Aliasing Error Modelling in Single-Input Single-Output Linear Estimation Systems

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Abstract. The problem of modelling the aliasing error in single-input single-output (SISO) linear systems with gridded input data is studied. First, a general linear estimation framework for SISO systems, based on the use of a multiresolution reference scaling kernel, is established, which includes the usual FFT-based numerical approximation of geodetic convolution integrals as a special case. The output signal error is modelled with the help of a spatio-statistical parameter (sampling phase) that depends on the resolution of the input data grid. A frequency domain algorithm is then developed which computes the decay rate of a certain output error functional with respect to the data resolution level, using the power spectra of the input signal, the chosen scaling estimation kernel, and the theoretic convolution kernel of the linear system. A simple numerical experiment is also included to compare the accuracy of the classic FFT approach in SISO approximation problems against the proposed generalization that utilizes an arbitrary reference scaling kernel.

1 Introduction – Problem Formulation

The theoretical solution of many geodetic problems can generally be described in terms of a single-input single-output (SISO) system, as the one shown in Fig. 1. A fully known (deterministic) signal is used as input to a certain 'black box', which produces a new and unique output signal according to some operatorial equation $f = \mathrm{S}\,g$. The nature of the input and output signals, as well as the properties of the system operator S, are dictated by the problem at hand.

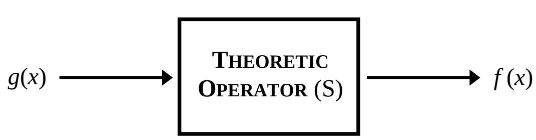

Fig. 1 A theoretical single-input/single-output system

Some typical geodetic examples include the problem of gravimetric geoid determination from Stokes' integral formula, the computation of various terrain dependent gravity field quantities (e.g. indirect effect, terrain correction, etc.) from the Earth's topography signal, and the analytic upward continuation of the anomalous potential using Poisson's kernel, among many others.

In practice, the system of Fig. 1 needs to be modified since we cannot access the full (continuous) input field $g(x)$, as required by the theoretical formulation of most physical geodesy problems. The input data is usually given in a discrete gridded form, and thus some external estimation model is always necessary to approximate the output signal $f(x)$ in a unique optimal sense. Such a situation is illustrated in the system of Fig. 2, where the input is now a series of gridded signal values $g(nh)_{n\in Z}$ at a uniform resolution level h, and the output corresponds to an estimate $\hat{f}(x)$ of the true field under consideration. In this case, the system operator will depend not only on the theoretic kernel S, but it will additionally incorporate some estimation model to overcome the underdeterminacy of the problem caused by the limited input signal information.

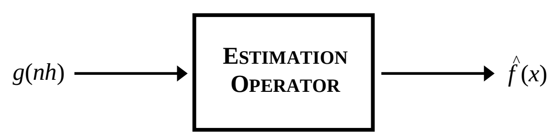

Fig. 2 An operational single-input/single-output system with gridded input data at resolution level h

The task of this paper is to study the output estimation error $\hat{f}(x) - f(x)$ of the SISO system, shown in Fig. 2, as a function of the input data resolution. In order to simplify the error analysis, two basic conditions will be imposed on the estimators of such systems, namely *linearity* and *translation-invariance*. Considering the convolution character of most theoretical models $f = \mathrm{S}\,g$ in physical geodesy, the latter assumption should not raise any major objections. Actually, in the authors' opinion the translation-invariance (T-I) restriction must always be assigned in every geodetic estimation method, even for non-linear schemes. Otherwise, how can one justify the dependence of the behaviour of the output signal estimate on the origin of the reference system used to describe the position of the input data values? At this point, we should state that by translation-invariance we mean the property of a system/algorithm/operator to shift its output (without altering its behaviour) when a corresponding shift in the input signal occurs. Note that for the case of the SISO system in Fig. 2, a data signal of the form $g(nh+\tau)_{n\in Z}$ does not necessarily correspond to a simple shifted version of the input signal $g(nh)_{n\in Z}$, unless we restrict the value of τ to be an integer multiple of the sampling resolution h. In the theoretical SISO system of Fig. 1, on the other hand, the admissible input shift value can be any real number.

For the purpose of this paper, the system in Fig. 2 will be further simplified according to the separable model shown in Fig. 3. The signal approximation will now consist of two basic steps that are connected in a linear cascading manner. The first step corresponds to a convolution-type interpolation procedure using the original gridded data in conjunction with some basic estimation kernel, whereas the second step involves the application of a convolution operator to the interpolated input field $\hat{g}(x)$ according to an underlying theoretical model $f = \mathrm{S}\, g$ (e.g. Stokes' integral formula). Note that this kind of system 'factorization' applies also for non translation-invariant linear estimation methods, such as collocation (minimum norm interpolation) in Hilbert spaces with non homogeneous reproducing kernels; see Moritz (1980).

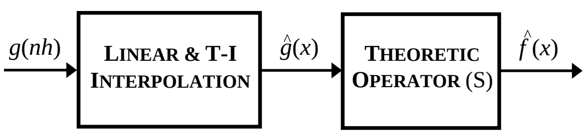

Fig. 3 Linear and translation-invariant single-input/single-output estimation system with gridded input data at resolution level h

Our focus will be on the development of an algorithmic procedure that can compute the decay rate of a suitable functional of the output signal error with respect to the data resolution h, for different choices of the interpolating kernel. The numerical approximation of convolution integral formulas $f = \mathrm{S}\, g$ using fast Fourier transform (FFT) techniques, which is routinely applied in gravity field applications with gridded data, will also be investigated as a special case of the system shown in Fig. 3. Due to space limitations and for the shake of notation economy, all the following discussions will be restricted in a one-dimensional setting. Two-dimensional planar generalizations are quite straightforward and they will soon be published elsewhere.

2 SISO Estimation System with a Multiresolution Reference Kernel

Before we study the output estimation error of the SISO system in Fig. 3, a more specific characterization of its interpolating component should first be made. Since it is assumed that the input data is always given in a gridded form with a uniform spatial resolution, we will adopt the general interpolatory model

$$\hat{g}(x) = \sum_n g(nh)\ \varphi(\frac{x}{h} - n) \tag{1}$$

where $\varphi(x)$ is some scaling approximation kernel whose spread adapts to the data grid resolution through an appropriate dilation. The use of convolution-based linear interpolating schemes of the form of Eq. (1) is very popular in the signal processing community, especially in view of their close connection with the multiresolution analysis and wavelet theory (Blu and Unser, 1999; Unser, 2000). The estimation kernel $\varphi(x)$ acts as a low-pass filter on the periodic (aliased) spectrum of the input sequence $g(nh)$, with its filter bandwidth being 'tuned' to the sampling interval h. Its actual choice can be quite arbitrary and the only essential restriction is the so-called Riesz condition (Unser and Daubechies, 1997)

$$0 < A \le \sum_{k\in Z} \left| \Phi(\omega + 2\pi k) \right|^2 \le B < \infty \tag{2}$$

where $\Phi(\omega)$ is the Fourier transform of $\varphi(x)$, and A, B denote finite constants. The above condition guarantees that Eq. (1) gives a unique and stable signal expansion (in the L^2 norm) for any set of square summable input values $\{g(nh)\}$. Note that Eq. (2) is not restrictive at all and it is satisfied by virtually any approximation kernel used in practice (e.g. sinc kernel, polynomial B-splines, etc.). The optimal determination of $\varphi(x)$ in the context of statistical collocation theory is discussed in Kotsakis (2000a). It is worth mentioning that the chosen scaling kernel does not need to be strictly interpolating (i.e. $\hat{g}(nh) \neq g(nh)$), and all the following discussions will equally hold for both interpolating and quasi-interpolating schemes.

The second component of the SISO system in Fig. 3 corresponds to a simple convolution operation of the form $\hat{f}(x) = s(x) * \hat{g}(x)$, where $s(x)$ is some theoretic kernel specified by the problem under consideration (e.g. Stokes' kernel). Taking into account Eq. (1), the final estimation formula for the output signal is

$$\hat{f}(x) = \sum_n g(nh) \left[\varphi(\frac{x}{h} - n) * s(x) \right] \tag{3}$$

or, equivalently, in the frequency domain

$$\hat{F}(\omega) = h \; \Phi(h\omega) \; S(\omega) \sum_n g(nh) \; e^{-inh\omega} \tag{4}$$

where $S(\omega)$ and $\hat{F}(\omega)$ denote the Fourier transforms of the theoretic kernel and the estimated output signal, respectively. Note that the integer index n in all above summations generally runs from $-\infty$ to $+\infty$. However, in cases of input signals with compact spatial support the previous series have a finite number of non-zero values, and the index n will accordingly be restricted within a finite range.

In practice, the evaluation of the estimated signal $\hat{f}(x)$ takes place only in a finite number of points, usually on the same points at which the gridded input data $g(nh)$ is given. Due to the convolution character of Eq. (3), such a numerical task can be very efficiently implemented using FFT methods and it requires the knowledge of the discrete values $\varphi(x/h) * s(x)\big|_{x=nh}$.

An interesting special case occurs if we set the scaling kernel in Eq. (3) equal to the Dirac delta function $\delta(x)$. Using the reproducing property of the delta function under the convolution product '$*$', the estimation formula for the output signal is simplified as follows:

$$\begin{aligned} \hat{f}(x) &= \sum_n g(nh) \left[\delta(\frac{x}{h} - n) * s(x) \right] \\ &= h \sum_n g(nh) \; s(x - nh) \end{aligned} \tag{5}$$

The last equation is just a simple discretization of the theoretical (continuous) convolution model implied in the SISO system of Fig. 1, i.e.

$$f(x) = s(x) * g(x) = \int_{-\infty}^{+\infty} g(y) \, s(x-y) \, dy \tag{6}$$

In fact, its use is equivalent to applying the familiar *parallelogram rule* for the numerical integration of the above formula. The fast evaluation of Eq. (5) at the input data points using FFT techniques has been the standard approximation framework for a variety of geodetic problems over the past fifteen years (Sideris, 1994). However, such an approach provides only a special, and rather crude, treatment within a more general convolution-based estimation algorithm according to Eq. (3). Furthermore, the use of a 'reference' scaling kernel $\varphi(x)$ for the gridded input data can be particularly beneficial if spectral analysis procedures are to be applied to the estimated input and output fields, and/or if a rigorous study of the SISO approximation problem within a Hilbert space framework is sought (Blu and Unser, 1999; Kotsakis, 2000b). In this paper, we will confine our attention on a general qualitative description of the approximation performance of Eq. (3) using various interpolating models (including the delta function) at different data resolution levels. Of special importance for this purpose is the definition of a suitable error functional, which is presented in the sequel.

3 Aliasing Error Modelling

Let us now study the signal error produced by the linear estimation formula in Eq. (3), i.e.

$$\begin{aligned} e(x) &= \hat{f}(x) - f(x) \\ &= \sum_n g(nh) \left[\varphi(\frac{x}{h} - n) * s(x) \right] - s(x) * g(x) \end{aligned} \tag{7}$$

as a function of the data resolution level h and the reference scaling kernel $\varphi(x)$. It will be assumed that the discrete input values $g(nh)$ are noise free, and thus the output error $e(x)$ contains only the (deterministic) aliasing effect due to the finite sampling resolution. For cases with noisy input data in SISO linear estimation systems, see Sideris (1995) and Kotsakis (2000b).

Obviously, a pointwise description of the aliasing error requires the pointwise knowledge of the full input field. Similarly, a spectral analysis for $e(x)$ also demands the a priori knowledge of $g(x)$. For practical applications, we need to develop alternative measures for studying the behaviour of the output signal error, whose evaluation is based on more 'accessible' (or more easily modelled) characteristics of the input field, such as its spatial covariance (CV) function or its power spectrum. For this purpose, we should express the output error as an explicit function of three distinct spatial parameters, according to the general form

$$e(x, x_o, h) = \hat{f}(x, x_o, h) - f(x)$$
$$= \sum_n g(nh - x_o)\left[\varphi(\frac{x - x_o}{h} - n) * s(x)\right] - s(x) * g(x) \quad (8)$$

where x denotes the spatial point location at which the error is evaluated, and the two additional parameters (x_o, h) represent the sampling phase and the sampling resolution of the input data set, respectively. The last two quantities are not completely independent and they always satisfy the condition $-h/2 \le x_o \le h/2$; for an illustrative explanation of the sampling phase concept, see Kotsakis (2000a). The initial error formula in Eq. (7) is just a special case of Eq. (8) for zero sampling phase in the input data set (i.e. the uniform sampling starts at the origin of the reference system).

Note that if we average the term $e(x, x_o, h)$ (or even its squared value) over all sampling phase values x_o at a certain resolution level, we would still need to know the complete input field in order to compute such a mean (in a spatio-statistical sense) signal error $\tilde{e}(x, h)$.

3.1 Error Modelling in the Space Domain

Using the general error signal from Eq. (8), we can define a *spatial error covariance function* at a certain data resolution level and sampling phase value. Such a covariance function has the usual stationary-like form

$$c_e(\xi, x_o, h) = \int_{-\infty}^{+\infty} e(x, x_o, h)\; e(x + \xi, x_o, h)\, dx \qquad (9a)$$

and its value at the origin ('error variance') corresponds to the square L^2 error norm for the associated data sampling parameters x_o and h, i.e.

$$c_e(0, x_o, h) = \int_{-\infty}^{+\infty} e^2(x, x_o, h)\, dx = \| e(x, x_o, h) \|^2_{L^2} \qquad (9b)$$

However, such an error CV function still requires the complete (pointwise) knowledge of the input field $g(x)$. In order to overcome this limitation, we can now define a *mean error covariance function* over all possible sampling phase values, according to the integral formula

$$\tilde{c}_e(\xi, h) = \frac{1}{h} \int_{-h/2}^{h/2} c_e(\xi, x_o, h)\, dx_o \qquad (10)$$

The value of this CV function at the origin is denoted by $\tilde{\sigma}^2(h)$ ('mean error variance') and it corresponds to the mean square L^2 error norm averaged over all sampling phases for a certain data resolution level, i.e.

$$\tilde{\sigma}^2(h) = \tilde{c}_e(0, h) = \frac{1}{h} \int_{-h/2}^{h/2} \| e(x, x_o, h) \|^2_{L^2}\, dx_o \quad (11)$$

The above resolution-dependent error functional is a very convenient measure for studying the average performance of the signal estimation formula in Eq. (3). It is actually closely related to the spatio-statistical interpretation of collocation according to Sansò (1980), for the special case of gridded input data (Kotsakis, 2000b). It has also been considered in an interesting recent study on generalized interpolation models by Blu and Unser (1999). In the following, we will use a frequency domain methodology for the computation of the mean error variance $\tilde{\sigma}^2(h)$ from the power spectra of the theoretic convolution kernel $s(x)$, the reference scaling kernel $\varphi(x)$, and the input signal $g(x)$.

3.2 Error Modelling in the Frequency Domain

The Fourier transform of the mean error CV function in Eq. (10), considered as a function of ξ only, can easily be expressed in the average form

$$\tilde{C}_e(\omega, h) = \frac{1}{h} \int_{-h/2}^{h/2} \left| E(\omega, x_o, h) \right|^2 dx_o \qquad (12)$$

where $E(\omega, x_o, h)$ is the Fourier transform of the error signal in Eq. (8) with respect to the parameter x. Hence, the mean error CV function is just the inverse Fourier transform of the mean error power spectrum, where the 'mean' in both domains is meant in a spatio-statistical sense over all sampling phase values at a given data resolution level h. Using Eqs. (8) and (12), we can also obtain the following equation for the mean error power spectrum of the estimated output signal (Kotsakis, 2000b):

$$\tilde{C}_e(\omega, h) = \left| S(\omega) \right|^2 \; \tilde{C}_e^{in}(\omega, h) \qquad (13)$$

where $\left| S(\omega) \right|^2$ is the power spectrum of the theoretic convolution kernel $s(x)$, and $\tilde{C}_e^{in}(\omega, h)$ denotes the mean (spatio-statistical) power spectrum of the aliasing error in the interpolated input field $\hat{g}(x)$ according to Eq. (1). The formula in Eq. (13) expresses a frequency-domain propagation law for the mean aliasing error in the SISO linear estimation system of Fig. 3. The input aliasing error $\tilde{C}_e^{in}(\omega,h)$ depends directly on the chosen scaling kernel $\varphi(x)$ and it is given by the general form

$$\tilde{C}_e^{in}(\omega,h) = \left| G(\omega) \right|^2 \left[1 - \Phi(h\omega) - \Phi^*(h\omega) \right] + \\ + \left| \Phi(h\omega) \right|^2 \sum_{k \in Z} \left| G(\omega + \frac{2\pi k}{h}) \right|^2 \tag{14}$$

where $\left| G(\omega) \right|^2$ is the power spectrum of the true input field $g(x)$, and the superscript * denotes complex conjugation. For a proof of Eq. (14), see Kotsakis (2000a).

The computation of the mean error variance for the estimated output field, which is defined by Eq. (11), can now be performed in the frequency domain according to the integral formula

$$\tilde{\sigma}^2(h) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{C}_e(\omega,h)\, d\omega \tag{15a}$$

since $\tilde{c}_e(\xi,h)$ and $\tilde{C}_e(\omega,h)$ form a Fourier transform pair. For more details and discussion, see Kotsakis (2000b, ch. 5). Similarly, the mean error variance of the interpolated input field $\hat{g}(x)$ will be given by a corresponding integral formula, as follows:

$$\tilde{\sigma}_{in}^2(h) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{C}_e^{in}(\omega,h)\, d\omega \tag{15b}$$

In order to demonstrate the behaviour of the output mean error variance $\tilde{\sigma}^2(h)$ for different choices of the reference scaling kernel, a simple example is presented in Fig. 4. The error curves shown in this figure correspond to the case where the power spectrum of the input signal $g(x)$ follows the simple model $(1+\omega^2)^{-1}$, and the theoretical convolution kernel $s(x)$ is the Gaussian function $exp(-x^2)$. Five different choices for the estimation kernel $\varphi(x)$ have been considered, namely: (i) linear orthonormal B-spline function, (ii) linear interpolating B-spline function, (iii) cubic interpolating B-spline function, (iv) sinc (Shannon) kernel, and (v) Dirac delta function. As it was previously discussed, the latter case (i.e. $\Phi(\omega)=1$) is identical with a simple FFT-based numerical approximation of convolution integrals using gridded input values (parallelogram rule for numerical integration). For the analytical space and frequency domain expressions of the various B-spline kernels, see Unser (1999).

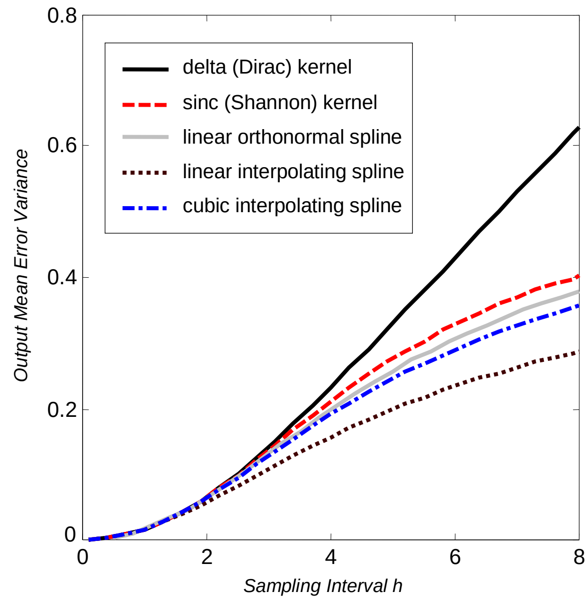

Fig. 4 Decay rate of the output mean error variance with respect to the input data resolution, using various reference scaling kernels $\varphi(x)$

3.3 Remarks

The above example gives an indication of the accuracy improvement in the convolution-based estimation algorithm of Eq. (3) due to the use of a proper scaling kernel $\varphi(x)$. Although the asymptotic decaying pattern of the mean error variance $\tilde{\sigma}^2(h)$ seems to be the same regardless of the reference estimation kernel, the performance of the FFT-based discrete methodology (i.e. *delta kernel case*, see Eq. (5)) worsens significantly as h increases. Of course, in order to evaluate the significance of such a difference in geodetic applications (e.g. local geoid determination from gravity data), we need to use realistic and properly scaled power spectra models for the input signal, as well as theoretic convolution kernels associated with actual geodetic problems. The result in Fig. 4 is merely a simple 1D numerical experiment, where the scales in both axes do not reflect any specific physical meaning.

Note that the square root of the mean error variance $\tilde{\sigma}^2(h)$ is not expressed in the same units as the actual output error $e(x,x_o,h)$, but it corresponds to a 'root-mean-square (rms)' value of the L^2 error norm in a spatio-statistical sense; see Eq. (11). For practical applications with compactly supported input fields $g(x)$, we can obtain a more useful average error estimate (expressed in the same units as the output signal) by simply dividing the value of $\tilde{\sigma}(h)$ by the spatial extent of the input data grid. For more details, see Kotsakis (2000b).

A brief note must be made regarding the consistency of the linear SISO system in Fig. 3, as the data resolution increases. Regardless of the choice of the reference scaling model $\varphi(x)$, the output signal error should eventually vanish in some sense, as the data sampling step *h* becomes infinitely small. Using as error measure at each data resolution level the spatio-statistical power spectrum $\tilde{C}_e(\omega,h)$, it is easily shown that

$$\lim_{h\to 0} \tilde{C}_e(\omega,h) = 0 \;\Leftrightarrow\; \lim_{h\to 0} \left| \Phi(h\omega) \right| = 1 \qquad (16)$$

The above condition imposes a very mild normalization constraint (i.e. $|\Phi(0)|=1$) for the admissibility of the scaling kernels that should be generally used in linear SISO estimation systems of the form shown in Fig. 3.

The study of the mean aliasing error as a function of the data resolution level can also be performed in the interpolated input signal $\hat{g}(x)$, according to the integral error formula of Eq. (15b). Note that in such case it is meaningless to consider the choice $\varphi(x)=\delta(x)$ for the reference scaling kernel, since no specific continuous model would be implied for the input field (i.e. $\hat{g}(x)$ is a sequence of Dirac impulses at the data points). Nevertheless, a considerable simplification of Eq. (15b) occurs if we restrict the scaling kernel to be a *symmetric* function. In this case, Eq. (15b) is equivalent to the formula

$$\tilde{\sigma}^2_{in}(h) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left| G(\omega)\right|^2 K(h\omega)\; d\omega \qquad (17a)$$

where the auxiliary kernel $K(\omega)$ depends only on the adopted interpolating model and it is given by the equation

$$K(\omega) = 1 - 2\,\Phi(\omega) + \sum_{k\in Z} \left(\Phi(\omega+2\pi k)\right)^2 \qquad (17b)$$

In the special case where the estimation kernel $\varphi(x)$ is also an *orthonormal* scaling function (Blu and Unser, 1999), the form of the auxiliary kernel $K(\omega)$ can be reduced to the very simple expression

$$K(\omega) = 2 - 2\,\Phi(\omega) \qquad (17c)$$

The detailed description and proof of the above error algorithm, as well as more discussion with specific examples, can be found in Kotsakis (2000b, ch. 5).

4 Conclusions

The aim of this paper was to introduce a novel analytic approach for studying the aliasing error in linear and translation-invariant SISO estimation systems, as a function of the input data resolution. Our methodology allows to compare the accuracy of the usual FFT-based numerical approximation of convolution integral formulas against more general linear estimation schemes, where a reference scaling kernel is employed for the gridded input data. Since the purpose of the paper was merely to give a short exposition of some general issues involved in aliasing error modelling for SISO linear systems, a more detailed theoretical treatment in a multi-dimensional setting is needed for geodetic applications.

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