Determination of the Achievable Accuracy of Relative GPS/Geoid Levelling in Northern Canada

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Abstract

It is well known that traditional spirit levelling, as a method for precise vertical positioning, suffers from a number of practical limitations caused by terrain roughness, harsh environmental conditions and restricted line-of-sight. In Canada, this is most evident when we look at the spatial distribution of vertical control stations, since the remote northern parts of the country are very poorly surveyed. A method that has been proven to be a useful and efficient alternative for vertical positioning in such environments is GPS-based levelling. A major advantage of GPS observations is that they are not affected (as much) by the practical limitations of spirit levelling. The achievable accuracy of this method, however, is still under question mainly because of datum inconsistencies and systematic errors inherent in the data. In this paper, a number of investigations are conducted to estimate the achievable accuracy of orthometric height determination in the northwestern parts of Canada, using GPS and geoid information in conjunction with various auxiliary parametric models (corrector surfaces) for describing datum offsets and systematic distortions. Specifically, the covariance (CV) matrix of the estimated parameters in the corrector surface model, and the combined relative accuracy of GPS and geoid data, are used to infer the accuracy of the orthometric height differences of newly established baselines in remote northern parts of Canada. A large test network consisting of the GPS benchmarks in western Canada is used for the computation of the covariance matrix of the estimated parameters in the corrector surface models, through a combined least-squares adjustment of GPS, levelling and geoid data. The results provide valuable insight into the role of the accuracy for the parameters in the corrector surface model for precise vertical positioning via GPS/geoid levelling.

1 Introduction

The purpose of this paper is to investigate the achievable accuracy of GPS/geoid levelling in northern Canada. There are several reasons for focusing our studies in northern Canada, including the numerous practical limitations that are involved in establishing vertical control and obtaining orthometric heights via differential spirit levelling, in such remote and largely unsurveyed territories. In an effort to obtain orthometric heights with respect to an established vertical datum, the concept of GPS-based levelling has been applied, which is based on the combination of three height types geometrically related by the following equation (Heiskanen and Moritz, 1967):

$$H_i = h_i - N_i \tag{1}$$

where h_i is the ellipsoidal height computed from GPS measurements, N_i is the geoidal undulation obtained from a gravimetric geoid model, and H_i refers to the Helmert orthometric height. In practice, the theoretical relationship given in Eq. (1) is not fulfilled, due to datum inconsistencies, data noise, and systematic biases inherent among the three height types. The major part of these discrepancies is usually attributed to the systematic effects and datum inconsistencies, which can be described by a *corrector surface* model such that

$$H_i = h_i - N_i - \mathbf{a}_i^T \mathbf{x} \tag{2}$$

where the bilinear term $\mathbf{a}_i^T \mathbf{x}$ describes the corrector surface.

The general idea described herein is to use the accuracy information of the different height types that is available from existing vertical control in densely surveyed areas (such as southwestern Canada) and propagate that information for determining the accuracy of the orthometric height difference for a newly established baseline in an area with no (or very limited) vertical control, such as northern Canada. The discussion begins with an overview of the formulations required for determining the accuracy of GPS/geoid levelling. This is followed by a description of how we simulate the accuracy for the different height data types. Finally, a number of numerical tests are performed, which provide interesting insight into the role of the corrector surface model accuracy for GPS/geoid levelling. Based on these results some conclusions are drawn and a brief mention for future work on this topic is provided.

2 GPS/Geoid Levelling Accuracy

For the purposes of this paper, we will focus on the relative model, where height differences are taken into account. Given the theoretical relationship among the three types of height data and the incorporation of an appropriate corrector surface model as shown in Eq. (2), the orthometric height difference ΔH_{kl} for a **new** baseline (k,l) as obtained from relative GPS/geoid levelling is given by

$$\Delta H_{kl} = \Delta h_{kl} - \Delta N_{kl} - (\mathbf{a}_l^T - \mathbf{a}_k^T)\hat{\mathbf{x}} \tag{3}$$

where $\hat{\mathbf{x}}$ is a vector containing the estimated parameters of the corrector surface model obtained through a combined adjustment of GPS/levelling/geoid data in a common control network, Δh_{kl} is the observed GPS height difference of the points (k,l), ΔN_{kl} is the geoid undulation difference, and $\mathbf{a}_l$, $\mathbf{a}_k$ correspond to the vectors of known coefficients of the selected parametric model (see below).

As it was mentioned previously, we are interested in the achievable *accuracy* of the orthometric height difference. By simply applying variance-covariance propagation to Eq. (3), the accuracy of GPS/geoid levelling for a new baseline can be obtained according to the following formula:

$$\sigma^2_{\Delta H_{kl}} = \sigma^2_{\Delta h_{kl}} + \sigma^2_{\Delta N_{kl}} + (\mathbf{a}_l^T - \mathbf{a}_k^T)\,\mathbf{C}_{\hat{\mathbf{x}}}\,(\mathbf{a}_l - \mathbf{a}_k) \tag{4}$$

where $\sigma^2_{\Delta h_{kl}}$ and $\sigma^2_{\Delta N_{kl}}$ represent the relative accuracy of the ellipsoidal and geoidal heights of the baseline, and $\mathbf{C}_{\hat{\mathbf{x}}}$ is the a-posteriori CV matrix of the estimated parameters in the corrector surface model. This CV matrix is obtained from a multi-data adjustment of relative GPS/geoid/levelling heights. Details of such combined adjustment problems are given in Kotsakis and Sideris (1999). The final form of $\mathbf{C}_{\hat{\mathbf{x}}}$ is given by

$$\mathbf{C}_{\hat{\mathbf{x}}} = \left[\mathbf{A}^T(\mathbf{C}_{\Delta h} + \mathbf{C}_{\Delta H} + \mathbf{C}_{\Delta N})^{-1}\mathbf{A}\right]^{-1} \tag{5}$$

where, $\mathbf{C}_{\Delta h}$, $\mathbf{C}_{\Delta H}$, and $\mathbf{C}_{\Delta N}$ denote the CV matrices for the relative GPS ellipsoidal heights, orthometric height differences and relative geoid heights in the control network, respectively. The design matrix $\mathbf{A}$ corresponds to the pre-selected parametric model. Its type varies in form and complexity depending on a number of factors. In this paper, three models have been tested ranging from a simple 3-parameter trigonometric model (its pointwise formulation is given in Heiskanen and Moritz, 1967, sec. 5-9) to a more complicated 6-parameter differential similarity transformation model (see Kotsakis et al., 2001). The general form of the corrector surface model for the relative case can be represented by

$$f_{ij} = \left(\mathbf{a}_j^T - \mathbf{a}_i^T\right)\mathbf{x} = \mathbf{a}_{ij}^T\mathbf{x} \tag{6}$$

where $\mathbf{a}_j, \mathbf{a}_i$ are vectors of known coefficients that depend on the horizontal location of the points (i,j) and $\mathbf{x}$ is a vector of unknown parameters. The three parametric models tested in this paper are

(a) 3-parameter model

$$\mathbf{a}_{ij} = \begin{bmatrix} \cos\varphi_j \cos\lambda_j - \cos\varphi_i \cos\lambda_i \\ \cos\varphi_j \sin\lambda_j - \cos\varphi_i \sin\lambda_i \\ \sin\varphi_j - \sin\varphi_i \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) 4-parameter model

$$\mathbf{a}_{ij} = \begin{bmatrix} \cos\varphi_j \cos\lambda_j - \cos\varphi_i \cos\lambda_i \\ \cos\varphi_j \sin\lambda_j - \cos\varphi_i \sin\lambda_i \\ \sin\varphi_j - \sin\varphi_i \\ \sin^2\varphi_j - \sin^2\varphi_i \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(c) 6-parameter model

$$\mathbf{a}_{ij} = \begin{bmatrix} \cos\varphi_j \cos\lambda_j - \cos\varphi_i \cos\lambda_i \\ \cos\varphi_j \sin\lambda_j - \cos\varphi_i \sin\lambda_i \\ \sin\varphi_j - \sin\varphi_i \\ \dfrac{\sin\varphi_j \cos\varphi_j \cos\lambda_j}{W_j} - \dfrac{\sin\varphi_i \cos\varphi_i \cos\lambda_i}{W_i} \\ \dfrac{\sin\varphi_j \cos\varphi_j \sin\lambda_j}{W_j} - \dfrac{\sin\varphi_i \cos\varphi_i \sin\lambda_i}{W_i} \\ \dfrac{\sin^2\varphi_j}{W_j} - \dfrac{\sin^2\varphi_i}{W_i} \end{bmatrix}$$

$$\mathbf{x} = [x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; x_5 \;\; x_6]^T$$

where, φ and λ are the horizontal geodetic coordinates of the network or baseline points, e is the eccentricity of a common datum ellipsoid and $W_{(\cdot)} = \sqrt{1-e^2 \sin^2 \varphi_{(\cdot)}}$. In the following three sub-sections, a description of how to compute the relative accuracy of the three height types, required as input for Eq. (5), is given.

2.1 Accuracy of the Levelling Data

The full covariance matrix for the orthometric height differences $\mathbf{C}_{\Delta H}$ in the multi-data adjustment of the control network, was computed the following formula:

$$\mathbf{C}_{\Delta H} = \mathbf{A}_{net}\mathbf{C}_H \mathbf{A}_{net}^T \tag{7}$$

where $\mathbf{A}_{net}$ is a design-type matrix (composed of –1, 1 and 0) corresponding to the baseline configuration of the multi-data network adjustment (see Fig. 2), and $\mathbf{C}_H$ is the covariance matrix for the absolute orthometric heights at all points in the test network. The latter quantity was simulated through a separate minimally constrained least-squares adjustment of the levelling part of the control network, as follows:

$$\mathbf{C}_H = \left(\mathbf{A}_{lev}^T \mathbf{P}_{lev} \mathbf{A}_{lev}\right)^{-1} \tag{8}$$

where $\mathbf{A}_{lev}$ is a design matrix (composed of –1, 1 and 0) corresponding to the baseline configuration of the levelling network adjustment (see Fig. 1), and $\mathbf{P}_{lev}$ is a diagonal weight matrix that takes into account the measuring accuracy of each levelling baseline in the test network. In this case, three different orders of accuracy were used for assigning the a-priori values $\sigma_{\Delta H}$ for each levelling baseline in the weight matrix $\mathbf{P}_{lev}$, namely $0.7mm\sqrt{d(km)}$, $1.3mm\sqrt{d(km)}$ and $2mm\sqrt{d(km)}$, referring to first, second and third order respectively. National standards for the accuracy of vertical control vary depending on the country. In our case, the U.S. standards were implemented, as they were readily available (NGS, 1994). For the case of levelling, larger baselines ($d > 80km$) usually constitute part of a national levelling campaign and adhere to first order levelling standards, followed by denser regional levelling campaigns ($30km < d \leq 80km$) of second order, and finally local levelling lines ($d \leq 30km$) which are of third order accuracy.

2.2 Accuracy of the GPS Data

A similar procedure to the one described in the previous section was followed in order to obtain the relative accuracy of the ellipsoidal heights $\mathbf{C}_{\Delta h}$ in the control test network used for the multi-data adjustment (see Eq. (5)). The final formulation for $\mathbf{C}_{\Delta h}$ is given by

$$\mathbf{C}_{\Delta h} = \mathbf{A}_{net}\mathbf{C}_h \mathbf{A}_{net}^T \tag{9}$$

where $\mathbf{A}_{net}$ is the same design matrix that was used in Eq. (7) and it corresponds to the baseline configuration of the multi-data network adjustment, and $\mathbf{C}_h$ is the covariance matrix for the ellipsoidal heights at all points of the test network. The latter was obtained through a separate minimally constrained least-squares adjustment for the GPS part of the control network, according to the formula:

$$\mathbf{C}_h = \left(\mathbf{A}_{GPS}^T \mathbf{P}_{GPS} \mathbf{A}_{GPS}\right)^{-1} \tag{10}$$

where $\mathbf{A}_{GPS}$ is a design matrix (composed only of –1, 1 and 0) corresponding to the baseline configuration of the GPS network adjustment (see Fig. 1). The stronger overall geometry used in the GPS-based network as compared to a levelling network is evident from Fig. 1, as the GPS network is composed of all of the baselines in the levelling network as well as additional baselines (dashed lines). The weaker geometrical configuration of the levelling network is due to the stringent line-of-sight restrictions of spirit levelling (Ollikainen, 1997). Here we see the advantages of using GPS from a network point of view, especially in rough terrain areas such as western Canada, where traditional spirit levelling is a difficult and laborious task, to say the least. Finally, $\mathbf{P}_{GPS}$ is a diagonal weight matrix that takes into account the measuring accuracy of the vertical component for each GPS baseline in the test network. In our analysis, ten different orders for the relative ellipsoidal height accuracy are used as defined by the U.S. standards (NGS, 1994). The a-priori values $\sigma_{\Delta h}$ in the weight matrix $\mathbf{P}_{GPS}$ were assigned based on the length of each GPS baseline in the test network, such that the accuracy of the baseline degrades as the baseline length increases (due to the spatial decorrelation of GPS errors, see Fotopoulos et al., 2001).

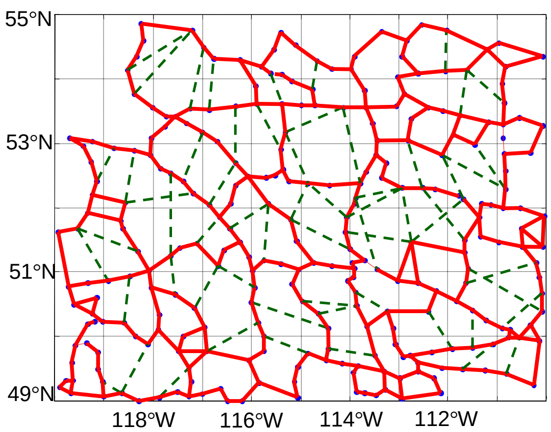

Fig. 1 Levelling (solid lines) and GPS (solid and dashed lines) network configurations

2.3 Accuracy of the Geoid Heights

Following the separate simulative adjustments of the levelling and GPS parts of the control network, full CV matrices $\mathbf{C}_{\Delta h}$ and $\mathbf{C}_{\Delta H}$ for the adjusted height differences were obtained. As stated previously, these CV matrices were used as input to a final integrated multi-data adjustment in order to obtain $\mathbf{C}_{\hat{\mathbf{x}}}$ (see Eq. (5)). The input accuracy for the geoid undulation differences $\mathbf{C}_{\Delta N}$ in the multi-data test network was approximated in a slightly different manner. A 1°×1° world-wide grid of the commission errors of the global geopotential model, EGM96 (Lemoine et al., 1998), as computed from the accuracy of the spherical harmonic coefficients up to degree and order 70 was used to interpolate the σ_N of each benchmark in the control test network. For our test network area (southwestern Canada), the 1°×1° grid resolution of the geoidal undulation errors was sufficient as the gravity coverage employed for this area in the computation of EGM96 was reasonably dense and homogeneously spaced (*ibid.*). It should be noted that, although a higher degree of expansion (i.e. 180 or 360) may theoretically recover higher frequency information (thus reducing the aliasing error), the noise is also increased as the number of coefficients increases. By using this global geoid error model and bilinearly interpolating the grid of commission errors for the points in the test network, a diagonal error CV matrix of the absolute geoid height errors $\mathbf{C}_N$ was obtained. The computation of the relative geoid height accuracy $\mathbf{C}_{\Delta N}$, required as input into Eq. (5), was obtained by propagating the absolute height accuracy as follows:

$$\mathbf{C}_{\Delta N} = \mathbf{A}_{net}\mathbf{C}_N\mathbf{A}_{net}^T \qquad (11)$$

where $\mathbf{A}_{net}$ is the same design matrix that was used in Eqs. (7) and (9), corresponding to the baseline configuration of the <u>multi-data network</u> adjustment. This results in a fully populated form of $\mathbf{C}_{\Delta N}$.

3 Description of the Numerical Tests

Although this study is based on the *accuracy* information of the different height data types and therefore does not make use of actual height data values, the tests were designed to mirror realistic conditions. The test network area used for the multi-data adjustment contains a subset of the GPS benchmarks in the southwestern part of Canada spanning parts of British Columbia and Alberta (275 points in total, see boxed network in Fig. 2). This area has a fairly good distribution of GPS benchmarks covering $49^\circ \le \varphi \le 55^\circ$ and $-120^\circ \le \lambda \le -110^\circ$.

Several test scenarios were conducted to determine the achievable accuracy of orthometric height differences via GPS/geoid levelling by varying the basic simulation parameters, such as:

- new baseline length
- location of new baseline with respect to original test network
- type of parametric model (see Sec. 2)
- form of the CV matrices $\mathbf{C}_{\Delta h}$, $\mathbf{C}_{\Delta H}$, and $\mathbf{C}_{\Delta N}$ required as input into Eq. (5)

It should be noted that sometimes, the estimation of the fully populated CV matrices may not be feasible or computationally efficient (due to the required inversion). In these cases, approximations of the CV matrices may be made in a diagonal form where only the variances of the baselines are included and the covariances between baselines are set to zero. In the following section, some of the key findings from the test scenarios mentioned above will be discussed.

4 Analysis of Results

Numerous tests were conducted by varying the new baseline length from a minimum of 10 km to a maximum of 100 km. As expected, it was found that the longer the length of the new baseline, the poorer the achievable ΔH accuracy from GPS/geoid levelling. This is partly due to the spatial decorrelation of GPS errors. The contribution of the corrector surface model error also increases as the baseline length increases due to its dependence with the horizontal locations of the points.

Overall, it was found that the major error contributor of the three height types to the final result was $\mathbf{C}_{\Delta N}$. As only a global geoid model was used to obtain the aforementioned CV matrix, the accuracy was significantly poorer than those selected for precise levelling and GPS heights. The error CV matrix for $\mathbf{C}_{\Delta N}$ may also be approximated by employing the internal precision of a local/regional geoid solution which is supported with very dense local gravity, height and density data. In such cases, the variances and covariances of the geoid height differences are usually smaller than those obtained from a global geoid error model. However, such a reliable model may not always be readily available, especially in remote areas.

By varying the type of corrector surface model used in the adjustment, the achievable accuracy also changed. For instance, for a 50 km baseline located in the north (φ = *60°N*, λ= *116°W*) with input accuracies of $0.6 \cdot \sqrt{d(km)}$ and *23* cm for GPS and

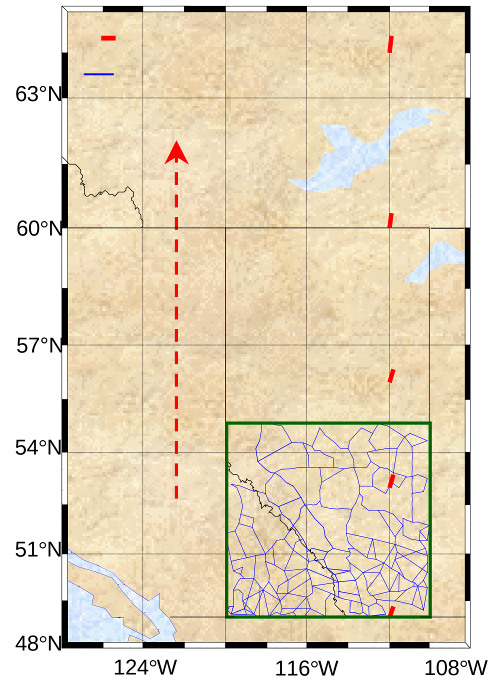

Fig. 2 Test network coverage area and locations of newly established baselines in the north

geoid height differences, respectively, the resultant relative orthometric height accuracy was 48.3 cm with the 3-parameter model and 52.4 cm with the 6-parameter model. Similarly, for a 100 km baseline the achievable accuracy degraded by ~ 13 cm with the 6-parameter model as compared to the 3 or even 4-parameter models. It is evident from these results that the more parameters in the model, the more amplified its error contribution is (see Table 1). This is interesting, as we are not evaluating the performance/fitting of different parametric models (no actual height data used), rather we focus on how the random errors flow through our model and affect the accuracy of the final value.

Table 1 provides a summary of some of the numerical results. The first column refers to the approximate latitude of a 40 km baseline *(λ ~ 112°W)*. The latitude varies as the newly established baseline is moved northward with respect to the test network area (see Fig. 2). The column labeled *3rd term* refers to the accuracy contribution of the corrector surface model (3rd term in Eq. (4)). There are three main groups of results, where *Full CVs* refers to fully populated covariance matrices for the three height types, *Diag. CVs* refers to diagonal covariance matrices for the three height types, and

Full & Diag. refers to fully populated CV matrices for levelling and GPS heights and a diagonal covariance matrix for the geoid heights. All results are based on input accuracies of $\sigma_{\Delta N} = 15$ cm and $\sigma_{\Delta h} = 0.15 \cdot \sqrt{d(km)}$ cm for Eq. (4).

Table 1. Results of Baselines Moving Northward from the Test Network Area (all values in cm)

φ	Full CVs		Diag. CVs		Full & Diag.	
	$\sigma_{\Delta H}$	3rd term	$\sigma_{\Delta H}$	3rd term	$\sigma_{\Delta H}$	3rd term
3 - Parameter Corrector Surface Model						
49°	17.8	0.1	18.0	2.7	18.0	2.7
53°	17.8	0.6	18.0	3.6	18.1	3.6
56°	17.8	1.1	18.8	6.0	18.8	6.0
60°	17.8	1.8	20.2	9.6	20.2	9.6
64°	17.9	2.5	22.2	13.3	22.2	13.3
6 - Parameter Corrector Surface Model						
49°	17.8	1.4	18.3	4.6	18.3	4.6
53°	17.8	0.8	18.3	4.5	18.3	4.5
56°	18.5	5.1	24.4	16.8	24.4	16.8
60°	24.2	16.4	54.5	51.6	54.6	51.6
64°	36.9	32.4	102.2	100.6	102.3	100.8

The results show that the achievable $\sigma_{\Delta H}$ for the new baseline was worse as it moved farther north from the control test network. This was mainly due to the increased error contribution of the corrector surface model parameters. Perhaps the most interesting result was the difference between using fully populated and (approximate) diagonal CV matrices for Δh, ΔH, and ΔN as input into Eq. (5). In studies where the newly established baseline was located *within* the test network coverage area, it was found that there was no difference between using fully populated versus (more approximate) diagonal CV matrices. However, in cases where the baseline is moved farther north (away from) the test network area, differences up to several tens of centimeters resulted. This result is quite significant as it indicates that approximate versions of the CV matrices should *not* be used, for new baselines situated *away from* the test network, in order to take advantage of the highest accuracy that GPS-based levelling provides.

5 Conclusions and Future Work

A method for evaluating the accuracy of relative GPS/geoid levelling was presented. Specifically, the achievable accuracy for tests in northern Canada was evaluated. The overall accuracy was affected by the relatively poor accuracy of the global geoid model (as compared to GPS and levelling accuracies). Of significance for future studies was the increased error contribution of the corrector surface based on the number of model parameters. Also, as the baseline was moved farther north, the significance of using fully populated versus diagonal CV matrices for the three height types became evident. In these cases, the achievable accuracy for ΔH varied as much as several tens of centimeters.

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