Q \tag{38}$$

Taking into account $\mathbf{C}_\mathbf{s}=\mathbf{K}$, $\mathbf{c}_P=\mathbf{k}_P$ and $\mathbf{c}_Q=\mathbf{k}_Q$, we can see that the propagated covariance in the stochastic approach contains an additional term (last line of Eq. (38)) expressing the uncertainty in the interpolation itself. Both approaches have their disadvantages. In the deterministic approach we assume that the chosen model (choice of type and number of base functions) describes perfectly the physical reality (no interpolation error!), which is not always correct. Hypothesis testing may be a helpful tool for the elimination of non-significant model parameters in this case, but the problem of choosing the proper type of base functions (e.g. polynomials, trigonometric functions, etc.) remains open. Even the same type of base functions expressed with the help of, e.g., plane horizontal coordinates $f_i(P)=f_i(x,y)$, gives different base functions for different choices of the coordinate system involved. As a consequence, the resulting interpolating function is not invariant under changes of the used coordinate system.

In the stochastic approach we assume that the chosen covariance function describes perfectly the stochastic behavior (e.g. smoothness) of the random field. However, the choice of the covariance function is to a certain degree arbitrary, since we do not have a sufficiently large data sample to secure a reliable empirical estimate. An advantage of the stochastic interpolation approach is its invariance under reference system transformations, when the covariance function depends only on the distance between the two relevant points $C(P,Q)=C(|PQ|)=C(r_{PQ})$.

In addition to the analytical exact least-squares interpolation and to the hybrid interpolation (with a deterministic or a stochastic interpretation), it is possible to combine the two in a single approach using the model

$$f(P)=\sum_i a_i\ f_i(P)\ +\ s(P) \tag{39}$$

where the function is decomposed into an analytic deterministic trend and an additional stochastic part. The model parameters $\{a_i\}$ are free to take any values, since they do not participate in the optimization principle. The assumption of a zero-mean random field for the validity of the minimum mean square error prediction requires the removal of the basic trend from the initial observed values, either simultaneously as seen in the above model, or a priori by a preceding analytical least-squares interpolation with a small number of base functions.

6 Implementation strategies for the interpolation of displacement fields

In general, spatial and/or temporal interpolation is required when the available data are not continuous. By “continuous” we mean data with a sufficiently high sampling resolution which can fully recover the full spectrum of the underlying function. In such cases, there is no need for interpolation per se, apart from the removal of the measurement noise with a filtering technique which indirectly implies a type of smoothing interpolation. The resulting filtered data are again discrete (with the same high sampling resolution) and not given in a continuous analytic form which allows the straightforward analytical computation of the required derivatives. Therefore, the elements of the deformation gradient or displacement gradient or velocity gradient matrix must be obtained by numerical differentiation techniques.

When the available data are not continuous, either in the time or the space domain, then the choice of the interpolation method depends on the characteristics of the underlying physical process. Experience from the daily solutions of permanent GPS stations has shown a linear temporal displacement with additive almost-white noise, while periodic terms of no relevance to crustal deformation have been properly computed and removed (e.g. Pietrantonio and Riguzzi 2004, Dong et al. 1998). This is the reason why ITRF models so far consist of initial coordinates and velocities only. Therefore, in the time domain one should use least-squares interpolation with a linear, or at most quadratic, analytic model.

Due to the lack of similar experiences with spatial data, one needs to experiment with either purely analytic models, or to use a combination of an analytic trend with an additive stochastic component in a hybrid interpolation scheme. The choice depends on the spectral behavior of the crustal deformation field in the particular application area. The analytic trend takes care of the lower frequencies of the field, whereas the stochastic signal represents, in a smooth way, the somewhat higher frequencies that may be present.

Another issue is whether the primary observations should be a-priori adjusted to compute coordinates and thus displacements, or interpolation and adjustment of the raw observations should be implemented simultaneously. The two-step procedure (adjustment, followed by interpolation) is usually dictated by the lack of access to the original data. From a strictly formal point of view, a simultaneous treatment is preferable provided the functional and stochastic models are correct. Since this is not the case, particularly for GPS data, a two-step procedure avoids interpreting observational noise as a physical crustal deformation process.

Apart from the above general guidelines, the data analysis strategy to be implemented depends on how observations are distributed in space and time, in relation to the overall design of the measurement procedure. In the sequel we discuss some specific data collection schemes that can appear in the geodetic practice.

Case 1: GPS repeated surveys

A network of stations is observed in repeated campaigns, within short-time interval (e.g. a week), so that they refer to discrete epochs. In this case, deformation analysis refers to the comparison of any two epochs which coincide with the actual observation epochs. Spatial-only interpolation is needed in order to compute the deformation gradient matrix at any point within the study area. However, since temporal coordinate variations are known to be of a linear nature, it may be advantageous to adjust all observation epochs simultaneously with a analytical linear interpolation $\mathbf{x}(P,t)=\mathbf{x}_0(P)+(t-t_0)\mathbf{v}(P)$, with respect to time. In this case, the station coordinates $\mathbf{x}(P_k,t_j)$ at the various observation epochs t_j are replaced by a much smaller number of unknowns, namely the coordinates $\mathbf{x}_0(P_k)=\mathbf{x}(P_k,t_0)$ at the reference epoch t_0 and the station velocities $\mathbf{v}(P_k)$. It remains to perform a spatial interpolation for the station velocities to obtain a velocity field from which displacement can be determined for any desired pair of epochs. Deformation analysis can be performed with the strain rate approach by using the velocity gradient instead of the displacement gradient.

Case 2: GPS "network scanning" surveys

Such situations arise when the number of available GPS receivers is not sufficient to cover with "simultaneous" observations the studied area. In order to complete the survey at a specific epoch, different "patches" of the monitoring network are sequentially observed until the entire area is properly covered. Repetitions of the above procedure at different time periods provide us with the necessary kinematic information for the estimation of the crustal deformation field.

In this case, temporal interpolation is required and it must be simultaneously performed with the least-squares data adjustment in order to obtain reference epoch coordinates and velocities values. The rest of the analysis follows the same guidelines as in the previous case.

Case 3: Monitoring by GPS permanent stations

In this case, the enormous amount of the available data makes the simultaneous adjustment for the determination of station coordinates and velocities impossible from a practical point of view. In practice daily adjustments are performed producing noisy coordinate estimates. A separate linear regression for every coordinate at every station provides the required parameters. In addition, it provides a great amount of residuals that can be used to estimate the variances and covariances for any pair of coordinates and velocity components of the same or different stations. Moreover, it is possible to detect temporal correlations by empirically estimating auto-covariance and cross-covariance functions. From limited experience, we may report that some small correlation exists for time intervals of one day (most probably due to unmodeled ionospheric effects), which becomes already negligible after two or more days. Thus, the quality assessment problem can be solved in a way completely independent of the formal covariance propagation which is known to be overly optimistic in the case of GPS data. As in the previous two cases, spatial interpolation of velocities must follow before performing the deformation analysis for a pair of epochs or a strain rate analysis.

Case 4: GPS permanent stations and repeated SAR surveys

There is little experience regarding the optimal merging of GPS and SAR data for deformation analysis (e.g. Bos et al. 2004, Lohman and Simons 2005). The simplest mind approach is to perform the analysis for the permanent GPS stations as in the previous case, and introduce the computed coordinates at the SAR campaign epoch as constraint values. Indeed, SAR is known to provide very good results in high spatial resolution while lacking a similar accuracy level in the longer wavelength parts. In this case the GPS information complements and enhances the information provided by SAR. When the deformation analysis involves comparison between time epochs that are covered by SAR campaigns, there is absolutely no need for interpolation in the spatial domain. For an arbitrary pair of epochs and/or strain rate deformation analysis, a temporal interpolation can be performed for all SAR points. The coordinates of each SAR densification point may not perfectly fit into a linear model (as in the case of GPS stations). The question that arises is whether a linear behavior should be imposed a-posteriori, or implemented in the simultaneous analysis of all SAR campaigns. The answer requires a deeper understanding of the data analysis methods used in the SAR data processing, which differs from the typical least-squares geodetic adjustment techniques (Hanssen 2001, 2005; Lohman and Simons 2005).

Case 5: GPS permanent stations and "permanent" SAR surveys

With a futuristic outlook, we envisage the case where SAR data sets are obtained with a very high temporal resolution (e.g. by involving a number of satellites). In this case, neither temporal or spatial interpolation is required, and one has to fully switch from the classical geodetic data analysis techniques to spectral methods which are continuous in principle but they are numerically implemented on dense discrete data. Instead of the problem of choosing the base functions or the covariance function, we face the problem of choosing the appropriate filter(s) which separate the noise attributed to observation errors from the signal attributed to crustal deformation.

7 Accuracy assessment of the results

The accuracy assessment of the results faces three fundamental problems:

(a) Quality assessment of the input data. This is a most critical problem for GPS derived coordinates which are accompanied by formal variance and covariance values that are commonly recognized to be too optimistic. The reason is that the raw data are supposed to be affected by uncorrelated noise while as in reality there are significant correlations due to physical processes (e.g. non-modeled atmospheric effects) or instrumental behavior (Williams 2003, Langbein and Johnson 1997, Mao et al. 1999). The hope for independent quality assessment lies in variance component estimation techniques, which take rather simplified forms when a linear trend is a realistic model for temporal evolution. The residuals after removal of the linear trend can be used to detect time-independent coordinate error variances and covariances or temporal correlations which however seem to be negligible for a two or more days interval.

(b) The independently assessed realistic covariance matrix of the input data must be propagated to the entries of the deformation gradient matrix. In this step the uncertainties due to the more-or-less arbitrary interpolation in between the discrete data points must be taken into account. This can be done only when a stochastic interpolation-prediction is employed, while in an analytic trend determination only one has to accept the interpolating smooth function as error-free. Both approaches (and therefore their combination) have their shortcomings. The trend removal depends on the particular model for the parameterization of the trend function. The stochastic prediction suffers from the arbitrariness of the chosen covariance function, summarized by the common variance of the function value at each point (stationarity assumption) and the correlation length under the isotropic hypothesis (distance where covariance drops at half the variance value).

(c) Covariance propagation is straightforward in the previous step where linear prediction algorithms are used and the predicted elements of the deformation gradient matrix $\mathbf{F}$ are linear functions of the input coordinates or displacements. This is not the case though for the computed deformation invariant parameters which, as functions of the singular values of the deformation gradient matrix, are highly non-linear functions of the elements of **F** (Soler and van Gelder 1991). More realistic covariance propagation can be assessed by Monte-Carlo simulation methods, or extension of the linear covariance propagation scheme where at least second order partial derivatives of the non-linear expressions are implemented.

8 Conclusions

The scope of this work was to give an overview of some theoretical and practical aspects that are critical in the geodetic-based determination of crustal deformation fields. The importance of studying the deformation of the Earth's crust needs hardly to be stressed, especially in those parts of the world where the tectonic activity can cause disastrous effects that affect the lives of millions of people. Geodesy, being the science of measuring and mapping the surface of the Earth, plays a key role in crustal deformation studies by determining the temporal variations of its shape/size at various spatial and time scales (Dermanis and Livieratos 1983). Our focus in this paper was placed on the description of the basic principles and the mathematical techniques for the estimation of crustal deformation parameters from geodetic data, as well as on some special types of applications with particular interest within the field of modern four-dimensional geodesy.

For reasons of economy in our presentation, we have deliberately avoided a detailed treatment of SAR data processing techniques for crustal deformation analysis. Some details for this interesting and relatively unexplored topic in the geodetic literature can be found in Hanssen (2001) and the references given therein. Also, the contribution of the temporal variations of the gravity field, in terms of corresponding mass changes and redistributions and/or changes in the Earth's flattening (i.e. dynamic form factor J_2) and rotation axis as determined, for example, by the CHAMP and GRACE missions, plays an important role for crustal deformation studies in large spatial scales. Such an issue also has not been discussed here.

We should finally note that a topic of special interest is the development of optimal methods and algorithms for the estimation of crustal deformation parameters using a combination of geodetic and non-geodetic (e.g. seismic soundings, strain-meter and tiltmeter readings) observations, thus leading to an integrated inter-disciplinary approach for crustal deformation analysis.

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