Chapter 2

Linear Differential Equations and The Laplace Transform

2.1 Introduction

2.1.1 First and Second-Order Equations

A number of situations of practical interest involve a *dynamic* variable[1], say x, whose rate of change is proportional to the present value of x, so that

$$\frac{dx}{dt} = kx,$$

for some real k. If x_0 is the starting value, the solution is then

$$x(t) = x_0 e^{kt}, \quad t \geq 0,$$

which, for $k < 0$ decays exponentially and, for $k > 0$, grows exponentially, in principle without bound. We say that for $k < 0$, the system is **stable**, while for $k > 0$ it is **unstable**. If $k = 0$, $x(t) = x_0$. Also, the larger $|k|$ is, the faster the transient (to zero in the stable case or to *infinity* in the unstable one.)

An even larger number of practical systems is adequately modeled by the second-order equation

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = 0, \tag{2.1}$$

[1] i.e. one changing over time

where $\omega_0 > 0$ is called the *natural frequency* and ζ the *damping ratio* of the system; it is usually taken to be non-negative, $\zeta \geq 0$, but **negative** damping is also possible, for instance in electronic circuits that include *negative resistance*.

The method for finding the solution of 2.1 for initial conditions x_0 and $\dot{x}_0$ involves the **auxilliary equation**

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0, \tag{2.2}$$

a second-order polynomial equation that has, therefore, two roots. We shall start calling the roots of auxilliary equations **poles**; this term will then generalize to equations of higher order.

Because of the special form that we used to write equation 2.1, the two roots are

$$s = \omega_0(-\zeta \pm \sqrt{\zeta^2 - 1}).$$

When there is no damping, $\zeta = 0$, we get the *harmonic oscillator* solution at the natural frequency ω_0:

$$x(t) = x_0 \cos(\omega_0 t) + \frac{\dot{x}_0}{\omega_0} \sin(\omega_0 t).$$

For small damping, $0 < \zeta < 1$, we have a pair of complex poles with nonzero imaginary part,

$$s = \omega_0(-\zeta \pm i\sqrt{1 - \zeta^2}),$$

and the solution is a sinusoidal wave of frequency

$$\omega = \omega_0\sqrt{1 - \zeta^2}$$

modulated by the exponential envelope $\pm e^{-\zeta\omega_0 t}$:

$$x(t) = e^{-\zeta\omega_0 t}(x_0 \cos(\omega t) + \frac{\dot{x}_0 + \zeta\omega_0 x_0}{\omega} \sin(\omega t)) \tag{2.3}$$

This is the **underdamped** case.

At the critical damping $\zeta_c = 1$, the solution is

$$x(t) = x_0 e^{-\omega_0 t} + (\dot{x}_0 + \omega_0 x_0) t e^{-\omega_0 t}.$$

Finally, the *overdamped* case gives two real poles

$$s = \zeta\omega_0(-1 \pm \sqrt{1 - \frac{1}{\zeta^2}})$$

and hence a solution that is a sum of real exponentials.

Exercise 2.1. Which configuration of poles of the general second-order system

$$\ddot{x} + a\dot{x} + bx = 0$$

is not possible for the system 2.1 above? *(Hint: Note the positive x term in 2.1.)*

It is very helpful to consider how the *poles* move in the complex plane $\mathbb{C}$ as the damping ζ increases from zero. The harmonic oscillator corresponds to poles on the imaginary axis; underdamped motion corresponds to a pair of complex conjugate poles, while critical damping means a double (by necessity) real pole. The overdamped poles move on the real axis and in opposite directions as ζ increases (see Figure 2.1.) One of the main topics of this

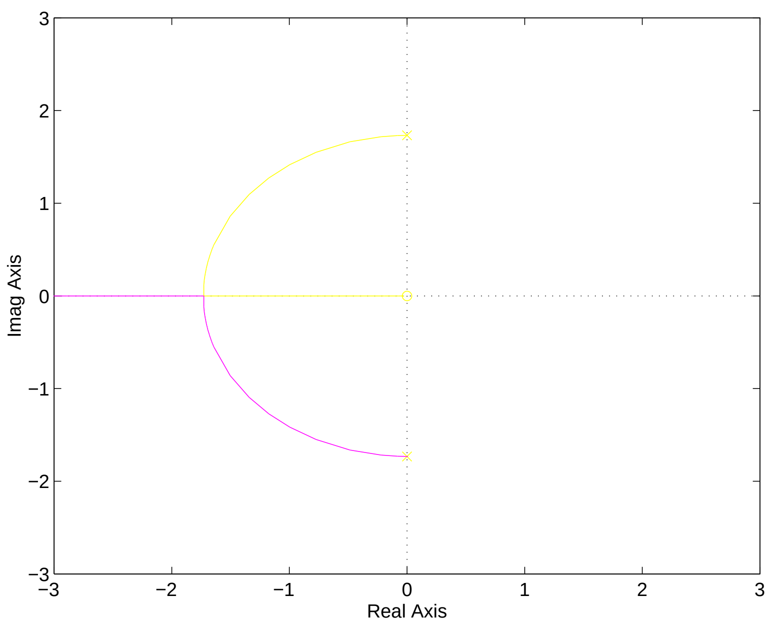

Figure 2.1: The movement of poles as a function of the damping ratio ζ

course, the *root locus* method, is in essence a generalization of the diagram of Figure 2.1. It shows the movement of poles as a parameter on which they depend is varied.

2.1.2 Higher-Order Equations and Linear System Models

There is no reason to limit ourselves to equations of second order. The general nth order homogeneous linear differential equation is

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = 0. \tag{2.4}$$

When the functions a_i are constant, we talk of *constant-coefficient linear differential equations*; this is the class of equations that will be of concern in this course.

In physical systems, the homogeneous equation describes the **unforced system**, roughly when external forces are zero. When we have a non-zero right-hand side, the *forced system* that is obtained corresponds to our notion of a **controlled system**.

The main point of the analysis we are about to present is that, in a natural sense to be made precise soon, **no new behaviour is observed** beyond that of first and second-order systems.[2] The key is a generalization of the auxilliary equation method that reveals the **modal structure** of linear systems. It is based on the Laplace transform.

In order to motivate the introduction of the Laplace transform, let us look at a **linear system** which, for now, will mean a system with input u and output y, the two being related through a linear, constant-coefficient ordinary differential equation (see Figure 8.1 of Chapter 1.) This implies that there is a *linear relation* involving the derivatives of u and y—up to a certain maximal order, say n. *Linear relation* here means the same as a *linear combination* of vectors in vector algebra. Thus we assume that the linear system is described by the equation

$$\sum_{i=0}^{n} a_{n-i}\frac{d^i y}{dt^i} = \sum_{j=0}^{m} b_{m-j}\frac{d^j y}{dt^j}, \tag{2.5}$$

where the coefficients $\{a_i\}$ and $\{b_j\}$ are constant and $a_0 \neq 0$. We assume that $n \geq m$ (for reasons that will become clear later.)

[2]This is not quite true, since we could have, for example, multiple complex pairs; these are quite rare in practice, though—non-generic in the modern terminology. The reader is advised to think of how a real polynomial can always be factorized into linear and quadratic factors, corresponding to real roots or pairs of complex roots, respectively.

Now suppose the input is given by the complex exponential function

$$u(t) = e^{st}$$

(remember how the auxilliary equation was derived.) The parameter s is called a *complex frequency*. Since taking the derivative of this waveform is the same as multiplying it by s, let us check that an output of the form

$$y(t) = g(s)e^{st},$$

for $g(s)$ some (in general, complex) number, actually *solves* the differential equation (more precisely is a particular integral of the d.e.) Substituting the given u and y into equation 2.5, we get

$$g(s)(\sum_{i=0}^{n} a_{n-i}s^i)e^{st} = (\sum_{j=0}^{m} b_{m-j}s^j)e^{st}. \tag{2.6}$$

Now, **provided s is not a zero of the algebraic equation**

$$\sum_{i=0}^{n} a_{n-i}s^i = 0$$

we can satisfy the above differential equation with

$$g(s) = \frac{\sum_{j=0}^{m} b_{m-j}s^j}{\sum_{i=0}^{n} a_{n-i}s^i}.$$

The rational function expression on the right-hand side is called the *transfer function* of the linear system. This is similar to the **eigenvalue/eigenvector** concept for linear transformations (matrices.) We say that the complex exponential e^{st} is an **eigenvector** of the linear system, with **eigenvalue** the complex number $g(s)$. The intervention of complex numbers in a *real* number context should not worry us more that the use of complex numbers in solving the eigenvalue problem in matrix analysis (necessitated by the fact that a polynomial with real coefficients may have complex roots.)

In the next section, we start by defining the Laplace transform and giving some of its properties. We then give the method for finding the general solution of equation 2.4.

Exercise 2.2. Find the complete solution $y = y_{CF} + y_{PI}$ of the second-order equation

$$\ddot{y} + a\dot{y} + by = u,$$

where $u(t) = e^{st}$ and s is not a root of the auxilliary equation. Under what conditions does the complementary function y_{CF} go to zero as $t \to \infty$?

Hence verify that the transfer function of the linear system with input u and output y described by above the linear differential equation is

$$g(s) = \frac{1}{s^2 + as + b}$$

and justify the statement that the output $g(s)u(t)$ is the steady-state response of the linear system to the input $u = e^{st}$.

2.2 The Laplace Transform Method

The reader may have already met **Fourier series** expansions of periodic signals $x(t+T) = x(t)$

$$x(t) = \sum_n c_i e^{2\pi int/T},$$

where the coefficients are given by

$$c_n = \frac{1}{T}\int_0^T x(t)e^{-2\pi int/T}dt$$

or the **Fourier transform** of finite energy signals,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

The Laplace transform is a more powerful technique and applies to a much larger class of signals than either of the above methods. The limitation is that we consider time functions on the half-infinite interval $[0, \infty)$ only, instead of the whole real axis.

Definition 2.1. If $f(t)$, $t \geq 0$ is a signal, the **Laplace transform** $\mathcal{L}(f)$ of f is defined by

$$\mathcal{L}(f)(s) = \int_0^{\infty} f(t)e^{-st}dt, \tag{2.7}$$

where s is a complex number for which the integral converges.

Roughly, the Laplace transform is a function of the complex argument s, well-defined at least in the region of the complex plane where the above integral converges. Since most of us are used to thinking of integrals in the *real* domain, this intrusion of complex numbers may seem confusing. If you are happier with real integrals, use the polar form

$$e^{st} = e^{(\sigma+i\omega)t} = e^{\sigma t}(\cos\omega t + i\sin\omega t) \tag{2.8}$$

and write the defining integral in the form

$$\mathcal{L}(f)(\sigma,\omega) = \int_0^\infty f(t)e^{-\sigma t}\cos(\omega t)dt + i\int_0^\infty f(t)e^{-\sigma t}\sin(\omega t)dt, \tag{2.9}$$

involving only real integrals. In practice this is unnecessary and one considers s as a *parameter* and integrates as if it were real. The only concern is then with whether the upper limit at ∞ is defined. Let us look at the simplest example.

Example 2.1. The Laplace transform of the real exponential function e^{at} (a real, but arbitrary) is

$$\mathcal{L}(e^{at})(s) = \int_0^\infty e^{-(s-a)t}dt = \lim_{T\to\infty}\left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^T. \tag{2.10}$$

Here the upper limit is

$$\lim_{t\to\infty} e^{-(\sigma-a)t}e^{-i\omega t}$$

which is zero provided $\sigma > a$.

The condition $\sigma > a$ defines a *half-plane* in $\mathbb{C}$, to the right of the line

$$\{s \in \mathbb{C}\,; \quad \Re(s) > a\}.$$

Note that, for a negative (a decaying exponential), this half-plane contains the imaginary axis. For a positive (an *'unstable'* exponential in positive time), the imaginary axis is not in the region of convergence of the integral. This is related to the fact that the Fourier transform is defined for a stable exponential, but not for an unstable one.

Now we can write down the Laplace transform of e^{at} by only evaluating the lower limit $t \to 0$:

$$\boxed{\mathcal{L}(e^{at}) = \frac{1}{s-a}} \tag{2.11}$$

Remark 2.1. Note that the right-hand side of equation 2.11 is in fact *defined for all complex numbers $s \neq a$* (the whole $\mathbb{C}$ plane except for a single point on the real axis.) This arises, of course, from the notion of *analytic continuation*, but we leave these considerations to a course in complex analysis. In this course, we shall quickly forget the convergence issue and work with the analytically continued transform, implicitly assuming that a suitable region can be found to make the evaluation at the upper limit vanish.

Exercise 2.3. By writing

$$\cos \omega_0 t = \frac{1}{2}(e^{i\omega_0 t} + e^{-i\omega_0 t})$$

or otherwise check that the Laplace transform of $\cos \omega_0 t$ is

$$\boxed{\mathcal{L}(\cos(\omega_0 t)) = \frac{s}{s^2 + \omega_0^2}} \tag{2.12}$$

(What is the region of convergence?)

Similarly, check that

$$\boxed{\mathcal{L}(\sin(\omega_0 t)) = \frac{\omega_0}{s^2 + \omega_0^2}} \tag{2.13}$$

2.2.1 Properties of the Laplace Transform

We have started building up a list of **Laplace transform pairs** which are going to be of use later on in solving differential equations and in handling control problems of interest. In order to add to the list, the best way forward is to first develop some basic properties of the Laplace transform and use them to derive transforms of larger and larger classes of functions. From now on, we shall use the notation

$$X(s) = \mathcal{L}(x(t))(s).$$

Since the Laplace transform is defined by the integral of x multiplied by the complex exponential function e^{-st}, it is clearly **linear** in x; in other words

$$\mathcal{L}(ax_1(t) + bx_2(t)) = a\mathcal{L}(x_1(t)) + b\mathcal{L}(x_2(t)) = aX_1(s) + bX_2(s) \tag{2.14}$$

Property 1 (Frequency-shift). *If $X(s)$ is the Laplace transform of $x(t)$, then, for a a real number,*

$$\mathcal{L}(e^{-at}x(t)) = X(s+a)$$

The proof is straightforward:

$$\mathcal{L}(e^{-at}x(t)) = \int_0^\infty x(t)e^{-(s+a)t}dt.$$

(Note that this changes the region of convergence.)

As an application, derive the Laplace transform pairs

$$\boxed{\mathcal{L}(e^{-at}\cos(\omega_0 t)) = \frac{s+a}{(s+a)^2+\omega_0^2}} \tag{2.15}$$

and

$$\boxed{\mathcal{L}(e^{-at}\sin(\omega_0 t)) = \frac{\omega_0}{(s+a)^2+\omega_0^2}} \tag{2.16}$$

Property 2 (Differentiation of Laplace Transform). *If $X(s)$ is the Laplace transform of $x(t)$, then*

$$\frac{dX}{ds} = \mathcal{L}(-tx(t)).$$

More generally,

$$\frac{d^nX(s)}{ds^n} = \mathcal{L}((-t)^n x(t)).$$

The proof is again direct:

$$\frac{d}{ds}\int_0^\infty x(t)e^{-st}dt = \int_0^\infty x(t)\frac{de^{-st}}{ds}dt = \int_0^\infty (-t)x(t)e^{-st}dt.$$

The $(-1)^n$ factor can be taken to the other side, if desired. Thus, for example,

$$\boxed{\mathcal{L}(te^{-at}) = \frac{1}{(s+a)^2}} \tag{2.17}$$

and, more generally,

$$\boxed{\mathcal{L}(t^ne^{-at}) = \frac{n!}{(s+a)^{n+1}}} \tag{2.18}$$

(Note how the minus signs conveniently disappear from the above formuæ.)

The case $a = 0$ is traditionally listed separately:

$$\boxed{\mathcal{L}(1) = \frac{1}{s}} \qquad \boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}} \tag{2.19}$$

Let us remember that what we mean by the constant function 1 is, in fact, only the part on the closed half-infinite line $[0, \infty)$. In the Engineering literature, this is called the **Heaviside unit step function** and is written

$$h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Some texts have the highly recommended habit of multiplying any function whose Laplace transform is to be taken by the Heaviside step function. Thus $\mathcal{L}(x(t))$ is really

$$\mathcal{L}(h(t)x(t)).$$

Since the lower limit of the integral defining the transform is zero and since h is identically equal to one on the positive real axis, this is harmless and prevents confusion between, for example, the cos function in $(-\infty, \infty)$ and the 'cos' function in the interval $[0, \infty)$ whose Laplace transform was found to be $s/(s^2 + \omega_0^2)$.

The above may seem a bit pedantic, until we get to the point where we address the problem of using the Laplace transform to solve differential equations *with given initial conditions*. For this, we use the next property of $\mathcal{L}$. Let $x(t)$ be a smooth real function, considered on the whole of the real axis; if we only consider the positive real axis, we require smoothness on $t > 0$, but we may have a jump at 0. Using the familiar right and left limits

$$x(0^+) = \lim_{\epsilon \to 0, \epsilon > 0} x(t + \epsilon)$$

and

$$x(0^-) = \lim_{\epsilon \to 0, \epsilon > 0} x(t - \epsilon),$$

we require

$$x(0) = x(0^+).$$

This condition is called **right-continuity**. The derivative of x is assumed to exist also for all $t \geq 0$.

Property 3 (Transform of Derivatives). *The Laplace transform of the derivative of a function $x(t)$ satisfying the conditions above is*

$$\mathcal{L}(\frac{dx}{dt}) = sX(s) - x(0). \tag{2.20}$$

More generally, if all the derivatives and limits exist and are right-continuous,

$$\mathcal{L}(\frac{d^n x}{dt^n}) = s^n X(s) - x(0)s^{n-1} - \dot{x}(0)s^{n-2} - \ldots - \frac{d^{n-2}x}{dt^{n-2}}(0)s - \frac{d^{n-1}x}{dt^{n-1}}(0). \tag{2.21}$$

The proof of 3 uses integration by parts.

$$\int_0^\infty \frac{dx}{dt} e^{-st} dt = \left[x(t)e^{-st}\right]_0^\infty - \int_0^\infty x(t)\frac{de^{-st}}{dt} dt$$

and the result follows since we have assumed (see remark 2.1) that s is chosen so that the upper limit of the fisrt term vanishes. (Also notice that the derivative is with respect to t this time.)

Exercise 2.4. Rederive the transform of $\sin \omega_0 t$ from the transform of $\cos \omega_0 t$ and the identity $d \cos x/dt = -\sin x$. Do not forget to use the initial value of cos at $t = 0$.

Property 4 (Positive Time Shift). *Suppose*

$$X(s) = \mathcal{L}(h(t)x(t)).$$

For any positive time τ, the transform of the function x shifted by time τ is

$$\mathcal{L}(h(t-\tau)x(t-\tau)) = e^{-\tau s} X(s). \tag{2.22}$$

Since τ is positive,

$$\mathcal{L}(h(t-\tau)x(t-\tau)) = \int_\tau^\infty x(t-\tau)e^{-st} dt$$

and the change of variable $u = t - \tau$ gives

$$\mathcal{L}(h(t-\tau)x(t-\tau)) = \int_0^\infty x(u)e^{-s(u+\tau)} du = e^{-s\tau} X(s),$$

as claimed.

2.2.2 Solving Linear Differential Equations

We now have all the tools we need to give a complete solution method for linear, constant-coefficient ordinary differential equations of arbitrary degree. The steps of the method will be made clear through an example.

Example 2.2. Consider the second-order differential equation

$$\ddot{y} + 2\dot{y} + 5y = u(t),$$

and the initial conditions

$$y_0 = 1, \quad \dot{y}_0 = -1.$$

Taking the Laplace transform of both sides and using the properties of the transform, we have

$$(s^2 Y(s) - sy_0 - \dot{y}_0) + 2(sY(s) - y_0) + 5Y(s) = U(s). \tag{2.23}$$

This algebraic (polynomial) relation between the transforms of y and u, $Y(s)$ and $U(s)$ is then solved for $Y(s)$ to give

$$Y(s) = \frac{U(s) + (s+2)y_0 + \dot{y}_0}{(s^2 + 2s + 5)} = \frac{U(s) + (s+1)}{(s^2 + 2s + 5)}. \tag{2.24}$$

If we are considering the response to a specific input, say $u(t) = 2t - 1$, then

$$U(s) = 2\frac{1}{s^2} - \frac{1}{s} = \frac{2-s}{s^2}$$

and

$$Y(s) = \frac{2 - s + s^2(s+1)}{s^2(s^2 + 2s + 5)} = \frac{s^3 + s^2 - s + 2}{s^2(s^2 + 2s + 5)}. \tag{2.25}$$

Finally, we invert the Laplace transform of Y to get the time signal $y(t)$, for $t > 0$. This involves breaking the denominator polynomial into its simple (irreducible) factors and then writing the ratio of polynomials as a sum of simpler terms, each involving just one of the simple factors of the denominator.

We shall look into this procedure in detail in the next section. For now, let us formulate the most general problem that can be treated using this method.

Definition 2.2. A rational function $\frac{n(s)}{d(s)}$ is called **proper** if the denominator degree is at least equal to the numerator degree, $\deg d(s) \geq \deg n(s)$.

A rational function is **strictly proper** if $\deg d(s) > \deg n(s)$.

Definition 2.3. A **linear differential relation** between the variables

$$x_1, \ldots, x_n$$

is the equating to zero of a finite linear sum involving the variables and their derivatives. We denote such a relation by

$$\mathcal{R}(x_1, \ldots, x_n) = 0.$$

This means an expression involving the variables and their derivatives, each term involving a single variable with a **constant coefficient**. Thus, a linear differential relation cannot involve *products* of the variables or their derivatives.

Since we assume that the sum is finite, there is a maximal derivative defined for each variable and hence a maximal degree of the relation. Thus, in the linear differential relation

$$\ddot{x}_2 + 3x_1^{(iv)} - x_2 + 10\dot{x}_1 = 0$$

the maximal degree of the relation is four. We write $\deg x_i$ for the maximal degree of the variable x_i in the relation.

It is easy to write down a linear differential relation in the Laplace transform domain, making use of the properties of the Laplace transform, in particular Property 3. Note that we do not assume zero initial conditions. Thus, $\mathcal{R}$ becomes

$$a_1(s)X_1(s) + a_2(s)X_2(s) + \ldots + a_n(s)X_n(s) = b(s).$$

Here $X_i(s)$ are the transforms of the variables $x_i(t)$, $a_i(s)$ are polynomials of degree at most equal to the maximal degree of the relation $\mathcal{R}$ and $b(s)$ is a polynomial arising from the initial conditions.

If we are given, say, N linear relations $\mathcal{R}_1, \ldots, \mathcal{R}_N$, the equivalent description in the s-domain is therefore nothing but a **system of linear equations** in the variables $X_1, \ldots, X_n$, the difference being that the coefficients are **polynomials**, rather than *scalars*. We can use the notation

$$A(s)X = \mathbf{b}(s), \tag{2.26}$$

where $A(s)$ is a matrix of dimension $N \times n$ with polynomial elements and $\mathbf{b}$ is an n-vector with polynomial entries.

For a *square system* (one where $N = n$), it is reasonable to assume that $\det A(s)$ is not the zero polynomial and that the solution

$$X(s) = A^{-1}(s)\mathbf{b}(s)$$

gives a rational function for each X_i that is **strictly proper**.
* Note that Property 3 shows that the initial condition terms in $\mathcal{L}(\frac{d^m x}{dt^m})$ are of degree $m-1$ at most. However, the strict properness we assumed above does not follow directly from this. The reader should try to find a counter-example.

Definition 2.4. A **linear control system** is a system of linear differential relations between a number m of input variables $u_1, \ldots, u_m$ and a number p of output variables $y_1, \ldots, y_p$.

A **scalar linear system**, or **single-input, single-output** (SISO) system, is a linear system involving a single input u and a single output y.

In terms of Laplace transforms, a linear control system is thus, for the moment, given by the following matrix relations:

$$D(s)Y(s) + N(s)U(s) = \mathbf{b}(s), \tag{2.27}$$

where we simply separated the terms involving the outputs from those involving the inputs in equation 2.26. It is commonly assumed that the matrix $D(s)$ is invertible.

If the initial conditions are zero, we get the so-called **transfer function matrix** description of the linear control system. This is

$$Y(s) = -D^{-1}(s)N(s)U(s),$$

which is a generalization to a multiple input and output system of the *transfer function* of a SISO system that is defined in the following Chapter. Staying with this level of generality clearly raises many interesting questions and leads to a rich theory; for the purposes of this introductory course, however, we mostly stick to SISO systems (see the Exercise section for some multi-variable problems.)

A SISO control system is described by the relation

$$Y(s) = \frac{n(s)U(s) + b(s)}{d(s)}. \tag{2.28}$$

We shall always assume that the linear system we consider is strictly proper and take n to be its degree (the degree of the output y.)

Let $\mathcal{R}(u, y)$ be a strictly proper scalar linear system. Suppose we want to find the output $y(t)$, $t \geq 0$ to the input[3] $u(t)$, $t \geq 0$, satisfying a set of initial conditions $y_0, \dot{y}_0, \dots, y_0^{(n-1)}$.

1. Take the Laplace transform of the linear differential relation defining the linear system. Recall that the relation is a constant-coefficient ode. Also note that the initial conditions are taken into account because of Property 3.

2. Solve the algebraic equation resulting from the first step for the transform $Y(s)$ of $y(t)$.

3. Apply the partial fractions method to write the expression for $Y(s)$ as a sum of terms whose inverse Laplace transform is known.

4. Finally, invert these simpler Laplace transforms to get the output $y(t)$ for $t \geq 0$.

Suitably generalized, the above method can also be applied to arbitrary linear control systems (involving more than one input and output variable.)

* **A note on (strict) properness** We have been rather remiss, so far, in not clearly justifying the assumption of properness made at various stages in the above exposition. The essential reason for these assumptions is that **real systems have dynamics**, in the sense that their state cannot be changed instantaneously. Thus, if I push a cart on a level floor, it will take time to change from the rest condition to a moving one. Similarly for other systems of physical origin. In addition, we have Property 3 that says that the initial condition terms in a linear equation in **one** variable will lead to a strictly proper expression for the Laplace transform of the variable, $Y(s) = \frac{b(s)}{d(s)}$.

We similarly expect that the effect of control cannot be instantaneous on the system it is controlling: a pilot cannot throttle up the engines with zero delay. At a deeper level, properness has to do with **causality**. Roughly, causality means that a system cannot at time t respond to the input values prior to t (see the advanced textbook [6].)

[3] We assume implicitly that this input has a known Laplace transform; often, we even assume this transform is a rational function.

2.3 Partial Fractions Expansions

2.3.1 Polynomials and Factorizations

The method we have just given for solving differential equations produces an algebraic expression for the output $Y(s)$ which is, in fact, a **rational function**[4] in s (provided the input transform $U(s)$ is a rational function.) As before, we shall make the assumption that this rational function is **proper**.

Thus, we have that

$$Y(s) = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n} = \frac{n(s)}{d(s)}, \tag{2.29}$$

with $m \leq n$ and we write n and d for the numerator and denominator polynomials.[5] In the case when $m = n$, polynomial division permits us to write $Y(s)$ as *a constant plus a strictly proper rational function.* For simplicity, we assume from now on that the ratio is actually *strictly proper*, $n > m$.

Let us look at the denominator polynomial

$$d(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n.$$

By assumption, it is a real polynomial of degree n and therefore has n roots, which we shall call **poles** in this context; they are either *real* or appear as *complex conjugate pairs*. Hence the factorized form of $p(s)$ is

$$d(s) = \prod_{\Im p_i = 0} (s - p_i) \prod_{\Im p_j \neq 0} (s - p_j)(s - \bar{p_j})$$

where we have grouped together the real roots and complex-pair roots. If we write

$$p_j = \alpha_j + i\beta_j$$

the factorization takes the form

$$d(s) = \prod_{\Im p_i = 0} (s - p_i) \prod_{\Im p_j \neq 0} ((s - \alpha_j)^2 + \beta_j^2).$$

[4] A *rational function* is a ratio of polynomials.

[5] Question: Is there a loss in generality in taking the leading coefficient of p to be equal to one?

We conclude that the real polynomial has two type of factors: **linear factors** corresponding to **real roots** and **quadratic factors** corresponding to a **complex conjugate pair of roots**.

Let us also note that each root p_i may appear more than once in the list of roots of $p(s)$; the **multiplicity** of p_i, k_i, counts how many times it appears as a root. Naturally,

$$\sum k_i = n.$$

2.3.2 The Partial Fractions Expansion Theorem

Proposition 2.1. *If the strictly proper rational function $n(s)/d(s)$ of equation 2.29 has more than one distinct roots, then it can be written as a sum of rational functions of denominator degree less than* $\deg d(s)$*, as follows:*

1. *To every* simple real root *p of the denominator $d(s)$, so that $d(s) = (s-p)q(s)$, with $q(p) \neq 0$, corresponds a single term of the form*

$$\frac{a}{s-p},$$

where

$$\boxed{a = \frac{n(p)}{q(p)} = \lim_{s \to p}(s-p)G(p)} \tag{2.30}$$

The coefficient a is known as the **residue** *of the simple pole at p.*

2. *To every* multiple real root *p of multiplicity k, ($k \geq 2$) of the denominator $d(s)$, so that $d(s) = (s-p)^k q(s)$, with $q(p) \neq 0$, correspond exactly k terms in the partial fractions expansion,*

$$\frac{c_1}{(s-p)} + \frac{c_2}{(s-p)^2} + \ldots + \frac{c_k}{(s-p)^k} \tag{2.31}$$

where, letting

$$h(s) = \frac{n(s)}{q(s)},$$

$$\boxed{\begin{array}{rcl} c_k & = & h(p) = \frac{n(p)}{q(p)} = \lim_{s \to p}(s-p)G(p) \\ c_{k-1} & = & h'(p) \\ & \cdots & \\ c_1 & = & \frac{1}{(k-1)!}\frac{d^{k-1}h}{ds^{k-1}}(p) \end{array}} \tag{2.32}$$

The coefficient c_k is known as the **residue** *of the pole at p.*

3. *To every* simple pair of complex conjugate roots $\alpha \pm i\beta$ *of the denominator* $d(s)$*, so that*

$$d(s) = ((s-\alpha)^2 + \beta^2)q(s),$$

with $q(\alpha \pm i\beta) \neq 0$*, correspond exactly two terms of the form*

$$\frac{A(s-\alpha)}{(s-\alpha)^2+\beta^2} + \frac{B\beta}{(s-\alpha)^2+\beta^2}, \tag{2.33}$$

where the coefficients A *and* B *are real and are found from the* **complex residue**

$$R = Res_{\ \alpha+i\beta} Y(s) = \frac{n(s)}{(s-\alpha+i\beta)q(s)}|_{s=\alpha+i\beta} \tag{2.34}$$

and are given by

$$\boxed{\begin{array}{rcl} A &=& 2\Re(R) \\ B &=& -2\Im(R) \end{array}} \tag{2.35}$$

4. *To every* multiple pair of complex poles $\alpha \pm i\beta$ *of multiplicity* k*,* $k \geq 2$*, correspond exactly* k *terms of the form*

$$\frac{A_1(s-\alpha)+B_1\beta}{(s-\alpha)^2+\beta^2} + \frac{A_2(s-\alpha)+B_2\beta}{((s-\alpha)^2+\beta^2)^2} + \ldots + \frac{A_k(s-\alpha)+B_k\beta}{((s-\alpha)^2+\beta^2)^k} \tag{2.36}$$

(We omit the formulæ for the coefficients (A_i, B_i) *as this case is rarely met in practice; they are derived by combining the above two cases.)*

The partial fractions expansion of $Y(s)$ *is exactly equal to the sum of the above terms.*

Remark 2.2. The above statement makes possible the *'inversion'* of the Laplace transform $Y(s)$ and casts it in a very convenient form, since the answers can be read directly off the Laplace transform table (see the final section.)

In particular, note that the complex pair yields PFE terms that have the obvious Laplace transform inverses

$$e^{-\alpha t}\cos\beta t = \mathcal{L}^{-1}(\frac{(s-\alpha)}{(s-\alpha)^2+\beta^2})$$

and

$$e^{-\alpha t}\sin\beta t = \mathcal{L}^{-1}(\frac{\beta}{(s-\alpha)^2+\beta^2})$$

Proof. The formula for the real residue a is familiar.

For a multiple real root p, break up the PFE into terms involving p and terms not involving it, so that we get

$$Y(s) = \frac{c_1(s-p)^{k-1} + c_2(s-p)^{k-2} + \cdots + c_{k-1}(s-p) + c_k}{(s-p)^k} + \frac{n'(s)}{q(s)}$$

for some polynomial $n'(s)$. Multiplying both sides by $(s-p)^k$, we see that, clearly, $c_k = (s-p)^k Y(s)|_{s=p} = h(p)$ and the other coefficients are found by taking successive derivatives.

The case of a simple complex pair is derived by writing the PFE terms for the two simple (but *complex*) poles $\alpha \pm i\beta$ and making use of the property of residues that states that the two residues corresponding to conjugate roots are themselves complex conjugates:[6]

$$\frac{R}{s-\alpha-i\beta} + \frac{\bar{R}}{s-\alpha+i\beta} = \frac{(R+\bar{R})(s-\alpha) + (R-\bar{R})i\beta}{(s-\alpha)^2 + \beta^2}$$

We now note that

$$R + \bar{R} = 2\Re(R), \quad \text{and} \quad R - \bar{R} = 2i\Im(R).$$

$\square$

It is time to give examples (they are taken from Kreyszig [12].)

Example 2.3. Find the output $y(t)$, $t \geq 0$, of the linear system described by the differential equation

$$\ddot{y} - 3\dot{y} + 2y = u(t)$$

when $u(t) = (4t)h(t)$ and the initial conditions are: $y_0 = 1, \dot{y}_0 = -1$.

Taking the Laplace transform of both sides and since

$$\mathcal{L}(4t) = \frac{4}{s^2},$$

we obtain

$$(s^2 - 3s + 2)Y(s) = \frac{4}{s^2} + s - 4 = \frac{4 + s^3 - 4s^2}{s^2}$$

[6] This relies on the original polynomials having **real** coefficients.

and finally

$$Y(s) = \frac{s^3 - 4s^2 + 4}{s^2(s-2)(s-1)}.$$

Using the PFE result, we have that

$$Y(s) = \frac{a}{s-2} + \frac{b}{s-1} + \frac{c_1}{s} + \frac{c_2}{s^2},$$

where

$$a = \frac{s^3 - 4s^2 + 4}{s^2(s-1)}|_{s=2} = -1$$

$$b = \frac{s^2 - 4s^2 + 4}{s^2(s-2)}|_{s=1} = -1$$

$$c_2 = \frac{s^3 - 4s^2 + 4}{s^2 - 3s + 2}|_{s=0} = 2$$

and

$$c_1 = \frac{d}{ds}\frac{s^3 - 4s^2 + 4}{s^2 - 3s + 2}|_{s=0} = 3.$$

The output is then

$$\boxed{y(t) = (3 + 2t) - e^{2t} - e^t} \tag{2.37}$$

(As a check, verify that $y_0 = 1$ and $\dot{y}_0 = -1$.)

Example 2.4 (Periodically Forced Harmonic Oscillator). Consider the linear system representing a harmonic oscillator with input that is sinusoidal, but of a frequency different from the natural frequency of the harmonic oscillator.

$$m\ddot{y} + ky = K_0 \sin \omega t,$$

where $\omega_0 = \sqrt{k/m} \neq \omega$. Divide by m and take the Laplace transform to get

$$(s^2 + \omega_0^2)Y(s) = \frac{(K_0/m)\omega}{s^2 + \omega^2}.$$

Letting $K = K_0/m$, we write

$$Y(s) = \frac{K\omega}{(s^2 + \omega_0^2)(s^2 + \omega^2)} = \frac{A_1 s + B_1 \omega_0}{s^2 + \omega_0^2} + \frac{A_2 s + B_2 \omega}{s^2 + \omega^2},$$

the PFE of Y. In order to find the coefficients A_i, B_i, we first compute the complex residues

$$R_0 = \frac{K\omega}{(s+i\omega_0)(s^2+\omega^2)}|_{s=i\omega_0} = \frac{K\omega}{(2i\omega_0)(-\omega_0^2+\omega^2)}$$

and

$$R = \frac{K\omega}{(s^2+\omega_0^2)(s+i\omega)}|_{s=i\omega} = \frac{K}{(\omega_0^2-\omega^2)(2i)}.$$

These are both *pure imaginary* and, since $1/i = -i$, we find that

$$A_1 = 0, \quad B_1 = \frac{K\omega/\omega_0}{\omega^2-\omega_0^2}$$

and

$$A_2 = 0, \quad B_2 = \frac{K}{\omega_0^2-\omega^2}$$

and finally

$$\boxed{y(t) = \frac{K}{\omega^2-\omega_0^2}\left(\frac{\omega}{\omega_0}\sin\omega_0 t - \sin\omega t\right).} \tag{2.38}$$

(Valid only for $\omega \neq \omega_0$.)

2.4 Summary

- The **Laplace transform** converts *linear differential equations* into *algebraic equations*. These are **linear** equations with polynomial coefficients. The solution of these linear equations therefore leads to **rational function** expressions for the variables involved.

- Under the assumption of properness, the resulting rational functions are written as a sum of simpler terms, using the method of Partial Fractions Expansion. This is a far-reaching formalization of the concept of *system decomposition into* **modes**.

- *Simple real roots* of the denominator correspond to exponential time functions. *A simple complex pair* corresponds to a modulated trigonometric function, $e^{at}\cos(\omega t + \phi)$.

- *Multiple roots* imply the multiplication of the time function corresponding to a simple root by a polynomial in t.

- A **linear control system** in the s-domain is simply a set of linear equations involving the sets of inputs and outputs.

- This course focuses mainly on *single-input, single-output* systems, described by the transfer function

$$\frac{Y(s)}{U(s)} = \frac{n(s)}{d(s)}.$$

2.5 Appendix: Table of Laplace Transform Pairs

Time Function	Laplace Transform
$h(t)$	$\frac{1}{s}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\frac{t^n}{n!}e^{-at}$	$\frac{1}{(s+a)^{n+1}}$
$\cos\omega_0 t$	$\frac{s}{s^2+\omega_0^2}$
$\sin\omega_0 t$	$\frac{\omega_0}{s^2+\omega_0^2}$
$e^{-at}\cos\omega_0 t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
$e^{-at}\sin\omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

2.6 Exercises

1. In this exercise, and in reference to the discussion in Section 2.1.2, we avoid using the complex exponential e^{st} and insist that the input be real.

   For the second-order linear system of Exercise 2.2, with $a \neq 0$, use the form

   $$\cos \omega t = \frac{1}{2}(e^{i\omega t} + e^{-\omega t})$$

   to show that the output $y(t)$ to the *real* input

   $$u(t) = \cos \omega t$$

   is

   $$y(t) = |g(i\omega)| \cos(\omega t + \angle g(i\omega)),$$

   where $|g(i\omega)|$ and $\angle g(i\omega)$ are the *amplitude* and *phase* of the transfer function $g(s)$ evaluated at $s = i\omega$. Thus, the output waveform is also sinusoidal of the same frequency; its amplitude is scaled by $|g(i\omega)|$ and there is a phase shift by $\angle g(i\omega)$.

2. Find the Laplace transform $X(s)$ of the following functions $x(t)$

   (a) The *square pulse* $x(t) = h(t) - h(t-1)$, where

   $$h(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

   is the unit step.

   (b) The *ramp function* $x(t) = t \cdot h(t)$

   (c) $x(t) = (1 - e^{-3t})h(t)$

   (d) $x(t) = t^2 e^{-4t} h(t)$

3. Assuming zero initial conditions, solve the following differential equations

   (a) $\frac{dx}{dt} + x = 2$

   (b) $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 1$

   (c) $\frac{dx}{dt} - 15x = \sin 4t$

(d) $\frac{dx}{dt} + 2y = 1, \quad \frac{dy}{dt} + 2x = 0$

4. A thermometer is thrust into a hot liquid of constant temperature ϕ^oC. At time $t = 0$, the temperature of the thermometer $\theta = 0^o$C. Newton's law of heating gives the temperature θ satisfying

$$\frac{d\theta}{dt} = \frac{1}{T}(\phi - \theta),$$

where T is the time constant. The thermometer takes one minute to reach 98% of the value ϕ.

(a) Find the time constant using the Laplace transform approach.

(b) If ϕ is time-varying and satisfies the equation

$$\frac{d\phi}{dt} = 10^o\text{C per minute},$$

how much error does the reading θ indicate as $t \to \infty$?

5. A model of water pollution in a river is given by the differential equation

$$\begin{aligned} \dot{x}_1 &= -1.7x_1 + 0.3x_2 \\ \dot{x}_2 &= -1.8x_2 + 1.5u, \end{aligned}$$

where x_1 =dissolved oxygen deficit (DOD) and x_2 =biochemical oxygen deficit (BOD) and the control u is the BOD content of effluent discharged into the river from the effluent treatment plant. The time scale is in days.

Calculate the response of DOD and BOD to a unit step input of effluent, $u(t) = h(t)$, assuming initial conditions

$$x_1(0) = A, \quad x_2(0) = B.$$

6. A Galitzin seismograph has dynamic behaviour modelled by

$$\begin{aligned} \ddot{x} + 2K_1\dot{x} + n_1^2 x &= \lambda\ddot{\xi} \\ \ddot{y} + 2K_2\dot{y} + n_2^2 y &= \mu\dot{x}, \end{aligned}$$

where y is the displacement of the mirror, x is the displacement of the pendulum and ξ is the ground displacement.

When $K_1 = K_2 = n_1 = n_2 = n > 0$,

(a) Determine the response $y(t)$ to a unit ground velocity shift, $\dot{\xi}(t) = h(t)$, assuming

$$\xi(0) = \dot{\xi}(0) = x(0) = \dot{x}(0) = y(0) = \dot{y}(0) = 0$$

(b) Show that the maximum value of $y(t)$ occurs at time $t = (3 - \sqrt{3})/n$ and the minimum value of $y(t)$ at time $t = (3 + \sqrt{3})/n$.