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Geoid Determination Using Different Gravity Reduction Techniques

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Abstract: The topographical masses outside the geoid have to be removed completely for its determination using Stokes's boundary value problem (BVP) approach. The mathematical and the physical treatment of this problem play an important role in the computation of a precise (local or regional) gravimetric geoid solution. There are various gravity reduction techniques used in physical geodesy to treat this problem. Bouguer reduction, residual terrain model (RTM) reduction, Airy-Heiskanen (AH), Pratt-Hayford (PH) and Vening Meinesz isostatic models, and the Helmert condensation method are mostly discussed. One of the most rugged areas of Canadian Rockies, which lies in latitude between 49°N and 54°N and in longitude between 124°W and 114°W, is selected to compute different gravimetric geoid solutions using the AH and PH topographic-isostatic reduction techniques, Helmert's second condensation method and the Rudzki method. The geoid is computed from Stokes's integral formula with the rigorous spherical kernel by the one dimensional fast Fourier transform algorithm, and the OSU91A model as reference global field. A digital terrain model of 15×15 arc seconds is used to compute the effect of topography and its indirect effect for the different reduction schemes. The results obtained from the gravimetric geoid solutions are finally compared with the GPS-levelling derived geoid undulations of this area.

Keywords. Geoid, indirect effects, Helmert, Pratt-Hayford, Airy-Heiskanen, Rudzki

1 Introduction

The topographical effect is one of the most important components in the solution of geodetic BVPs, and should be treated properly in the determination of a precise geoid. The classical solution of geodetic BVP using Stokes's formula for geoid determination assumes that there should be no masses outside the geoid. The gravity measurements should be referred to the geoid, which requires the actual Earth's topography to be regularized. There are several reduction techniques, which differ depending on how these topographical masses outside the geoid are dealt with. Each gravity reduction scheme treats the topography in a different way. In theory the gravimetric solution for geoid determination using different mass reduction methods should give the same result, provided the indirect effect is taken into account properly (Heiskanen and Moritz, 1967).

In practice, the choice of reduction method depends on the magnitude of the indirect effects, smoothness and smallness of reduced gravity anomalies, and the geophysical meaning of its reduction method (Heiskanen and Moritz, 1967). The complete Bouguer reduction removes all the topographic masses above the geoid giving smooth anomalies but introducing excessively large indirect effects. Topographic-isostatic reductions remove topographical masses with their isostatic compensation, and fulfill all the requirements of a good reduction technique. Helmert's second method of condensation is mostly used in practice throughout the world as a mass reduction scheme for geoid determination. In this reduction method, the topographic masses between geoid and the Earth's surface are condensed on the geoid forming a surface layer. The direct topographic effects and indirect effects using this condensation reduction method have been discussed in the literature; see for example, Heiskanen and Moritz, (1967), Wichiencharoen (1982), Vanicek and Kleusberg (1987), Wang and Rapp (1990), Sideris (1990), Martinec and Vanicek (1993), Martinec et al. (1993) and Heck (1993).

The purpose of this paper is to study gravimetric geoid determination using different reduction methods, in planar approximation, in a rugged area of the Canadian Rockies. Although the topographic and isostatic reductions are used quite often, the Rudzki reduction is not. The inversion method of Rudzki shifts all the masses into the interior of the geoid in such a way that there is no indirect effect. This reduction method has no geophysical meaning (Heiskanen and Moritz, 1967). The TC program written by Forsberg (1984) is modified for this paper to obtain different gravimetric quantities. Details on the computation of different topographic-isostatic anomalies using TC program is also given by Abd-Elmotaal (1998).

2 Computational methodology

Global geopotential model, local gravity information and digital terrain model represent the low, medium and high frequency part of the gravity signal, respectively. Gravimetric geoid solution is carried out using remove-restore technique in this investigation for all gravity reduction methods. Each method treats the topography in a different way.

First, the gravity anomaly is reduced in a remove step using a mass reduction scheme to formulate boundary values on the geoid, which can be expressed as

$$\Delta g = \Delta g \;\; - \Delta g_T - \Delta g_{GM} \tag{1}$$

where Δg is the gravity anomaly, Δg_T is the attraction change due to the removal of topography with compensation or condensation depending on the reduction method used and Δg_{GM} is the reference gravity anomaly from geopotential model The total geoid obtained as the result of the restore step (in remove-restore technique) can then be expressed as

$$N = N_{GM} + N_{\Delta g} + N_{ind} \tag{2}$$

where N_{GM} denotes the long wavelength part of the geoid obtained from a geopotential model, $N_{\Delta g}$ represents residual geoid obtained by using Δg from equation (1) in Stokes's formula and N_{ind} is the indirect effect on geoid, which depends on the mass reduction method used. Stokes's integral formula with the rigorous spherical kernel by the one-dimensional fast Fourier transform algorithm is used in this paper (Haagmans et al., 1993). The formulas for the computation of Δg_{GM} and N_{GM} are given in Heiskanen and Moritz (1967).

The indirect effect on geoid, N_{ind} in equation (2), can be computed from Bruns's formula, as follows:

$$N_{ind} = \frac{\Delta T}{\gamma} \tag{3}$$

where ΔT is the change in the potential at the geoid, which depends on the reduction method used and is described in details in the following sections. The indirect effect on gravity, which reduces gravity anomaly from geoid to the co-geoid, can be expressed as a simple free-air reduction (Heiskanen and Moritz, 1967)

$$\delta g = +0.3086 N_{ind} \text{ mGal} \tag{4}$$

2.1 Airy-Heiskanen, Pratt-Hayford and Rudzki topographic models

The AH model is based on the principle that mountains are floating on the material of higher density forming roots under mountains and anti-roots under the oceans. The condition of floating equilibrium for the continents can be formulated as (Heiskanen and Moritz, 1967)

$$t\Delta\rho = h\rho_0 \tag{5}$$

where t is the root depth, $\Delta\rho$ is the density contrast between the crust and the upper mantle, which is taken equal to 0.6 g/cm^3 for this test, h is the height of the topography and ρ_0 is the standard crust density of 2.67 g/cm^3. The normal thickness of the Earth's crust is assumed to be 30 km. The attraction change Δg_T in equation (1) can be expressed as

$$\Delta g_T = A_T - A_C \tag{6}$$

where A_T is the attraction of all topographic masses above the geoid and A_C is the attraction of the compensating masses based on the AH principle. The two components in the right hand side of Equation (6) can be expressed as

$$A_T(i, j) = -G \iint_E \int_0^{h(x,y)} \frac{\rho(x, y, z)(h_{ij} - z)}{r^3(x_i - x, y_j - y, h_{ij} - z)} dx dy dz,$$

$$A_C(i, j) = -G \iint_E \int_{-D-t-h(i,j)}^{-D-h(i,j)} \frac{\Delta\rho(x, y, z)(h_{ij} - z)}{r^3(x_i - x, y_j - y, h_{ij} - z)} dx dy dz, \tag{7}$$

G is Newton's gravitational constant, ρ (x, y, z) is the topographical density function, Δρ is the density contrast, D is the normal thickness of Earth's crust and E denotes the integration area. r (x, y, z) is the distance kernel defined as

$$r(x, y, z) = (x^2 + y^2 + z^2)^{1/2} . \tag{8}$$

Equation (7) can be numerically integrated using rectangular prisms with the computation point coinciding with the origin of the coordinate system as (Nagy, 1966):

$$A_T(i, j) = -G\rho \;|||\; x\ln(y + r) + y\ln(x + r) - z\arctan\frac{xy}{zr} \Big|_{x_1}^{x_2}\Big|_{y_1}^{y_2}\Big|_{0}^{h(x,y)}$$

$$A_C(i, j) = -G\Delta\rho \;|||\; x\ln(y + r) + y\ln(x + r) - z\arctan\frac{xy}{zr} \Big|_{x_1}^{x_2}\Big|_{y_1}^{y_2}\Big|_{-D-t-h_{(i,j)}}^{-D-h_{(i,j)}} \tag{9}$$

The difference between the gravitational potential of the actual topographical masses and that of the compensating masses, ΔT in equation (3), can be expressed as

$$\Delta T = T_T - T_C \tag{10}$$

where T_T and T_C are the gravitational potentials of the topography and compensating masses respectively. The individual components in the right hand side of equation (10) can be expressed as

$$T_T(i, j) = -G\iint_E \int_0^{h(x,y)} \frac{\rho(x, y, z)}{r\;(x_i - x, y_j - y, h_{ij} - z)} dx\,dy\,dz,$$

$$T_C(i, j) = -G\iint_E \int_{-D-t}^{-D} \frac{\Delta\rho(x, y, z)}{r\;(x_i - x, y_j - y, h_{ij} - z)} dx\,dy\,dz, \tag{11}$$

Equation (11) can also be numerically integrated integration using rectangular prisms with the computation point coinciding with the origin of the coordinate system (Nagy, 1966):

$$T_T(i, j) = G\rho \;|||\; xy\ln(z + r) + xz\ln(y + z) + yz\ln(x + r) - \frac{x^2}{2}\tan^{-1}\Big(\frac{yz}{xr}\Big) - \frac{y^2}{2}\tan^{-1}\Big(\frac{xz}{yr}\Big) - \frac{z^2}{2}\tan^{-1}\Big(\frac{xy}{zr}\Big) \Big|_{x_1}^{x_2}\Big|_{y_1}^{y_2}\Big|_{0}^{h(x,y)}$$

$$T_C(i, j) = G\Delta\rho \;|||\; xy\ln(z + r) + xz\ln(y + z) + yz\ln(x + r) - \frac{x^2}{2}\tan^{-1}\Big(\frac{yz}{xr}\Big) - \frac{y^2}{2}\tan^{-1}\Big(\frac{xz}{yr}\Big) - \frac{z^2}{2}\tan^{-1}\Big(\frac{xy}{zr}\Big) \Big|_{x_1}^{x_2}\Big|_{y_1}^{y_2}\Big|_{-D-t}^{-D} \tag{12}$$

The PH topographic model is based on the principle that the mass of each column of same cross section is equal and there is constant density under the level of compensation. The topographic masses are distributed between the level of compensation and the sea level. The original density of the column according to this principle for the continental case can be expressed as (Heiskanen and Moritz, 1967)

$$\rho = \frac{D}{D + h}\rho_0 \tag{13}$$

where D is the compensation depth, which is assumed equal to 100 km for this test. The Bouguer reduction is usually carried out using constant density and thus the density defect is computed by

$$\Delta\rho = \frac{h}{D}\rho_0 \tag{14}$$

The attraction change and potential change based on this reduction method can be formulated in the same way as in equation (6) and in equation (10) respectively, where A_C will represent the attraction and T_c the potential of the compensating masses according to the PH theory. The attraction $A_C(i, j)$ and the potential of the compensating masses $T_C(i, j)$ can be given in the same way as in equations (9) and (12) :

$$A_C(i, j) = -G\Delta\rho \;|||\; x\ln(y + r) + y\ln(x + r) - z\arctan\frac{xy}{zr} \Big|_{x_1}^{x_2}\Big|_{y_1}^{y_2}\Big|_{-D-h_{(i,j)}}^{-h_{(i,j)}}$$

$$T_C(i, j) = G\Delta\rho \;|||\; xy\ln(z + r) + xz\ln(y + z) + yz\ln(x + r) - \frac{x^2}{2}\tan^{-1}\Big(\frac{yz}{xr}\Big) - \frac{y^2}{2}\tan^{-1}\Big(\frac{xz}{yr}\Big) - \frac{z^2}{2}\tan^{-1}\Big(\frac{xy}{zr}\Big) \Big|_{x_1}^{x_2}\Big|_{y_1}^{y_2}\Big|_{-D}^{-0} \tag{15}$$

The inversion reduction of Rudzki shifts all the topographic masses inside the geoid so that equation (10) becomes zero, which means the co-geoid of Rudzki coincides with the geoid (Heiskanen and Moritz, 1967). The density of the inverted masses is equal to that of topography and the thickness of the inverted masses is equal to the height of the topography in planar approximation. A_c in equation (6) is the attraction due to the inverted masses in this method and can be expressed as follows

$$A_C(i, j) = -G\rho \,|||\, x\ln(y+r) + y\ln(x+r) - z\arctan\frac{xy}{zr}\Big|_{x_1}^{x_2}\Big|_{y_1}^{y_2}\Big|_{-h_{(x,y)}-h_{(i,j)}}^{-h_{(i,j)}} \quad (16)$$

2.2 Hemert's second method of condensation

Hemert's second method of condensation can be regarded as PH isostatic reduction when the depth of compensation goes to zero (Heiskanen and Moritz, 1967). This method is mostly used in practice. The attraction change in equation (6) based on this condensation scheme can be expressed in planar approximation as

$$\Delta g_T = A_T - A_C = -c \quad (17)$$

where c is classical terrain correction, which here represents the difference between the attraction of topography computed on the surface of the topography and the attraction due to the condensed masses computed on the geoid. This condensation method is not a gravity smoothing terrain reduction and thus the use of other reduction methods is recommended to smooth gravity anomalies for gridding. The indirect effect on gravity for this reduction scheme can be given as (Sideris and She 1995)

$$\delta g \approx \frac{2\pi G\rho h^2}{R} \quad (18)$$

The indirect effect on geoid can be formulated in planar approximation as (Wichiencharoen, 1982)

$$N_{ind} = -\frac{\pi G\rho}{\gamma} h_{(i,j)}^2 - \frac{G\rho}{6\gamma} \iint \frac{h^3 - h_{(i,j)}^3}{r_0^3} dxdy \quad (19)$$

where r_0 is the planar distance between computation and running point.

3 Numerical tests

One of the most rugged areas in the Canadian Rockies bounded by latitude between 49ºN and 54ºN and longitude between 124º W and 114ºW is selected to compute different gravimetric geoid solutions with AH and PH isostatic reductions, the Rudzki inversion method and Helmert's second method of condensation. A total of 9477 measured gravity values are used for this test, the distribution of which is given in figure 1. The normal gradient of 0.3086 mGal is used for the computation of Free-air anomalies. The standard constant density of 2.67 g/cm^3 is assumed. The digital terrain model of 15″ grid resolution is used to compute the attraction of the topography, attraction of compensating or condensed masses and the indirect effects. There is a maximum elevation of 3840 m with standard deviation of 543 m in the test area. The attraction of the topography and the attraction of the compensating masses are computed using a radius of 300 km around the computation point. The reference gravity field is computed from OSU91A geopotential model complete to degree and order 360.

The statistics of gravity anomalies for different topographic-isostatic reductions are presented in table 1. Topographic-isostatic anomalies based on AH and PH theory are the smoothest, with a standard deviation around 18 mGal. The gravity anomalies based on Helmert's second method of condensation have highest range between maximum and minimum values as well as standard deviation compared to those of other reduction methods. The removal of global reference field does improve the statistics of reduced gravity anomalies for Helmert and Rudzki inversion schemes but not for Bouguer, AH and PH topographic-isostatic methods. Faye and

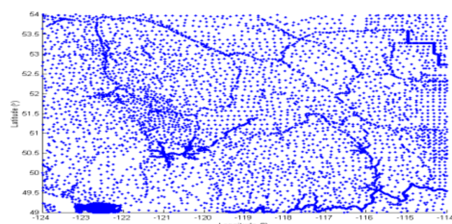

Fig 1. The distribution of gravity points in the test area of Canadian Rockies

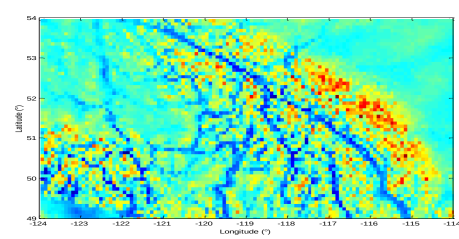

Fig 2. Faye gravity anomalies

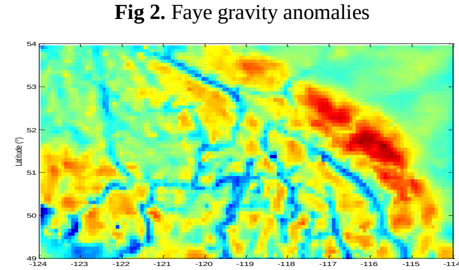

Fig 3. Rudzki gravity anomalies

Table 1. The statistics of gravity anomalies (mGal)

Reduction Scheme	Gravity anomalies	Max	Min	Mean	STD
Faye (Helmert)	$\Delta g_{FA} + c$	245.56	-150.59	-13.65	57.41
	$\Delta g_{FA} + c - OSU91A$	198.07	-193.44	-6.75	48.03
Refined Bouguer	$\Delta g_{FA} - \Delta g_B + c$	-4.67	-212.03	-109.23	43.62
	$\Delta g_{FA} - \Delta g_B + c - OSU91A$	20.75	-284.62	-102.33	69.76
Airy-Heiskanen	$\Delta g_{FA} - A_{AH}$	56.46	-200.58	-24.32	18.70
	$\Delta g_{FA} - A_{AH} - OSU91A$	59.52	-224.73	-17.43	33.33
Pratt-Hayford	$\Delta g_{FA} - A_{PH}$	50.71	-204.17	-28.97	18.22
	$\Delta g_{FA} - A_{PH} - OSU91A$	56.94	-228.32	-22.07	34.74
Rudzki	$\Delta g_{FA} - A_R$	125.56	-176.38	-16.58	36.03
	$\Delta g_{FA} - A_R - OSU91A$	76.22	-200.53	-9.67	25.68

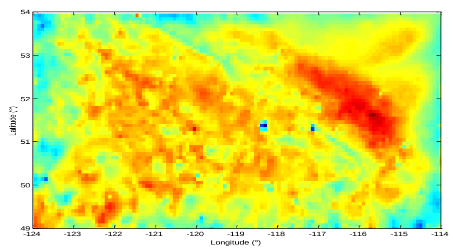

Fig 4. Airy-Heiskanen gravity anomalies

Rudzki anomalies are highly correlated with topography while the correlation of AH and PH anomalies with topography is not much. Figure 2, 3 and 4 show Faye, Rudzki and AH gravity anomalies respectively.

The indirect effect on gravity for Helmert, AH and PH models is considered before applying Stokes' formula for these reductions. The statistics of this effect is given in table 2. The indirect effect on geoid undulation for PH isostatic reduction changes the geoid as much as nearly 10 m while that for Helmert method changes 47 cm. Table 2 shows the statistics of indirect effects on geoid for different reductions. Figure 5 shows the indirect effect on geoid for Partt-Hayford model. The indirect effect for Helmert method is computed considering terms, a regular and an irregular part, of equation (19). The maximum indirect effect for all reductions is seen in mountains.

Table 2. Indirect effects on gravity (mGal) and on geoid undulation (m)

Geoid Model	Indirect Effect	Max	Min	Mean	STD
Helmert	gravity	0.26	0.00	0.04	0.03
	geoid	0.01	-0.47	-0.12	0.08
Pratt Hayford	gravity	3.08	0.18	1.35	0.75
	geoid	9.97	0.59	4.36	2.41
Airy Heiskanen	gravity	2.61	0.09	1.04	0.64
	geoid	8.46	0.31	3.36	2.06

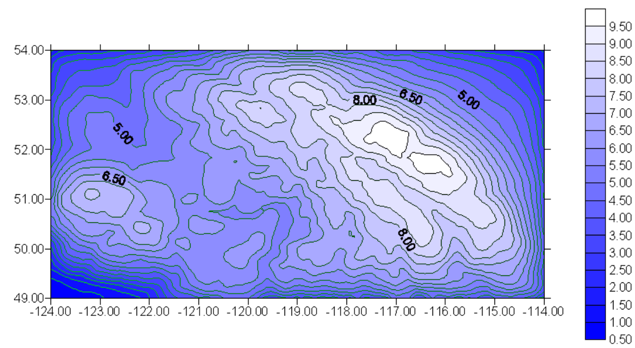

Fig 5. The indirect effect on geoid for PH model (m)

A total of 258 GPS benchmarks available in the test area is used to fit gravimetric geoid solutions with GPS-leveling geoid, the distribution of which is given in figure 6. There are no GPS leveling points above the elevation of 2000 m. A Four - parameter trend surface is applied to fit gravimetric geoid solutions with GPS-leveling. The statistics of the difference of gravimetric geoid undulations with GPS-leveling before and after fit are given in table 3. Gravimetric geoid determination based on Rudzki inversion topographic reduction shows the smallest differences from GPS-leveling before fit. The absolute magnitude of maximum, minimum and mean values of the difference between isostatic gravimetric solutions based on AH and PH with GPS-leveling before fit is much higher compared to those of Rudzki and Helmert methods but their standard deviation is less than Helmert's method.

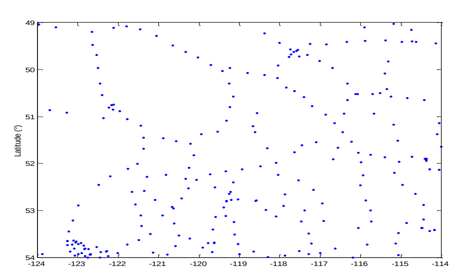

Fig 6. Distribution of GPS-leveling points

Table 3. Statistics of different gravimetric geoid solutions (m) (values in the parentheses are before fit)

Geoid Model	Max	Min	Mean	STD
Helmert	0.54 (-0.69)	-0.75 (-3.48)	0.00 (-2.22)	0.25 (0.58)
Airy-Heiskanen	0.66 (-5.25)	-1.04 (-7.63)	0.00 (-5.99)	0.35 (0.45)
Pratt Hayford	0.64 (-5.85)	-1.02 (-8.21)	0.00 (-6.64)	0.35 (0.44)
Rudzki	0.47 (0.47)	-0.97 (2.63)	0.00 (-1.52)	0.24 (0.43)

4 Conclusions

Different mass reduction schemes, Helmert second method of condensation, AH and PH isostatic models and Rudzki inversion method have been applied in the classical solution of BVP using Stokes's approach in one of the most rugged areas of Canadian Rockies. Helmert anomalies present the most rough gravity field for this area while topographic-isostatic anomalies show smoother field as well as less correlation with topography. The removal of reference field does not improve the statistics of isostatic anomalies of AH and PH models but does improve that of Helmert and Rudzki reductions.

Indirect effect on geoid undulation with topographic-isostatic models changes the geoid surface as much as 10 m. Indirect effect with Helmert method changes the geoid undulation maximum of 47 cm.

Rudzki-geoid shows best overall statistics in the differences from GPS-levelling and should, therefore, become a standard method for geoid determination. There is a large difference in the statistics between absolute geoid of AH and PH models with GPS-levelling geoid before fit. These large biases possibly indicate that the reference gravity field from OSU91A geoptential model does not agree with topographic-isostatic models in the test area. More studies regarding the improvement of absolute geoid for these models should be carried out. Helmert and Rudzki geoid show similar characteristics after the fit and show better results compared to those based on AH and PH models.

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