Categories and Homological Algebra

Course at Paris VI University, 2004/2005

Pierre Schapira

April 7, 2005

Contents

Introduction

The aim of these Notes is to introduce the reader to the language of categories with emphazis on homological algebra.

We treat with some details basic homological algebra, that is, categories of complexes in additive and abelian categories and construct with some care the derived functors. We also introduce the reader to the more sophisticated concepts of triangulated and derived categories. Our exposition on these topics is rather sketchy, and the reader is encouraged to consult the literature.

These Notes are extracted from [12]. Other references are [14], [2] for the general theory of categories, [6], [17] and [11], Ch I for homological algebra, including derived categories. The book [13] provides a nice elementary introduction to the classical homological algebra. For further developements, see [9], [12].

Let us briefly describe the contents of these Notes.

Chapter 1 is a survey of linear algebra over a ring. It serves as a guide for the theory of additive and abelian categories. First, we study the functors Hom and $\otimes$ on the category $\mathrm{Mod}(A)$ of modules over a (non necessarily commutative) ring A. Then we introduce the inductive and projective limits of modules and study the exactness of the functors $\varinjlim$ and $\varprojlim$. Finally we introduce Koszul complexes.

In **Chapter 2** we expose the basic language of categories and functors, including the Yoneda Lemma, and the notions of representable and adjoint functors.

In **Chapter 3** we construct the projective and inductive limits and, as a particular case, the kernels and cokernels, products and coproducts. We introduce the notions filtrant category and cofinal functors, and study with some care filtrant inductive limits in the category **Set** of sets. Finally, we define right or left exact functors and give some examples.

Chapter 4 is devoted to the study of additive categories and complexes in such categories. We expose some basic constructions such as the shift functor, the mapping cone, the simple complex associated with a double complex and we introduce the notion of morphism homotopic to zero. As a first application, we show how the Koszul complex associated with n linear maps may be obtained as the mapping cone of a endomorphism of a Koszul complex associated with $n-1$ linear maps. We also construct complexes associated with functors defined on simplicial sets and give a criterion for such complexes to be homotopic to zero.

In **Chapter 5** we treat abelian categories. The toy model of such categories is the category $\mathrm{Mod}(A)$ of modules over a ring A and for sake of simplicity, we shall always argue as if we were working in a full abelian subcategory of a category $\mathrm{Mod}(A)$. We explain the notions of exact sequences, give some basic lemmas such as "the five lemma" and "the snake lemma", and study injective resolutions. We apply these results to construct the derived functors of a left exact functor (or bifunctor), assuming the category admits enough injectives. As an application we get the functors Ext and Tor.

In **Chapter 6**, we construct the localization of a category with respect to a family of morphisms $\mathcal{S}$ satisfying suitable conditions and we construct the localization of functors. Localization of categories appears in particular in the construction of derived categories.

In **Chapter 7**, we introduce triangulated categories. The main result, which is stated without proof, is that the homotopy category $K(\mathcal{C})$ associated with an additive category $\mathcal{C}$, is triangulated. We also localize triangulated categories and triangulated functors.

In **Chapter 8**, we construct the derived category of an abelian category $\mathcal{C}$, by localizing the category $K(\mathcal{C})$ with respect to the quasi-isomorphisms. We also construct the right derived functor of a left exact functor.

Caution. In these Notes, we do not mention the problem of universes. To be correct, we should have taken care of the universes in which we were working. For example, given a universe $\mathcal{U}$, when taking inductive or projective limits indexed by a category I with values a category $\mathcal{C}$, if $\mathcal{C}$ is a $\mathcal{U}$-category, then the category I should be "$\mathcal{U}$-small". In particular, the localization of a $\mathcal{U}$-category may fail to be a $\mathcal{U}$-category and we should consider a bigger universe. We hope that, as far as we are concerned in these Notes, these questions may be skipped.

Conventions. In these Notes, all rings are unital and associative but not necessarily commutative. The operations, the zero element, and the unit are denoted by $+, \cdot, 0, 1$, respectively. However, we shall often write for short ab instead of $a \cdot b$.

All along these Notes, k will denote a *commutative* ring. (Sometimes, k will be a field.)

A k-algebra A is a ring endowed with a morphism of rings $\varphi : k \to A$

such that the image of k is contained in the center of A. Note that a ring A is always a $\mathbb{Z}$-algebra.

We denote by $\emptyset$ the empty set and by $\{\mathrm{pt}\}$ a set with one element.

We denote by $\mathbb{N}$ the set of non-negative integers, $\mathbb{N} = \{0, 1, \dots\}$.

Chapter 1

Linear algebra over a ring

This chapter is a short review of basic and classical notions of commutative algebra.
Some references: [1], [2].

1.1 Modules and linear maps

Let A be a ring. Since we do not assume A is commutative, we have to distinguish between left and right structures. Unless otherwise specified, a module M over A means a left A-module. In the sequel, we shall assume that A is a k-algebra, where k is a commutative ring. Notice that this hypothesis is not restrictive since any ring A is a $\mathbb{Z}$-algebra, and, in case A is commutative, an A-algebra.

Recall that an A-module M is an additive group (whose operations and zero element are denoted $+, 0$) endowed with an external law $A \times M \to M$ satisfying:

$$\begin{cases} (ab)m = a(bm) \\ (a+b)m = am + bm \\ a(m+m') = am + am' \\ 1 \cdot m = m \end{cases}$$

where $a, b \in A$ and $m, m' \in M$.

Note that M inherits a structure of a k-module via φ. In the sequel, if there is no risk of confusion, we shall not write φ.

We denote by $A^{\rm op}$ the ring A with the opposite structure. Hence the product ab in $A^{\rm op}$ is the product ba in A and an $A^{\rm op}$-module is a right A-module.

Note that if the ring A is a field (here, a field is always commutative), then an A-module is nothing but a vector space.

Examples 1.1.1. (i) The first example of a ring is $\mathbb{Z}$, the ring of integers. Since a field is a ring, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are rings. If A is a commutative ring, then $A[x_1, \ldots, x_n]$, the ring of polynomials in n variables with coefficients in A, is also a commutative ring. It is a sub-ring of $A[[x_1, \ldots, x_n]]$, the ring of formal powers series with coefficients in A.
(ii) Let k be a field. Then for $n > 1$, the ring $M_n(k)$ of square matrices of rank n with entries in k is non commutative.
(iii) Let k be a field of characteristic 0 (i.e., k contains $\mathbb{Q}$). The *Weyl algebra* in n variables, denoted $W_n(k)$, is the non commutative ring of polynomials in the variables x_i, ∂_j $(1 \leq i, j \leq n)$ with coefficients in k, and relations :

$$[x_i, x_j] = 0,\ [\partial_i, \partial_j] = 0,\ [\partial_j, x_i] = \delta^i_j$$

where $[p, q] = pq - qp$ and δ^i_j the Kronecker symbol.

A morphism $f : M \to N$ of A-modules is an A-linear map, i.e. f satisfies:

$$\begin{cases} f(m + m') = f(m) + f(m') \\ f(am) = af(m) \end{cases}$$

where $m, m' \in M, a \in A$.

A morphism f is an isomorphism if there exists a morphism $g : N \to M$ with $f \circ g = \mathrm{id}_N, g \circ f = \mathrm{id}_M$.

If f is bijective, it is easily checked that the inverse map $f^{-1} : N \to M$ is itself A-linear. Hence f is an isomorphism if and only if f is A-linear and bijective.

The notions of submodule and quotient module will not be recalled here. Let us only say that their constructions are similar to the corresponding ones on vector spaces.

Let I be a set, and let $(M_i)_{i\in I}$ be a family of A-modules indexed by I. Recall that the product $\prod_i M_i$ is the set of families $\{(x_i)_{i\in I}\}$ with $x_i \in M_i$, and this set naturally inherits a structure of an A-module.

The direct sum $\bigoplus_i M_i$ is the submodule of $\prod_i M_i$ consisting of families $\{(x_i)_{i\in I}\}$ with $x_i = 0$ for all but a finite number of $i \in I$. In particular, if the set I is finite, the natural injection $\bigoplus_i M_i \to \prod_i M_i$ is an isomorphism. There are natural injective morphisms:

$$\varepsilon_k : M_k \to \bigoplus_i M_i$$

and natural surjective morphisms:

$$\pi_k : \prod_i M_i \to M_k.$$

We shall sometimes identify M_k to its image in $\bigoplus_i M_i$ by ε_k.

If $M_i = M$ for all $i \in I$, one writes:

$$M^{(I)} := \bigoplus_i M_i, \qquad M^I := \prod_i M_i.$$

A submodule of the A-module A is called an ideal of A. Note that if A is a field, it has no non trivial ideal, i.e. its only ideals are $\{0\}$ and A. If $A = \mathbb{C}[x]$, then $I = \{P \in \mathbb{C}[x]; P(0) = 0\}$ is a non trivial ideal.

An A-module M is free of rank one if it is isomorphic to A, and M is free if it is isomorphic to a direct sum $\bigoplus_{i\in I} L_i$, each L_i being free of rank one. If card (I) is finite, say r, then r is uniquely determined and one says M is free of rank r.

Let $f : M \to N$ be a morphism of A-modules. One sets :

$$\begin{aligned} \operatorname{Ker} f &= \{m \in M; \quad f(m) = 0\} \\ \operatorname{Im} f &= \{n \in N; \quad \text{there exists } m \in M, \quad f(m) = n\}. \end{aligned}$$

These are submodules of M and N respectively, called the kernel and the image of f, respectively. One also introduces the cokernel of f as the quotient :

$$\operatorname{Coker} f = N/\operatorname{Im} f,$$

and the coimage of f, as :

$$\operatorname{Coim} f = M/\operatorname{Ker} f.$$

Since the natural morphism $\operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism, one shall not use Coim when dealing with A-modules.

Example 1.1.2. The Weyl algebra $W_n(k)$ may be regarded as the (non commutative) ring of differential operators with coefficients in $k[x_1, \ldots, x_n]$, and $k[x_1, \ldots, x_n]$ becomes a left $W_n(k)$-module: x_i acts by multiplication and ∂_i is the derivation with respect to x_i. Consider the left $W_n(k)$-linear map $W_n(k) \to k[x_1, \ldots, x_n]$, $W_n(k) \ni P \mapsto P(1) \in k[x_1, \ldots, x_n]$. This map is clearly surjective and its kernel is generated by $(\partial_1, \cdots, \partial_n)$. Hence, one has the isomorphism of left $W_n(k)$-modules:

$$W_n(k)/\sum_j W_n(k)\partial_j \xrightarrow{\sim} k[x_1, \ldots, x_n]. \tag{1.1}$$

If $(M_i)_{i\in I}$ is a family of submodules of an A-module M, one denotes by $\sum_i M_i$ the submodule of M obtained as the image of the natural morphism $\bigoplus_i M_i \to M$. This is also the module generated in M by the set $\bigcup_i M_i$. One calls this module the sum of the M_i's in M.

1.2 Complexes

Definition 1.2.1. A complex $M^\bullet$ of A-modules is a sequence of modules $M^j, j \in \mathbb{Z}$ and A-linear maps $d_M^j : M^j \to M^{j+1}$ such that $d_M^j \circ d_M^{j-1} = 0$ for all j.

One writes a complex as:

$$M^\bullet : \cdots \to M^j \xrightarrow{d_M^j} M^{j+1} \to \cdots$$

If there is no risk of confusion, one writes M instead of $M^\bullet$. One also often write d^j instead of d_M^j.

A morphism of complexes $f : M \to N$ is a commutative diagram:

$$\begin{tikzcd}
\arrow[r] & M^{k-1} \arrow[r, "d_M^{k-1}"] \arrow[d, "f^{k-1}"] & M^k \arrow[r] \arrow[d, "f^k"] & {} \\
\arrow[r] & N^{k-1} \arrow[r, "d_N^{k-1}"'] & N^k \arrow[r] & {}
\end{tikzcd}$$

Remark 1.2.2. One also encounters finite sequences of morphisms

$$M^j \xrightarrow{d^j} M^{j+1} \to \cdots \to M^{j+k}$$

such that $d^n \circ d^{n-1} = 0$ when it is defined. In such a case we also call such a sequence a complex by identifying it to the complex

$$\cdots \to 0 \to M^j \xrightarrow{d^j} M^{j+1} \to \cdots \to M^{j+k} \to 0 \to \cdots .$$

In particular, $M' \xrightarrow{f} M \xrightarrow{g} M''$ is a complex if $g \circ f = 0$.

Consider a sequence

(1.2) $M' \xrightarrow{f} M \xrightarrow{g} M'',$ with $g \circ f = 0$. (Hence, this sequence is a complex.)

Definition 1.2.3. (i) The sequence (1.2) is exact if $\operatorname{Im} f \xrightarrow{\sim} \operatorname{Ker} g$.

(ii) More generally, a complex $M^j \to \cdots \to M^{j+k}$ is exact if any sequence $M^{n-1} \to M^n \to M^{n+1}$ extracted from this complex is exact.

(iii) An exact complex $0 \to M' \to M \to M'' \to 0$ is called a short exact sequence.

Example 1.2.4. Let $A = k[x_1, x_2]$ and consider the sequence:

$$0 \to A \xrightarrow{d^0} A^2 \xrightarrow{d^1} A \to 0$$

where $d^0(P) = (x_1P, x_2P)$ and $d^1(Q, R) = x_2Q - x_1R$. One checks immediately that $d^1 \circ d^0 = 0$: the sequence above is a complex.

One defines the k-th cohomology object of a complex $M^\bullet$ as:

$$H^k(M^\bullet) = \operatorname{Ker} d^k / \operatorname{Im} d^{k-1}.$$

Hence, a complex $M^\bullet$ is exact if all its cohomology objects are zero, that is, $\operatorname{Im} d^{k-1} = \operatorname{Ker} d^k$ for all k.

If $f^\bullet : M^\bullet \to N^\bullet$ is a morphism of complexes, then for each j, f^j sends $\operatorname{Ker} d^j_{M^\bullet}$ to $\operatorname{Ker} d^j_{N^\bullet}$ and sends $\operatorname{Im} d^{j-1}_{M^\bullet}$ to $\operatorname{Im} d^{j-1}_{N^\bullet}$. Hence it defines the morphism

$$H^j(f^\bullet) : H^j(M^\bullet) \to H^j(N^\bullet).$$

One says that f is a *quasi-isomorphism* (a qis, for short) if $H^j(f)$ is an isomorphism for all j.

As a particular case, consider a complex $M^\bullet$ of the type:

$$0 \to M^0 \xrightarrow{f} M^1 \to 0.$$

Then $H^0(M^\bullet) = \operatorname{Ker} f$ and $H^1(M^\bullet) = \operatorname{Coker} f$.

To a morphism $f : M \to N$ one then associates the two short exact sequences :

$$\begin{gathered} 0 \to \operatorname{Ker} f \to M \to \operatorname{Im} f \to 0, \\ 0 \to \operatorname{Im} f \to N \to \operatorname{Coker} f \to 0, \end{gathered}$$

and f is an isomorphism if and only if $\operatorname{Ker} f = \operatorname{Coker} f = 0$. In this case one writes :

$$f : M \xrightarrow{\sim} N.$$

One says f is a monomorphism (resp. epimorphism) if $\operatorname{Ker} f$ (resp. $\operatorname{Coker} f$) $= 0$.

Proposition 1.2.5. *Consider an exact sequence*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0. \tag{1.3}$$

Then the following conditions are equivalent:

(a) *there exists* $h: M'' \to M$ *such that* $g \circ h = \mathrm{id}_{M''}$,

(b) *there exists* $k: M \to M'$ *such that* $k \circ f = \mathrm{id}_{M'}$

(c) *there exists* $h: M'' \to M$ *and* $k: M \to M'$ *such that such that* $\mathrm{id}_M = f \circ k + h \circ g$,

(d) *there exists* $\varphi = (k, g): M \to M' \oplus M''$ *and* $\psi = (f + h): M' \oplus M'' \to M$, *such that* φ *and* ψ *are isomorphisms inverse to each other. In other words, the exact sequence (1.3) is isomorphic to the exact sequence* $0 \to M' \to M' \oplus M'' \to M'' \to 0$.

Proof. (a) $\Rightarrow$ (c). Since $g = g \circ h \circ g$, we get $g \circ (\mathrm{id}_M - h \circ g) = 0$, which implies that $\mathrm{id}_M - h \circ g$ factors through $\operatorname{Ker} g$, that is, through M'. Hence, there exists $k: M \to M'$ such that $\mathrm{id}_M - h \circ g = f \circ k$.
(b) $\Rightarrow$ (c). The proof is similar and left to the reader.
(c) $\Rightarrow$ (a). Since $g \circ f = 0$, we find $g = g \circ h \circ g$, that is $(g \circ h - \mathrm{id}_{M''}) \circ g = 0$. Since g is onto, this implies $g \circ h - \mathrm{id}_{M''} = 0$.
(c) $\Rightarrow$ (b). The proof is similar and left to the reader.
(d) $\Leftrightarrow$ (a)&(b)&(c) is obvious. q.e.d.

Definition 1.2.6. In the above situation, one says that the exact sequence (1.3) splits.

If A is a field, all exact sequences split, but this is not the case in general. For example, the exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

does not split.

1.3 Hom and Tens

In this section, A denotes a k-algebra. Let M and N be two A-modules. One denotes by $\operatorname{Hom}_A(M, N)$ the set of A-linear maps $f: M \to N$. This is clearly a k-module. In fact one defines the action of k on $\operatorname{Hom}_A(M, N)$ by setting: $(\lambda f)(m) = \lambda(f(m))$. Hence $(\lambda f)(am) = \lambda f(am) = \lambda a f(m) = a\lambda f(m) = a(\lambda f(m))$, and $\lambda f \in \operatorname{Hom}_A(M, N)$.

We shall often set for short

$$\operatorname{Hom}(M, N) = \operatorname{Hom}_k(M, N).$$

Notice that if K is a k-module, then $\operatorname{Hom}(K, M)$ is an A-module.

There is a natural isomorphism $\operatorname{Hom}_A(A, M) \simeq M$: to $u \in \operatorname{Hom}_A(A, M)$ one associates $u(1)$ and to $m \in M$ one associates the linear map $A \to M, a \mapsto am$. More generally, if I is an ideal of A then $\operatorname{Hom}_A(A/I, M) \simeq \{m \in M; Im = 0\}$.

Let $g : K \to L$ be an A-linear map. Composition to the left by g gives a k-linear map :

$$\begin{array}{rcl} \operatorname{Hom}_A(M, g) : \operatorname{Hom}_A(M, K) & \xrightarrow{g\circ} & \operatorname{Hom}_A(M, L) \\ (M \xrightarrow{h} K) & \mapsto & (M \xrightarrow{h} K \xrightarrow{g} L). \end{array}$$

One sees that $\operatorname{Hom}_A(M, \bullet)$ sends the A-module K to the the k-module $\operatorname{Hom}_A(M, K)$, and sends $\operatorname{Hom}_A(K, L)$ to $\operatorname{Hom}(\operatorname{Hom}_A(M, K), \operatorname{Hom}_A(M, L))$. As we shall see in Chapter 2, $\operatorname{Hom}_A(M, \bullet)$ is a *functor* from the category $\operatorname{Mod}(A)$ of A-modules to the category $\operatorname{Mod}(k)$ of k-modules.

Similarly, $\operatorname{Hom}_A(\bullet, N)$ is a contravariant functor (it reverses the direction of arrows) from the category $\operatorname{Mod}(A)$ to the category $\operatorname{Mod}(k)$. If K is an A-module, $\operatorname{Hom}_A(K, N)$ is a k-module, and if $g : K \to L$ is A-linear, composition to the right by g gives a k-linear map :

$$\begin{array}{rcl} \operatorname{Hom}_A(g, N) : \operatorname{Hom}_A(L, N) & \xrightarrow{\circ g} & \operatorname{Hom}_A(K, N) \\ (L \xrightarrow{h} N) & \mapsto & (K \xrightarrow{g} L \xrightarrow{h} N). \end{array}$$

Hence $\operatorname{Hom}_A(\bullet, N)$ sends $\operatorname{Hom}_A(K, L)$ to $\operatorname{Hom}(\operatorname{Hom}_A(L, N), \operatorname{Hom}_A(K, N))$.

Clearly, the two functors $\operatorname{Hom}_A(M, \bullet)$ and $\operatorname{Hom}_A(\bullet, N)$ commute to finite direct sums or finite products, *i.e.,*

$$\begin{array}{rcl} \operatorname{Hom}_A(K \oplus L, N) & \simeq & \operatorname{Hom}_A(K, N) \times \operatorname{Hom}_A(L, N) \\ \operatorname{Hom}_A(M; K \times L) & \simeq & \operatorname{Hom}_A(M, K) \times \operatorname{Hom}_A(M, L). \end{array}$$

One says that these functors are *additive.*

Proposition 1.3.1. (a) *Let* $0 \to M' \xrightarrow{f} M \xrightarrow{g} M''$ *be a complex of* A*-modules. The assertions below are equivalent.*

(i) *the sequence is exact,*

(ii) M' *is isomorphic by* f *to* $\operatorname{Ker} g$*,*

(iii) *any morphism* $h : L \to M$ *such that* $g \circ h = 0$*, factorizes uniquely through* M' (i.e. $h = f \circ h'$*, with* $h' : L \to M'$)*. This is visualized by*

$$\begin{tikzcd} & & L \arrow[dl, dotted, "h'"'] \arrow[d, "h"] \arrow[dr, "0"] & \\ 0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \end{tikzcd}$$

(iv) *for any module L, the sequence of k-modules*

$$(1.4) \qquad 0 \to \operatorname{Hom}_A(L, M') \to \operatorname{Hom}_A(L, M) \to \operatorname{Hom}_A(L, M'')$$

is exact.

(b) *Let $M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be a complex of A-modules. The assertions below are equivalent.*

(i) *the sequence is exact,*

(ii) *M'' is isomorphic by g to* $\operatorname{Coker} f$*,*

(iii) *any morphism $h : M \to L$ such that $h \circ f = 0$, factorizes uniquely through M''* (i.e. *$h = h'' \circ g$, with $h'' : M'' \to L$). This is visualized by*

$$\begin{tikzcd} M' \arrow[r, "f"] \arrow[dr, "0"'] & M \arrow[r, "g"] \arrow[d, "h"] & M'' \arrow[r] \arrow[dl, dotted, "h''"] & 0 \\ & L & & \end{tikzcd}$$

(iv) *for any module L, the sequence of k-modules*

$$(1.5) \qquad 0 \to \operatorname{Hom}_A(M'', L) \to \operatorname{Hom}_A(M, L) \to \operatorname{Hom}_A(M', L)$$

is exact.

Proof. (a) (i) $\Leftrightarrow$ (ii) is obvious, as well as (ii) $\Leftrightarrow$ (iii), since any linear map $h : L \to M$ such that $g \circ h = 0$ factorizes uniquely through $\operatorname{Ker} g$, and this characterizes $\operatorname{Ker} g$. Finally, (iii) $\Leftrightarrow$ (iv) is tautological.
(b) The proof is similar. q.e.d.

As we shall see in Chapter 2, the fact that (a) (i) implies (a) (iv) (resp. (b) (i) implies (b) (iv)) is formulated as: "$\operatorname{Hom}_A(\,\bullet\,, L)$ (resp. $\operatorname{Hom}_A(L, \bullet))$ is a left exact functor".

Note that if $A = k$ is a field, then $\operatorname{Hom}_k(M, k)$ is the algebraic dual of M, the vector space of linear functional on M, usually denoted by M^*. If M is finite dimensional, then $M \simeq M^{**}$. If $u : L \to M$ is a linear map, the map $\operatorname{Hom}_k(u, k) : M^* \to L^*$ is usually denoted by ${}^t u$ and called the transpose of u.

Example 1.3.2. The functors $\operatorname{Hom}_A(\,\bullet\,, L)$ and $\operatorname{Hom}_A(M, \bullet\,)$ are not "right exact" in general. In fact choose $A = k[x]$, with k a field, and consider the exact sequence of A-modules:

$$(1.6) \qquad 0 \to A \xrightarrow{\cdot x} A \to A/Ax \to 0$$

(where $\cdot x$ means multiplication by x). Apply $\mathrm{Hom}_A(\cdot, A)$ to this sequence. We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax, A) \to A \xrightarrow{x\cdot} A \to 0$$

which is not exact since $x\cdot$ is not surjective. On the other hand, since $x\cdot$ is injective and $\mathrm{Hom}_A(\,\bullet\,, A)$ is left exact, we find that $\mathrm{Hom}_A(A/Ax, A) = 0$.

Similarly, apply $\mathrm{Hom}_A(A/Ax, \bullet\,)$ to the exact sequence (1.6). We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax, A) \to \mathrm{Hom}_A(A/Ax, A) \to \mathrm{Hom}_A(A/Ax, A/Ax) \to 0.$$

Since $\mathrm{Hom}_A(A/Ax, A) = 0$ and $\mathrm{Hom}_A(A/Ax, A/Ax) \neq 0$, this sequence is not exact.

Notice moreover that the functor $\mathrm{Hom}_A(\,\bullet\,, \bullet\,)$ being additive, it sends split exact sequences to split exact sequences. This shows again that (1.6) does not split.

Proposition 1.3.3. *Let $f : M \to N$ be a morphism of A-modules. The conditions below are equivalent:*

(i) *f is an isomorphism,*

(ii) *for any A-module L, the map $\mathrm{Hom}_A(L, M) \xrightarrow{f\circ} \mathrm{Hom}_A(L, N)$ is an isomorphism,*

(iii) *for any A-module L, the map $\mathrm{Hom}_A(N, L) \xrightarrow{\circ f} \mathrm{Hom}_A(M, L)$ is an isomorphism.*

Proof. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are obvious.
(ii) $\Rightarrow$ (i). Choose $L = A$.
(iii) $\Rightarrow$ (i). By choosing $L = M$ and $\mathrm{id}_M \in \mathrm{Hom}_A(M, M)$ we find that there exists $g : N \to M$ such that $g \circ f = \mathrm{id}_M$. Hence, f is injective and moreover, by Proposition 1.2.5 there exists an isomorphism $N \simeq M \oplus P$. Therefore, $\mathrm{Hom}_A(P, L) \simeq 0$ for all module L, hence $\mathrm{Hom}_A(P, P) \simeq 0$, and this implies $P \simeq 0$. q.e.d.

Tensor product

The tensor product, that we shall construct below, solves a "universal problem". Namely, consider a right A-module N, a left A-module M, and a k-module L. Let us say that a map $f : N \times M \to L$ is (A, k)-bilinear if f is additive with respect to each of its arguments and satisfies $f(na, m) = f(n, am)$, $f(n(\lambda), m) = \lambda(f(n, m))$ for all $(n, m) \in N \times M$ and $a \in A, \lambda \in k$.

We shall construct a k-module denoted $N \otimes_A M$ such that f factors uniquely through the bilinear map $N \times M \to N \otimes_A M$ followed by a k-linear map $N \otimes_A M \to L$. This is visualized by:

$$\begin{tikzcd} N \times M \arrow[r] \arrow[dr, "f"'] & N \otimes_A M \arrow[d, dotted] \\ & L \end{tikzcd}$$

First, remark that one may identify a set I to a subset of $k^{(I)}$ as follows: to $i \in I$, we associate $\{l_j\}_{j\in I} \in k^{(I)}$ given by

$$l_j = \begin{cases} 1 \text{ if } & j = i, \\ 0 \text{ if } & j \neq i. \end{cases} \tag{1.7}$$

The tensor product $N \otimes_A M$ is the k-module defined as the quotient of $k^{(N\times M)}$ by the submodule generated by the following elements (where $n, n' \in N, m, m' \in M, a \in A, \lambda \in k$ and $N \times M$ is identified to a subset of $k^{(N\times M)}$):

$$\begin{cases} (n + n', m) - (n, m) - (n', m) \\ (n, m + m') - (n, m) - (n, m') \\ (na, m) - (n, am) \\ \lambda(n, m) - (n\lambda, m). \end{cases}$$

The image of (n, m) in $N \otimes_A M$ is denoted $n \otimes m$. Hence an element of $N \otimes_A M$ may be written (not uniquely!) as a finite sum $\sum_j n_j \otimes m_j$, $n_j \in N, m_j \in M$ and:

$$\begin{cases} (n + n') \otimes m = n \otimes m + n' \otimes m \\ n \otimes (m + m') = n \otimes m + n \otimes m' \\ na \otimes m = n \otimes am \\ \lambda(n \otimes m) = n\lambda \otimes m = n \otimes \lambda m. \end{cases}$$

Consider an A-linear map $f : M \to L$. It defines a linear map $\mathrm{id}_N \times f : N \times M \to N \times L$, hence a (A, k)-bilinear map $N \times M \to N \otimes_A L$, and finally a k-linear map

$$\mathrm{id}_N \otimes f : N \otimes_A M \to N \otimes_A L.$$

One constructs similarly $g \otimes \mathrm{id}_M$ associated to $g : N \to L$.

Note that if A is commutative, there is an isomorphism: $N \otimes_A M \simeq M \otimes_A N$, given by $n \otimes m \mapsto m \otimes n$ and moreover the tensor product is associative, that is, if L, M, N are A-modules, there are natural isomorphisms $L \otimes_A (M \otimes_A N) \simeq (L \otimes_A M) \otimes_A N$. One simply writes $L \otimes_A M \otimes_A N$.

Tensor product commutes to direct sum, that is, there are natural isomorphisms:

$$\begin{aligned}(N \oplus N') \otimes_A M &\simeq (N \otimes_A M) \oplus (N' \otimes_A M),\\ N \otimes_A (M \oplus M') &\simeq (N \otimes_A M) \oplus (N \otimes_A M').\end{aligned}$$

There is a natural isomorphism $A \otimes_A M \simeq M$. We shall often write for short

$$M \otimes_k N = M \otimes N.$$

Sometimes, one has to consider various rings. Consider two k-algebras, A_1 and A_2. Then $A_1 \otimes A_2$ has a natural structure of a k-algebra, by setting

$$(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2.$$

An $(A_1 \otimes A_2^{\mathrm{op}})$-module M is also called a (A_1, A_2)-bimodule (a left A_1-module and right A_2-module). Note that the actions of A_1 and A_2 on M commute, that is,

$$a_1 a_2 m = a_2 a_1 m,\ a_1 \in A_1,\ a_2 \in A_2,\ m \in M.$$

Let A_1, A_2, A_3, A_4 denote four k-algebras.

Proposition 1.3.4. *Let ${}_iM_j$ be an $(A_i \otimes A_j^{\mathrm{op}})$-module. Then*

$$\begin{aligned}{}_1M_2 \otimes_{A_2} {}_2M_3 &\text{ is an } (A_1 \otimes A_3^{\mathrm{op}})\text{-module},\\ \mathrm{Hom}_{A_1}({}_1M_2, {}_1M_3) &\text{ is an } (A_2 \otimes A_3^{\mathrm{op}})\text{-module},\end{aligned}$$

and there is a natural isomorphism of $A_4 \otimes A_3^{\mathrm{op}}$-modules

$$\mathrm{Hom}_{A_1}({}_1M_4, \mathrm{Hom}_{A_2}({}_2M_1, {}_2M_3)) \simeq \mathrm{Hom}_{A_2}({}_2M_1 \otimes_{A_1} {}_1M_4, {}_2M_3). \tag{1.8}$$

In particular, if A is a k-algebra, M, N are left A-modules and L is a k-module, we have the isomorphisms

$$\begin{aligned}\mathrm{Hom}_A(L \otimes_k N, M) &\simeq \mathrm{Hom}_A(N, \mathrm{Hom}_k(L, M))\\ &\simeq \mathrm{Hom}_k(L, \mathrm{Hom}_A(N, M)).\end{aligned} \tag{1.9}$$

The first isomorphism is translated (see Chapter 2 below) by saying that the functors $L \otimes_k \bullet$ and $\mathrm{Hom}_k(L, \bullet)$ are adjoint.

Proof. We shall only prove (1.9) in the particular case where $A = k$. In this case, $\mathrm{Hom}_A(L \otimes_k N, M)$ is nothing but the k-module of k-bilinear maps from $L \times N$ to M, and a k-bilinear map from $L \times N$ to M defines uniquely a linear map from L to $\mathrm{Hom}_A(N, M)$ and conversely. q.e.d.

Proposition 1.3.5. *If $M' \to M \to M'' \to 0$ is an exact sequence of left A-modules, then the sequence of k-modules $N \otimes_A M' \to N \otimes_A M \to N \otimes_A M'' \to 0$ is exact.*

Proof. By Proposition 1.3.1 (b), it is enough to check that for any k-module L, the sequence

$$0 \to \mathrm{Hom}_k(N \otimes_A M'', L) \to \mathrm{Hom}_k(N \otimes_A M, L) \to \mathrm{Hom}_k(N \otimes_A M', L)$$

is exact. This sequence is isomorphic to the sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_k(M'', \mathrm{Hom}_A(N, L)) \to \mathrm{Hom}_k(M, \mathrm{Hom}_A(N, L)) \\ \to \mathrm{Hom}_k(M', \mathrm{Hom}_A(N, L)) \end{aligned}$$

and it remains to apply Proposition 1.3.1 ((b), (i) $\Rightarrow$ (ii)). q.e.d.

One says (see Chapter 2 below) that $\bullet \otimes_A M$ (resp. $N \otimes_A \bullet$) is a right exact functor from $\mathrm{Mod}(A^{\mathrm{op}})$ (resp. $\mathrm{Mod}(A)$) to $\mathrm{Mod}(k)$.

Example 1.3.6. $\bullet \otimes_A M$ is not left exact in general. In fact, consider the commutative ring $A = \mathbb{C}[x]$ and the exact sequence of A-modules:

$$0 \to A \xrightarrow{x\cdot} A \to A/xA \to 0.$$

Apply $\bullet \otimes_A A/Ax$. We get the sequence:

$$0 \to A/Ax \xrightarrow{x\cdot} A/Ax \to A/xA \otimes_A A/Ax \to 0$$

Multiplication by x is 0 on A/Ax. Hence this sequence is the same as:

$$0 \to A/Ax \xrightarrow{0} A/Ax \to A/Ax \otimes_A A/Ax \to 0$$

which shows that $A/Ax \otimes_A A/Ax \simeq A/Ax$ and moreover that this sequence is not exact.

Generators and relations

Suppose one is interested in studying a system of linear equations

$$\sum_{j=1}^{N_0} p_{ij} u_j = v_i, \quad (i = 1, \dots, N_1) \tag{1.10}$$

where the p_{ij}'s belong to the ring A and u_j, v_i belong to some left A-module L. Using matrix notations, one can write equations (1.10) as

$$Pu = v \tag{1.11}$$

where P is the matrix (p_{ij}) with N_1 rows and N_0 columns, defining the A-linear map $P\cdot : L^{N_0} \to L^{N_1}$. Now consider the right A-linear map

$$\cdot P : A^{N_1} \to A^{N_0}, \tag{1.12}$$

where $\cdot P$ operates on the right and the elements of A^{N_0} and A^{N_1} are written as rows. Let $(e_1, \dots, e_{N_0})$ and $(f_1, \dots, f_{N_1})$ denote the canonical basis of A^{N_0} and A^{N_1}, respectively. One gets:

$$f_i \cdot P = \sum_{j=1}^{N_0} p_{ij} e_j, \quad (i = 1, \dots, N_1). \tag{1.13}$$

Hence $\operatorname{Im} P$ is generated by the elements $\sum_{j=1}^{N_0} p_{ij} e_j$ for $i = 1, \dots, N_1$. Denote by M the quotient module $A^{N_0}/A^{N_1} \cdot P$ and by $\psi : A^{N_0} \to M$ the natural A-linear map. Let $(u_1, \dots, u_{N_0})$ denote the images by ψ of $(e_1, \dots, e_{N_0})$. Then M is a left A-module with generators $(u_1, \dots, u_{N_0})$ and relations $\sum_{j=1}^{N_0} p_{ij} u_j = 0$ for $i = 1, \dots, N_1$. By construction, we have an exact sequence of left A-modules:

$$A^{N_1} \xrightarrow{\cdot P} A^{N_0} \xrightarrow{\psi} M \to 0. \tag{1.14}$$

Applying the left exact functor $\operatorname{Hom}_A(\bullet, L)$ to this sequence, we find the exact sequence of k-modules:

$$0 \to \operatorname{Hom}_A(M, L) \to L^{N_0} \xrightarrow{P\cdot} L^{N_1}. \tag{1.15}$$

Hence, the k-module of solutions of the homogeneous equations associated to (1.10) is described by $\operatorname{Hom}_A(M, L)$.

Injective and projective modules

Definition 1.3.7. (i) An A-module I is injective if for any exact sequence $M' \to M \to M'' \to 0$ in $\mathrm{Mod}(A)$, the sequence $0 \to \operatorname{Hom}_A(M'', I) \to \operatorname{Hom}_A(M, I) \to \operatorname{Hom}_A(M', I)$ is exact in $\mathrm{Mod}(k)$.

(ii) An A-module P is projective if for any exact sequence $0 \to M' \to M \to M''$ in $\mathrm{Mod}(A)$, the sequence $0 \to \operatorname{Hom}_A(P, M') \to \operatorname{Hom}_A(P, M) \to \operatorname{Hom}_A(P, M'')$ is exact in $\mathrm{Mod}(k)$.

(iii) An A^{op}-module N is flat if for any exact sequence $0 \to M' \to M \to M''$ in $\mathrm{Mod}(A)$, the sequence $0 \to N \otimes_A M' \to N \otimes_A M \to N \otimes_A M''$ is exact in $\mathrm{Mod}(k)$, and similarly if N is an A-mdules, replacing the exact sequence in $\mathrm{Mod}(A)$ with an exact sequence in $\mathrm{Mod}(A^{\mathrm{op}})$.

(iv) If N is flat and moreover $N \otimes_A M = 0$ (or $M \otimes_A N = 0$) implies $M = 0$, one says that N is faithfully flat.

Proposition 1.3.8. *An A-module I is injective if and only if for any solid diagram in which the row is exact:*

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "k"'] & M \arrow[dl, dotted, "h"'] \\ & I & \end{tikzcd}$$

the dotted arrow may be completed, making the diagram commutative.

Proof. (i) Assume that I is injective and let M'' denote the cokernel of the map $M' \to M$. Applying $\mathrm{Hom}_A(\,\bullet\,, I)$ to the sequence $0 \to M' \to M \to M''$, one gets the exact sequence:

$$\mathrm{Hom}_A(M'', I) \to \mathrm{Hom}_A(M, I) \xrightarrow{\circ f} \mathrm{Hom}_A(M', I) \to 0.$$

Thus there exists $h : M \to I$ such that $h \circ f = k$.
(ii) Conversely, consider an exact sequence $0 \to M' \xrightarrow{f} M \xrightarrow{g} M''$. Set $L = \mathrm{Coker}\, f$ and consider the exact sequence $0 \to M' \xrightarrow{f} M \xrightarrow{h} L \to 0$. Then the sequence

$$0 \to \mathrm{Hom}_A(L, I) \xrightarrow{\circ h} \mathrm{Hom}_A(M, I) \xrightarrow{\circ f} \mathrm{Hom}_A(M', I) \to 0$$

is exact by Proposition 1.3.1 and the hypothesis. To conclude, remark that since $g\colon M \to M''$ factorizes through $h\colon M \to L$, then $\circ g\colon \mathrm{Hom}_A(M'', I) \to \mathrm{Hom}_A(M, I)$ factorizes through $\circ h\colon \mathrm{Hom}_A(L, I) \to \mathrm{Hom}_A(M, I)$. It follows that the sequence

$$\mathrm{Hom}_A(M'', I) \xrightarrow{\circ g} \mathrm{Hom}_A(M, I) \xrightarrow{\circ f} \mathrm{Hom}_A(M', I) \to 0$$

is exact. q.e.d.

By reversing the arrows, we get a similar result assuming P is projective. In other words, P is projective if and only if for any solid diagram in which the row is exact:

$$\begin{tikzcd} & P \arrow[dl, dotted, "h"'] \arrow[d, "k"] & \\ M \arrow[r, "f"] & M'' \arrow[r] & 0 \end{tikzcd}$$

the dotted arrow may be completed, making the diagram commutative.

A free module is projective and a projective module is flat (see Exercise 1.2). If $A = k$ is a field, all modules are both injective and projective.

1.4 Limits

Definition 1.4.1. Let I be a set.

(i) An order $\leq$ on I is a relation which satisfies: (a) $i \leq i$, (b) $i \leq j$ & $j \leq k$ implies $i \leq k$, (c) $i \leq j$ and $j \leq i$ implies $i = j$.

(ii) The opposite order $(I, \leq^{\mathrm{op}})$ is defined by $i \leq^{\mathrm{op}} j$ if and only if $j \leq i$.

(iii) An order is discrete if $i \leq j$ implies $i = j$.

The following definition will be of constant use.

Definition 1.4.2. Let $(I, \leq)$ be an ordered set.

(i) One says that $(I, \leq)$ is filtrant (one also says "directed") if for any $i, j \in I$ there exists k with $i \leq k$ and $j \leq k$.

(ii) Let $J \subset I$ be a subset. One says that J is cofinal to I if for any $i \in I$ there exists $j \in J$ with $i \leq j$.

Let $(I, \leq)$ be a ordered set and let A be a ring. A projective system (N_i, v_{ij}) of A-modules indexed by $(I, \leq)$ is the data for each $i \in I$ of an A-module N_i and for each pair i, j with $i \leq j$ of an A-linear map $v_{ij} : N_j \to N_i$, such that for all i, j, k with $i \leq j$ and $j \leq k$:

$$\begin{aligned} v_{ii} &= \mathrm{id}_{N_i} \\ v_{ij} \circ v_{jk} &= v_{ik}. \end{aligned}$$

Consider the "universal problem": to find an A-module N and linear maps $v_i : N \to N_i$ satisfying $v_{ij} \circ v_j = v_i$ for all $i \leq j$, such that for any A-module L and linear maps $g_i : L \to N_i$, satisfying $v_{ij} \circ g_j = g_i$ for all $i \leq j$, there is a unique linear map $g : L \to N$ such that $g_i = v_i \circ g$ for all i. If such a family (N, v_i) exists (and we shall show below that it does), it is unique up to unique isomorphism and one calls it the projective limit of the projective system (N_i, v_{ij}), denoted $\varprojlim\limits_i N_i$.

An inductive system (M_i, u_{ji}) of A-modules indexed by $(I, \leq)$ is the data for each $i \in I$ of an A-module M_i and for each pair i, j with $i \leq j$ of an A-linear map $u_{ji} : M_i \to M_j$, such that for all i, j, k with $i \leq j$ and $j \leq k$:

$$\begin{aligned} u_{ii} &= \mathrm{id}_{M_i} \\ u_{kj} \circ u_{ji} &= u_{ki}. \end{aligned}$$

Note that a projective system indexed by $(I, \leq)$ is nothing but an inductive system indexed by $(I, \leq^{\mathrm{op}})$.

Consider the "universal problem": to find an A-module M and linear maps $u_i : M_i \to M$ satisfying $u_j \circ u_{ji} = u_i$ for all $i \leq j$, such that for any A-module L and linear maps $f_i : M_i \to L$ satisfying $f_j \circ u_{ji} = f_i$ for all $i \leq j$, there is a unique linear map $f : M \to L$ such that $f_i = f \circ u_i$ for all i. If such a family $(M, u_i)_i$ exists (and we shall show below that it does), it is unique up to unique isomorphism and one calls it the inductive limit of the inductive system (M_i, u_{ji}), denoted $\varinjlim_i M_i$.

Theorem 1.4.3. (i) *The projective limit of the projective system (N_i, v_{ij}) is the A-module*

$$\varprojlim_i N_i = \{(x_i)_i \in \prod_i N_i; v_{ij}(x_j) = x_i \textit{ for all } i \leq j\}.$$

The maps $v_i : \varprojlim_j N_j \to N_i$ are the natural ones.

(ii) *The inductive limit of the inductive system (M_i, u_{ij}) is the A-module*

$$\varinjlim_i M_i = (\bigoplus_{i \in I} M_i)/N$$

where N is the submodule of $\bigoplus_{i \in I} M_i$ generated by $\{x_i - u_{ji}(x_i); x_i \in M_i, i \leq j\}$. The maps $u_i : M_i \to \varinjlim_j M_j$ are the natural ones.

Note that if I is discrete, then $\varinjlim_i M_i = \bigoplus_i M_i$ and $\varprojlim_i N_i = \prod_i N_i$.

The proof is straightforward.

The universal properties on the projective and inductive limit are better formulated by the isomorphisms which characterize $\varprojlim_i N_i$ and $\varinjlim_i M_i$:

$$\mathrm{Hom}_A(L, \varprojlim_i N_i) \xrightarrow{\sim} \varprojlim_i \mathrm{Hom}_A(L, N_i), \tag{1.16}$$

$$\mathrm{Hom}_A(\varinjlim_i M_i, L) \xrightarrow{\sim} \varprojlim_i \mathrm{Hom}_A(M_i, L). \tag{1.17}$$

There are also natural morphisms

$$\varinjlim_i \mathrm{Hom}_A(L, M_i) \to \mathrm{Hom}_A(L, \varinjlim_i M_i) \tag{1.18}$$

$$\varinjlim_i \mathrm{Hom}_A(N_i, L) \to \mathrm{Hom}_A(\varprojlim_i N_i, L). \tag{1.19}$$

One should be aware morphisms (1.18) and (1.19) are not isomorphisms in general (see Example 1.4.12 below).

Proposition 1.4.4. *Let $M_i' \xrightarrow{f_i} M_i \xrightarrow{g_i} M_i''$ be a family of exact sequences of A-modules, indexed by the set I. Then the sequence*

$$\prod_i M_i' \to \prod_i M_i \to \prod_i M_i''$$

is exact.

The proof is left as an (easy) exercise.

Proposition 1.4.5. (i) *Consider a projective system of exact sequences of A-modules: $0 \to N_i' \xrightarrow{f_i} N_i \xrightarrow{g_i} N_i''$. Then the sequence $0 \to \varprojlim_i N_i' \xrightarrow{f} \varprojlim_i N_i \xrightarrow{g} \varprojlim_i N_i''$ is exact.*

(ii) *Consider an inductive system of exact sequences of A-modules: $M_i' \xrightarrow{f_i} M_i \xrightarrow{g_i} M_i'' \to 0$. Then the sequence $\varinjlim_i M_i' \xrightarrow{f} \varinjlim_i M_i \xrightarrow{g} \varinjlim_i M_i'' \to 0$ is exact.*

Proof. (i) Since $\varprojlim_i N_i'$ is a submodule of $\prod_i N_i'$, the fact that f is injective follows from Proposition 1.4.4. Let $(x_i)_i \in \varprojlim_i N_i$ with $g((x_i)_i) = 0$. Then $g_i(x_i) = 0$ for all i, and there exists a unique $x_i' \in N_i'$ such that $x_i = f_i(x_i')$. One checks immedialtely that the element $(x_i')_i$ belongs to $\varprojlim_i N_i'$.

(ii) Let L be an A-module. The sequence

$$0 \to \operatorname{Hom}_A(\varinjlim_i M_i'', L) \to \operatorname{Hom}_A(\varinjlim_i M_i, L) \to \operatorname{Hom}_A(\varinjlim_i M_i', L)$$

is isomorphic to the sequence

$$0 \to \varprojlim_i \operatorname{Hom}_A(M_i'', L) \to \varprojlim_i \operatorname{Hom}_A(M_i, L) \to \varprojlim_i \operatorname{Hom}_A(M_i', L)$$

and this sequence is exact by (i) and Proposition 1.3.1. Then the result follows, again by Proposition 1.3.1. q.e.d.

One says that "the functor $\varinjlim$ is right exact", and "the functor $\varprojlim$ is left exact". We shall give a precise meaning to these sentences in Chapter 2.

Lemma 1.4.6. *Assume I is a filtrant ordered set and let $M = \varinjlim_i M_i$.*

(i) *Let $x_i \in M_i$. Then $u_i(x_i) = 0 \Leftrightarrow$ there exists $k \geq i$ with $u_{ki}(x_i) = 0$.*

(ii) *Let $x \in M$. Then there exists $i \in I$ and $x_i \in M_i$ with $u_i(x_i) = x$.*

Proof. We keep the notations of Theorem 1.4.3 (ii).
(i) Let N' denote the subset of $\oplus_i M_i$ consisting of finite sums $\sum_{j\in J} x_j$, $x_j \in M_j$ such that there exists $k \geq j$ for all $j \in J$ with $\sum_{j\in J} u_{kj}(x_j) = 0$. Since I is filtrant, N' is a submodule of $\oplus_i M_i$. Moreover, $N = N'$. It remains to notice that

$$N' \cap M_i = \{x_i \in M_i;\ \text{there exists } k \geq i \text{ with } u_{ki}(x_i) = 0\}.$$

(ii) Let $x \in M$. There exist a finite set $J \subset I$ and $x_j \in M_j$ such that $x = \sum_{j\in J} u_j(x_j)$. Choose i with $i \geq j$ for all $j \in J$. Then

$$x = \sum_{j\in J} u_k u_{ij}(x_j) = u_i(\sum_{j\in J} u_{ij}(x_j)).$$

Setting $x_i = \sum_{j\in J} u_{ij}(x_j)$, the result follows. q.e.d.

Example 1.4.7. Let X be a topological space, $x \in X$ and denote by I_x the set of open neighborhoods of x in X. We endow I_x with the order: $U \leq V$ if $V \subset U$. Given U and V in I_x, and setting $W = U \cap V$, we have $U \leq W$ and $V \leq W$. Therefore, I_x is filtrant.

Denote by $\mathcal{C}^0(U)$ the $\mathbb{C}$-vector space of complex valued continuous functions on U. The restriction maps $\mathcal{C}^0(U) \to \mathcal{C}^0(V), V \subset U$ define an inductive system of $\mathbb{C}$-vector spaces indexed by I_x. One sets

$$\mathcal{C}^0_{X,x} = \varinjlim_{U\in I_x} \mathcal{C}^0(U). \tag{1.20}$$

An element φ of $\mathcal{C}^0_{X,x}$ is called a germ of continuous function at 0. Such a germ is an equivalence class $(U, \varphi_U)/\sim$ with U a neighborhood of x, φ_U a continuous function on U, and $(U, \varphi_U) \sim 0$ if there exists a neighborhood V of x with $V \subset U$ such that the restriction of φ_U to V is the zero function. Hence, a germ of function is zero at x if this function is identically zero in a neighborhood of x.

Proposition 1.4.8. *Consider an inductive system of exact sequences of A-modules indexed by a filtrant ordered set I: $M'_i \xrightarrow{f_i} M_i \xrightarrow{g_i} M''_i$. Then the sequence*

$$\varinjlim_i M'_i \xrightarrow{f} \varinjlim_i M_i \xrightarrow{g} \varinjlim_i M''_i$$

is exact.

Proof. Let $x \in \varinjlim_i M_i$ with $g(x) = 0$. There exists $x_i \in M_i$ with $u_i(x_i) = x$, and there exists $j \geq i$ such that $u_{ji}(g_i(x_i)) = 0$. Hence $g_j(u_{ji}(x_i)) = u_{ji}(f_i(x_i)) = 0$, which implies that there exists $x'_j \in M'_j$ such that $u_{ji}(x_i) = f_j(x'_j)$. Then $x' = u'_j(x'_j)$ satisfies $f(x') = f(u'_j(x'_j)) = u_j f_j(x'_j) = u_j u_{ji}(x_i) = x$. q.e.d.

Proposition 1.4.9. *Assume $J \subset I$ and assume that J is filtrant and cofinal to I.*

(i) *Let (M_i, u_{ij}) be an inductive system of A-modules indexed by I. Then the natural morphism $\varinjlim_{j\in J} M_j \to \varinjlim_{i\in I} M_i$ is an isomorphism.*

(ii) *Let (M_i, v_{ji}) be a projective system of A-modules indexed by I. Then the natural morphism $\varprojlim_{i\in I} M_i \to \varprojlim_{j\in J} M_j$ is an isomorphism.*

The proof is left as an exercise.

In particular, assume $I = \{0, 1\}$ with $0 < 1$. Then the inductive limit of the inductive system $u_{10} : M_0 \to M_1$ is M_1, and the projective limit of the projective system $v_{01} : M_1 \to M_0$ is M_1.

Remark 1.4.10. (i) If all M_i's are submodules of a module M, and if the maps $u_{ji} : M_i \to M_j, (i \leq j)$ are the natural injective morphisms, then $\varinjlim_i M_i \simeq \bigcup_i M_i$.
(ii) If all M_i's are submodules of a module M, and if the maps $v_{ij} : M_j \to M_i, (i \leq j)$ are the natural injective morphisms, then $\varprojlim_i M_i \simeq \bigcap_i M_i$.

Let us study the relations of $\otimes$ and inductive limits. Let (M_i, u_{ji}) be an inductive system of A-modules, N a right A-module. The family of morphisms $M_i \to \varinjlim_i M_i$ defines the family of morphisms $N \otimes_A M_i \to N \otimes_A \varinjlim_i M_i$, hence the morphism

$$\varinjlim_i (N \otimes_A M_i) \to N \otimes_A \varinjlim_i M_i. \tag{1.21}$$

Proposition 1.4.11. *The morphism* (1.21) *is an isomorphism.*

Proof. Let L be a k-module. Consider the chain of isomorphisms

$$\begin{aligned}\operatorname{Hom}_k(N \otimes_A \varinjlim_i M_i, L) &\simeq \operatorname{Hom}_A(\varinjlim_i M_i, \operatorname{Hom}_k(N, L)) \\ &\simeq \varprojlim_i \operatorname{Hom}_A(M_i, \operatorname{Hom}_k(N, L)) \\ &\simeq \varprojlim_i \operatorname{Hom}_k(N \otimes_A M_i, L) \\ &\simeq \operatorname{Hom}_k(\varinjlim_i (N \otimes_A M_i), L).\end{aligned}$$

Then the result follows from Proposition 1.3.3. q.e.d.

Example 1.4.12. Let k be a commutative ring and consider the k-algebra $A := k[x]$. Denote by $I = A \cdot x$ the ideal generated by x. Notice that $A/I^{n+1} \simeq k[x]^{\leq n}$, where $k[x]^{\leq n}$ denotes the k-module consisting of polynomials of degree less than or equal to n.
(i) For $p \leq n$ there are monomorphisms $u_{pn} : k[x]^{\leq p} \rightarrowtail k[x]^{\leq n}$ which define an inductive system of k-modules. One has the isomorphism

$$k[x] = \varinjlim_n k[x]^{\leq n}.$$

Notice that $\mathrm{id}_{k[x]} \notin \varinjlim_n \operatorname{Hom}_k(k[x], k[x]^{\leq n})$. This shows that the morphism (1.18) is not an isomorphism in general.
(ii) For $p \leq n$ there are epimorphisms $v_{pn} : A/I^n \twoheadrightarrow A/I^p$ which define a projective system of A-modules whose projective limit is $k[[x]]$, the ring of formal series with coefficients in k.
(iii) For $p \leq n$ there are monomorphisms $I^n \rightarrowtail I^p$ which define a projective system of A-modules whose projective limit is 0.
(iv) We thus have a projective system of complexes of A-modules

$$L_n^\bullet : 0 \to I^n \to A \to A/I^n \to 0.$$

Taking the projective limit, we get the complex $0 \to 0 \to k[x] \to k[[x]] \to 0$ which is no more exact.

Recall (Proposition 1.4.4) that a product of exact sequences of A-modules is an exact sequence. Let us give another criterion in order that the projective limit of an exact sequence remains exact. This is a particular case of the so-called “Mittag-Leffler” condition (see [8]).

Proposition 1.4.13. *Let* $0 \to \{M'_n\} \xrightarrow{f_n} \{M_n\} \xrightarrow{g_n} \{M''_n\} \to 0$ *be an exact sequence of projective systems of* A*-modules indexed by* $\mathbb{N}$*. Assume that for each* n*, the map* $M'_{n+1} \to M'_n$ *is surjective. Then the sequence*

$$0 \to \varprojlim_n M'_n \xrightarrow{f} \varprojlim_n M_n \xrightarrow{g} \varprojlim_n M''_n \to 0$$

is exact.

Proof. Let us denote for short by v_p the morphisms $M_p \to M_{p-1}$ which define the projective system $\{M_p\}$, and similarly for v'_p, v''_p.

Let $\{x''_p\}_p \in \varprojlim_n M''_n$. Hence $x''_p \in M''_p$, and $v''_p(x''_p) = x''_{p-1}$.

We shall first show that $v_n : g_n^{-1}(x''_n) \to g_{n-1}^{-1}(x''_{n-1})$ is surjective. Let $x_{n-1} \in g_{n-1}^{-1}(x''_{n-1})$. Take $x_n \in g_n^{-1}(x''_n)$. Then $g_{n-1}(v_n(x_n) - x_{n-1})) = 0$. Hence $v_n(x_n) - x_{n-1} = f_{n-1}(x'_{n-1})$. By the hypothesis $f_{n-1}(x'_{n-1}) = f_{n-1}(v'_n(x'_n))$ for some x'_n and thus $v_n(x_n - f_n(x'_n)) = x_{n-1}$.

Then we can choose $x_n \in g_n^{-1}(x''_n)$ inductively such that $v_n(x_n) = x_{n-1}$.
q.e.d.

1.5 Koszul complexes

If L is a finite free k-module of rank n, one denotes by $\bigwedge^j L$ the k-module consisting of j-multilinear alternate forms on the dual space L^* and calls it the j-th exterior power of L. (Recall that $L^* = \mathrm{Hom}_k(L, k)$.)

Note that $\bigwedge^1 L \simeq L$ and $\bigwedge^n L \simeq k$. One sets $\bigwedge^0 L = k$.

If $(e_1, \ldots, e_n)$ is a basis of L and $I = \{i_1 < \cdots < i_j\} \subset \{1, \ldots, n\}$, one sets

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_j}.$$

For a subset $I \subset \{1, \ldots, n\}$, one denotes by $|I|$ its cardinal. The family of e_I's with $|I| = j$ is a basis of the free module $\bigwedge^j L$.

Let M be an A-module and let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be n endomorphisms of M over A which commute with one another:

$$[\varphi_i, \varphi_j] = 0, \ 1 \le i, j \le n.$$

(Recall the notation $[a, b] := ab - ba$.) Set $M^{(j)} = M \otimes \bigwedge^j k^n$. Hence $M^{(0)} = M$ and $M^{(n)} \simeq M$. Denote by $(e_1, \ldots, e_n)$ the canonical basis of k^n. Hence, any element of $M^{(j)}$ may be written uniquely as a sum

$$m = \sum_{|I|=j} m_I \otimes e_I.$$

One defines $d \in \operatorname{Hom}_A(M^{(j)}, M^{(j+1)})$ by:

$$d(m \otimes e_I) = \sum_{i=1}^{n} \varphi_i(m) \otimes e_i \wedge e_I$$

and extending d by linearity. Using the commutativity of the φ_i's one checks easily that $d \circ d = 0$. Hence we get a complex, called a Koszul complex and denoted $K^\bullet(M, \varphi)$:

$$0 \to M^{(0)} \xrightarrow{d} \cdots \to M^{(n)} \to 0.$$

When $n = 1$, the cohomology of this complex gives the kernel and cokernel of φ_1. More generally,

$$\begin{aligned} H^0(K^\bullet(M, \varphi)) &\simeq \operatorname{Ker} \varphi_1 \cap \ldots \cap \operatorname{Ker} \varphi_n, \\ H^n(K^\bullet(M, \varphi)) &\simeq M/(\varphi_1(M) + \cdots + \varphi_n(M)). \end{aligned}$$

Definition 1.5.1. (i) If for each j, $\quad 1 \leq j \leq n$, φ_j is injective as an endomorphism of $M/(\varphi_1(M) + \cdots + \varphi_{j-1}(M))$, one says $(\varphi_1, \ldots, \varphi_n)$ is a regular sequence.

(ii) If for each j, $1 \leq j \leq n$, φ_j is surjective as an endomorphism of $\operatorname{Ker} \varphi_1 \cap \ldots \cap \operatorname{Ker} \varphi_{j-1}$, one says $(\varphi_1, \ldots, \varphi_n)$ is a coregular sequence.

Theorem 1.5.2. (i) *Assume that* $(\varphi_1, \ldots, \varphi_n)$ *is a regular sequence. Then* $H^j(K^\bullet(M, \varphi)) \simeq 0$ *for* $j \neq n$.

(ii) *Assume that* $(\varphi_1, \ldots, \varphi_n)$ *is a coregular sequence. Then* $H^j(K^\bullet(M, \varphi)) \simeq 0$ *for* $j \neq 0$.

Proof. The proof will be given in Section 5.2. Here, we restrict ourselves to the simple case $n = 2$ for coregular sequences. Hence we consider the complex:

$$0 \to M \xrightarrow{d} M \times M \xrightarrow{d} M \to 0$$

where $d(x) = (\varphi_1(x), \varphi_2(x))$, $d(y, z) = \varphi_2(y) - \varphi_1(z)$ and we assume φ_1 is surjective on M, φ_2 is surjective on $\operatorname{Ker} \varphi_1$.

Let $(y, z) \in M \times M$ with $\varphi_2(y) = \varphi_1(z)$. We look for $x \in M$ solution of $\varphi_1(x) = y, \quad \varphi_2(x) = z$. First choose $x' \in M$ with $\varphi_1(x') = y$. Then $\varphi_2 \circ \varphi_1(x') = \varphi_2(y) = \varphi_1(z) = \varphi_1 \circ \varphi_2(x')$. Thus $\varphi_1(z - \varphi_2(x')) = 0$ and there exists $t \in M$ with $\varphi_1(t) = 0, \quad \varphi_2(t) = z - \varphi_2(x')$. Hence $y = \varphi_1(t + x'), \quad z = \varphi_2(t + x')$ and $x = t + x'$ is a solution to our problem. q.e.d.

Example 1.5.3. Let k be a field of characteristic 0 and let $A = k[x_1, \dots, x_n]$.
(i) Denote by $x_i\cdot$ the multiplication by x_i in A. We get the complex:

$$0 \to A^{(0)} \xrightarrow{d} \cdots \to A^{(n)} \to 0$$

where:

$$d(\sum_I a_I \otimes e_I) = \sum_{j=1}^{n} \sum_I x_j \cdot a_I \otimes e_j \wedge e_I.$$

The sequence $(x_1\cdot, \dots, x_n\cdot)$ is a regular sequence in A, considered as an A-module. Hence the Koszul complex is exact except in degree n where its cohomology is isomorphic to k.
(ii) Denote by ∂_i the partial derivation with respect to x_i. This is a k-linear map on the k-vector space A. Hence we get a Koszul complex

$$0 \to A^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} A^{(n)} \to 0$$

where:

$$d(\sum_I a_I \otimes e_I) = \sum_{j=1}^{n} \sum_I \partial_j(a_I) \otimes e_j \wedge e_I.$$

The sequence $(\partial_1\cdot, \dots, \partial_n\cdot)$ is a coregular sequence, and the above complex is exact except in degree 0 where its cohomology is isomorphic to k. Writing dx_j instead of e_j, we recognize the “de Rham complex”.

Example 1.5.4. Let $W = W_n(k)$ be the Weyl algebra introduced in Example 1.1.2, and denote by $\cdot\partial_i$ the multiplication on the right by ∂_i. Then $(\cdot\partial_1, \dots, \cdot\partial_n)$ is a regular sequence on W (considered as an W-module) and we get the Koszul complex:

$$0 \to W^{(0)} \xrightarrow{\delta} \cdots \to W^{(n)} \to 0$$

where:

$$\delta(\sum_I a_I \otimes e_I) = \sum_{j=1}^{n} \sum_I a_I \cdot \partial_j \otimes e_j \wedge e_I.$$

This complex is exact except in degree n where its cohomology is isomorphic to $k[x]$ (see Exercise 1.3).

Remark 1.5.5. One may also encounter co-Koszul complexes. For $I = (i_1, \dots, i_k)$, introduce

$$e_j \lfloor e_I = \begin{cases} 0 & \text{if } j \notin \{i_1, \dots, i_k\} \\ (-1)^{l+1} e_{I_{\hat{l}}} := (-1)^{l+1} e_{i_1} \wedge \ldots \wedge \widehat{e_{i_l}} \wedge \ldots \wedge e_{i_k} & \text{if } e_{i_l} = e_j \end{cases}$$

where $e_{i_1} \wedge \ldots \wedge \widehat{e_{i_l}} \wedge \ldots \wedge e_{i_k}$ means that e_{i_l} should be omitted in $e_{i_1} \wedge \ldots \wedge e_{i_k}$. Define δ by:

$$\delta(m \otimes e_I) = \sum_{j=1}^{n} \varphi_j(m) e_j \lfloor e_I.$$

Here again one checks easily that $\delta \circ \delta = 0$, and we get the complex:

$$K_\bullet(M, \varphi) : 0 \to M^{(n)} \xrightarrow{\delta} \cdots \to M^{(0)} \to 0,$$

This complex is in fact isomorphic to a Koszul complex. Consider the isomorphism

$$* : \bigwedge^{j} k^n \xrightarrow{\sim} \bigwedge^{n-j} k^n$$

which associates $\varepsilon_I m \otimes e_{\hat{I}}$ to $m \otimes e_I$, where $\hat{I} = (1, \ldots, n) \setminus I$ and ε_I is the signature of the permutation which sends $(1, \ldots, n)$ to $I \sqcup \hat{I}$ (any $i \in I$ is smaller than any $j \in \hat{I}$). Then, up to a sign, $*$ interchanges d and δ.

Exercises to Chapter 1

Exercise 1.1. Consider two complexes of A-modules $M_1' \to M_1 \to M_1''$ and $M_2' \to M_2 \to M_2''$. Prove that the two sequences are exact if and only if the sequence $M_1' \oplus M_2' \to M_1 \oplus M_2 \to M_1'' \oplus M_2''$ is exact.

Exercise 1.2. (i) Prove that a free module is projective and flat.
(ii) Prove that a module P is projective if and only if it is a direct summand of a free module (i.e. there exists a module K such that $P \oplus K$ is free).
(iii) Deduce that projective modules are flat.

Exercise 1.3. Let k be a field of characteristic 0, $W := W_n(k)$ the Weyl algebra in n variables.
(i) Denote by $x_i \cdot : W \to W$ the multiplication on the left by x_i on W (hence, the $x_i\cdot$'s are morphisms of right W-modules). Prove that $\varphi = (x_1\cdot, \ldots, x_n\cdot)$ is a regular sequence and calculate $H^j(K^\bullet(W, \varphi))$.
(ii) Denote $\cdot\partial_i$ the multiplication on the right by ∂_i on W. Prove that $\psi = (\cdot\partial_1, \ldots, \cdot\partial_n)$ is a regular sequence and calculate $H^j(K^\bullet(W, \psi))$.
(iii) Now consider the left $W_n(k)$-module $\mathcal{O} := k[x_1, \ldots, x_n]$ and the k-linear map $\partial_i : \mathcal{O} \to \mathcal{O}$ (derivation with respect to x_i). Prove that $\lambda = (\partial_1, \ldots, \partial_n)$ is a coregular sequence and calculate $H^j(K^\bullet(\mathcal{O}, \lambda))$.

Exercise 1.4. Let $A = W_2(k)$ be the Weyl algebra in two variables. Construct the Koszul complex associated to $\varphi_1 = \cdot x_1$, $\varphi_2 = \cdot\partial_2$ and calculate its cohomology.

Exercise 1.5. If M is a $\mathbb{Z}$-module, set $M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.
(i) Prove that $\mathbb{Q}/\mathbb{Z}$ is injective in $\mathrm{Mod}(\mathbb{Z})$.
(ii) Prove that the map $\mathrm{Hom}_{\mathbb{Z}}(M, N) \to \mathrm{Hom}_{\mathbb{Z}}(N^\vee, M^\vee)$ is injective for any $M, N \in \mathrm{Mod}(\mathbb{Z})$.
(iii) Prove that if P is a right projective A-module, then $P^\vee$ is left A-injective.
(iv) Let M be an A-module. Prove that there exists an injective A-module I and a monomorphism $M \to I$.
(Hint: (iii) Use formula (1.9). (iv) Prove that $M \mapsto M^{\vee\vee}$ is an injective map using (ii), and replace M with $M^{\vee\vee}$.)

Exercise 1.6. Let k be a field, $A = k[x, y]$ and consider the A-module $M = \bigoplus_{i\geq 1} k[x]t^i$, where the action of $x \in A$ is the usual one and the action of $y \in A$ is defined by $y \cdot x^n t^{j+1} = x^n t^j$ for $j \geq 1$, $y \cdot x^n t = 0$. Define the endomorphisms of M, $\varphi_1(m) = x \cdot m$ and $\varphi_2(m) = y \cdot m$. Calculate the cohomology of the Kozsul complex $K^\bullet(M, \varphi)$.

Exercise 1.7. Let I be a filtrant ordered set and let $M_i, i \in I$ be an inductive sytem of k-modules indexed by I. Let $M = \bigsqcup M_i/\sim$ where $\bigsqcup$ denotes the set-theoretical disjoint union and $\sim$ is the relation $M_i \ni x_i \sim y_j \in M_j$ if there exists $k \geq i, k \geq j$ such that $u_{ki}(x_i) = u_{kj}(y_j)$.

Prove that M is naturally a k-module and is isomorphic to $\varinjlim_i M_i$.

Exercise 1.8. Let I be a filtrant ordered set and let $A_i, i \in I$ be an inductive sytem of rings indexed by I.
(i) Prove that $A := \varinjlim_i A_i$ is naturally endowed with a ring structure.
(ii) Define the notion of an inductive system M_i of A_i-modules, and define the A-module $\varinjlim_i M_i$.
(iii) Let N_i (resp. M_i) be an inductive system of right (resp. left) A_i modules. Prove the isomorphism

$$\varinjlim_i (N_i \otimes_{A_i} M_i) \xrightarrow{\sim} \varinjlim_i N_i \otimes_A \varinjlim_i M_i.$$

Chapter 2

The language of categories

In this chapter we introduce some basic notions of category theory which are of constant use in various fields of Mathematics, without spending too much time on this language. After giving the main definitions on categories and functors, we prove the Yoneda Lemma. We also introduce the notions of representable functors and adjoint functors.
Some references: [14], [2], [13], [6], [11], [12].

2.1 Categories and functors

Definition 2.1.1. A category $\mathcal{C}$ consists of:

(i) a family $\mathrm{Ob}(\mathcal{C})$, the objects of $\mathcal{C}$,

(ii) for each $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, the morphisms from X to Y,

(iii) for any $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a map: $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, called the composition and denoted $(f, g) \mapsto g \circ f$,

these data satisfying:

(a) $\circ$ is associative,

(b) for each $X \in \mathrm{Ob}(\mathcal{C})$, there exists $\mathrm{id}_X \in \mathrm{Hom}\,(X, X)$ such that for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$, $f \circ \mathrm{id}_X = f$, $\mathrm{id}_X \circ g = g$.

Note that $\mathrm{id}_X \in \mathrm{Hom}\,(X, X)$ is characterized by the condition in (b).

Remark 2.1.2. There are some set-theoretical dangers, and one should mention in which "universe" we are working. For sake of simplicity, we shall not enter in these considerations here. (See remark 3.4.10.)

Notation 2.1.3. One often writes $X \in \mathcal{C}$ instead of $X \in \mathrm{Ob}(\mathcal{C})$ and $f : X \to Y$ instead of $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$. One calls X the source and Y the target of f.

A morphism $f : X \to Y$ is an *isomorphism* if there exists $g : X \leftarrow Y$ such that $f \circ g = \mathrm{id}_Y$, $g \circ f = \mathrm{id}_X$. In such a case, one writes $f : X \xrightarrow{\sim} Y$ or simply $X \simeq Y$. Of course g is unique, and one also denotes it by f^{-1}.

A morphism $f : X \to Y$ is a *monomorphism* (resp. an *epimorphism*) if for any morphisms g_1 and g_2, $f \circ g_1 = f \circ g_2$ (resp. $g_1 \circ f = g_2 \circ f$) implies $g_1 = g_2$. One sometimes writes $f : X \rightarrowtail Y$ or else $X \hookrightarrow Y$ (resp. $f : X \twoheadrightarrow Y$) to denote a monomorphism (resp. an epimorphism).

Two morphisms f and g are parallel if they have the same sources and targets, visualized by $f, g : X \rightrightarrows Y$.

One introduces the *opposite category* $\mathcal{C}^{\mathrm{op}}$:

$$\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C}), \quad \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X).$$

A category $\mathcal{C}'$ is a *subcategory* of $\mathcal{C}$, denoted $\mathcal{C}' \subset \mathcal{C}$, if: $\mathrm{Ob}(\mathcal{C}') \subset \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}'}(X,Y) \subset \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for any $X, Y \in \mathcal{C}'$ and the composition $\circ$ in $\mathcal{C}'$ is induced by the composition in $\mathcal{C}$. One says that $\mathcal{C}'$ is a *full* subcategory if for all $X, Y \in \mathcal{C}'$, $\mathrm{Hom}_{\mathcal{C}'}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$.

A category is *discrete* if the only morphisms are the identity morphisms. Note that a set is naturally identified with a discrete category.

A category $\mathcal{C}$ is *finite* if the family of all morphisms in $\mathcal{C}$ (hence, in particular, the family of objects) is a finite set.

A category $\mathcal{C}$ is a *groupoid* if all morphisms are isomorphisms.

Examples 2.1.4. (i) $\mathbf{Set}$ is the category of sets and maps, $\mathbf{Set}^f$ is the full subcategory consisting of finite sets.
(ii) $\mathbf{Rel}$ is defined by: $\mathrm{Ob}(\mathbf{Rel}) = \mathrm{Ob}(\mathbf{Set})$ and $\mathrm{Hom}_{\mathbf{Rel}}(X,Y) = \mathcal{P}(X \times Y)$, the set of subsets of $X \times Y$. The composition law is defined as follows. If $f : X \to Y$ and $g : Y \to Z$, $g \circ f$ is the set

$$\{(x,z) \in X \times Z;\ \text{there exists } y \in Y \text{ with } (x,y) \in f, (y,z) \in g\}.$$

Of course, $\mathrm{id}_X = \Delta \subset X \times X$, the diagonal of $X \times X$.
Notice that $\mathbf{Set}$ is a subcategory of $\mathbf{Rel}$, not a full subcategory.
(iii) Let A be a ring. The category of left A-modules and A-linear maps is denoted $\mathrm{Mod}(A)$. In particular $\mathrm{Mod}(\mathbb{Z})$ is the category of abelian groups.

We shall often use the notations $\mathbf{Ab}$ instead of $\mathrm{Mod}(\mathbb{Z})$ and $\mathrm{Hom}_A(\cdot,\cdot)$ instead of $\mathrm{Hom}_{\mathrm{Mod}(A)}(\cdot,\cdot)$.

One denotes by $\mathrm{Mod}^f(A)$ the full subcategory of $\mathrm{Mod}(A)$ consisting of finitely generated A-modules.

(iv) $C(\mathrm{Mod}(A))$ is the category whose objects are the complexes of A-modules and morphisms, morphisms of such complexes.
(v) One associates to a pre-ordered set $(I, \leq)$ a category, still denoted by I for short, as follows. $\mathrm{Ob}(I) = I$, and the set of morphisms from i to j has a single element if $i \leq j$, and is empty otherwise. Note that I^{op} is the category associated with I endowed with the opposite order.
(vi) We denote by **Top** the category of topological spaces and continuous maps.

Definition 2.1.5. Let I be a category.

(i) One defines the category $\mathrm{Mor}(I)$ by

$$\begin{aligned} \mathrm{Ob}(\mathrm{Mor}(I)) &= \{(i,j,s); i,j \in \mathcal{I}, s \in \mathrm{Hom}_I(i,j), \\ \mathrm{Hom}_{\mathrm{Mor}(I)}((s : i \to j), (s' : i' \to j')) &= \{u : i \to i', v : j \to j'; v \circ s = s' \circ u\}. \end{aligned}$$

(ii) One defines the category $\mathrm{Mor}_0(I)$ by

$$\begin{aligned} \mathrm{Ob}(\mathrm{Mor}_0(I)) &= \{(i,j,s); i,j \in \mathcal{I}, s \in \mathrm{Hom}_I(i,j), \\ \mathrm{Hom}_{\mathrm{Mor}_0(I)}((s : i \to j), (s' : i' \to j')) &= \{u : i \to i', v : j' \to j; s = v \circ s' \circ u\}. \end{aligned}$$

The morphisms in $\mathrm{Mor}(I)$ (resp. $\mathrm{Mor}_0(I)$) are visualized by the commutative diagram on the left (resp. on the right) below:

$$\begin{tikzcd} i \arrow[r, "s"] \arrow[d, "u"'] & j \arrow[d, "v"] \\ i' \arrow[r, "s'"] & j', \end{tikzcd} \qquad \begin{tikzcd} i \arrow[r, "s"] \arrow[d, "u"'] & j \\ i' \arrow[r, "s'"] & j'. \arrow[u, "w"'] \end{tikzcd}$$

Definition 2.1.6. (i) An object $P \in \mathcal{C}$ is called initial if for all $X \in \mathcal{C}, \mathrm{Hom}_{\mathcal{C}}(P, X) \simeq \{\mathrm{pt}\}$. One often denotes by $\emptyset_{\mathcal{C}}$ an initial object in $\mathcal{C}$.

(ii) One says that P is terminal if P is initial in $\mathcal{C}^{\mathrm{op}}$, *i.e.,* for all $X \in \mathcal{C}, \mathrm{Hom}_{\mathcal{C}}(X, P) \simeq \{\mathrm{pt}\}$. One often denotes by $\mathrm{pt}_{\mathcal{C}}$ a terminal object in $\mathcal{C}$.

(iii) One says that P is a zero-object if it is both initial and terminal. In such a case, one often denotes it by 0. If $\mathcal{C}$ has a zero object, for any object $X \in \mathcal{C}$, the morphism obtained as the composition $X \to 0 \to X$ is still denoted by $0 \colon X \to X$.

Note that initial (resp. terminal) objects are unique up to unique isomorphisms.

Examples 2.1.7. (i) In the category **Set**, $\emptyset$ is initial and $\{\mathrm{pt}\}$ is terminal.
(ii) The zero module 0 is a zero-object in $\mathrm{Mod}(A)$.
(iii) The category associated with the ordered set $(\mathbb{Z}, \leq)$ has neither initial nor terminal object.

Definition 2.1.8. Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. A functor $F : \mathcal{C} \to \mathcal{C}'$ consists of a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$ and for all $X, Y \in \mathcal{C}$, of a map still denoted by F: $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ such that

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}, \quad F(f \circ g) = F(f) \circ F(g).$$

A contravariant functor from $\mathcal{C}$ to $\mathcal{C}'$ is a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}'$. In other words, it satisfies $F(g \circ f) = F(f) \circ F(g)$. If one wishes to put the emphasis on the fact that a functor is not contravariant, one says it is covariant.

One denotes by $\mathrm{op} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ the contravariant functor, associated with $\mathrm{id}_{\mathcal{C}^{\mathrm{op}}}$.

Definition 2.1.9. (i) One says that F is faithful (resp. full, resp. fully faithful) if for X, Y in $\mathcal{C}$

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$$

is injective (resp. surjective, resp. bijective).

(ii) One says that F is essentially surjective if for each $Y \in \mathcal{C}'$ there exists $X \in \mathcal{C}$ and an isomorphism $F(X) \simeq Y$.

One defines the product of two categories $\mathcal{C}$ and $\mathcal{C}'$ by :

$$\begin{aligned} \mathrm{Ob}(\mathcal{C} \times \mathcal{C}') &= \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}') \\ \mathrm{Hom}_{\mathcal{C} \times \mathcal{C}'}((X, X'), (Y, Y')) &= \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}'}(X', Y'). \end{aligned}$$

A bifunctor $F : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$ is a functor on the product category. This means that for $X \in \mathcal{C}$ and $X' \in \mathcal{C}'$, $F(X, \cdot) : \mathcal{C}' \to \mathcal{C}''$ and $F(\cdot, X') : \mathcal{C} \to \mathcal{C}''$ are functors, and moreover for any morphisms $f : X \to Y$ in $\mathcal{C}$, $g : X' \to Y'$ in $\mathcal{C}'$, the diagram below commutes:

$$\begin{tikzcd} F(X, X') \arrow[r, "F(X,g)"] \arrow[d, "F(f,X')"'] & F(X, Y') \arrow[d, "F(f,Y')"] \\ F(Y, X') \arrow[r, "F(Y,g)"] & F(Y, Y') \end{tikzcd}$$

In fact, $(f, g) = (\mathrm{id}_Y, g) \circ (f, \mathrm{id}_{X'}) = (f, \mathrm{id}_{Y'}) \circ (\mathrm{id}_X, g)$.

Examples 2.1.10. (i) $\mathrm{Hom}_{\mathcal{C}}(\cdot,\cdot) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ is a bifunctor.
(ii) If A is a k-algebra, $\cdot \otimes_A \cdot \colon \mathrm{Mod}(A^{\mathrm{op}}) \times \mathrm{Mod}(A) \to \mathrm{Mod}(k)$ and $\mathrm{Hom}_A(\cdot,\cdot)\colon \mathrm{Mod}(A)^{\mathrm{op}} \times \mathrm{Mod}(A) \to \mathrm{Mod}(k)$ are bifunctors.
(iii) Let A be a ring. Then $H^j(\cdot) : C(\mathrm{Mod}(A)) \to \mathrm{Mod}(A)$ is a functor.
(iv) The forgetful functor $for : \mathrm{Mod}(A) \to \mathbf{Set}$ associates to an A-module M the set M, and to a linear map f the map f.

The following categories often appear in Category Theory.
Let $\mathcal{C}$, $\mathcal{C}'$ be categories and $F\colon \mathcal{C} \to \mathcal{C}'$ a functor. Let $Z \in \mathcal{C}'$.

Definition 2.1.11. (i) The category $\mathcal{C}_Z$ is defined as follows:

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}_Z) &= \{(X,u); X \in \mathcal{C}, u\colon F(X) to Y\}, \\ \mathrm{Hom}_{\mathcal{C}_Z}((X_1,u_1),(X_2,u_2)) &= \{v\colon X_1 \to X_2; u_1 = u_2 \circ F(v)\}. \end{aligned}$$

(ii) The category $\mathcal{C}^Z$ is defined as follows:

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}^Z) &= \{(X,u); X \in \mathcal{C}, u\colon Y \to F(X))\}, \\ \mathrm{Hom}_{\mathcal{C}^Z}((X_1,u_1),(X_2,u_2)) &= \{v\colon X_1 \to X_2; u_2 = u_1 \circ F(v)\}. \end{aligned}$$

Note that the natural functors $(X,u) \mapsto X$ from $\mathcal{C}_Z$ and $\mathcal{C}^Z$ to $\mathcal{C}$ are faithful.

The morphisms in $\mathcal{C}_Z$ (resp. $\mathcal{C}^Z$) are visualized by the commutative diagram on the left (resp. on the right) below:

$$\begin{tikzcd} F(X_1) \arrow[r, "u_1"] \arrow[d, "F(v)"'] & Z \\ F(X_2) \arrow[ur, "u_2"'] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "u_1"] \arrow[dr, "u_2"'] & F(X_1) \arrow[d, "F(v)"] \\ & F(X_2) \end{tikzcd}$$

2.2 Morphisms of functors

Definition 2.2.1. Let F_1, F_2 are two functors from $\mathcal{C}$ to $\mathcal{C}'$. A morphism of functors $\theta : F_1 \to F_2$ is the data for all $X \in \mathcal{C}$ of a morphism $\theta(X) : F_1(X) \to F_2(X)$ such that for all $f : X \to Y$, the diagram below commutes:

$$\begin{tikzcd} F_1(X) \arrow[r, "\theta(X)"] \arrow[d, "F_1(f)"'] & F_2(X) \arrow[d, "F_2(f)"] \\ F_1(Y) \arrow[r, "\theta(Y)"] & F_2(Y) \end{tikzcd}$$

A morphism of functors is visualized by a diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{C}' \end{tikzcd}$$

Hence, by considering the family of functors from $\mathcal{C}$ to $\mathcal{C}'$ and the morphisms of such functors, we get a new category.

Notation 2.2.2. We denote by $\mathrm{Fct}(\mathcal{C}, \mathcal{C}')$ the category of functors from $\mathcal{C}$ to $\mathcal{C}'$. One may also use the shorter notation $(\mathcal{C}')^{\mathcal{C}}$.

Examples 2.2.3. Let k be a field and consider the functor

$$\begin{aligned} {}^*\colon \mathrm{Mod}(k)^{\mathrm{op}} &\to \mathrm{Mod}(k), \\ V &\mapsto V^* = \mathrm{Hom}_k(V, k). \end{aligned}$$

Then there is a natural morphism of functors $\mathrm{id} \to {}^* \circ {}^*$ in $\mathrm{Fct}(\mathrm{Mod}(k), \mathrm{Mod}(k))$.

(ii) We shall encounter morphisms of functors when considering pairs of adjoint functors (see (2.5)).

In particular we have the notion of an isomorphism of categories. If F is an isomorphism of categories, then there exists $G : \mathcal{C}' \to \mathcal{C}$ such that for all $X \in \mathcal{C}$, $G \circ F(X) = X$. In practice, such a situation rarely occurs and is not really interesting. There is an weaker notion that we introduce below.

Definition 2.2.4. A functor $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories if there exists $G : \mathcal{C}' \to \mathcal{C}$ such that: $G \circ F$ is isomorphic to $\mathrm{id}_{\mathcal{C}}$ and $F \circ G$ is isomorphic to $\mathrm{id}_{\mathcal{C}'}$.

Theorem 2.2.5. *The functor $F : \mathcal{C} \to \mathcal{C}'$ is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

If two categories are equivalent, all results and concepts in one of them have their counterparts in the other one. This is why this notion of equivalence of categories plays an important role in Mathematics.

Examples 2.2.6. (i) Let k be a field and let $\mathcal{C}$ denote the category defined by $\mathrm{Ob}(\mathcal{C}) = \mathbb{N}$ and $\mathrm{Hom}_{\mathcal{C}}(n, m) = M_{m,n}(k)$, the space of matrices of type (m, n) with entries in a field k (the composition being the usual composition of matrices). Define the functor $F : \mathcal{C} \to \mathrm{Mod}^f(k)$ as follows. To $n \in \mathbb{N}$, $F(n)$ associates $k^n \in \mathrm{Mod}^f(k)$ and to a matrix of type (m, n), F associates the induced linear map from k^n to k^m. Clearly F is fully faithful, and since any finite dimensional vector space admits a basis, it is isomorphic to k^n for

some n, hence F is essentially surjective. In conclusion, F is an equivalence of categories.
(ii) let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. There is an equivalence

$$\mathrm{Fct}(\mathcal{C},\mathcal{C}')^{\mathrm{op}} \simeq \mathrm{Fct}(\mathcal{C}^{\mathrm{op}},(\mathcal{C}')^{\mathrm{op}}). \tag{2.1}$$

(iii) Let I, J and $\mathcal{C}$ be categories. There are equivalences

$$\mathrm{Fct}(I\times J,\mathcal{C}) \simeq \mathrm{Fct}(J,\mathrm{Fct}(I,\mathcal{C})) \simeq \mathrm{Fct}(I,\mathrm{Fct}(J,\mathcal{C})). \tag{2.2}$$

2.3 The Yoneda lemma

Definition 2.3.1. Let $\mathcal{C}$ be a category. One defines the categories

$$\begin{aligned}\mathcal{C}^\wedge &= \mathrm{Fct}(\mathcal{C}^{\mathrm{op}},\mathbf{Set}),\\ \mathcal{C}^\vee &= \mathrm{Fct}(\mathcal{C}^{\mathrm{op}},\mathbf{Set}^{\mathrm{op}}),\end{aligned}$$

and the functors

$$\begin{aligned}\mathrm{h}_\mathcal{C} &: \mathcal{C}\to\mathcal{C}^\wedge, \quad X\mapsto \mathrm{Hom}_\mathcal{C}(\cdot,X),\\ \mathrm{k}_\mathcal{C} &: \mathcal{C}\to\mathcal{C}^\vee, \quad X\mapsto \mathrm{Hom}_\mathcal{C}(X,\cdot).\end{aligned}$$

By (2.1) there is a natural isomorphism

$$\mathcal{C}^\vee \simeq \mathcal{C}^{\mathrm{op}\wedge\mathrm{op}} \tag{2.3}$$

Proposition 2.3.2. (The Yoneda lemma.)

(i) *For $A\in\mathcal{C}^\wedge$ and $X\in\mathcal{C}$,* $\mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{h}_\mathcal{C}(X),A)\simeq A(X)$.

(ii) *For $B\in\mathcal{C}^\vee$ and $X\in\mathcal{C}$,* $\mathrm{Hom}_{\mathcal{C}^\vee}(B,\mathrm{k}_\mathcal{C}(X))\simeq B(X)$.

Moreover, these isomorphisms are functorial with respect to X, A, B, that is, they define isomorphisms of functors from $\mathcal{C}^{\mathrm{op}}\times\mathcal{C}^\wedge$ to **Set** *or from $\mathcal{C}^{\vee\mathrm{op}}\times\mathcal{C}$ to* **Set**.

Proof. By (2.3) is enough to prove one of the two statements. Let us prove (i).

One constructs the morphism $\varphi\colon \mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{h}_\mathcal{C}(X),A)\to A(X)$ by the chain of morphisms: $\mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{h}_\mathcal{C}(X),A)\to\mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}_\mathcal{C}(X,X),A(X))\to A(X)$, where the last map is associated with id_X.

To construct $\psi : A(X)\to\mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{h}(X)_\mathcal{C},A)$, it is enough to associate with $s\in A(X)$ and $Y\in\mathcal{C}$ a map from $\mathrm{Hom}_\mathcal{C}(Y,X)$ to $A(Y)$. It is defined by the chain of maps $\mathrm{Hom}_\mathcal{C}(Y,X)\to\mathrm{Hom}_{\mathbf{Set}}(A(X),A(Y))\to A(Y)$ where the last map is associated with $s\in A(X)$.

One checks that φ and ψ are inverse to each other. q.e.d.

Corollary 2.3.3. *The two functors* $\mathrm{h}_{\mathcal{C}}$ *and* $\mathrm{k}_{\mathcal{C}}$ *are fully faithful.*

Proof. For X and Y in $\mathcal{C}$, one has $\mathrm{Hom}_{\mathcal{C}^\wedge}(\mathrm{h}_{\mathcal{C}}(X), \mathrm{h}(Y)) \simeq \mathrm{h}_{\mathcal{C}}(Y)(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$. q.e.d.

One calls $\mathrm{h}_{\mathcal{C}}$ and $\mathrm{k}_{\mathcal{C}}$ the Yoneda embeddings. Hence, one may consider $\mathcal{C}$ as a full subcategory of $\mathcal{C}^\wedge$ or of $\mathcal{C}^\vee$.

Corollary 2.3.4. *Let* $\mathcal{C}$ *be a category and let* $f : X \to Y$ *be a morphism in* $\mathcal{C}$*.*

(i) *Assume that for any* $Z \in \mathcal{C}$*, the map* $\mathrm{Hom}_{\mathcal{C}}(Z, X) \xrightarrow{f\circ} \mathrm{Hom}_{\mathcal{C}}(Z, Y)$ *is bijective. Then* f *is an isomorphism.*

(ii) *Assume that for any* $Z \in \mathcal{C}$*, the map* $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \xrightarrow{\circ f} \mathrm{Hom}_{\mathcal{C}}(X, Z)$ *is bijective. Then* f *is an isomorphism.*

Proof. (i) By the hypothesis, $\mathrm{h}_{\mathcal{C}}(f) : \mathrm{h}_{\mathcal{C}}(X) \to \mathrm{h}_{\mathcal{C}}(Y)$ is an isomorphism in $\mathcal{C}^\wedge$. Since $\mathrm{h}_{\mathcal{C}}$ is fully faithful, this implies that f is an isomorphism.
(ii) follows by reversing the arrows, that is, by replacing $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$. q.e.d.

In the sequel, we shall often identify $\mathcal{C}$ to a full subcategory of $\mathcal{C}^\wedge$ or $\mathcal{C}^\vee$ and we shall not write the functor $\mathrm{h}_{\mathcal{C}}$ and $\mathrm{k}_{\mathcal{C}}$.

2.4 Representable functors

Definition 2.4.1. One says that a functor F from $\mathcal{C}^{\mathrm{op}}$ to $\mathbf{Set}^{\mathrm{op}}$ (resp. $\mathcal{C}^{\mathrm{op}}$ to $\mathbf{Set}$) is representable if $F \simeq \mathrm{k}_{\mathcal{C}}(X)$ (resp. $\mathrm{h}_{\mathcal{C}}(X)$) for some $X \in \mathcal{C}$. Such an object X is called a representative of F.

It is important to notice that the isomorphism $F \simeq \mathrm{h}_{\mathcal{C}}(X)$ (resp. $F \simeq \mathrm{k}_{\mathcal{C}}(X)$) determines X up to unique isomorphism.

Representable functors provides a categorical language to deal with universal problems. Let us illustrate this by an example.

Example 2.4.2. Let k be a commutative ring and let M, N, L be three k-modules. Denote by $B(N \times M, L)$ the set of k-bilinear maps from $N \times M$ to L. Then the functor $F \colon L \mapsto B(N \times M, L)$ is representable by $N \otimes_k M$, since $F(L) = B(N \times M, L) \simeq \mathrm{Hom}_k(N \otimes M, L)$.

Definition 2.4.3. Let $F \colon \mathcal{C} \to \mathcal{C}'$ and $G \colon \mathcal{C}' \to \mathcal{C}$ be two functors. One says that (F, G) is a pair of adjoint functors or that F is a left adjoint to G, or that G is a right adjoint to F if there exists an isomorphism of bifunctors:

$$\mathrm{Hom}_{\mathcal{C}'}(F(\cdot), \cdot) \simeq \mathrm{Hom}_{\mathcal{C}}(\cdot, G(\cdot)) \tag{2.4}$$

If G is an adjoint to F, then G is unique up to isomorphism. In fact, $G(Y)$ is a representative of the functor $X \mapsto \mathrm{Hom}_{\mathcal{C}}(F(X), Y)$.

The isomorphism (2.4) gives the isomorphisms

$$\mathrm{Hom}_{\mathcal{C}'}(F \circ G(\cdot), \cdot) \simeq \mathrm{Hom}_{\mathcal{C}}(G(\cdot), G(\cdot)),$$
$$\mathrm{Hom}_{\mathcal{C}'}(F(\cdot), F(\cdot)) \simeq \mathrm{Hom}_{\mathcal{C}}(\cdot, G \circ F(\cdot)),$$

from which we deduce the morphisms of functors

$$F \circ G \to \mathrm{id}_{\mathcal{C}'}, \qquad \mathrm{id}_{\mathcal{C}} \to G \circ F. \tag{2.5}$$

Example 2.4.4. Let A be a k-algebra. Let $K \in \mathrm{Mod}(k)$ and let $M, N \in \mathrm{Mod}(A)$. The formula:

$$\mathrm{Hom}_{A}(N \otimes K, M) \simeq \mathrm{Hom}_{A}(N, \mathrm{Hom}\,(K, M)).$$

tells us that the functors $\cdot \otimes K$ and $\mathrm{Hom}\,(K, \cdot)$ from $\mathrm{Mod}(A)$ to $\mathrm{Mod}(A)$ are adjoint.

In the preceding situation, denote by *for* : $\mathrm{Mod}(A) \to \mathrm{Mod}(k)$ the "forgetful functor" which, to an A-module M associates the underlying k-module. Applying the above formula with $N = A$, we get

$$\mathrm{Hom}_{A}(A \otimes K, M) \simeq \mathrm{Hom}\,(K, \mathit{for}(M)).$$

Hence, the functors $A \otimes \cdot$ (extension of scalars) and *for* are adjoint.

Exercises to Chapter 2

Exercise 2.1. Prove that the categories $\mathbf{Set}$ and $\mathbf{Set}^{\mathrm{op}}$ are not equivalent. (Hint: if $F : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ were such an equivalence, then $F(\emptyset) \simeq \{\mathrm{pt}\}$ and $F(\{\mathrm{pt}\}) \simeq \emptyset$. Now compare $\mathrm{Hom}_{\mathbf{Set}}(\{\mathrm{pt}\}, X)$ and $\mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(F(\{\mathrm{pt}\}), F(X))$ when X is a set with two elements.)

Exercise 2.2. Let $F \colon \mathcal{C} \to \mathcal{C}'$ be a fully faithful functor and let f be a morphism in $\mathcal{C}$. Prove that if $F(f)$ is an isomorphism, then f is an isomorphism.

Exercise 2.3. Prove that the category $\mathcal{C}$ is equivalent to the opposite category $\mathcal{C}^{\mathrm{op}}$ in the following cases:
(i) $\mathcal{C}$ denotes the category of finite abelian groups,
(ii) $\mathcal{C}$ is the category $\mathbf{Rel}$ of relations.

Exercise 2.4. (i) Prove that in the category $\mathbf{Set}$, a morphism f is a monomorphism (resp. an epimorphism) if and only if it is injective (resp. surjective).
(ii) Prove that in the category of rings, the morphism $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism.

Exercise 2.5. Let $\mathcal{C}$ be a category. We denote by $\mathrm{id}_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ the identity functor of $\mathcal{C}$ and by $\mathrm{End}\,(\mathrm{id}_{\mathcal{C}})$ the set of endomorphisms of the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, that is,

$$\mathrm{End}\,(\mathrm{id}_{\mathcal{C}}) = \mathrm{Hom}\,_{\mathrm{Fct}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}).$$

Prove that the composition law on $\mathrm{End}\,(\mathrm{id}_{\mathcal{C}})$ is commutative.

Exercise 2.6. In the category **Top**, give an example of a morphism which is both a monomorphism and an epimorphism and which is not an isomorphism. (Hint: consider a continuous injective map $f \colon X \hookrightarrow Y$ with dense image.)

Chapter 3

Limits

Inductive and projective limits are at the heart of category theory. They are an essential tool, if not the only one, to construct new objects and new functors. Inductive and projective limits in categories are constructed by using *projective* limits in the category **Set** of sets. In this chapter we define these limits and give many examples. We also closely analyze some related notions, in particular those of cofinal categories, filtrant categories and exact functors. Special attention will be paid to filtrant inductive limits in the category **Set**.

3.1 Limits

In the sequel, I will denote a category. Let $\mathcal{C}$ be a category. A functor $\alpha\colon I \to \mathcal{C}$ (resp. $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$) is sometimes called an inductive (resp. projective) system in $\mathcal{C}$ indexed by I.

For example, if $(I, \leq)$ is a pre-ordered set, I the associated category, an inductive system indexed by I is the data of a family $(X_i)_{i\in I}$ of objects of $\mathcal{C}$ and for all $i \leq j$, a morphism $X_i \to X_j$ with the natural compatibility conditions.

Assume first that $\mathcal{C}$ is the category **Set** and let us consider projective systems. In other words, β is an object of $I^\wedge$. Denote by $\beta_\circ$ the constant functor from I^{op} to **Set**, defined by $\beta_\circ(i) = \{\mathrm{pt}\}$ for all $i \in I$. One defines the projective limit of β as

$$\varprojlim \beta = \mathrm{Hom}_{I^\wedge}(\beta_\circ, \beta). \tag{3.1}$$

The family of morphisms:

$$\mathrm{Hom}_{I^\wedge}(\beta_\circ, \beta) \to \mathrm{Hom}_{\mathbf{Set}}(\beta_\circ(i), \beta(i)) = \beta(i), \quad i \in I,$$

defines the map $\varprojlim \beta \to \prod_i \beta(i)$, and one checks immediately that:

$$\varprojlim \beta = \{\{x_i\}_i \in \prod_i \beta(i); \beta(s)(x_j) = x_i \text{ for all } s \in \operatorname{Hom}_I(i,j)\}.$$

The next result is obvious.

Lemma 3.1.1. *Let $\beta\colon I^{\mathrm{op}} \to \mathbf{Set}$ be a functor and let $X \in \mathbf{Set}$. There is a natural isomorphism*

$$\operatorname{Hom}_{\mathbf{Set}}(X, \varprojlim \beta) \xrightarrow{\sim} \varprojlim \operatorname{Hom}_{\mathbf{Set}}(X, \beta),$$

where $\operatorname{Hom}_{\mathbf{Set}}(X,\beta)$ denotes the functor $I^{\mathrm{op}} \to \mathbf{Set}$, $i \mapsto \operatorname{Hom}_{\mathbf{Set}}(X, \beta(i))$.

Now let α (resp. β) be a functor from I (resp. I^{op}) to $\mathcal{C}$. For $X \in \mathcal{C}$, $\operatorname{Hom}_{\mathcal{C}}(\alpha, X)$ and $\operatorname{Hom}_{\mathcal{C}}(X, \beta)$ are functors from I^{op} to $\mathbf{Set}$. We can then define inductive and projective limits as functors from $\mathcal{C}$ or $\mathcal{C}^{\mathrm{op}}$ to $\mathbf{Set}$ as follows.

Definition 3.1.2. (i) One defines $\varinjlim \alpha \in \mathcal{C}^\vee$ and $\varprojlim \beta \in \mathcal{C}^\wedge$ by the formulas

$$\varinjlim \alpha \;:=\; X \mapsto \varprojlim \operatorname{Hom}_{\mathcal{C}}(\alpha, X) = \varprojlim (\mathrm{h}_{\mathcal{C}}(X) \circ \alpha), \tag{3.2}$$

$$\varprojlim \beta \;:=\; X \mapsto \varprojlim \operatorname{Hom}_{\mathcal{C}}(X, \beta) = \varprojlim (\mathrm{k}_{\mathcal{C}}(X) \circ \beta). \tag{3.3}$$

(ii) If these functors are representable, one keeps the same notations to denote their representative in $\mathcal{C}$, and one calls these representative the inductive or projective limit, respectively.

(iii) If every functor from I (resp. I^{op}) to $\mathcal{C}$ admits an inductive (resp. projective) limit, one says that $\mathcal{C}$ admits inductive (resp. projective) limits indexed by I.

(iv) One says that a category $\mathcal{C}$ admits finite projective (resp. inductive) limits if it admits projective (resp. inductive) limits indexed by finite categories.

When $\mathcal{C} = \mathbf{Set}$ this definition of $\varprojlim \beta$ coincides with the former one, in view of Lemma 3.1.1. Hence, the category $\mathbf{Set}$ admits projective limits.

Proposition 3.1.3. *The category* $\mathbf{Set}$ *admits inductive limits. More precisely, if I is a category and $\alpha\colon I \to \mathbf{Set}$ is a functor, then*

$$\varinjlim \alpha \;\simeq\; \Big(\bigsqcup_{i \in I} \alpha(i)\Big)/\sim \quad \textit{where } \sim \textit{ is the equivalence relation } \text{generated } \textit{by}$$
$$\alpha(i) \ni x \sim y \in \alpha(j) \textit{ if there exists } s\colon i \to j \textit{ with } \alpha(s)(x) = y.$$

In particular, the coproduct in **Set** is the disjoint union, $\coprod = \bigsqcup$.

Proof. Let $S \in$ **Set**. By the definition of the projective limit in **Set** we get:

$$\begin{aligned}\varprojlim \operatorname{Hom}(\alpha, S) \ &\simeq\ \{\{p(i,x)\}_{i\in I, x\in\alpha(i)}; p(i,x)\in S, p(i,x)=p(j,y)\\ &\qquad \text{if there exists } s\colon i\to j \text{ with } \alpha(s)(x)=y\}.\end{aligned}$$

The result follows. q.e.d.

Notation 3.1.4. In the category **Set** one uses the notation $\bigsqcup$ rather than $\coprod$.

By Definition 3.1.2, if $\varinjlim \alpha$ or $\varprojlim \beta$ are representable, one gets:

$$\operatorname{Hom}_{\mathcal{C}}(\varinjlim \alpha, X) \ \simeq\ \varprojlim \operatorname{Hom}_{\mathcal{C}}(\alpha, X), \tag{3.4}$$

$$\operatorname{Hom}_{\mathcal{C}}(X, \varprojlim \beta) \ \simeq\ \varprojlim \operatorname{Hom}_{\mathcal{C}}(X, \beta). \tag{3.5}$$

Note that the right-hand sides are the projective limits in **Set**.

Assume that $\varinjlim \alpha$ exists $\mathcal{C}$. One gets:

$$\varprojlim \operatorname{Hom}_{\mathcal{C}}(\alpha, \varinjlim \alpha) \simeq \operatorname{Hom}_{\mathcal{C}}(\varinjlim \alpha, \varinjlim \alpha)$$

and the identity of $\varinjlim \alpha$ defines a family of morphisms

$$\rho_i\colon \alpha(i) \to \varinjlim \alpha.$$

Consider a family of morphisms $\{f_i\colon \alpha(i)\to X\}_{i\in I}$ in $\mathcal{C}$ satisfying the compatibility conditions

$$f_i = f_j \circ f(s) \text{ for all } s \in \operatorname{Hom}_I(i,j). \tag{3.6}$$

This family of morphisms is nothing but an element of $\varprojlim\limits_i \operatorname{Hom}(\alpha(i), X)$, hence by (3.4), an element of $\operatorname{Hom}(\varinjlim \alpha, X)$. Therefore, $\varinjlim \alpha$ is characterized by the "universal property":

$$\begin{cases}\text{for all } X\in\mathcal{C} \text{ and all family of morphisms } \{f_i\colon \alpha(i)\to X\}_{i\in I} \text{ in}\\ \mathcal{C} \text{ satisfying (3.6), all morphisms } f_i\text{'s factorize uniquely through}\\ \varinjlim \alpha.\end{cases} \tag{3.7}$$

Similarly, assume that $\varprojlim \beta$ exists in $\mathcal{C}$. One gets:

$$\varprojlim \operatorname{Hom}_{\mathcal{C}}(\varprojlim \beta, \beta) \simeq \operatorname{Hom}_{\mathcal{C}}(\varprojlim \beta, \varprojlim \beta)$$

and the identity of $\varprojlim \beta$ defines a family of morphisms

$$\rho_i \colon \varprojlim \beta \to \beta(i).$$

Consider a family of morphisms $\{f_i \colon X \to \beta(i)\}_{i \in I}$ in $\mathcal{C}$ satisfying the compatibility conditions

$$f_j = f_i \circ f(s) \text{ for all } s \in \mathrm{Hom}_I(i,j). \tag{3.8}$$

This family of morphisms is nothing but an element of $\varprojlim_i \mathrm{Hom}\,(X, \beta(i))$, hence by (3.5), an element of $\mathrm{Hom}\,(X, \varprojlim \beta, X)$. Therefore, $\varprojlim \beta$ is characterized by the "universal property":

$$\begin{cases} \text{for all } X \in \mathcal{C} \text{ and all family of morphisms } \{f_i \colon X \to \beta(i)\}_{i \in I} \text{ in} \\ \mathcal{C} \text{ satisfying (3.8), all morphisms } f_i\text{'s factorize uniquely through} \\ \varprojlim \beta. \end{cases} \tag{3.9}$$

Inductive and projective limits are visualized by the diagrams:

$$\begin{tikzcd}
\alpha(i) \arrow[dd, "\alpha(s)"'] \arrow[dr, "\rho_i"'] \arrow[drr, "f_i"] & & \\
 & \varinjlim \alpha \arrow[r, dotted] & X \\
\alpha(j) \arrow[ur, "\rho_j"] \arrow[urr, "f_j"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \beta(i) \\
X \arrow[r, dotted] \arrow[urr, "f_i"] \arrow[drr, "f_j"'] & \varprojlim \beta \arrow[ur, "\rho_i"'] \arrow[dr, "\rho_j"] & \\
 & & \beta(j) \arrow[uu, "\beta(s)"']
\end{tikzcd}$$

If $\varphi \colon J \to I$, $\alpha \colon I \to \mathcal{C}$ and $\beta \colon I^{\mathrm{op}} \to \mathcal{C}$ are functors, we have natural morphisms:

$$\varinjlim\,(\alpha \circ \varphi) \to \varinjlim \alpha, \tag{3.10}$$
$$\varprojlim\,(\beta \circ \varphi) \leftarrow \varprojlim \beta. \tag{3.11}$$

This follows immediately of (3.6) and (3.8).

Proposition 3.1.5. *Let I be a category and assume that $\mathcal{C}$ admits inductive limits* (*resp. projective limits*) *indexed by I. Then for any category J, the category $\mathcal{C}^J$ admits inductive limits* (*resp. projective limits*) *indexed by I. Moreover, if $\alpha \colon I \to \mathcal{C}^J$* (*resp. $\beta \colon I^{\mathrm{op}} \to \mathcal{C}^J$*) *is a functor, then its inductive* (*resp. projective*) *limit is defined by*

$$\begin{aligned} (\varinjlim \alpha)(j) &= \varinjlim\,(\alpha(j)),\ j \in J \\ (\textit{resp. } (\varprojlim \beta)(j) &= \varprojlim\,(\beta(j)),\ j \in J). \end{aligned}$$

The proof is obvious.

Corollary 3.1.6. *The categories $\mathcal{C}^\wedge$ and $\mathcal{C}^\vee$ admit projective and inductive limits.*

If $\varphi\colon J \to I$, $\alpha\colon I \to \mathcal{C}$ and $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$ are functors, we have natural morphisms:

$$\varinjlim (\alpha \circ \varphi) \to \varinjlim \alpha, \tag{3.12}$$
$$\varprojlim (\beta \circ \varphi) \leftarrow \varprojlim \beta. \tag{3.13}$$

This follows immediately from the universal properties (3.6) and (**??**).

One can consider inductive or projective limits associated with bifunctors.

Recall the equivalence of categories (2.2) and consider a bifunctor $\alpha\colon I \times J \to \mathcal{C}$. It defines a functor $\alpha_J\colon I \to \mathcal{C}^J$ as well as a functor $\alpha_I\colon J \to \mathcal{C}^I$. One easily checks that

$$\varinjlim \alpha \simeq \varinjlim (\varinjlim \alpha_J) \simeq \varinjlim (\varinjlim \alpha_I). \tag{3.14}$$

Similarly, if $\beta\colon I^{\mathrm{op}} \times J^{\mathrm{op}} \to \mathcal{C}$ is a bifunctor, then β defines a functor $\beta_J\colon I^{\mathrm{op}} \to \mathcal{C}^{J^{\mathrm{op}}}$ and a functor $\beta_I\colon J^{\mathrm{op}} \to \mathcal{C}^{I^{\mathrm{op}}}$ and one has the isomorphisms

$$\varprojlim \beta \simeq \varprojlim \varprojlim \beta_J \simeq \varprojlim \varprojlim \beta_I. \tag{3.15}$$

In other words:

$$\begin{aligned}
\varinjlim_{i,j} \alpha(i,j) &\simeq \varinjlim_j(\varinjlim_i(\alpha(i,j)) \simeq \varinjlim_i \varinjlim_j(\alpha(i,j)),\\
\varprojlim_{i,j} \beta(i,j) &\simeq \varprojlim_j \varprojlim_i(\beta(i,j)) \simeq \varprojlim_i \varprojlim_j(\beta(i,j)).
\end{aligned}$$

3.2 Examples

Empty limits.

If I is the empty category and $\alpha\colon I \to \mathcal{C}$ is a functor, then $\varinjlim \alpha$ exists in $\mathcal{C}$ if and only if $\mathcal{C}$ has an initial object $\emptyset_{\mathcal{C}}$, and in this case $\varinjlim \alpha \simeq \emptyset_{\mathcal{C}}$. Similarly, $\varprojlim \alpha$ exists in $\mathcal{C}$ if and only if $\mathcal{C}$ has a terminal object $\mathrm{pt}_{\mathcal{C}}$, and in this case $\varprojlim \alpha \simeq \mathrm{pt}_{\mathcal{C}}$.

Terminal object

If I admits a terminal object, say i_o and $\alpha\colon I \to \mathcal{C}$ (resp. $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$) is a functor, then

$$\varinjlim \alpha \simeq \alpha(i_o),$$
$$\varprojlim \beta \simeq \beta(i_o).$$

This follows immediately of (3.6) and (3.8).

Sums and products

Consider a discrete category I.

Definition 3.2.1. (i) When the category I is discrete, inductive and projective limits are called coproduct and products, denoted $\coprod$ and $\prod$, respectively. Hence, writing $\alpha(i) = X_i$ or $\beta(i) = X_i$, we get for $Y \in \mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}}(Y, \prod_i X_i) \simeq \prod_i \mathrm{Hom}_{\mathcal{C}}(Y, X_i),$$
$$\mathrm{Hom}_{\mathcal{C}}(\coprod_i X_i, Y) \simeq \prod_i \mathrm{Hom}_{\mathcal{C}}(X_i, Y).$$

(ii) If I is discrete with two objects, a functor $I \to \mathcal{C}$ is the data of two objects X_0 and X_1 in $\mathcal{C}$ and their coproduct and product (if they exist) are denoted by $X_0 \coprod X_1$ and $X_0 \prod X_1$, respectively. Moreover, one usually writes $X_0 \sqcup X_1$ and $X_0 \times X_1$ instead of $X_0 \coprod X_1$ and $X_0 \prod X_1$, respectively.

Hence, if $\alpha\colon I \to \mathcal{C}$ is a functor, with I discrete, one writes $\coprod \alpha$ (resp. $\prod \alpha$) or $\coprod_{i\in I} \alpha(i)$ (resp. $\prod_{i\in I} \alpha(i)$) to denote its limit. One says that $\coprod_{i\in I} \alpha(i)$ (resp. $\prod_{i\in I} \alpha(i)$) is the coproduct (resp. product) of the $\alpha(i)$'s.

If $\alpha(i) = X$ for all $i \in I$, one simply denotes this limit by $X^{\coprod I}$ (resp. $X^{\prod I}$). One also writes $X^{(I)}$ and X^I instead of $X^{\coprod I}$ and $X^{\prod I}$, respectively.

Example 3.2.2. In the category **Set**, we have for $I, X, Z \in \mathbf{Set}$:

$$\begin{aligned} X^{(I)} &\simeq I \times X, \\ X^I &\simeq \mathrm{Hom}_{\mathbf{Set}}(I, X), \\ \mathrm{Hom}_{\mathbf{Set}}(I \times X, Z) &\simeq \mathrm{Hom}_{\mathbf{Set}}(I, \mathrm{Hom}_{\mathbf{Set}}(X, Z)), \\ &\simeq \mathrm{Hom}_{\mathbf{Set}}(X, Z)^I. \end{aligned}$$

The coproduct and product of two objects are visualized by the diagrams:

$$\begin{tikzcd}
X_0 \arrow[dr] \arrow[drr] & & \\
& X_0 \sqcup X_1 \arrow[r, dotted] & X \\
X_1 \arrow[ur] \arrow[urr] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & X_0 \\
X \arrow[r, dotted] \arrow[urr] \arrow[drr] & X_0 \times X_1 \arrow[ur] \arrow[dr] & \\
& & X_1
\end{tikzcd}$$

In other words, any pair of morphisms from (resp. to) X_0 and X_1 to (resp. from) X factors uniquely through $X_0 \sqcup X_1$ (resp. $X_0 \times X_1$). If $\mathcal{C}$ is the category **Set**, $X_0 \sqcup X_1$ is the disjoint union and $X_0 \times X_1$ is the product of the two sets X_0 and X_1.

Cokernels and kernels

Consider the category I with two objects and two parallel morphisms other than identities, visualized by

$$\bullet \rightrightarrows \bullet$$

A functor $\alpha \colon I \to \mathcal{C}$ is characterized by two parallel arrows in $\mathcal{C}$:

$$f, g : X_0 \rightrightarrows X_1 \tag{3.16}$$

In the sequel we shall identify such a functor with the diagram (3.16).

Definition 3.2.3. Consider two parallel arrows $f, g : X_0 \rightrightarrows X_1$ in $\mathcal{C}$.

(i) A co-equalizer (one also says a cokernel), if it exists, is an inductive limit of this functor. It is denoted by $\operatorname{Coker}(f, g)$.

(ii) An equalizer (one also says a kernel), if it exists, is a projective limit of this functor. It is denoted by $\operatorname{Ker}(f, g)$.

(iii) A sequence $X_0 \rightrightarrows X_1 \to Z$ (resp. $Z \to X_0 \rightrightarrows X_1$) is exact if Z is isomorphic to the co-equalizer (resp. equalizer) of $X_0 \rightrightarrows X_1$.

(iv) Assume that the category $\mathcal{C}$ admits a zero-object 0. Let $f \colon X \to Y$ be a morphism in $\mathcal{C}$. A cokernel (resp. a kernel) of f, if it exists, is a cokernel (resp. a kernel) of $f, 0 \colon X \rightrightarrows Y$. It is denoted $\operatorname{Coker}(f)$ (resp. $\operatorname{Ker}(f)$).

The co-equalizer L is visualized by the diagram:

$$\begin{tikzcd} X_0 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr] & X_1 \arrow[r, "k"] \arrow[d, "h"] & L \arrow[dl, dotted] \\ & X & \end{tikzcd}$$

which means that any morphism $h\colon X_1 \to X$ such that $h \circ f = h \circ g$ factors uniquely through k.

Note that

(3.17) $$k \text{ is an epimorphism.}$$

Indeed, consider a pair of parallel arrows $a, b\colon L \rightrightarrows X$ such that $a \circ k = b \circ k = h$. Then $h \circ f = a \circ k \circ f = a \circ k \circ g = b \circ k \circ g = h \circ g$. Hence h factors uniquely through k, and this implies $a = b$.

Dually, the equalizer K is visualized by the diagram:

$$\begin{tikzcd} K \arrow[r, "h"] & X_0 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X_1 \\ & X \arrow[ul, dotted] \arrow[u, "h"] \arrow[ur] & \end{tikzcd}$$

and

(3.18) $$h \text{ is a monomorphism.}$$

We have seen that coproducts and co-equalizers (resp. products and equalizers) are particular cases of inductive (resp. projective) limits. We shall show that conversely, one can construct inductive (resp. projective) limits using coproducts and co-equalizers (resp. products and equalizers), when such objects exist.

Denote by I_d the discrete category associated with I, and recall that $\mathrm{Mor}(I)$ denote the set of morphisms in I. There are two natural functors (source and target) from $\mathrm{Mor}(I)$ to I:

$$\sigma\colon \mathrm{Mor}(I) \to I, (s\colon i \to j) \mapsto i,$$
$$\tau\colon \mathrm{Mor}(I) \to I, (s\colon i \to j) \mapsto j.$$

If $\alpha\colon I \to \mathcal{C}$ is a functor and $s\colon i \to j$ a morphism in I, we get two morphisms

$$\alpha(i) \underset{\alpha(s)}{\overset{\mathrm{id}_{\alpha(i)}}{\rightrightarrows}} \alpha(i) \sqcup \alpha(j)$$

from which we deduce two morphisms $\alpha(\sigma(s)) \rightrightarrows \coprod_{i\in I}\alpha(i)$. These morphisms define the two morphisms

$$\coprod_{s\in \mathrm{Mor}(I)} \alpha(\sigma(s)) \underset{b}{\overset{a}{\rightrightarrows}} \coprod_{i\in I}\alpha(i). \tag{3.19}$$

Similarly, if $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$ is a functor and $s\colon i \to j$, we get two morphisms

$$\beta(i)\times\beta(j) \underset{\beta(s)}{\overset{\mathrm{id}_{\beta(i)}}{\rightrightarrows}} \beta(i)$$

from which we deduce two morphisms $\prod_{i\in I}\beta(i) \rightrightarrows \beta(\sigma(s))$. These morphisms define the two morphisms

$$\prod_{i\in I}\beta(i) \underset{b}{\overset{a}{\rightrightarrows}} \prod_{s\in \mathrm{Mor}(I)} \beta(\sigma(s)). \tag{3.20}$$

Proposition 3.2.4. (i) $\varinjlim \alpha$ *is the co-equalizer of* (a,b) *in* (3.19),

(ii) $\varprojlim \beta$ *is the equalizer of* (a,b) *in* (3.20).

Proof. Replacing $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$, it is enough to prove (ii).

When $\mathcal{C} = \mathbf{Set}$, (ii) is nothing but the definition of projective limits in **Set**.

Therefore if $Z \in \mathbf{Set}$, then $\varprojlim \mathrm{Hom}_{\mathcal{C}}(Z,\beta)$ is the equalizer of

$$\prod_{i\in I}\mathrm{Hom}_{\mathcal{C}}(Z,\beta(i)) \underset{b}{\overset{a}{\rightrightarrows}} \prod_{s\in \mathrm{Mor}(I)} \mathrm{Hom}_{\mathcal{C}}(Z,\beta(\sigma(s))).$$

The result follows. q.e.d.

Corollary 3.2.5. *A category $\mathcal{C}$ admits finite projective limits if and only if it satisfies:*

(i) *$\mathcal{C}$ admits a terminal object,*

(ii) *for any $X, Y \in \mathrm{Ob}(\mathcal{C})$, the product $X \times Y$ exists in $\mathcal{C}$,*

(iii) *for any parallel arrows in $\mathcal{C}$, $f, g\colon X \rightrightarrows Y$, the equalizer exists in $\mathcal{C}$.*

Moreover, if $\mathcal{C}$ admits finite projective limits, a functor $F\colon \mathcal{C} \to \mathcal{C}'$ commutes with such limits if and only if it commutes with the terminal object, (finite) products and kernels.

There is a similar result for finite inductive limits, replacing a terminal object by an initial object, products by coproducts and equalizers by co-equalizers.

3.3 Exact functors

Definition 3.3.1. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor.

(i) Let I be a category and assume that $\mathcal{C}$ admits inductive limits indexed by I. One says that F commutes with such limits if for any $\alpha\colon I \to \mathcal{C}$, $\varinjlim (F \circ \alpha)$ exits in $\mathcal{C}'$ and is represented by $F(\varinjlim \alpha)$.

(ii) Similarly if I is a category and $\mathcal{C}$ admits projective limits indexed by I, one says that F commutes with such limits if for any $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$, $\varprojlim (F \circ \beta)$ exists and is represented by $F(\varprojlim \beta)$.

Note that if $\mathcal{C}$ and $\mathcal{C}'$ admit inductive (resp. projective) limits indexed by I, there is a natural morphism $\varinjlim (F \circ \alpha) \to F(\varinjlim \alpha)$ (resp. $F(\varprojlim \beta) \to \varprojlim (F \circ \beta)$).

Example 3.3.2. Let k be a field, $\mathcal{C} = \mathcal{C}' = \mathrm{Mod}(k)$, and let $X \in \mathcal{C}$. Then the functor $\mathrm{Hom}_k(X, \cdot)$ does not commute with inductive limit if X is infinite dimensional.

Definition 3.3.3. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor.

(i) Assume that $\mathcal{C}$ admits finite projective limits. One says that F is left exact if it commutes with such limits.

(ii) Assume that $\mathcal{C}$ admits finite inductive limits. One says that F is right exact if it commutes with such limits.

(iii) One says that F is exact if it is both left and right exact.

Proposition 3.3.4. *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a functor. Assume that*

(i) *F admits a left adjoint $G\colon \mathcal{C}' \to \mathcal{C}$,*

(ii) *$\mathcal{C}$ admits projective limits indexed by a category I.*

Then F commutes with projective limits indexed by I, that is, $F(\varprojlim_i \beta(i)) \simeq \varprojlim_i F(\beta(i))$.

Proof. Let $\beta\colon I^{\rm op}\to\mathcal{C}$ be a projective system indexed by I and let $Y\in\mathcal{C}'$. One has the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}'}(Y,F(\varprojlim_i\beta(i))) &\simeq \mathrm{Hom}_{\mathcal{C}}(G(Y),\varprojlim_i\beta(i))\\
&\simeq \varprojlim_i\mathrm{Hom}_{\mathcal{C}}(G(Y),\beta(i))\\
&\simeq \varprojlim_i\mathrm{Hom}_{\mathcal{C}'}(Y,F(\beta(i)))\\
&\simeq \mathrm{Hom}_{\mathcal{C}'^\wedge}(Y,\varprojlim_i F(\beta(i))).
\end{aligned}$$

Then the result follows by the Yoneda lemma. q.e.d.

Of course there is a similar result for inductive limits. If $\mathcal{C}$ admits inductive limits indexed by I and F admits a right adjoint, then F commutes with such limits.

Proposition 3.3.5. (i) *Let $\mathcal{C}$ be a category which admits finite inductive and finite projective limits. Then the functor* $\mathrm{Hom}_{\mathcal{C}}\colon\mathcal{C}^{\rm op}\times\mathcal{C}\to\mathbf{Set}$ *is left exact.*

(ii) *Let $F\colon\mathcal{C}\to\mathcal{C}'$ be a functor. If F admits a right* (*resp. left*) *adjoint, then F is right* (*resp. left*) *exact.*

(iii) *Let I and $\mathcal{C}$ be two categories and assume that $\mathcal{C}$ admits inductive* (*resp. projective*(*limits indexed by I. Then the functor $\varinjlim\colon\mathrm{Fct}(I,\mathcal{C})\to\mathcal{C}$* (*resp. $\varprojlim\colon\mathrm{Fct}(J^{\rm op},\mathcal{C})\to\mathcal{C}$*) *is right* (*resp. left*) *exact.*

(iv) *Let I be a discrete category. Then the functor $\prod\colon\mathrm{Mod}(k)^I\to\mathrm{Mod}(k)$ is exact.*

Proof. (i) follows immediately from (3.4) and (3.5).
(ii) is a particular case of Proposition 3.3.4.
(iii) Use the isomorphism (3.14) or (3.15).
(iv) is well-known and obvious. q.e.d.

3.4 Filtrant inductive limits

We shall generalize some notions of Definition 1.4.2 as well as Lemma 1.4.6 and Proposition 1.4.8.

Definition 3.4.1. A category I is called filtrant if it satisfies the conditions (i)–(iii) below.

(i) I is non empty,

(ii) for any i and j in I, there exists $k \in I$ and morphisms $i \to k, j \to k$,

(iii) for any parallel morphisms $f, g\colon i \rightrightarrows j$, there exists a morphism $h\colon j \to k$ such that $h \circ f = h \circ g$.

One says that I is cofiltrant if I^{op} is filtrant.

The conditions (ii)–(iii) of being filtrant are visualized by the diagrams:

$$\begin{tikzcd} i \arrow[dr, dotted] & & i \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, dotted] & j \arrow[d, dotted] \\ & k & & k \\ j \arrow[ur, dotted] & & & \end{tikzcd}$$

Of course, if $(I, \leq)$ is a non-empty directed ordered set, then the associated category I is filtrant.

We shall first study filtrant inductive limits in the category **Set**.

Proposition 3.4.2. *Let $\alpha\colon I \to \mathbf{Set}$ be a functor, with I filtrant. Define the relation $\sim$ on $\coprod_i \alpha(i)$ by $\alpha(i) \ni x_i \sim x_j \in \alpha(j)$ if there exists $s\colon i \to k$ and $t\colon j \to k$ such that $\alpha(s)(x_i) = \alpha(t)(x_j)$. Then*

(i) *the relation $\sim$ is an equivalence relation,*

(ii) $\varinjlim \alpha \simeq \coprod_i \alpha(i)/\sim$.

Proof. (i) Let $x_j \in \alpha(i_j)$, $j = 1, 2, 3$ with $x_1 \sim x_2$ and $x_2 \sim x_3$. There exist morphisms visualized by the diagram:

$$\begin{tikzcd} i_1 \arrow[r, "s_1"] & j_1 \arrow[dr, "u_1"] & & \\ i_2 \arrow[ur, "s_2"] \arrow[dr, "t_2"'] & & k_1 \arrow[r, "v"] & l \\ i_3 \arrow[r, "t_3"] & j_2 \arrow[ur, "u_2"'] & & \end{tikzcd}$$

such that $\alpha(s_1)x_1 = \alpha(s_2)x_2$, $\alpha(t_2)x_2 = \alpha(t_3)x_3$, and $v \circ u_1 \circ s_2 = v \circ u_2 \circ t_2$. Set $w_1 = v \circ u_1 \circ s_1$, $w_2 = v \circ u_1 \circ s_2 = v \circ u_2 \circ t_2$ and $w_3 = v \circ u_2 \circ t_3$. Then $\alpha(w_1)x_1 = \alpha(w_2)x_2 = \alpha(w_3)x_3$. Hence $x_1 \sim x_3$.
(ii) follows from Proposition 3.1.3. q.e.d.

Corollary 3.4.3. *Let $\alpha\colon I \to \mathbf{Set}$ be a functor, with I filtrant.*

(i) *Let S be a finite subset in $\varinjlim \alpha$. Then there exists $i \in I$ such that S is contained in the image of $\alpha(i)$ by the natural map $\alpha(i) \to \varinjlim \alpha$.*

(ii) *Let $i \in I$ and let x and y be elements of $\alpha(i)$ with the same image in $\varinjlim \alpha$. Then there exists $s\colon i \to j$ such that $\alpha(s)(x) = \alpha(s)(y)$ in $\alpha(j)$.*

The proof is left as an exercise.

Corollary 3.4.4. *Let A be a ring and denote by for the forgetful functor $\mathrm{Mod}(A) \to \mathbf{Set}$. Then the functor for commutes with filtrant inductive limits. In other words, if I is filtrant and $\alpha\colon I \to \mathrm{Mod}(A)$ is a functor, then*

$$for \circ (\varinjlim_i \alpha(i)) = \varinjlim_i (for \circ \alpha(i)).$$

Inductive limits with values in **Set** indexed by filtrant categories commute with finite projective limits. More precisely:

Proposition 3.4.5. *For a filtrant category I, a finite category J and a functor $\alpha\colon I \times J^{\mathrm{op}} \to \mathbf{Set}$, one has $\varinjlim_i \varprojlim_j \alpha(i,j) \xrightarrow{\sim} \varprojlim_j \varinjlim_i \alpha(i,j)$. In other words, the functor*

$$\varinjlim : \mathrm{Fct}(I, \mathbf{Set}) \to \mathbf{Set}$$

commutes with finite projective limits.

Proof. It is enough to prove that $\varinjlim$ commutes with equalizers and with finite products. This verification is left to the reader. q.e.d.

Applying this result together with Corollary 3.4.4, we obtain:

Corollary 3.4.6. *Let A be a ring and let I be a filtrant category. Then the functor $\varinjlim : \mathrm{Mod}(A)^I \to \mathrm{Mod}(A)$ is exact.*

Cofinal functors

Definition 3.4.7. Let I be a filtrant category and let $\varphi\colon J \to I$ be a fully faithful functor. One says that J is cofinal to I (or that $\varphi : J \to I$ is cofinal) if for any $i \in I$ there exists $j \in J$ and a morphism $i \to \varphi(j)$.

Note that the hypothesis implies that J is filtrant.

Proposition 3.4.8. *Assume I is filtrant, $\varphi\colon J \to I$ is fully faithful and $J \to I$ is cofinal. Let $\alpha\colon I \to \mathcal{C}$ (resp. $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$) be a functor. Then the natural morphism $\varinjlim (\alpha \circ \varphi) \to \varinjlim \alpha$ (resp. $\varprojlim \beta \to \varprojlim (\beta \circ \varphi^{\mathrm{op}})$) is an isomorphism in $\mathcal{C}^\vee$ (resp. in $\mathcal{C}^\wedge$).*

The proof is left as an exercise.

Remark 3.4.9. In Definition 3.4.7, we have assumed that I is filtrant, but there exists a general definition of cofinal functor which do not make this hypothesis and for which the conclusion of Proposition 3.4.8 remains true. (See Exercise 3.8 for an example.)

Remark 3.4.10. In these notes, we have skipped problems related to questions of cardinality and universes, but we should have not. Indeed, the reader will assume that all categories ($\mathcal{C}$, $\mathcal{C}'$ etc.) belong to a given universe $\mathcal{U}$ and that all limits are indexed by $\mathcal{U}$-small categories (I, J, etc.). (We do not give the meaning of "universe" and "small" here.)

Let us give an example which show that, otherwise, we may have troubles. Let $\mathcal{C}$ be a category which admits products and assume there exist $X, Y \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has more than one element. Set $M = \mathrm{Mor}(\mathcal{C})$, where $\mathrm{Mor}(\mathcal{C})$ denotes the "set" of all morphisms in $\mathcal{C}$, and let $\pi = \mathrm{card}(M)$, the cardinal of the set M. We have $\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y)^M$ and therefore $\mathrm{card}(\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \geq 2^\pi$. On the other hand, $\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \subset \mathrm{Mor}(\mathcal{C})$ which implies $\mathrm{card}(\mathrm{Hom}_{\mathcal{C}}(X, Y^M) \leq \pi$.

The "contradiction" comes from the fact that $\mathcal{C}$ does not admit products indexed by such a big set as $\mathrm{Mor}(\mathcal{C})$, or else, $\mathrm{Mor}(\mathcal{C})$ is not small (in general) in the universe to which $\mathcal{C}$ belongs. (The remark was found in [?].)

Exercises to Chapter 3

Exercise 3.1. Let $X, Y \in \mathcal{C}$ and consider the category $\mathcal{D}$ whose arrows are triplets $Z \in \mathcal{C}, f : Z \to X, g : Z \to Y$, the morphisms being the natural one. Prove that this category admits a terminal object if and only if the product $X \times Y$ exists in $\mathcal{C}$, and that in such a case this terminal object is isomorphic to $X \times Y, X \times Y \to X, X \times Y \to Y$. Deduce that if $X \times Y$ exists, it is unique up to unique isomorphism.

Exercise 3.2. (i) Let I be a (non necessarily finite) set and $(X_i)_{i \in I}$ a family of sets indexed by I. Show that $\coprod_i X_i$ is the disjoint union of the sets X_i.
(ii) Construct the natural map $\coprod_i \mathrm{Hom}_{\mathbf{Set}}(Y, X_i) \to \mathrm{Hom}_{\mathbf{Set}}(Y, \coprod_i X_i)$ and prove it is injective.
(iii) Prove that the map $\coprod_i \mathrm{Hom}_{\mathbf{Set}}(X_i, Y) \to \mathrm{Hom}_{\mathbf{Set}}(\prod_i X_i, Y)$ is not injective in general.

Exercise 3.3. Let I and $\mathcal{C}$ be two categories and denote by Δ the functor from $\mathcal{C}$ to $\mathcal{C}^I$ which, to $X \in \mathcal{C}$, associates the constant functor $\Delta(X)\colon I \ni i \mapsto X \in \mathcal{C}$, $(i \to j) \in \mathrm{Mor}(I) \mapsto \mathrm{id}_X$. Assume that any functor from I to $\mathcal{C}$ admits an inductive limit.
(i) Prove that $\varinjlim : \mathcal{C}^I \to \mathcal{C}$ is a functor.
(ii) Prove the formula (for $\alpha : I \to \mathcal{C}$ and $Y \in \mathcal{C}$):

$$\mathrm{Hom}_{\mathcal{C}}(\varinjlim_i \alpha(i), Y) \simeq \mathrm{Hom}_{\,\mathrm{Fct}(I,\mathcal{C})}(\alpha, \Delta(Y)).$$

(iii) Replacing I with the opposite category, deduce the formula (assuming projective limits exist):

$$\mathrm{Hom}_{\mathcal{C}}(X, \varprojlim_i G(i)) \simeq \mathrm{Hom}_{\,\mathrm{Fct}(I^{\mathrm{op}},\mathcal{C})}(\Delta(X), G).$$

Exercise 3.4. Let $\mathcal{C}$ be a category which admits filtrant inductive limits. One says that an object X of $\mathcal{C}$ is of finite type (resp. of finite presentation) if for any functor $\alpha\colon I \to \mathcal{C}$ with I filtrant, the natural map $\varinjlim \mathrm{Hom}_{\mathcal{C}}(X, \alpha) \to \mathrm{Hom}_{\mathcal{C}}(X, \varinjlim \alpha)$ is injective (resp. bijective).
(i) Show that this definition coincides with the classical one when $\mathcal{C} = \mathrm{Mod}(A)$, for a ring A.
(ii) Does this definition coincide with the classical one when $\mathcal{C}$ denotes the category of commutative algebras?

Exercise 3.5. Let $\mathcal{C}$ be a category and recall that the category $\mathcal{C}^\wedge$ admits inductive limits. One denotes by "$\varinjlim$" the inductive limit in $\mathcal{C}^\wedge$. Let k be a field and let $\mathcal{C} = \mathrm{Mod}(k)$. Prove that the Yoneda functor $\mathrm{h}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}^\wedge$ does not commute with inductive limits.

Exercise 3.6. Consider the category I with three objects $\{a, b, c\}$ and two morphisms other than the identities, vizualized by the diagram

$$a \leftarrow c \to b.$$

Let $\mathcal{C}$ be a category. A functor $\beta\colon I^{\mathrm{op}} \to \mathcal{C}$ is nothing but the data of three objects X, Y, Z and two morphisms vizualized by the diagram

$$X \xrightarrow{f} Z \xleftarrow{g} Y.$$

The fiber product $X \times_Z Y$ of X and Y over Z, if it exists, is the projective limit of β.

(i) Assume that $\mathcal{C}$ admits products (of two objects) and kernels. Prove that the sequence

$$X \times_Z Y \to X \rightrightarrows Y$$

is exact. Here, the two morphisms $X \rightrightarrows Y$ are given by f, g.
(ii) Prove that $\mathcal{C}$ admits finite projective limits if and only if it admits fiber products and a terminal object.

Exercise 3.7. Let I and $\mathcal{C}$ be two categories and let $F, G : I \rightrightarrows \mathcal{C}$ be two functors. Prove the isomorphism:

$$\begin{aligned}&\mathrm{Hom}_{\,\mathrm{Fct}(I,\mathcal{C})}(F,G) \simeq \\ &\qquad \mathrm{Ker}\big(\prod_{i\in I}\mathrm{Hom}_{\mathcal{C}}(F(i),G(i)) \rightrightarrows \prod_{(j\to k)\in\mathrm{Mor}(I)}\mathrm{Hom}_{\mathcal{C}}(F(j),G(k))\big).\end{aligned}$$

Here, the double arrow is associated with the two maps:

$$\prod_{i\in I}\mathrm{Hom}_{\mathcal{C}}(F(i),G(i)) \to \mathrm{Hom}_{\mathcal{C}}(F(j),G(j)) \to \mathrm{Hom}_{\mathcal{C}}(F(j),G(k)),$$
$$\prod_{i\in I}\mathrm{Hom}_{\mathcal{C}}(F(i),G(i)) \to \mathrm{Hom}_{\mathcal{C}}(F(k),G(k)) \to \mathrm{Hom}_{\mathcal{C}}(F(j),G(k)).$$

Equivalently, with the notations of Example 2.1.4 (vi), prove the isomorphism

$$\mathrm{Hom}_{\,\mathrm{Fct}(I,\mathcal{C})}(F,G) \xrightarrow{\sim} \varprojlim_{(i\to j)\in\mathrm{Mor}_0(I)} \mathrm{Hom}_{\mathcal{C}}(F(i),G(j)). \tag{3.21}$$

Exercise 3.8. Prove Proposition 3.4.8.

Exercise 3.9. Let I be a category, J a full subcategory. Assume that for any $i \in I$, there is a unique $j \in J$ and a unique morphism $j \to i$ (in other words, for any $i \in I$, the category J^i is reduced to $\{\mathrm{pt}\}$, the discrete category with a single objet).
(i) Prove that J is discrete.
(ii) Let $\alpha\colon I \to \mathcal{C}$ be a functor. Prove that the natural morphism $\varinjlim(\alpha\circ\varphi) \to \varinjlim\alpha$ (resp. $\varprojlim\beta \to \varprojlim(\beta\circ\varphi^{\mathrm{op}})$) is an isomorphism in $\mathcal{C}^\vee$ (resp. in $\mathcal{C}^\wedge$). In other words, Proposition 3.4.8 holds in this case.

Chapter 4

Additive categories

Many results or constructions in the category $\mathrm{Mod}(A)$ of modules over a ring A have their counterparts in other contexts, such as finitely generated A-modules, or graded modules over a graded ring, or sheaves of A-modules, etc. Hence, it is natural to look for a common language which avoids to repeat the same arguments. This is the language of additive and abelian categories.

In this chapter, we give the main properties of additive categories. We expose some basic constructions and notions on complexes such as the shift functor, the homotopy, the mapping cone and the simple complex associated with a double complex. We also construct complexes associated with simplicial objects in an additive category and give a criterion for such a complex to be homotopic to zero.

4.1 Additive categories

Definition 4.1.1. A category $\mathcal{C}$ is additive if it satisfies conditions (i)-(v) below:

(i) for any $X, Y \in \mathcal{C}$, $\quad \mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathbf{Ab}$,

(ii) the composition law $\circ$ is bilinear,

(iii) there exists a zero object in $\mathcal{C}$,

(iv) the category $\mathcal{C}$ admits finite coproducts,

(v) the category $\mathcal{C}$ admits finite products.

Note that $\mathrm{Hom}_{\mathcal{C}}(X,Y) \neq \emptyset$ since it is a group. Note that $\mathrm{Hom}_{\mathcal{C}}(X,0) = \mathrm{Hom}_{\mathcal{C}}(0,X) = 0$ for all $X \in \mathcal{C}$.

Notation 4.1.2. If X and Y are two objects of $\mathcal{C}$, one denotes by $X \oplus Y$ (instead of $X \sqcup Y$) their coproduct, and calls it their direct sum. One denotes as usual by $X \times Y$ their product. This change of notations is motivated by the fact that if A is a ring, the forgetful functor $\mathrm{Mod}(A) \to \mathbf{Set}$ does not commute with coproducts.

By the definition of a coproduct and a product in a category, for each $Z \in \mathcal{C}$, there is an isomorphism in $\mathrm{Mod}(\mathbb{Z})$:

$$\mathrm{Hom}_{\mathcal{C}}(X, Z) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \simeq \mathrm{Hom}_{\mathcal{C}}(X \oplus Y, Z), \tag{4.1}$$

$$\mathrm{Hom}_{\mathcal{C}}(Z, X) \times \mathrm{Hom}_{\mathcal{C}}(Z, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(Z, X \times Y). \tag{4.2}$$

Lemma 4.1.3. *Let $\mathcal{C}$ be a category satisfying conditions (i) to (iii) in Definition 4.1.1. Then conditions (iv) and (v) are equivalent. Moreover, each of these conditions is equivalent to*

(vi) For any two objects X and Y there exits an object Z and morphisms $i_1 : X \to Z$, $p_1 : Z \to X$, $i_2 : Y \to Z$, $p_2 : Z \to Y$, satisfying

$$p_1 \circ i_1 = \mathrm{id}_X, \quad p_1 \circ i_2 = 0 \tag{4.3}$$

$$p_2 \circ i_2 = \mathrm{id}_Y, \quad p_2 \circ i_1 = 0, \tag{4.4}$$

$$i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_Z . \tag{4.5}$$

Proof. (iv) $\Rightarrow$ (vi). Choosing $Z = X \oplus Y$ in (4.1), the identity of $X \oplus Y$ defines i_1 and i_2. Choosing $Z = X$, the identity of X and the zero morphism $Y \to X$ define $p_1 \colon X \oplus Y \to X$ satisfying $p_1 \circ i_1 = \mathrm{id}_X$ and $p_1 \circ i_2 = 0$. One gets similarly p_2 satisfying (4.4) and (4.5) follows.
(vi) $\Rightarrow$ (v). Let $W \in \mathcal{C}$ and consider morphisms $f \colon W \to X$ and $g \colon W \to Y$. Set $h := i_1 \circ f \oplus i_2 \circ g$. Then $h \colon W \to X \oplus Y$ satisfies $p_1 \circ h = f$ and $p_2 \circ h = g$ and such an h is unique. Hence $X \oplus Y \simeq X \times Y$.
Replacing $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$, we get (v) $\Rightarrow$ (vi) $\Rightarrow$ (iv) q.e.d.

Example 4.1.4. (i) If A is a ring, $\mathrm{Mod}(A)$ and $\mathrm{Mod}^f(A)$ are additive categories.
(ii) **Ban**, the category of $\mathbb{C}$-Banach spaces and linear continuous maps is additive.
(iii) If $\mathcal{C}$ is additive, then $\mathcal{C}^{\mathrm{op}}$ is additive.
(iv) Let I be category. If $\mathcal{C}$ is additive, the category $\mathcal{C}^I$ of functors from I to $\mathcal{C}$, is additive.
(v) If $\mathcal{C}$ and $\mathcal{C}'$ are additive, then $\mathcal{C} \times \mathcal{C}'$ are additive.

Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor of additive categories. One says that F is additive if for $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$ is a morphism of groups. One can prove the following

Proposition 4.1.5. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor of additive categories. Then F is additive if and only if it commutes with direct sum, that is, for X and Y in $\mathcal{C}$:*

$$\begin{aligned} F(0) &\simeq 0 \\ F(X \oplus Y) &\simeq F(X) \oplus F(Y). \end{aligned}$$

Unless otherwise specified, functors between additive categories will be assumed to be additive.
Generalization: Let k be a *commutative* ring. One defines the notion of a k-additive category by assuming that for X and Y in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a k-module and the composition is k-bilinear.

4.2 Complexes in additive categories

Let $\mathcal{C}$ denote an additive category.

Definition 4.2.1. (i) A differential object $(X^\bullet, d_X^\bullet)$ in $\mathcal{C}$ is a sequence of objects X^k and morphisms d^k ($k \in \mathbb{Z}$):

$$\cdots \to X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1} \to \cdots . \tag{4.6}$$

(ii) A complex is a differential object $(X^\bullet, d_X^\bullet)$ such that such that $d^k \circ d^{k-1} = 0$ for all $k \in \mathbb{Z}$.

A morphism of differential objects $f^\bullet \colon X^\bullet \to Y^\bullet$ is visualized by a commutative diagram:

$$\begin{tikzcd} \cdots \arrow[r] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\ \cdots \arrow[r] & Y^n \arrow[r, "d_Y^n"] & X^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

One defines naturally the direct sum of two differential objects. Hence, we get a new additive category, the category $Diff(\mathcal{C})$ of differential objects in $\mathcal{C}$. One denotes by $C(\mathcal{C})$ the full additive subcategory of $Diff(\mathcal{C})$ consisting of complexes.

From mow on, we shall concentrate our study on the category $C(\mathcal{C})$.

A complex is bounded (resp. bounded below, bounded above) if $X^n = 0$ for $|n| >> 0$ (resp. $n << 0$, $n >> 0$). One denotes by $C^*(\mathcal{C})(* = b, +, -)$ the full additive subcategory of $C(\mathcal{C})$ consisting of bounded complexes (resp. bounded below, bounded above).

One considers $\mathcal{C}$ as a full subcategory of $C^b(\mathcal{C})$ by identifying an object $X \in \mathcal{C}$ with the complex $X^\bullet$ "concentrated in degree 0":

$$X^\bullet := \cdots \to 0 \to X \to 0 \to \cdots$$

where X stands in degree 0.

Shift functor

Let $X \in C(\mathcal{C})$ and $k \in \mathbb{Z}$. One defines the shifted complex $X[k]$ by:

$$\begin{cases} (X[k])^n = X^{n+k} \\ d^n_{X[k]} = (-1)^k d^{n+k}_X \end{cases}$$

If $f : X \to Y$ is a morphism in $C(\mathcal{C})$ one defines $f[k] : X[k] \to Y[k]$ by $(f[k])^n = f^{n+k}$.

The shift functor $X \mapsto X[1]$ is an automorphism (*i.e.* an invertible functor) of $C(\mathcal{C})$.

Homotopy

Let $\mathcal{C}$ denote an additive category.

Definition 4.2.2. (i) A morphism $f : X \to Y$ in $C(\mathcal{C})$ is homotopic to zero if for all k there exists a morphism $s^k : X^k \to Y^{k-1}$ such that:

$$f^k = s^{k+1} \circ d^k_X + d^{k-1}_Y \circ s^k.$$

Two morphisms $f, g : X \to Y$ are homotopic if $f - g$ is homotopic to zero.

(ii) An object X in $C(\mathcal{C})$ is homotopic to 0 if id_X is homotopic to zero.

A morphism homotopic to zero is visualized by the diagram (which is not commutative):

$$\begin{tikzcd}
X^{k-1} \arrow[r] & X^k \arrow[r, "d^k_X"] \arrow[dl, "s^k"'] \arrow[d, "f^k"] & X^{k+1} \arrow[dl, "s^{k+1}"'] \\
Y^{k-1} \arrow[r, "d^{k-1}_Y"'] & Y^k \arrow[r] & Y^{k+1}
\end{tikzcd}$$

Note that an additive functor sends a morphism homotopic to zero to a morphism homotopic to zero.

Example 4.2.3. The complex $0 \to X' \to X' \oplus X'' \to X'' \to 0$ is homotopic to zero.

Mapping cone

Definition 4.2.4. Let $f : X \to Y$ be a morphism in $C(\mathcal{C})$. The mapping cone of f, denoted $\mathrm{Mc}(f)$, is the object of $C(\mathcal{C})$ defined by:

$$\begin{aligned} \mathrm{Mc}(f)^k &= (X[1])^k \oplus Y^k \\ d^k_{\mathrm{Mc}(f)} &= \begin{pmatrix} d^k_{X[1]} & 0 \\ f^{k+1} & d^k_Y \end{pmatrix} \end{aligned}$$

Of course, before to state this definition, one should check that $d^{k+1}_{\mathrm{Mc}(f)} \circ d^k_{\mathrm{Mc}(f)} = 0$. Indeed:

$$\begin{pmatrix} -d^{k+2}_X & 0 \\ f^{k+2} & d^{k+1}_Y \end{pmatrix} \circ \begin{pmatrix} -d^{k+1}_X & 0 \\ f^{k+1} & d^k_Y \end{pmatrix} = 0$$

Notice that although $\mathrm{Mc}(f)^k = (X[1])^k \oplus Y^k$, $\mathrm{Mc}(f)$ is not isomorphic to $X[1] \oplus Y$ in $C(\mathcal{C})$ unless f is the zero morphism.

There are natural morphisms of complexes

$$\alpha(f) : Y \to \mathrm{Mc}(f), \quad \beta(f) : \mathrm{Mc}(f) \to X[1].$$

and $\beta(f) \circ \alpha(f) = 0$.

If $F : \mathcal{C} \to \mathcal{C}'$ is an additive functor, then $F(\mathrm{Mc}(f)) \simeq \mathrm{Mc}(F(f))$.

4.3 Simplicial constructions

We shall define the simplicial category and use it to construct complexes and homotopies in additive categories.

Definition 4.3.1. (a) The simplicial category, denoted by $\mathbf{\Delta}$, is the category whose objects are the finite totally ordered sets and the morphisms are the order-preserving maps.

(b) We denote by $\widetilde{\mathbf{\Delta}}_i$ the subcategory of $\mathbf{\Delta}$ such that $\mathrm{Ob}(\widetilde{\mathbf{\Delta}}_i) = \mathrm{Ob}(\mathbf{\Delta})$, the morphisms being the injective order-preserving maps.

For integers n, m denote by $[n, m]$ the totally ordered set $\{k \in \mathbb{Z}; n \leq k \leq m\}$.

Proposition 4.3.2. (i) *the natural functor $\mathbf{\Delta} \to \mathbf{Set}^f$ is faithful,*

(ii) *the full subcategory of $\mathbf{\Delta}$ consisting of objects $\{[0, n]\}_{n \geq -1}$ is equivalent to $\mathbf{\Delta}$,*

(iii) $\mathbf{\Delta}$ *admits an initial object, namely* $\emptyset$*, and a terminal object, namely* $\{0\}$.

The proof is obvious.

Let us denote by

$$d_i^n \colon [0,n] \to [0,n+1] \qquad (0 \le i \le n+1)$$

the injective order-preserving map which does not take the value i. In other words

$$d_i^n(k) = \begin{cases} k & \text{for } k < i, \\ k+1 & \text{for } k \ge i. \end{cases}$$

One checks immediately that

$$d_j^{n+1} \circ d_i^n = d_i^{n+1} \circ d_{j-1}^n \text{ for } 0 \le i < j \le n+2. \tag{4.7}$$

Indeed, both morphisms are the unique injective order-preserving map which does not take the values i and j.

The category $\widetilde{\mathbf{\Delta}}_i$ is visualized by

$$\emptyset \xrightarrow{d_0^{-1}} \{0\} \overset{\xrightarrow{d_0^0}}{\underset{\xrightarrow{d_1^0}}{}} \{0,1\} \begin{matrix} \xrightarrow{d_0^1} \\ \xrightarrow{d_1^1} \\ \xrightarrow{d_2^1} \end{matrix} \{0,1,2\} \begin{matrix} \dashrightarrow \\ \dashrightarrow \\ \dashrightarrow \\ \dashrightarrow \end{matrix} \tag{4.8}$$

Let $\mathcal{C}$ be an additive category and $F \colon \widetilde{\mathbf{\Delta}}_i \to \mathcal{C}$ a functor. We set for $n \in \mathbb{Z}$:

$$F^n = \begin{cases} F([0,n]) & \text{for } n \ge -1, \\ 0 & \text{otherwise,} \end{cases}$$

$$d_F^n \colon F^n \to F^{n+1}, \quad d_F^n = \sum_{i=0}^{n+1} (-)^i F(d_i^n).$$

Consider the differential object

$$F^\bullet := \cdots \to 0 \to F^{-1} \xrightarrow{d_F^{-1}} F^0 \xrightarrow{d_F^0} F^1 \to \cdots \to F^n \xrightarrow{d_F^n} \cdots . \tag{4.9}$$

Theorem 4.3.3. (i) *The differential object $F^\bullet$ is a complex.*

(ii) *Assume that there exist morphisms $s_F^n \colon F^n \to F^{n-1}$ $(n \ge 0)$satisfying:*

$$\begin{cases} s_F^{n+1} \circ F(d_0^n) = \mathrm{id}_{F^n} & \textit{for } n \ge -1, \\ s_F^{n+1} \circ F(d_{i+1}^n) = F(d_i^{n-1}) \circ s_F^n & \textit{for } i > 0, n \ge 0. \end{cases}$$

Then $F^\bullet$ is homotopic to zero.

Proof. (i) By (4.7), we have

$$\begin{aligned}
d_F^{n+1} \circ d_F^n &= \sum_{j=0}^{n+2} \sum_{i=0}^{n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) \\
&= \sum_{0 \le j \le i \le n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) + \sum_{0 \le i < j \le n+2} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) \\
&= \sum_{0 \le j \le i \le n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) + \sum_{0 \le i < j \le n+2} (-)^{i+j} F(d_i^{n+1} \circ d_{j-1}^n) \\
&= 0\,.
\end{aligned}$$

Here, we have used

$$\begin{aligned}
\sum_{0 \le i < j \le n+2} (-)^{i+j} F(d_i^{n+1} \circ d_{j-1}^n) &= \sum_{0 \le i < j \le n+1} (-)^{i+j+1} F(d_i^{n+1} \circ d_j^n) \\
&= \sum_{0 \le j \le i \le n+1} (-)^{i+j+1} F(d_j^{n+1} \circ d_i^n).
\end{aligned}$$

(ii) We have

$$\begin{aligned}
& s_F^{n+1} \circ d_F^n + d_F^{n-1} \circ s^n \\
&\quad = \sum_{i=0}^{n+1} (-1)^i s_F^{n+1} \circ F(d_i^n) + \sum_{i=0}^{n} (-1)^i F(d_i^{n-1} \circ s_F^n) \\
&\quad = s_F^{n+1} \circ F(d_0^n) + \sum_{i=0}^{n} (-1)^{i+1} s_F^{n+1} \circ F(d_{i+1}^n) + \sum_{i=0}^{n} (-1)^i F(d_i^{n-1} \circ s_F^n) \\
&\quad = \mathrm{id}_{F^n} + \sum_{i=0}^{n} (-1)^{i+1} F(d_i^{n-1} \circ s_F^n) + \sum_{i=0}^{n} (-1)^i F(d_i^{n-1} \circ s_F^n) \\
&\quad = \mathrm{id}_{F^n}.
\end{aligned}$$

q.e.d.

4.4 Double complexes

Let $\mathcal{C}$ be as above an additive category. A double complex $(X^{\bullet,\bullet}, d_X)$ in $\mathcal{C}$ is the data of

$$\{X^{n,m}, d'^{n,m}_X, d''^{n,m}_X; (n,m) \in \mathbb{Z} \times \mathbb{Z}\}$$

where $X^{n,m} \in \mathcal{C}$ and the "differentials" $d'^{n,m}_X : X^{n,m} \to X^{n+1,m}$, $d''^{n,m}_X : X^{n,m} \to X^{n,m+1}$ satisfy:

$$d'^2_X = d''^2_X = 0,\ d' \circ d'' = d'' \circ d'. \tag{4.10}$$

One can represent a double complex by a commutative diagram:

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & X^{n,m} \arrow[r, "d''^{n,m}"] \arrow[d, "d'^{n,m}"'] & X^{n,m+1} \arrow[r] \arrow[d, "d'^{n,m+1}"'] & {} \\
{} \arrow[r] & X^{n+1,m} \arrow[r, "d''^{n+1,m}"'] \arrow[d] & X^{n+1,m+1} \arrow[r] \arrow[d] & {} \\
 & {} & {} &
\end{tikzcd}$$

One defines naturally the notion of a morphism of double complexes, and one obtains the additive category $C^2(\mathcal{C})$ of double complexes.

There are two functors $F_I, F_{II} : C^2(\mathcal{C}) \to C(C(\mathcal{C}))$ which associate to a double complex X the complex whose objects are the rows (resp. the columns) of X. These two functors are clearly isomorphisms of categories.

Now consider the finiteness condition:

$$\text{for all } p \in \mathbb{Z}, \quad \{(m,n) \in \mathbb{Z} \times \mathbb{Z}; X^{n,m} \neq 0, m+n=p\} \text{ is finite} \tag{4.11}$$

and denote by $C_f^2(\mathcal{C})$ the full subcategory of $C^2(\mathcal{C})$ consisting of objects X satisfying (4.11). To such an X one associates its "total complex" $\mathrm{tot}(X)$ by setting:

$$\begin{aligned}
\mathrm{tot}(X)^p &= \oplus_{m+n=p} X^{n,m}, \\
d^p_{\mathrm{tot}(X)}|_{X^{n,m}} &= d'^{n,m} + (-1)^n d''^{n,m}.
\end{aligned}$$

This is visualized by the diagram:

$$\begin{tikzcd}
X^{n,m} \arrow[r, "(-)^n d''"] \arrow[d, "d'"'] & X^{n,m+1} \\
X^{n+1,m} &
\end{tikzcd}$$

Proposition 4.4.1. *The differential object* $\{\mathrm{tot}(X)^p, d^p_{\mathrm{tot}(X)}\}_{p\in\mathbb{Z}}$ *is a complex* (i.e. $d^{p+1}_{\mathrm{tot}(X)} \circ d^p_{\mathrm{tot}(X)} = 0$) *and* $\mathrm{tot} : C_f^2(\mathcal{C}) \to C(\mathcal{C})$ *is a functor of additive categories.*

Proof. For $(n,m) \in \mathbb{Z} \times \mathbb{Z}$, one has

$$\begin{aligned}
d \circ d(X^{n,m}) &= d'' \circ d''(X^{n,m}) + d' \circ d'(X^{n,m}) \\
&\quad + (-)^n d'' \circ d'(X^{n,m}) + (-)^{n+1} d' \circ d''(X^{n,m}) \\
&= 0.
\end{aligned}$$

It is left to the reader to check that tot is an additive functor. q.e.d.

Example 4.4.2. Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $C(\mathcal{C})$. Consider the double complex $Z^{\bullet,\bullet}$ such that $Z^{-1,\bullet} = X^\bullet$, $Z^{0,\bullet} = Y^\bullet$, $Z^{i,\bullet} = 0$ for $i \neq -1, 0$, with differentials $f^j : Z^{-1,j} \to Z^{0,j}$. Then

$$\operatorname{tot}(Z^{\bullet,\bullet}) \simeq \operatorname{Mc}(f^\bullet). \tag{4.12}$$

Bifunctor

Let $\mathcal{C}, \mathcal{C}'$ and $\mathcal{C}''$ be additive categories and let $F : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$ be an additive bifunctor (i.e., $F(\,\bullet\,,\,\bullet\,)$ is additive with respect to each argument). It defines an additive bifunctor $C^2(F) : C(\mathcal{C}) \times C(\mathcal{C}') \to C^2(\mathcal{C}'')$. In other words, if $X \in C(\mathcal{C})$ and $X' \in C(\mathcal{C}')$ are complexes, then $C^2(F)(X, X')$ is a double complex.

Examples 4.4.3. (i) Consider the bifunctor $\operatorname{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \operatorname{Mod}(\mathbb{Z})$. We shall write $\operatorname{Hom}_{\mathcal{C}}^{\bullet,\bullet}$ instead of $C^2(\operatorname{Hom}_{\mathcal{C}})$. If X and Y are two objects of $C(\mathcal{C})$, one has

$$\begin{aligned} &\operatorname{Hom}_{\mathcal{C}}^{\bullet,\bullet}(X,Y)^{n,m} && = \operatorname{Hom}_{\mathcal{C}}(X^{-m}, Y^n), \\ d'^{n,m} &= \operatorname{Hom}_{\mathcal{C}}(X^{-m}, d_Y^n), && d''^{n,m} = \operatorname{Hom}_{\mathcal{C}}((-)^n d_X^{-n-1}, Y^m). \end{aligned}$$

Note that $\operatorname{Hom}_{\mathcal{C}}^{\bullet,\bullet}(X,Y)$ is a double complex in the category **Ab**, which should not be confused with the group $\operatorname{Hom}_{C(\mathcal{C})}(X,Y)$.

(ii) Consider the bifunctor $\bullet \otimes \bullet : \operatorname{Mod}(A^{\mathrm{op}}) \times \operatorname{Mod}(A) \to \operatorname{Mod}(\mathbb{Z})$. We shall simply write $\otimes$ instead of $C^2(\otimes)$. Hence, for $X \in C^-(\operatorname{Mod}(A^{\mathrm{op}}))$ and $Y \in C^-(\operatorname{Mod}(A))$, one has

$$(X \otimes Y)^{n,m} = X^n \otimes Y^m,$$

$$d'^{n,m} = d_X^n \otimes Y^m, d'\prime^{n,m} = X^n \otimes d_Y^m.$$

Definition 4.4.4. Let $X \in C^-(\mathcal{C})$ and $Y \in C^+(\mathcal{C})$. One sets

$$\operatorname{Hom}_{\mathcal{C}}^{\bullet}(X,Y) = \operatorname{tot}(\operatorname{Hom}_{\mathcal{C}}^{\bullet,\bullet}(X,Y)). \tag{4.13}$$

Exercises to Chapter 4

Exercise 4.1. Let $\mathcal{C}$ be an additive category and let $X \in C(\mathcal{C})$.

(i) Prove that $d_X\colon X \to X[1]$ defines a morphism in $C(\mathcal{C})$.

(ii) Prove that $d_X\colon X \to X[1]$ is homotopic to zero.

Exercise 4.2. Let $\mathcal{C}$ be an additive category, $f, g\colon X \rightrightarrows Y$ two morphisms in $C(\mathcal{C})$. Prove that f and g are homotopic if and only if there exists a commutative diagram in $C(\mathcal{C})$

$$\begin{tikzcd} Y \arrow[r, "\alpha(f)"'] \arrow[d, equal] & \mathrm{Mc}(f) \arrow[r, "\beta(f)"'] \arrow[d, "u"] & X[1] \arrow[d, equal] \\ Y \arrow[r, "\alpha(g)"'] & \mathrm{Mc}(f) \arrow[r, "\beta(g)"'] & X[1]. \end{tikzcd}$$

In such a case, prove that u is an isomorphism in $C(\mathcal{C})$.

Exercise 4.3. Let $\mathcal{C}$ be an additive category and let $f\colon X \to Y$ be a morphism in $C(\mathcal{C})$.
Prove that the following conditions are equivalent:

(a) f is homotopic to zero,

(b) f factors through $\alpha(\mathrm{id}_X)\colon X \to \mathrm{Mc}(\mathrm{id}_X)$,

(c) f factors through $\beta(\mathrm{id}_Y)[-1]\colon \mathrm{Mc}(\mathrm{id}_Y)[-1] \to Y$,

(d) f decomposes as $X \to Z \to Y$ with Z a complex homotopic to zero.

Exercise 4.4. A category with translation $(\mathcal{A}, T)$ is a category $\mathcal{A}$ together with an equivalence $T\colon \mathcal{A} \to \mathcal{A}$. A differential object (X, d_X) in a category with translation $(\mathcal{A}, T)$ is an object $X \in \mathcal{A}$ together with a morphism $d_X\colon X \to T(X)$. A morphism $f\colon (X, d_X) \to (Y, d_Y)$ of differential objects is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "d_X"] \arrow[d, "f"] & TX \arrow[d, "T(f)"] \\ Y \arrow[r, "d_Y"] & TY. \end{tikzcd}$$

One denotes by $\mathcal{A}_d$ the subcategory of $(\mathcal{A}, T)$ consisting of differential objects and morphisms of such objects. If $\mathcal{A}$ is additive, one says that a differential object (X, d_X) in $(\mathcal{A}, T)$ is a complex if the composition $X \xrightarrow{d_X} T(X) \xrightarrow{T(d_X)} T^2(X)$ is zero. One denotes by $\mathcal{A}_c$ the full subcategory of $\mathcal{A}_d$ consisting of complexes.
(i) Let $\mathcal{C}$ be a category. Denote by $\mathbb{Z}_d$ the set $\mathbb{Z}$ considered as a discrete category and still denote by $\mathbb{Z}$ the ordered set $(\mathbb{Z}, \leq)$ considered as a category. Prove that $\mathcal{C}^{\mathbb{Z}} := \mathrm{Fct}(\mathbb{Z}_d, \mathcal{C})$ is a category with translation.
(ii) Show that the category $\mathrm{Fct}(\mathbb{Z}, \mathcal{C})$ may be identified to the category of differential objects in $\mathcal{C}^{\mathbb{Z}}$.

(iii) Let $\mathcal{C}$ be an additive category. Show that the notions of differential objects and complexes given above coincide with those in Definition 4.2.1 when choosing $\mathcal{A} = C(\mathcal{C})$ and $T = [1]$.

Exercise 4.5. Consider the catgeory $\mathbf{\Delta}$ and for $n > 0$, denote by

$$s_i^n : [0, n] \to [0, n-1] \qquad (0 \leq i \leq n-1)$$

the surjective order-preserving map which takes the same value at i and $i+1$. In other words

$$s_i^n(k) = \begin{cases} k & \text{for } k \leq i, \\ k-1 & \text{for } k > i. \end{cases}$$

Checks the relations:

$$\begin{cases} s_j^n \circ s_i^{n+1} = s_{i-1}^n \circ s_j^{n+1} & \text{for } 0 \leq j < i \leq n, \\ s_j^{n+1} \circ d_i^n = d_i^{n-1} \circ s_{j-1}^n & \text{for } 0 \leq i < j \leq n, \\ s_j^{n+1} \circ d_i^n = \mathrm{id}_{[0,n]} & \text{for } 0 \leq i \leq n+1, i = j, j+1, \\ s_j^{n+1} \circ d_i^n = d_{i-1}^{n-1} \circ s_j^n & \text{for } 1 \leq j+1 < i \leq n+1. \end{cases}$$

Chapter 5

Abelian categories

In this chapter, we give the main properties of abelian categories and expose some basic constructions on complexes in such categories, such as the snake Lemma. We explain the notion of injective resolutions and apply it to the construction of derived functors, with applications to the functors Ext and Tor.

For sake of simplicity, we shall always argue as if we were working in a full abelian subcategory of $\mathrm{Mod}(A)$ for a ring A. (See Convention 5.1.1 below.) Some important historical references are the book [4] and the paper [7].

5.1 Abelian categories

Convention 5.1.1. In these Notes, when dealing with an abelian category $\mathcal{C}$ (see Definition 5.1.4 below), we shall assume that $\mathcal{C}$ is a full abelian subcategory of a category $\mathrm{Mod}(A)$ for some ring A. This makes the proofs much easier and moreover there exists a famous theorem (due to Freyd & Mitchell) that asserts that this is in fact always the case (up to equivalence of categories).

From now on, $\mathcal{C}$, $\mathcal{C}'$ will denote additive categories.

Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Recall that if $\operatorname{Ker} f$ exists, it is unique up to unique isomorphism, and for any $W \in \mathcal{C}$, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(W, \operatorname{Ker} f) \to \mathrm{Hom}_{\mathcal{C}}(W, X) \xrightarrow{f} \mathrm{Hom}_{\mathcal{C}}(W, Y) \tag{5.1}$$

is exact in $\mathrm{Mod}(\mathbb{Z})$.

Similarly, if $\operatorname{Coker} f$ exists, then for any $W \in \mathcal{C}$, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(\operatorname{Coker} f, W) \to \mathrm{Hom}_{\mathcal{C}}(Y, W) \xrightarrow{f} \mathrm{Hom}_{\mathcal{C}}(X, W) \tag{5.2}$$

is exact in $\mathrm{Mod}(\mathbb{Z})$.

Example 5.1.2. Let A be a ring, I an ideal which is not finitely generated and let $M = A/I$. Then the natural morphism $A \to M$ in $\mathrm{Mod}^f(A)$ has no kernel.

Let $\mathcal{C}$ be an additive category which admits kernels and cokernels. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. One defines:

$$\begin{aligned}\mathrm{Coim}\, f &= \mathrm{Coker}\, h, \text{ where } h : \mathrm{Ker}\, f \to X \\ \mathrm{Im}\, f &= \mathrm{Ker}\, k, \text{ where } k : Y \to \mathrm{Coker}\, f.\end{aligned}$$

Consider the diagram:

$$\begin{tikzcd}
\mathrm{Ker}\, f \arrow[r, "h"] & X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[r, "k"] & \mathrm{Coker}\, f \\
& \mathrm{Coim}\, f \arrow[ur, dotted, "\tilde{f}"] \arrow[r, dotted, "u"] & \mathrm{Im}\, f \arrow[u] &
\end{tikzcd}$$

Since $f \circ h = 0$, f factors uniquely through $\tilde{f}$, and $k \circ f$ factors through $k \circ \tilde{f}$. Since $k \circ f = k \circ \tilde{f} \circ s = 0$ and s is an epimorphism, we get that $k \circ \tilde{f} = 0$. Hence $\tilde{f}$ factors through $\mathrm{Ker}\, k = \mathrm{Im}\, f$. We have thus constructed a canonical morphism:

$$\mathrm{Coim}\, f \xrightarrow{u} \mathrm{Im}\, f. \tag{5.3}$$

Examples 5.1.3. (i) If A is a ring and f is a morphism in $\mathrm{Mod}(A)$, then (5.3) is an isomorphism.
(ii) The category **Ban** admits kernels and cokernels. If $f : X \to Y$ is a morphism of Banach spaces, define $\mathrm{Ker}\, f = f^{-1}(0)$ and $\mathrm{Coker}\, f = Y/\overline{\mathrm{Im}\, f}$ where $\overline{\mathrm{Im}\, f}$ denotes the closure of the space $\mathrm{Im}\, f$. It is well-known that there exist continuous linear maps $f \colon X \to Y$ which are injective, with dense and non closed image. For such an f, $\mathrm{Ker}\, f = \mathrm{Coker}\, f = 0$ although f is not an isomorphism. Thus $\mathrm{Coim}\, f \simeq X$ and $\mathrm{Im}\, f \simeq Y$. Hence, the morphism (5.3) is not an isomorphism.

Definition 5.1.4. Let $\mathcal{C}$ be an additive category. One says that $\mathcal{C}$ is abelian if:

(i) any $f : X \to Y$ admits a kernel and a cokernel,

(ii) for any morphism f in $\mathcal{C}$, the natural morphism $\mathrm{Coim}\, f \to \mathrm{Im}\, f$ is an isomorphism.

In an abelian category, a morphism f is a monomorphism (resp. an epimorphism) if and only if $\mathrm{Ker}\, f \simeq 0$ (resp. $\mathrm{Coker}\, f \simeq 0$). If f is both a monomorphism and an epimorphism, it is an isomorphism.

Examples 5.1.5. (i) If A is a ring, $\mathrm{Mod}(A)$ is an abelian category.
(ii) If A is noetherian, then $\mathrm{Mod}^f(A)$ is abelian.
(iii) The category **Ban** admits kernels and cokernels but is not abelian. (See Examples 5.1.3 (ii).)
(iv) Let I be category. Then if $\mathcal{C}$ is abelian, the category $\mathcal{C}^I$ of functors from I to $\mathcal{C}$, is abelian. For example, if $F, G : I \to \mathcal{C}$ are two functors and $\varphi : F \to G$ is a morphism of functors, define the functor $\mathrm{Ker}\,\varphi$ by $\mathrm{Ker}\,\varphi(X) = \mathrm{Ker}(F(X) \to G(X))$. Then clearly, $\mathrm{Ker}\,\varphi$ is a kernel of φ. One defines similarly the cokernel.
(v) If $\mathcal{C}$ is abelian, then the opposite category $\mathcal{C}^{\mathrm{op}}$ is abelian.

Unless otherwise specified, we assume until the end of this chapter that $\mathcal{C}$ is abelian.

One naturally extends Definition 1.2.1 to abelian categories. Consider a sequence of morphisms $X' \xrightarrow{f} X \xrightarrow{g} X''$ with $g \circ f = 0$ (sometimes, one calls such a sequence a complex). It defines a morphism $\mathrm{Coim}\, f \to \mathrm{Ker}\, g$, hence, $\mathcal{C}$ being abelian, a morphism $\mathrm{Im}\, f \to \mathrm{Ker}\, g$.

Definition 5.1.6.
(i) One says that a sequence $X' \xrightarrow{f} X \xrightarrow{g} X''$ with $g \circ f = 0$ is exact if $\mathrm{Im}\, f \xrightarrow{\sim} \mathrm{Ker}\, g$.

(ii) More generally, a sequence of morphisms $X^p \xrightarrow{d^p} \cdots \to X^n$ with $d^{i+1} \circ d^i = 0$ for all $i \in [p, n-1]$ is exact if $\mathrm{Im}\, d^i \xrightarrow{\sim} \mathrm{Ker}\, d^{i+1}$ for all $i \in [p, n-1]$.

(iii) A short exact sequence is an exact sequence $0 \to X' \to X \to X'' \to 0$

Any morphism $f : X \to Y$ may be decomposed into short exact sequences:

$$0 \to \mathrm{Ker}\, f \to X \to \mathrm{Im}\, f \to 0$$

$$0 \to \mathrm{Im}\, f \to Y \to \mathrm{Coker}\, f \to 0.$$

Proposition 5.1.7. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be a short exact sequence in $\mathcal{C}$. Then the conditions* (i) *to* (iii) *are equivalent.*

(i) *there exists $h : X'' \to X$ such that $g \circ h = \mathrm{id}_{X''}$,*

(ii) *there exists $k : X \to X'$ such that $k \circ f = \mathrm{id}_{X'}$,*

(iii) *there exists $\varphi = (k, g)$ and $\psi = (f + h)$ such that $X \xrightarrow{\varphi} X' \oplus X''$ and $X' \oplus X'' \xrightarrow{\psi} X$ are isomorphisms inverse to each other,* The proof is similar to the case of A-modules and is left as an exercise.

If the conditions of the above proposition are satisfied, one says that the sequence splits.

Note that an additive functor of abelian categories sends split exact sequences into split exact sequences.

Lemma 5.1.8. *(The "five lemma".) Consider a commutative diagram:*

$$\begin{tikzcd} X^0 \arrow[r, "\alpha_0"] \arrow[d, "f^0"'] & X^1 \arrow[r, "\alpha_1"] \arrow[d, "f^1"'] & X^2 \arrow[r, "\alpha_2"] \arrow[d, "f^2"'] & X^3 \arrow[d, "f^3"'] \\ Y^0 \arrow[r, "\beta_0"'] & Y^1 \arrow[r, "\beta_1"'] & Y^2 \arrow[r, "\beta_2"'] & Y^3 \end{tikzcd}$$

and assume that the rows are exact sequences.

(i) *If f^0 is an epimorphism and f^1, f^3 are monomorphisms, then f^2 is a monomorphism.*

(ii) *If f^3 is a monomorphism, and f^0, f^2 are epimorphisms, then f^1 is an epimorphism.*

According to Convention 5.1.1, we shall assume that $\mathcal{C}$ is a full abelian subcategory of $\mathrm{Mod}(A)$ for some ring A. Hence we may choose elements in the objects of $\mathcal{C}$.

Proof. (i) Let $x_2 \in X_2$ and assume that $f^2(x_2) = 0$. Then $f^3 \circ \alpha_2(x_2) = 0$ and f^3 being a monomorphism, this implies $\alpha_2(x_2) = 0$. Since the first row is exact, there exists $x_1 \in X_1$ such that $\alpha_1(x_1) = x_2$. Set $y_1 = f^1(x_1)$. Since $\beta_1 \circ f^1(x_1) = 0$ and the second row is exact, there exists $y_0 \in Y^0$ such that $\beta_0(y_0) = f^1(x_1)$. Since f^0 is an epimorphism, there exists $x_0 \in X^0$ such that $y_0 = f^0(x_0)$. Since $f^1 \circ \alpha_0(x_0) = f^1(x_1)$ and f^1 is a monomorphism, $\alpha_0(x_0) = x_1$. Therefore, $x_2 = \alpha_1(x_1) = 0$.
(ii) is nothing but (i) in $\mathcal{C}^{\mathrm{op}}$. q.e.d.

Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor of abelian categories. Since F is additive, $F(0) \simeq 0$ and $F(X \oplus Y) \simeq F(X) \oplus F(Y)$. In other words, F commutes with finite direct sums (and with finite products).

Let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor. Recall that F is left exact if and only if it commutes with kernels, that is, if and only if for any exact sequence in $\mathcal{C}$, $0 \to X' \to X \to X''$ the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact in $\mathcal{C}'$.

Similarly, F is right exact if and only if it commutes with cokernels, that is, if and only if for any exact sequence in $\mathcal{C}$, $X' \to X \to X'' \to 0$ the sequence $F(X') \to F(X) \to F(X'') \to 0$ is exact.

Lemma 5.1.9. *Let $F\mathcal{C} \to \mathcal{C}'$ be an additive functor.*

(i) *F is left exact if and only if for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact.*

(ii) *F is exact if and only if for any exact sequence $X' \to X \to X''$ in $\mathcal{C}$, the sequence $F(X') \to F(X) \to F(X'')$ is exact.*

The proof is left as an exercise.

Examples 5.1.10. (i) Let $\mathcal{C}$ be an abelian category. The functor $\mathrm{Hom}_{\mathcal{C}}$ from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to $\mathrm{Mod}(\mathbb{Z})$ is left exact.
(ii) Let A be a k-algebra. Let M and N in $\mathrm{Mod}(A)$. It follows from (i) that the functors Hom_A from $\mathrm{Mod}(A)^{\mathrm{op}} \times \mathrm{Mod}(A)$ to $\mathrm{Mod}(k)$ is left exact.
The functors $\otimes_A$ from $\mathrm{Mod}(A^{\mathrm{op}}) \times \mathrm{Mod}(A)$ to $\mathrm{Mod}(k)$ is right exact.
If A is a field, all the above functors are exact.
(iii) Let I and $\mathcal{C}$ be two categories with $\mathcal{C}$ abelian. Assume that $\mathcal{C}$ admits inductive limits. Recall that the functor $\varinjlim : \mathrm{Fct}(I, \mathcal{C}) \to \mathcal{C}$ is right exact. If $\mathcal{C} = \mathrm{Mod}(A)$ and I is filtrant, then the functor $\varinjlim$ is exact.

Similarly, if $\mathcal{C}$ admits projective limits, the functor $\varprojlim : \mathrm{Fct}(I^{\mathrm{op}}, \mathcal{C}) \to \mathcal{C}$ is left exact. If $\mathcal{C} = \mathrm{Mod}(A)$ and I is discrete, the functor $\varprojlim$ (that is, the functor $\prod$) is exact.

5.2 Complexes in abelian categories

We assume that $\mathcal{C}$ is abelian. Notice first that the categories $C^*(\mathcal{C})$ are clearly abelian for $* = \emptyset, +, -, b$. For example, if $f : X \to Y$ is a morphism in $C(\mathcal{C})$, the complex Z defined by $Z^n = \mathrm{Ker}(f^n : X^n \to Y^n)$, with differential induced by those of X, will be a kernel for f, and similarly for $\mathrm{Coker}\, f$.

Let $X \in C(\mathcal{C})$. One defines the following objects of $\mathcal{C}$:

$$
\begin{aligned}
Z^k(X) &:= \mathrm{Ker}\, d_X^k \\
B^k(X) &:= \mathrm{Im}\, d_X^{k-1} \\
H^k(X) &:= Z^k(X)/B^k(X) \quad (:= \mathrm{Coker}(B^k(X) \to Z^k(X)))
\end{aligned}
$$

One calls $H^k(X)$ the k-th cohomology object of X. If $f : X \to Y$ is a morphism in $C(\mathcal{C})$, then it induces morphisms $Z^k(X) \to Z^k(Y)$ and $B^k(X) \to B^k(Y)$, thus a morphism $H^k(f) : H^k(X) \to H^k(Y)$. Clearly,$H^k(X \oplus Y) \simeq H^k(X) \oplus H^k(Y)$. Hence we have obtained an additive functor:

$$H^k(\,\cdot\,) : C(\mathcal{C}) \to \mathcal{C}.$$

Notice that:

$$H^k(X) = H^0(X[k]).$$

Lemma 5.2.1. *Let $\mathcal{C}$ be an abelian category and let $f : X \to Y$ be a morphism in $C(\mathcal{C})$ homotopic to zero. Then $H^k(f) : H^k(X) \to H^k(Y)$ is the 0 morphism.*

Proof. Let $f^k = s^{k+1} \circ d_X^k + d_Y^{k-1} \circ s^k$. Then $d_X^k = 0$ on $\operatorname{Ker} d_X^k$ and $d_Y^{k-1} \circ s^k = 0$ on $\operatorname{Ker} d_Y^k / \operatorname{Im} d_Y^{k-1}$. Hence $H^k(f) : \operatorname{Ker} d_X^k / \operatorname{Im} d_X^{k-1} \to \operatorname{Ker} d_Y^k / \operatorname{Im} d_Y^{k-1}$ is the zero morphism. q.e.d.

Definition 5.2.2. One says that a morphism $f : X \to Y$ in $C(\mathcal{C})$ is a quasi-isomorphism (a qis, for short) if $H^k(f)$ is an isomorphism for all $k \in \mathbb{Z}$. In such a case, one says that X and Y are quasi-isomorphic.

In particular, X is qis to 0 means that the complex X is exact.

Remark 5.2.3. By Lemma 5.2.1, a complex homotopic to 0 is qis to 0, but the converse is false. For example, a short exact sequence does not necessarily split. One shall be aware that the property for a complex of being homotopic to 0 is preserved when applying an additive functor, contrarily to the property of being qis to 0.

Remark 5.2.4. Consider a bounded complex $X^\bullet$ and denote by $Y^\bullet$ the complex given by $Y^j = H^j(X^\bullet), d_Y^j \equiv 0$. One has:

$$Y^\bullet = \oplus_i H^i(X^\bullet)[-i]. \tag{5.4}$$

The complexes $X^\bullet$ and $Y^\bullet$ have the same cohomology objects. In other words, $H^j(Y^\bullet) \simeq H^j(X^\bullet)$. However, in general these isomorphisms are neither induced by a morphism from $X^\bullet \to Y^\bullet$, nor by a morphism from $Y^\bullet \to X^\bullet$, and the two complexes $X^\bullet$ and $Y^\bullet$ are not quasi-isomorphic.

There are exact sequences

$$X^{k-1} \to \operatorname{Ker} d_X^k \to H^k(X) \to 0,$$
$$0 \to H^k(X) \to \operatorname{Coker} d_X^{k-1} \to X^{k+1},$$

which give rise to the exact sequence:

$$0 \to H^k(X) \to \operatorname{Coker}(d_X^{k-1}) \xrightarrow{d_X^k} \operatorname{Ker} d_X^{k+1} \to H^{k+1}(X) \to 0. \tag{5.5}$$

Lemma 5.2.5. (The snake lemma.) *Consider the commutative diagram in $\mathcal{C}$ below with exact rows:*

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r] \arrow[d, "\gamma"'] & 0 \\ 0 \arrow[r] & X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' & \end{tikzcd}$$

Then it gives rise to an exact sequence:

$$\operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \xrightarrow{\varphi} \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma.$$

The proof is similar to that of Lemma 5.1.8 and is left as an exercise.

Theorem 5.2.6. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $C(\mathcal{C})$.*

(i) *For each $k \in \mathbb{Z}$, the sequence $H^k(X') \to H^k(X) \to H^k(X'')$ is exact.*

(ii) *For each $k \in \mathbb{Z}$, there exists $\delta^k : H^k(X'') \to H^{k+1}(X')$ making the sequence:*

$$H^k(X) \to H^k(X'') \xrightarrow{\delta^k} H^{k+1}(X') \to H^{k+1}(X) \tag{5.6}$$

exact. Moreover, one can construct δ^k functorial with respect to short exact sequences of $C(\mathcal{C})$.

Proof. The exact sequence in $C(\mathcal{C})$ gives rise to commutative diagrams with exact rows:

$$\begin{tikzcd} & \operatorname{Coker} d_{X'}^{k-1} \arrow[r, "f"'] \arrow[d, "d_{X'}^k"'] & \operatorname{Coker} d_X^{k-1} \arrow[r, "g"'] \arrow[d, "d_X^k"'] & \operatorname{Coker} d_{X''}^{k-1} \arrow[r] \arrow[d, "d_{X''}^k"'] & 0 \\ 0 \arrow[r] & \operatorname{Ker} d_{X'}^{k+1} \arrow[r, "f"'] & \operatorname{Ker} d_X^{k+1} \arrow[r, "g"'] & \operatorname{Ker} d_{X''}^{k+1} & \end{tikzcd}$$

Then using the exact sequence (5.5), the result follows from Lemma 5.2.5. q.e.d.

Remark 5.2.7. Let us denote for a while by $\delta^k(f,g)$ the map δ^k constructed in Theorem 5.2.6. Then one can prove that $\delta^k(-f,g) = \delta^k(f,-g) = -\delta^k(f,g)$.

Corollary 5.2.8. *In the situation of Theorem 5.2.6, if two of the complexes X', X, X'' are exact, so is the third one.*

Corollary 5.2.9. *Let $f : X \to Y$ be a morphism in $C(\mathcal{C})$. Then there is a long exact sequence*

$$\cdots \to H^k(X) \xrightarrow{H^k(f)} H^k(Y) \to H^{k+1}(\mathrm{Mc}(f)) \to \cdots$$

Proof. There are natural morphisms $Y \to \mathrm{Mc}(f)$ and $\mathrm{Mc}(f) \to X[1]$ which give rise to an exact sequence in $C(\mathcal{C})$:

$$0 \to Y \to \mathrm{Mc}(f) \to X[1] \to 0. \tag{5.7}$$

Applying Theorem 5.2.6, one finds a long exact sequence

$$\cdots \to H^k(X[1]) \xrightarrow{\delta^k} H^{k+1}(Y) \to H^{k+1}(\mathrm{Mc}(f)) \to \cdots .$$

One can prove that the morphism $\delta^k : H^{k+1}(X) \to H^{k+1}(Y)$ is $H^{k+1}(f)$ up to a sign. q.e.d.

Double complexes

Let $\mathcal{C}$ denote as above an abelian category.

Theorem 5.2.10. *Let $X^{\bullet,\bullet}$ be a double complex such that all rows $X^{j,\bullet}$ and columns $X^{\bullet,j}$ are 0 for $j < 0$ and are exact for $j > 0$.*

Then $H^p(X^{0,\bullet}) \simeq H^p(X^{\bullet,0}) \simeq H^p(\mathrm{tot}(X^{\bullet,\bullet}))$ for all p.

Proof. We shall only describe the first isomorphism $H^p(X^{0,\bullet}) \simeq H^p(X^{\bullet,0})$ in the case where $\mathcal{C} = \mathrm{Mod}(A)$, by the so-called "Weil procedure".

Let $x^{p,0} \in X^{p,0}$, with $d'x^{p,0} = 0$ which represents $y \in H^p(X^{\bullet,0})$. Define $x^{p,1} = d''x^{p,0}$. Then $d'x^{p,1} = 0$, and the first column being exact, there exists $x^{p-1,1} \in X^{p-1,1}$ with $d'x^{p-1,1} = x^{p,1}$. One can iterate this procedure until getting $x^{0,p} \in X^{0,p}$. Since $d'd''x^{0,p} = 0$, and d' is injective on $X^{0,p}$ for $p > 0$ by the hypothesis, we get $d''x^{0,p} = 0$. The class of $x^{0,p}$ in $H^p(X^{0,\bullet})$ will be the image of y by the Weil procedure. Of course, one has to check that this image does not depend of the various choices we have made, and that it induces an isomorphism.

This can be visualized by the diagram:

$$\begin{tikzcd}
 & & & & x^{0,p} \arrow[r, "d''"] \arrow[d, "d'"'] & 0 \\
 & & & x^{1,p-2} \arrow[r, "d''"] \arrow[d, dotted] & x^{1,p-1} & \\
 & & & {} & & \\
 & x^{p-1,1} \arrow[r, dotted] \arrow[d, "d'"'] & {} & & & \\
x^{p,0} \arrow[r, "d''"] \arrow[d, "d'"'] & x^{p,1} & & & & \\
0 & & & & &
\end{tikzcd}$$

q.e.d.

Proposition 5.2.11. *Let $X^{\bullet,\bullet}$ be a double complex such that all rows $X^{j,\bullet}$ and columns $X^{\bullet,j}$ are 0 for $j<0$. Assume that all rows (resp. all columns) of $X^{\bullet,\bullet}$ are exact. Then the complex $\mathrm{tot}(X^{\bullet,\bullet})$ is exact.*

The proof is left as an exercise. Note that if there are only two rows let's say in degrees -1 and 0, then the result follows from Theorem 5.6.4

5.3 Application to Koszul complexes

Consider a Koszul complex, as in §1.5. Keeping the notations of this section, set $\varphi' = \{\varphi_1, \dots, \varphi_{n-1}\}$ and denote by d' the differential in $K^\bullet(M, \varphi')$. Then φ_n defines a morphism

$$\widetilde{\varphi}_n : K^\bullet(M, \varphi') \to K^\bullet(M, \varphi') \tag{5.8}$$

Proposition 5.3.1. *The complex $K^\bullet(M, \varphi)[1]$ is isomorphic to the mapping cone of $-\widetilde{\varphi}_n$.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathrm{Mc}(\widetilde{\varphi}_n)^p \arrow[r, "d_M^p"'] \arrow[d, "\lambda^p"'] & \mathrm{Mc}(\widetilde{\varphi}_n)^{p+1} \arrow[d, "\lambda^{p+1}"'] \\
K^{p+1}(M, \varphi) \arrow[r, "d_K^{p+1}"'] & K^{p+2}(M, \varphi)
\end{tikzcd}$$

given explicitly by:

$$\begin{tikzcd}[column sep=huge]
(M \otimes \bigwedge^{p+1}\mathbb{Z}^{n-1}) \oplus (M \otimes \bigwedge^{p}\mathbb{Z}^{n-1}) \arrow[r, "{\begin{pmatrix} -d' & 0 \\ -\varphi_n & d' \end{pmatrix}}"'] \arrow[d, "\mathrm{id}\oplus(\mathrm{id}\otimes e_n\wedge)"'] & M(\otimes \bigwedge^{p+2}\mathbb{Z}^{n-1}) \oplus (M \otimes \bigwedge^{p+1}\mathbb{Z}^{n-1}) \arrow[d, "\mathrm{id}\oplus(\mathrm{id}\otimes e_n\wedge)"'] \\
M \otimes \bigwedge^{p+1}\mathbb{Z}^{n} \arrow[r, "-d"'] & M \otimes \bigwedge^{p+2}\mathbb{Z}^{n}
\end{tikzcd}$$

Then

$$\begin{aligned}
&d_M^p(a\otimes e_J + b\otimes e_K) = -d'(a\otimes e_J) + (d'(b\otimes e_K) - \varphi_n(a)\otimes e_J),\\
&\lambda^p(a\otimes e_J + b\otimes e_K) = a\otimes e_J + b\otimes e_n\wedge e_K.
\end{aligned}$$

(i) The vertical arrows are isomorphisms. Indeed, let us treat the first one. It is described by:

$$\sum_J a_J\otimes e_J + \sum_K b_K\otimes e_K \mapsto \sum_J a_J\otimes e_J + \sum_K b_K\otimes e_n\wedge e_K \tag{5.9}$$

with $|J| = p+1$ and $|K| = p$. Any element of $M\otimes\bigwedge^{p+1}\mathbb{Z}^n$ may uniquely be written as in the right hand side of (5.9).
(ii) The diagram commutes. Indeed,

$$\begin{aligned}
\lambda^{p+1}\circ d_M^p(a\otimes e_J + b\otimes e_K) &= -d'(a\otimes e_J) + e_n\wedge d'(b\otimes e_K) - \varphi_n(a)\otimes e_n\wedge e_J\\
&= -d'(a\otimes e_J) - d'(b\otimes e_n\wedge e_K) - \varphi_n(a)\otimes e_n\wedge e_J,\\
d_K^{p+1}\circ\lambda^p(a\otimes e_J + b\otimes e_K) &= -d(a\otimes e_J + b\otimes e_n\wedge e_K)\\
&= -d'(a\otimes e_J) - \varphi_n(a)\otimes e_n\wedge e_J - d'(b\otimes e_n\wedge e_K).
\end{aligned}$$

q.e.d.

Proposition 5.3.2. *There exists a long exact sequence*

$$\cdots \to H^j(K^\bullet(M,\varphi')) \xrightarrow{\varphi_n} H^j(K^\bullet(M,\varphi')) \to H^{j+1}(K^\bullet(M,\varphi)) \to \cdots \tag{5.10}$$

Proof. Apply Proposition 5.3.1 and Corollary 5.2.9. q.e.d.

We can now give a proof to Theorem 1.5.2. Assume for example that $(\varphi_1,\dots,\varphi_n)$ is a regular sequence, and let us argue by induction on n. The cohomology of $K^\bullet(M,\varphi')$ is thus concentrated in degree $n-1$ and is isomorphic to $M/(\varphi_1(M)+\cdots+\varphi_{n-1}(M))$. By the hypothesis, φ_n is injective on this group, and Theorem 1.5.2 follows.

5.4 Injective objects

Definition 5.4.1. (i) An object I of $\mathcal{C}$ is injective if $\mathrm{Hom}_{\mathcal{C}}(\bullet, I)$ is an exact functor.

(ii) One says that $\mathcal{C}$ has enough injectives if for any $X \in \mathcal{C}$ there exists a monomorphism $X \rightarrowtail I$ with I injective.

(iii) An object P is projective in $\mathcal{C}$ iff it is injective in $\mathcal{C}^{\mathrm{op}}$, i.e. if the functor $\mathrm{Hom}_{\mathcal{C}}(P, \bullet)$ is exact.

(iv) One says that $\mathcal{C}$ has enough projectives if for any $X \in \mathcal{C}$ there exists an epimorphism $P \twoheadrightarrow X$ with P projective.

Example 5.4.2. Let A be a ring. An A-module M is called injective (resp. projective) if it is so in the category $\mathrm{Mod}(A)$. If M is free then it is projective. More generally, if there exists an A-module N such that $M \oplus N$ is free then M is projective (see Exercise 1.2). One immediately deduces that the category $\mathrm{Mod}(A)$ has enough projectives. One can prove that $\mathrm{Mod}(A)$ has enough injectives (see Exercise 1.5).
If k is a field, then any object of $\mathrm{Mod}(k)$ is both injective and projective.

Proposition 5.4.3. *The object $I \in \mathcal{C}$ is injective if and only if, for any $X, Y \in \mathcal{C}$ and any diagram in which the row is exact:*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "f"] \arrow[d, "k"'] & Y \arrow[dl, dotted, "h"] \\ & I & \end{tikzcd}$$

the dotted arrow may be completed, making the solid diagram commutative.

The proof is similar to that of Proposition 1.3.8.

Lemma 5.4.4. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathcal{C}$, and assume that X' is injective. Then the sequence splits.*

Proof. Applying the preceding result with $k = \mathrm{id}_{X'}$, we find $h \colon X \to X'$ such that $k \circ f = \mathrm{id}_{X'}$. Then apply Proposition 5.1.7. q.e.d.

It follows that if $F \colon \mathcal{C} \to \mathcal{C}'$ is an additive functor of abelian categories, and the hypotheses of the lemma are satisfied, then the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ splits and in particular is exact.

Lemma 5.4.5. *Let X', X'' belong to $\mathcal{C}$. Then $X' \oplus X''$ is injective if and only if X' and X'' are injective.*

Proof. It is enough to remark that for two additive functors of abelian categories F and G, $X \mapsto F(X) \oplus G(X)$ is exact if and only if F and G are exact. q.e.d.

Applying Lemmas 5.4.4 and 5.4.5, we get:

Proposition 5.4.6. *Let $0 \to X' \to X \to X'' \to 0$ be an exact sequence in $\mathcal{C}$ and assume X' and X are injective. Then X'' is injective.*

5.5 Resolutions

In this section, $\mathcal{C}$ denotes an abelian category and $\mathcal{I}_{\mathcal{C}}$ its full additive subcategory consisting of injective objects. We shall asume

(5.11) the abelian category $\mathcal{C}$ admits enough injectives.

Definition 5.5.1. Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{C}$. We say that $\mathcal{J}$ is cogenerating if for all X in $\mathcal{C}$, there exist $Y \in \mathcal{J}$ and a monomorphism $X \rightarrowtail Y$.

Note that the category of injective objects is cogenerating iff $\mathcal{C}$ has enough injectives.

Notations 5.5.2. Consider an exact sequence in $\mathcal{C}$, $0 \to X \to J^0 \to \cdots \to J^n \to \cdots$ and denote by $J^\bullet$ the complex $0 \to J^0 \to \cdots \to J^n \to \cdots$. We shall say for short that $0 \to X \to J^\bullet$ is a resolution of X. If the J^k's belong to $\mathcal{J}$, we shall say that this is a $\mathcal{J}$-resolution of X. When $\mathcal{J}$ denotes the category of injective objects one says this is an injective resolution.

Proposition 5.5.3. *Assume $\mathcal{J}$ is cogenerating. Then for any $X \in \mathcal{C}$, there exists a $\mathcal{J}$-resolution of X.*

Proof. We proceed by induction. Assume to have constructed:

$$0 \to X \to J^0 \to \cdots \to J^n$$

For $n = 0$ this is the hypothesis. Set $B^n = \mathrm{Coker}(J^{n-1} \to J^n)$ (with $J^{-1} = X$). Then $J^{n-1} \to J^n \to B^n \to 0$ is exact. Embed B^n in an object of $\mathcal{J}$: $0 \to B^n \to J^{n+1}$. Then $J^{n-1} \to J^n \to J^{n+1}$ is exact, and the induction proceeds. q.e.d.

Proposition 5.5.4. (i) *Let $f^\bullet : X^\bullet \to I^\bullet$ be a morphism in $C^+(\mathcal{C})$. Assume $I^\bullet$ belongs to $\mathcal{C}^+(\mathcal{I}_\mathcal{C})$ and $X^\bullet$ is exact. Then $f^\bullet$ is homotopic to* 0.

(ii) *Let $I^\bullet \in C^+(\mathcal{I}_\mathcal{C})$ and assume $I^\bullet$ is exact. Then $I^\bullet$ is homotopic to* 0.

Proof. (i) Consider the diagram:

$$\begin{tikzcd}
X^{k-2} \arrow[r] & X^{k-1} \arrow[r] \arrow[dl, dotted, "s^{k-1}"'] \arrow[d, "f^{k-1}"] & X^k \arrow[r] \arrow[dl, dotted, "s^k"'] \arrow[d, "f^k"] & X^{k+1} \arrow[dl, dotted, "s^{k+1}"'] \\
I^{k-2} \arrow[r] & I^{k-1} \arrow[r] & I^k \arrow[r] & I^{k+1}
\end{tikzcd}$$

We shall construct by induction morphisms s^k satisfying:

$$f^k = s^{k+1} \circ d_X^k + d_I^{k-1} \circ s^k.$$

For $j << 0$, $s^j = 0$. Assume we have constructed the s^j for $j \leq k$. Define $g^k = f^k - d_I^{k-1} \circ s^k$. One has

$$\begin{aligned}
g^k \circ d_X^{k-1} &= f^k \circ d_X^{k-1} - d_I^{k-1} \circ s^k \circ d_X^{k-1} \\
&= f^k \circ d_X^{k-1} - d_I^{k-1} \circ f^{k-1} + d_I^{k-1} \circ d_I^{k-2} \circ s^{k-1} \\
&= 0.
\end{aligned}$$

Hence, g^k factorizes through $X^k / \operatorname{Im} d_X^{k-1}$. Since the complex $X^\bullet$ is exact, the sequence $0 \to X^k / \operatorname{Im} d_X^{k-1} \to X^{k+1}$ is exact. Consider

$$\begin{tikzcd}
0 \arrow[r] & X^k / \operatorname{Im} d_X^{k-1} \arrow[r] \arrow[d, "g^k"'] & X^{k+1} \arrow[dl, dotted, "s^{k+1}"'] \\
& I^k &
\end{tikzcd}$$

The dotted arrow may be completed by Proposition 5.4.3.
(ii) Apply the result of (i) with $X^\bullet = I^\bullet$ and $f = \mathrm{id}_X$. q.e.d.

Proposition 5.5.5. (i) *Let $f : X \to Y$ be a morphism in $\mathcal{C}$, let $0 \to X \to X^\bullet$ be a resolution of X and let $0 \to Y \to J^\bullet$ be a complex with the J^k's injective. Then there exists a morphism $f^\bullet : X^\bullet \to J^\bullet$ making the diagram below commutative:*

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "f"'] & X^\bullet \arrow[d, "f^\bullet"'] \\
0 \arrow[r] & Y \arrow[r] & J^\bullet
\end{tikzcd}$$

(ii) *The morphism* $f^\bullet$ *in* $C(\mathcal{C})$ *constructed in* (i) *is unique up to homotopy.*

Proof. (i) Let us denote by d_X (resp. d_Y) the differential of the complex $X^\bullet$ (resp. $J^\bullet$), by d_X^{-1} (resp. d_Y^{-1}) the morphism $X \to X^0$ (resp. $Y \to J^0$) and set $f^{-1} = f$.

We shall construct the f^n's by induction. Morphism f^0 is obtained by Proposition 5.4.3. Assume we have constructed $f^0, \dots, f^n$. Let $g^n = d_Y^n \circ f^n : X^n \to J^{n+1}$. The morphism g^n factorizes through $h^n : X^n / \operatorname{Im} d_X^{n-1} \to J^{n+1}$. Since $X^\bullet$ is exact, the sequence $0 \to X^n / \operatorname{Im} d_X^{n-1} \to X^{n+1}$ is exact. Since J^{n+1} is injective, h^n extends as $f^{n+1} : X^{n+1} \to J^{n+1}$.
(ii) We may assume $f = 0$ and we have to prove that in this case $f^\bullet$ is homotopic to zero. Since the sequence $0 \to X \to X^\bullet$ is exact, this follows from Proposition 5.5.4 (i), replacing the exact sequence $0 \to Y \to J^\bullet$ by the complex $0 \to 0 \to J^\bullet$. q.e.d.

5.6 Derived functors

In this section, $\mathcal{C}$ and $\mathcal{C}'$ will denote abelian categories and $F : \mathcal{C} \to \mathcal{C}'$ a left exact functor. We shall make the hypothesis

$$\text{the category } \mathcal{C} \text{ admits enough injectives.} \tag{5.12}$$

Lemma 5.6.1. (i) *Let* $X \in \mathcal{C}$ *and let* $I_X^\bullet$ *be an injective resolution of* X*. Then* $H^k(F(I_X^\bullet))$ *does not depend on the choice of the injective resolution* $I_X^\bullet$ *up to unique isomorphism.*

(ii) *Let* $f : X \to Y$ *be a morphism in* $\mathcal{C}$*, let* $I_X^\bullet$ *and* $I_Y^\bullet$ *be injective resolutions of* X *and* Y *and let* $f^\bullet : I_X^\bullet \to I_Y^\bullet$ *be a morphism of complexes such as in Proposition 5.5.5. Then* $H^k(F(f^\bullet)) : H^k(F(I_X^\bullet)) \to H^k(F(I_Y^\bullet))$ *does not depend on the choice of* $f^\bullet$*.*

Proof. (i) Consider two injective resolutions $I_X^\bullet$ and $J_X^\bullet$ of X. By Proposition 5.5.5 applied to id_X, there exists a morphism $f^\bullet : I_X^\bullet \to J_X^\bullet$ and this morphism is unique up to homotopy. Hence, there exists a unique morphism $H^k(f^\bullet) : H^k(I_X^\bullet) \to H^k(J_X^\bullet)$. Similarly, there exists $g^\bullet : J_X^\bullet \to I_X^\bullet$ and $H^k(g^\bullet)$ is unique. By choosing $J_X^\bullet = I_X^\bullet$, we find

$$H^k(f^\bullet) \circ H^k(g^\bullet) = H^k(f^\bullet \circ g^\bullet) = \mathrm{id}_{I_X^\bullet},$$
$$H^k(g^\bullet) \circ H^k(f^\bullet) = H^k(f^\bullet \circ g^\bullet) = \mathrm{id}_{J_X^\bullet}.$$

Hence, $H^k(f^\bullet)$ is an isomorphism and it is unique.
(ii) The proof is similar. q.e.d.

In particular, we get that if $g\colon Y \to Z$ is another morphism in $\mathcal{C}$ and $I_Z^\bullet$ is an injective resolutions of Z, then

$$H^k(F(g^\bullet \circ f^\bullet)) = H^k(F((g \circ f)^\bullet)).$$

Definition 5.6.2. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a left exact functor of abelian categories and assume that $\mathcal{C}$ has enough injectives. The k-th right derived functor of F is defined as follws. One sets $R^kF(X) = H^k(F(I_X^\bullet))$ for $X \in \mathcal{C}$ and $R^kF(f) = H^k(F(f^\bullet))$ for f a morphism in $\mathcal{C}$.

Note that R^kF is an additive functor from $\mathcal{C}$ to $\mathcal{C}'$ and

$$\begin{array}{rcl} R^kF(X) & \simeq & 0 \text{ for } k < 0, \\ R^0F(X) & \simeq & F(X), \\ \text{if } F \text{ is exact } R^kF(X) & \simeq & 0 \text{ for } k \neq 0, \\ \text{if } X \text{ is injective } R^kF(X) & \simeq & 0 \text{ for } k \neq 0. \end{array}$$

The first assertion is obvious since $I_X^k = 0$ for $k < 0$, and the second one follows from the fact that F being left exact, then $\mathrm{Ker}(F(I_X^0) \to F(I_X^1)) \simeq F(\mathrm{Ker}(I_X^0 \to I_X^1)) \simeq F(X)$. The third assertion is clear since F being exact, it commutes with $H^j(\,\bullet\,)$. The last assertion is obvious by the construction of $R^jF(X)$.

Definition 5.6.3. An object X of $\mathcal{C}$ such that $R^kF(X) \simeq 0$ for all $k > 0$ is called F-acyclic.

Hence, injective objects are F-acyclic for all left exact functors F.

Theorem 5.6.4. *Let* $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ *be an exact sequence in* $\mathcal{C}$*. Then there exists a long exact sequence:*

$$0 \to F(X') \to F(X) \to \cdots \to R^kF(X') \to R^kF(X) \to R^kF(X'') \to \cdots$$

Sketch of the proof. One constructs an exact sequence of complexes $0 \to X'^\bullet \to X^\bullet \to X''^\bullet \to 0$ whose objects are injective and this sequence is quasi-isomorphic to the sequence $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ in $C(\mathcal{C})$. Since the objects X'^j are injectice, we get a short exact sequence in $C(\mathcal{C}')$:

$$0 \to F(X'^\bullet) \to F(X^\bullet) \to F(X''^\bullet) \to 0$$

Then one applies Theorem 5.2.6. q.e.d.

Definition 5.6.5. Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{C}$. One says that $\mathcal{J}$ is F-injective if:

(i) $\mathcal{J}$ is cogenerating,

(ii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ with $X' \in \mathcal{J}, X \in \mathcal{J}$, then $X'' \in \mathcal{J}$,

(iii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ with $X' \in \mathcal{J}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

By considering $\mathcal{C}^{\mathrm{op}}$, one obtains the notion of an F-projective subcategory, F being right exact.

Proposition 5.6.6. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor and denote by $\mathcal{I}_F$ the full subcategory of $\mathcal{C}$ consisting of F-acyclic objects. Then $\mathcal{I}_F$ is F-injective.*

Proof. Since injective objects are F-acyclic, hypothesis (5.12) implies that $\mathcal{I}_F$ is co-generating. The conditions (ii) and (iii) in Definition 5.6.5 are satisfied by Theorem 5.6.4. q.e.d.

Examples 5.6.7. (i) If $\mathcal{C}$ has enough injectives, the category $\mathcal{I}$ of injective objects is F-acyclic for all left exact functors F.

(ii) Let A be a ring and let N be a right A-module. The full additive subcategory of $\mathrm{Mod}(A)$ consisting of flat A-modules is projective with respect to the functor $N \otimes_A \bullet$.

Lemma 5.6.8. *Assume $\mathcal{J}$ is F-injective and let $X^\bullet \in C^+(\mathcal{J})$ be a complex qis to zero* (i.e. *$X^\bullet$ is exact*)*. Then $F(X^\bullet)$ is qis to zero.*

Proof. We decompose $X^\bullet$ into short exact sequences (assuming that this complex starts at step 0 for convenience):

$$\begin{aligned}
&0 \to X^0 \to X^1 \to Z^1 \to 0\\
&0 \to Z^1 \to X^2 \to Z^2 \to 0\\
&\cdots\\
&0 \to Z^{n-1} \to X^n \to Z^n \to 0
\end{aligned}$$

By induction we find that all the Z^j's belong to $\mathcal{J}$, hence all the sequences:

$$0 \to F(Z^{n-1}) \to F(X^n) \to F(Z^n) \to 0$$

are exact. Hence the sequence

$$0 \to F(X^0) \to F(X^1) \to \cdots$$

is exact. q.e.d.

Theorem 5.6.9. *Assume $\mathcal{J}$ is F-injective and contains the category $\mathcal{I}_{\mathcal{C}}$ of injective objects. Let $X \in \mathcal{C}$ and let $0 \to X \to J^\bullet$ be a resolution of X with $J^k \in \mathcal{J}$. Then for each k, there is an isomorphism $R^kF(X) \simeq H^k(F(J^\bullet))$.*

Proof. Let $0 \to X \to J^\bullet$ be a $\mathcal{J}$-resolution of X and let $0 \to X \to I^\bullet$ be an injective resolution of X. Applying Proposition 5.5.5, there exists $f : J^\bullet \to I^\bullet$ making the diagram below commutative

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "\mathrm{id}"] & J^0 \arrow[r, "d_J^0"] \arrow[d, "f^0"] & J^1 \arrow[r, "d_J^1"] \arrow[d, "f^1"] & \cdots \\
0 \arrow[r] & X \arrow[r] & I^0 \arrow[r, "d_I^0"] & I^1 \arrow[r, "d_I^1"] & \cdots
\end{tikzcd}$$

Define the complex $K^\bullet = \mathrm{Mc}(f)$, the mapping cone of f. By the hypothesis, $K^\bullet$ belongs to $C^+(\mathcal{J})$ and this complex is qis to zero by Corollary 5.2.8. By Lemma 5.6.8, $F(K^\bullet)$ is qis to zero.

On the other-hand, $F(\mathrm{Mc}(f))$ is isomorphic to $\mathrm{Mc}(F(f))$, the mapping cone of $F(f) : F(J^\bullet) \to F(I^\bullet)$. Applying Theorem 5.2.6 to this sequence, we find a long exact sequence

$$\cdots \to H^n(F(J^\bullet)) \to H^n(F(I^\bullet)) \to H^n(F(K^\bullet)) \to \cdots$$

Since $F(K^\bullet)$ is qis to zero, the result follows. q.e.d.

By this result, one sees that in order to calculate the k-th derived functor of F at X, the recipe is as follows. Consider a resolution $0 \to X \to J^\bullet$ of X by objects of $\mathcal{J}$, then apply F to the complex $J^\bullet$, and take the k-th cohomology object.

Proposition 5.6.10. *Let $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}''$ be left exact functors of abelian categories. We assume that $\mathcal{C}$ and $\mathcal{C}'$ have enough injectives.*

(i) *If G is exact, then $R^j(G \circ F) \simeq G \circ R^jF$.*

(ii) *Assume that F is exact. There is a natural morphism $R^j(G \circ F) \to (R^jG) \circ F$.*

(iii) *Let $\mathcal{J}'$ be a G-injective subcategory of $\mathcal{C}'$ and assume that F sends the injective objects of $\mathcal{C}$ in $\mathcal{J}'$. If $X \in \mathcal{C}$ satisfies $R^kF(X) = 0$ for $k \neq 0$, then $R^j(G \circ F)(X) \simeq R^jG(F(X))$.*

(iv) *In particular, let $\mathcal{J}'$ be a G-injective subcategory of $\mathcal{C}'$ and assume that F is exact and sends the injective objects of $\mathcal{C}$ in $\mathcal{J}'$. Then $R^j(G \circ F) \simeq R^jG \circ F$.*

Proof. Let $X \in \mathcal{C}$ and let $0 \to X \to I_X^\bullet$ be an injective resolution of X. Then $R^j(G \circ F)(X) \simeq H^j(G \circ F(I_X^\bullet))$.
(i) If G is exact, the right-hand side is isomorphic to $G(H^j(F(I_X^\bullet))$.
(ii) Consider an injective resolution $0 \to F(X) \to J_{F(X)}^\bullet$ of $F(X)$. By Proposition 5.5.5, there exists a morphism $F(I_X^\bullet) \to J_{F(X)}^\bullet$. Applying G we get a morphism of complexes: $(G \circ F)(I_X^\bullet) \to G(J_{F(X)}^\bullet)$. Since $H^j((G \circ F)(I_X^\bullet)) \simeq R^j(G \circ F)(X)$ and $H^j(G(J_{F(X)}^\bullet)) \simeq R^jG(F(X))$, we get the result.
(iii) By the hypothesis, $F(I_X^\bullet)$ is qis to $F(X)$ and belongs to $C^+(\mathcal{J}')$. Hence $R^jG(F(X)) \simeq H^j(G(F(I_X^\bullet)))$.
(iv) is a particular case of (iii). q.e.d.

5.7 Bifunctors

Now consider an additive bifunctor $F : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$ of abelian categories, and assume: F is left exact with respect of each of its arguments (i.e., $F(X, \bullet)$ and $F(\bullet, Y)$ are left exact).

Let $\mathcal{I}_\mathcal{C}$ (resp. $\mathcal{I}_{\mathcal{C}'}$) denote the full additive subcategory of $\mathcal{C}$ (resp. $\mathcal{C}'$) consisting of injective objects.

Definition 5.7.1. (a) The pair $(\mathcal{I}_\mathcal{C}, \mathcal{C}')$ is F-injective if $\mathcal{C}$ admits enough injective and for all $I \in \mathcal{I}_\mathcal{C}$, $F(I, \bullet)$ is exact.

(b) If $(\mathcal{I}_\mathcal{C}, \mathcal{C}')$ is F-injective, we denote by $R^kF(X, Y)$ the k-th derived functor of $F(\bullet, Y)$ at X, *i.e.,* $R^kF(X, Y) = R^kF(\bullet, Y)(X)$.

(This definiton will be generalized in Definition 8.4.1.)

Proposition 5.7.2. *Assume that $(\mathcal{I}_\mathcal{C}, \mathcal{C}')$ is F-injective.*

(i) *Let $0 \to X' \to X \to X'' \to 0$ be an exact sequence in $\mathcal{C}$ and let $Y \in \mathcal{C}'$. Then there is a long exact sequence in $\mathcal{C}''$:*

$$\cdots \to R^{k-1}F(X'', Y) \to R^kF(X', Y) \to R^kF(X, Y) \to R^kF(X'', Y) \to$$

(ii) *Let $0 \to Y' \to Y \to Y'' \to 0$ be an exact sequence in $\mathcal{C}'$ and let $X \in \mathcal{C}$. Then there is a long exact sequence in $\mathcal{C}''$:*

$$\cdots \to R^{k-1}F(X, Y'') \to R^kF(X, Y') \to R^kF(X, Y) \to R^kF(X, Y'') \to$$

Proof. (i) is a particular case of Theorem 5.6.4.

(ii) Let $0 \to X \to I^\bullet$ be an injective resolution of X. By the hypothesis, the sequence in $C(\mathcal{C}'')$:

$$0 \to F(I^\bullet, Y') \to F(I^\bullet, Y) \to F(I^\bullet, Y'') \to 0$$

is exact. By Theorem 5.2.6, it gives rise to the desired long exact sequence. q.e.d.

Proposition 5.7.3. *Assume that both $(\mathcal{I}_\mathcal{C}, \mathcal{C}')$ and $(\mathcal{C}, \mathcal{I}_{\mathcal{C}'})$ are F-injective. Then for $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$, we have the isomorphism: $R^kF(X, Y) := R^kF(\,\bullet\,, Y)(X) \simeq R^kF(X, \,\bullet\,)(Y)$.*

Moreover if $I_X^\bullet$ is an injective resolution of X and $I_Y^\bullet$ an injective resolution of Y, then $R^kF(X, Y) \simeq \mathrm{tot}H^k(F(I_X^\bullet, I_Y^\bullet)$.

Proof. Let $0 \to X \to I_X^\bullet$ and $0 \to Y \to I_Y^\bullet$ be injective resolutions of X and Y, respectively. Consider the double complex:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & F(I_X^0, Y) \arrow[r] \arrow[d] & F(I_X^1, Y) \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & F(X, I_Y^0) \arrow[r] \arrow[d] & F(I_X^0, I_Y^0) \arrow[r] \arrow[d] & F(I_X^1, I_Y^0) \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & F(X, I_Y^1) \arrow[r] \arrow[d] & F(I_X^0, I_Y^1) \arrow[r] \arrow[d] & F(I_X^1, I_Y^1) \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} &
\end{tikzcd}$$

The cohomology of the first row (resp. column) calculates $R^kF(\,\bullet\,, Y)(X)$ (resp. $R^kF(X, \,\bullet\,)(Y)$). Since the other rows and columns are exact by the hypotheses, the result follows from Theorem 5.2.10. q.e.d.

Example 5.7.4. Assume $\mathcal{C}$ has enough injectives. Then

$$R^k\mathrm{Hom}_\mathcal{C} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Ab}$$

exists and is calculated as follows. Let $X \in \mathcal{C}$, $Y \in \mathcal{C}$. There exists a qis in $C^+(\mathcal{C})$, $Y \to I^\bullet$, the I^j's being injective. Then:

$$R^k\mathrm{Hom}_\mathcal{C}(X, Y) \simeq H^k(\mathrm{Hom}_\mathcal{C}(X, I^\bullet)).$$

If $\mathcal{C}$ has enough projectives, and $P^\bullet \to X$ is a qis in $C^-(\mathcal{C})$, the P^j's being projective, one also has:

$$\begin{aligned} R^k\mathrm{Hom}_\mathcal{C}(X, Y) &\simeq H^k\mathrm{Hom}_\mathcal{C}(P^\bullet, Y) \\ &\simeq H^k\mathrm{tot}(\mathrm{Hom}_\mathcal{C}(P^\bullet, I^\bullet)). \end{aligned}$$

If $\mathcal{C}$ has enough injectives or enough projectives, one sets:

$$\mathrm{Ext}^k_{\mathcal{C}}(\,\bullet\,,\,\bullet\,) = R^k\mathrm{Hom}_{\mathcal{C}}(\,\bullet\,,\,\bullet\,). \tag{5.13}$$

For example, let $A = k[x, y]$, $M = k \simeq A/xA+yA$ and let us calculate the groups $\mathrm{Ext}^j_A(M, A)$. Since injective resolutions are not easy to calculate, it is much simpler to calculate a free (hence, projective) resolution of M. Since (x, y) is a regular sequence of endomorphisms of A (viewed as an A-module), M is quasi-isomorphic to the complex:

$$M^\bullet : 0 \to A \xrightarrow{u} A^2 \xrightarrow{v} A \to 0,$$

where $u(a) = (ya, -xa)$, $v(b, c) = xb + yc$ and the module A on the right stands in degree 0. Therefore, $\mathrm{Ext}^j_A(M, N)$ is the j-th cohomology object of the complex $\mathrm{Hom}_A(M^\bullet, N)$, that is:

$$0 \to N \xrightarrow{v'} N^2 \xrightarrow{u'} N \to 0,$$

where $v' = \mathrm{Hom}\,(v, N)$, $u' = \mathrm{Hom}\,(u, N)$ and the module N on the left stands in degree 0. Since $v'(n) = (xn, yn)$ and $u'(m, l) = ym - xl$, we find again a Koszul complex. Choosing $N = A$, its cohomology is concentrated in degree 2. Hence, $\mathrm{Ext}^j_A(M, A) \simeq 0$ for $j \neq 2$ and $\simeq k$ for $j = 2$.

Example 5.7.5. Let A be a k-algebra. Since the category $\mathrm{Mod}(A)$ admits enough projective objects, the bifunctor

$$\bullet \otimes \bullet\, : \mathrm{Mod}(A^{\mathrm{op}}) \times \mathrm{Mod}(A) \to \mathrm{Mod}(k)$$

admits derived functors, denoted $\mathrm{Tor}^A_{-k}(\,\bullet\,,\,\bullet\,)$ or else, $\mathrm{Tor}^k_A(\,\bullet\,,\,\bullet\,)$.

If $Q^\bullet \to N \to 0$ is a projective resolution of the A^{op}-module N, or $P^\bullet \to M \to 0$ is a projective resolution of the A-module M, then :

$$\begin{aligned} \mathrm{Tor}^A_k(N, M) &\simeq H^{-k}(Q^\bullet \otimes_A M) \\ &\simeq H^{-k}(N \otimes_A P^\bullet) \\ &\simeq H^{-k}(\mathrm{tot}(Q^\bullet \otimes_A P^\bullet)). \end{aligned}$$

Exercises to Chapter 5

Exercise 5.1. Let $\mathcal{C}$ be an abelian category which admits inductive limits and such that filtrant inductive limits are exact. Let $\{X_i\}_{i\in I}$ be a family of objects of $\mathcal{C}$ indexed by a set I and let $i_0 \in I$. Prove that the natural morphism $X_{i_0} \to \bigoplus_{i\in I} X_i$ is a mornomorphism.

Exercise 5.2. Let $\mathcal{C}$ be an abelian category.
(i) Prove that a complex $0 \to X \to Y \to Z$ is exact iff and only if for any object $W \in \mathcal{C}$ the complex of abelian groups $0 \to \mathrm{Hom}_{\mathcal{C}}(W,X) \to \mathrm{Hom}_{\mathcal{C}}(W,Y) \to \mathrm{Hom}_{\mathcal{C}}(W,Z)$ is exact.
(ii) By reversing the arrows, state and prove a similar statement for a complex $X \to Y \to Z \to 0$.

Exercise 5.3. Let $\mathcal{C}$ be an abelian category. A square is a commutative diagram:

$$\begin{tikzcd} V \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z. \end{tikzcd}$$

A square is Cartesian if moreover the sequence $0 \to V \to X \times Y \to Z$ is exact, that is, if $V \simeq X \times_Z Y$ (recall that $X \times_Z Y = \mathrm{Ker}(f-g)$, where $f-g : X \oplus Y \to Z$). A square is co-Cartesian if the sequence $V \to X \oplus Y \to Z \to 0$ is exact, that is, if $Z \simeq X \oplus_V Y$ (recall that $X \oplus_Z Y = \mathrm{Coker}(f'-g')$, where $f'-g' : V \to X \times Y$).
(i) Assume the square is Cartesian and f is an epimorphism. Prove that f' is an epimorphism.
(ii) Assume the square is co-Cartesian and f' is a monomorphism. Prove that f is a monomorphism.

Exercise 5.4. Let $\mathcal{C}$ be an abelian category and consider two sequences of morphisms $X'_i \xrightarrow{f_i} X_i \xrightarrow{g_i} X''_i$, $i = 1,2$ with $g_i \circ f_i = 0$. Set $X' = X'_1 \oplus X'_2$, and define similarly X, X'' and f, g. Prove that the two sequences above are exact if and only if the sequence $X' \xrightarrow{f} X \xrightarrow{g} X''$ is exact.

Exercise 5.5. Let $\mathcal{C}$ be an abelian category and consider a commutative diagram of complexes

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X'_0 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & X''_0 \arrow[d] \\ 0 \arrow[r] & X'_1 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X''_1 \arrow[d] \\ 0 \arrow[r] & X'_2 \arrow[r] & X_2 \arrow[r] & X''_2 \end{tikzcd}$$

Assume that all rows are exact as well as the second and third column. Prove that all columns are exact.

Exercise 5.6. Let $\mathcal{C}$ be an abelian category and let $X^{\bullet,\bullet}$ be a double complex with $X^{i,j} = 0$ for $i < -1$ or $j < -1$. Assume all rows and all columns of $X^{\bullet,\bullet}$ are exact, and denote by $Y^{\bullet,\bullet}$ the double complex obtained by replacing $X^{-1,j}$ and $X^{i,-1}$ by 0 for all j and all i. Prove that there is a qis $X^{-1,-1} \to \mathrm{tot}(Y^{\bullet,\bullet})$.

Exercise 5.7. Let $\mathcal{C}$ be an abelian category. To $X \in C^b(\mathcal{C})$, one associates the new complex $H^\bullet(X) = \bigoplus H^j(X)[-j]$ with 0-differential. In other words

$$H^\bullet(X) := \qquad \cdots \to H^i(X) \xrightarrow{0} H^{i+1}(X) \xrightarrow{0} \cdots$$

(i) Prove that $H^\bullet : C^b(\mathcal{C}) \to C^b(\mathcal{C})$ is a well-defined additive functor.
(ii) Give examples which show that in general, $H^\bullet$ is neither right nor left exact.

Exercise 5.8. Let $\varphi = (\varphi_1, \dots, \varphi_n)$ be n commuting endomorphisms of an A-module M. Let $\varphi' = (\varphi_1, \dots, \varphi_{n-p})$ and $\varphi'' = (\varphi_{n-p+1}, \dots, \varphi_n)$.
Calculate the cohomology of $K^\bullet(M, \varphi)$ assuming that φ' is a regular sequence and φ'' is a coregular sequence.

Exercise 5.9. Let $A = k[x_1, x_2]$. Consider the A-modules: $M' = A/(Ax_1 + Ax_2)$, $M = A/(Ax_1^2 + Ax_1x_2)$, $M'' = A/(Ax_1)$.
(i) Show that the monomorphism $Ax_1 \hookrightarrow A$ induces a monomorphism $M' \hookrightarrow M$and deduce an exact sequence of A-modules $0 \to M' \to M \to M'' \to 0$.
(ii) By considering the action of x_1 on these three modules, show that the sequence above does not split.
(iii) Construct free resolutions of M' and M''.
(iv) Calculate $\mathrm{Ext}^j_A(M, A)$ for all j .

Exercise 5.10. Let $\mathcal{C}$ and $\mathcal{C}'$ be two abelian categories. We assume that $\mathcal{C}'$ admits inductive limits and filtrant inductive limits are exact in $\mathcal{C}'$. Let $\{F_i\}_{i \in I}$ be an inductive system of left exact functors from $\mathcal{C}$ to $\mathcal{C}'$, indexed by a filtrant category$I$.
(i) Prove that $\varinjlim_i F_i$ is a left exact functor.
(ii) Prove that for each $k \in \mathbb{Z}$, $\{R^k F_i\}_{i \in I}$ is an inductive system of functors and $R^k(\varinjlim_i F_i) \simeq \varinjlim_i R^k F_i$.

Exercise 5.11. Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor of abelian categories. Let $\mathcal{J}$ be an F-injective subcategory of $\mathcal{C}$, and let $Y^\bullet$ be an object of $C^+(\mathcal{J})$. Assume that $H^k(Y^\bullet) = 0$ for all $k \neq p$ for some $p \in \mathbb{Z}$, and let $X = H^p(Y^\bullet)$. Prove that $R^k F(X) \simeq H^{k+p}(F(Y^\bullet))$.

Exercise 5.12. We consider the following situation: $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}''$ are left exact functors of abelian categories having enough injectives, $\mathcal{J}'$ is an G-injective subcategory of $\mathcal{C}'$ and F sends injective objects of $\mathcal{C}$ in $\mathcal{J}'$.
(i) Let $X \in \mathcal{C}$ and assume that there is $q \in \mathbb{N}$ with $R^kF(X) = 0$ for $k \neq q$. Prove that $R^j(G \circ F)(X) \simeq R^{j-q}G(R^qF(X))$. (Hint: use Exercise 5.11.)
(ii) Assume now that $R^jF(X) = 0$ for $j \neq 0, 1$. Prove that there is a long exact sequence:

$$\cdots \to R^{k-1}G(R^1F(X)) \to R^k(G \circ F)(X) \to R^kG(F(X)) \to \cdots$$

(Hint: construct an exact sequence $0 \to X \to X^0 \to X^1 \to 0$ with X^0 injective and X^1 F-acyclic.)

Exercise 5.13. In the situation of Proposition 5.6.10, let $X \in \mathcal{C}$ and assume that $R^jF(X) \simeq 0$ for $j < n$. Prove that $R^n(F' \circ F)(X) \simeq F'(R^nF(X))$.

Exercise 5.14. Let $\mathcal{C}, \mathcal{C}'$ and $\mathcal{C}''$ be abelian categories, $G : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$ an exact bifunctor. Let $0 \to X \to I^\bullet$ and $0 \to Y \to J^\bullet$ be resolutions of $X \in \mathcal{C}$ and $Y \in \mathcal{C}'$ respectively. Prove that $0 \to G(X, Y) \to \mathrm{tot}(G(I^\bullet, J^\bullet))$ is a resolution of $G(X, Y)$. (Hint: use Exercise 5.6.)

Exercise 5.15. Here, we shall use the notation $H^\bullet$ introduced in Exercise 5.7. Assume that k is a field and consider the complexes in $\mathrm{Mod}(k)$:

$$\begin{aligned} X^\bullet :=& \quad X^0 \xrightarrow{f} X^1, \\ Y^\bullet :=& \quad Y^0 \xrightarrow{g} Y^1 \end{aligned}$$

and the double complex

$$X^\bullet \otimes Y^\bullet := \quad \begin{tikzcd} X^0 \otimes Y^0 \arrow[r, "f\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes g"'] & X^1 \otimes Y^0 \arrow[d, "\mathrm{id}\otimes g"'] \\ X^0 \otimes Y^1 \arrow[r, "f\otimes\mathrm{id}"] & X^1 \otimes Y^1. \end{tikzcd}$$

(i) Prove that $\mathrm{tot}(X^\bullet \otimes Y^\bullet)$ and $\mathrm{tot}(H^\bullet(X^\bullet) \otimes Y^\bullet)$ have the same cohomology objects.
(ii) Deduce that $\mathrm{tot}(X^\bullet \otimes Y^\bullet)$ and $\mathrm{tot}(H^\bullet(X^\bullet) \otimes H^\bullet(Y^\bullet))$ have the same cohomology objects.

Exercise 5.16. Assume that k is a field. Let $X^\bullet$ and $Y^\bullet$ be two objects of $C^b(\mathrm{Mod}(k))$. Prove the isomorphism

$$\begin{aligned} H^p(\mathrm{tot}(X^\bullet \otimes Y^\bullet)) &\simeq \bigoplus_{i+j=p} H^i(X^\bullet) \otimes H^j(Y^\bullet) \\ &\simeq H^p(\bigoplus_i H^i(X^\bullet)[-i] \otimes \bigoplus_j H^j(Y^\bullet)[-j]). \end{aligned}$$

Here, we use the convention that:

$$\begin{aligned} &(A \oplus B) \otimes (C \oplus D) \simeq (A \otimes C) \oplus (A \otimes D) \oplus (B \otimes C) \oplus (B \otimes D) \\ &A[i] \otimes B[j] \sim A \otimes B[i+j]. \end{aligned}$$

(Hint: use the result of Exercise 5.15.)

Chapter 6

Localization

Consider a category $\mathcal{C}$ and a family $\mathcal{S}$ of morphisms in $\mathcal{C}$. The aim of localization is to find a new category $\mathcal{C}_\mathcal{S}$ and a functor $Q : \mathcal{C} \to \mathcal{C}_\mathcal{S}$ which sends the morphisms belonging to $\mathcal{S}$ to isomorphisms in $\mathcal{C}_\mathcal{S}$, $(Q, \mathcal{C}_\mathcal{S})$ being "universal" for such a property.

In this chapter, we shall construct the localization of a category when $\mathcal{S}$ satisfies suitable conditions and the localization of functors.

Localization of categories appears in particular in the construction of derived categories.

A classical reference is [5].

6.1 Localization of categories

Let $\mathcal{C}$ be a category and let $\mathcal{S}$ be a family of morphisms in $\mathcal{C}$.

Definition 6.1.1. A localizaton of $\mathcal{C}$ by $\mathcal{S}$ is the data of a category $\mathcal{C}_\mathcal{S}$ and a functor $Q : \mathcal{C} \to \mathcal{C}_\mathcal{S}$ satisfying:

(a) for all $s \in \mathcal{S}$, $Q(s)$ is an isomorphism,

(b) for any functor $F : \mathcal{C} \to \mathcal{A}$ such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$, there exists a functor $F_\mathcal{S} : \mathcal{C}_\mathcal{S} \to \mathcal{A}$ and an isomorphism $F \simeq F_\mathcal{S} \circ Q$,

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{A} \\ \mathcal{C}_\mathcal{S} \arrow[ur, dotted, "F_\mathcal{S}"'] & \end{tikzcd}$$

(c) if G_1 and G_2 are two objects of $\mathrm{Fct}(\mathcal{C}_\mathcal{S}, \mathcal{A})$, then the natural map

$$(6.1)\quad \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C}_\mathcal{S},\mathcal{A})}(G_1, G_2) \to \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C},\mathcal{A})}(G_1 \circ Q, G_2 \circ Q)$$

is bijective.

Note that (c) means that the functor $\circ Q : \mathrm{Fct}(\mathcal{C}_{\mathcal{S}}, \mathcal{A}) \to \mathrm{Fct}(\mathcal{C}, \mathcal{A})$ is fully faithful. This implies that $F_{\mathcal{S}}$ in (b) is unique up to unique isomorphism.

Proposition 6.1.2. (i) *If $\mathcal{C}_{\mathcal{S}}$ exists, it is unique up to equivalence of categories.*

(ii) *If $\mathcal{C}_{\mathcal{S}}$ exists, then, denoting by $\mathcal{S}^{\mathrm{op}}$ the image of $\mathcal{S}$ in $\mathcal{C}^{\mathrm{op}}$ by the functor* op, *$(\mathcal{C}^{\mathrm{op}})_{\mathcal{S}^{\mathrm{op}}}$ exists and there is an equivalence of categories:*

$$(\mathcal{C}_{\mathcal{S}})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})_{\mathcal{S}^{\mathrm{op}}}.$$

Proof. (i) is obvious.
(ii) Assume $\mathcal{C}_{\mathcal{S}}$ exists. Set $(\mathcal{C}^{\mathrm{op}})_{\mathcal{S}^{\mathrm{op}}} := (\mathcal{C}_{\mathcal{S}})^{\mathrm{op}}$ and define $Q^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to (\mathcal{C}^{\mathrm{op}})_{\mathcal{S}^{\mathrm{op}}}$ by $Q^{\mathrm{op}} = \mathrm{op} \circ Q \circ \mathrm{op}$. Then properties (a), (b) and (c) of Definition 6.1.1 are clearly satisfied. q.e.d.

Definition 6.1.3. One says that $\mathcal{S}$ is a right multiplicative system if it satisfies the axioms S1-S4 below.

S1 For all $X \in \mathcal{C}$, $\mathrm{id}_X \in \mathcal{S}$.

S2 For all $f \in \mathcal{S}, g \in \mathcal{S}$, if $g \circ f$ exists then $g \circ f \in \mathcal{S}$.

S3 Given two morphisms, $f : X \to Y$ and $s : X \to X'$ with $s \in \mathcal{S}$, there exist $t : Y \to Y'$ and $g : X' \to Y'$ with $t \in \mathcal{S}$ and $g \circ s = t \circ f$. This can be visualized by the diagram:

$$\begin{tikzcd} X' & & & X' \arrow[r, dotted, "g"'] & Y' \\ X \arrow[u, "s"] \arrow[r, "f"'] & Y & \Rightarrow & X \arrow[u, "s"] \arrow[r, "f"'] & Y \arrow[u, dotted, "t"'] \end{tikzcd}$$

S4 Let $f, g : X \to Y$ be two parallel morphisms. If there exists $s \in \mathcal{S} : W \to X$ such that $f \circ s = g \circ s$ then there exists $t \in \mathcal{S} : Y \to Z$ such that $t \circ f = t \circ g$. This can be visualized by the diagram:

$$\begin{tikzcd} W \arrow[r, "s"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, dotted, "t"] & Z \end{tikzcd}$$

Notice that these axioms are quite natural if one wants to invert the elements of $\mathcal{S}$. In other words, if the element of $\mathcal{S}$ would be invertible, then these axioms would clearly be satisfied.

Remark 6.1.4. Axioms S1-S2 asserts that $\mathcal{S}$ is the family of morphisms of a subcategory $\widetilde{\mathcal{S}}$ of $\mathcal{C}$ with $\mathrm{Ob}(\widetilde{\mathcal{S}}) = \mathrm{Ob}(\mathcal{C})$.

Remark 6.1.5. One defines the notion of a left multiplicative system $\mathcal{S}$ by reversing the arrows. This means that the condition S3 is replaced by: given two morphisms, $f : X \to Y$ and $t : Y' \to Y$, with $t \in \mathcal{S}$, there exist $s : X' \to X$ and $g : X' \to Y'$ with $s \in \mathcal{S}$ and $t \circ g = f \circ s$. This can be visualized by the diagram:

$$\begin{tikzcd} & Y' \arrow[d, "t"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} X' \arrow[r, dotted, "g"] \arrow[d, dotted, "s"'] & Y' \arrow[d, "t"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and S4 is replaced by: if there exists $t \in \mathcal{S} : Y \to Z$ such that $t \circ f = t \circ g$ then there exists $s \in \mathcal{S} : W \to X$ such that $f \circ s = g \circ s$. This is visualized by the diagram

$$\begin{tikzcd} W \arrow[r, dotted, "s"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "t"] & Z \end{tikzcd}$$

In the literature, one often calls a multiplicative system a system which is both right and left multiplicative.

Many multiplicative systems that we shall encounter satisfy a useful property that we introduce now.

Definition 6.1.6. Let $\mathcal{S}$ be a right multiplicative system. One says that $\mathcal{S}$ is saturated if it satisfies

S5 for any morphisms $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$ such that $g \circ f$ and $h \circ g$ belong to $\mathcal{S}$, the morphism f belongs to $\mathcal{S}$.

Definition 6.1.7. Assume that $\mathcal{S}$ satisfies the axioms S1-S2 and let $X \in \mathcal{C}$. One defines the categories $\mathcal{S}_X$ and $\mathcal{S}^X$ as follows.

$$\begin{aligned} \mathrm{Ob}(\mathcal{S}^X) &= \{s : X \to X'; s \in \mathcal{S}\} \\ \mathrm{Hom}_{\mathcal{S}^X}((s : X \to X'), (s : X \to X'')) &= \{h : X' \to X''; h \circ s = s'\} \\ \mathrm{Ob}(\mathcal{S}_X) &= \{s : X' \to X; s \in \mathcal{S}\} \\ \mathrm{Hom}_{\mathcal{S}_X}((s : X' \to X), (s' : X'' \to X)) &= \{h : X'' \to X'; s' \circ h = s\}. \end{aligned}$$

Proposition 6.1.8. *Assume that $\mathcal{S}$ is a right (resp. left) multiplicative system. Then the category $\mathcal{S}^X$ (resp. $\mathcal{S}_X^{\mathrm{op}}$) is filtrant.*

Proof. By reversing the arrows, both results are equivalent. We treat the case of $\mathcal{S}^X$.
(a) Let $s : X \to X'$ and $s' : X \to X''$ belong to $\mathcal{S}$. By S3, there exists $t : X' \to X'''$ and $t' : X'' \to X'''$ such that $t' \circ s' = t \circ s$, and $t \in \mathcal{S}$. Hence, $t \circ s \in \mathcal{S}$ by S2 and $(X \to X''')$ belongs to $\mathcal{S}^X$.
(b) Let $s : X \to X'$ and $s' : X \to X''$ belong to $\mathcal{S}$, and consider two morphisms $f, g : X' \to X''$, with $f \circ s = g \circ s = s'$. By S4 there exists $t : X'' \to W, t \in \mathcal{S}$ such that $t \circ f = t \circ g$. Hence $t \circ s' : X \to W$ belongs to $\mathcal{S}^X$. q.e.d.

One defines the functors:

$$\begin{aligned} &\alpha_X : \mathcal{S}^X \to \mathcal{C} \qquad (s : X \to X') \mapsto X', \\ &\beta_X : \mathcal{S}_X^{\mathrm{op}} \to \mathcal{C} \qquad (s : X' \to X) \mapsto X'. \end{aligned}$$

We shall concentrate on right multiplicative system.

Definition 6.1.9. Let $\mathcal{S}$ be a right multiplicative system, and let $X, Y \in \mathrm{Ob}(\mathcal{C})$. We set

$$\mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X, Y) = \varinjlim_{(Y \longrightarrow Y') \in \mathcal{S}^Y} \mathrm{Hom}_{\mathcal{C}}(X, Y').$$

Lemma 6.1.10. *Assume that $\mathcal{S}$ is a right multiplicative system. Let $Y \in \mathcal{C}$ and let $s : X \to X' \in \mathcal{S}$. Then s induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X', Y) \xrightarrow[\circ s]{\sim} \mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X, Y).$$

Proof. (i) The map $\circ s$ is surjective. This follows from S3, as visualized by the diagram in which $s, t, t' \in \mathcal{S}$:

$$\begin{tikzcd} X' \arrow[r, dotted] & Y'' & \\ X \arrow[u, "s"] \arrow[r, "f"'] & Y' \arrow[u, dotted, "t'"] & Y \arrow[l, "t"'] \end{tikzcd}$$

(ii) The map $\circ s$ is injective. This follows from S4, as visualized by the diagram in which $s, t, t' \in \mathcal{S}$:

$$\begin{tikzcd} X \arrow[r, "s"] & X' \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y' \arrow[r, dotted, "t'"] & Y'' \\ & & Y \arrow[u, "t"] & \end{tikzcd}$$

q.e.d.

Using Lemma 6.1.10, we define the composition

$$\mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X,Z) \tag{6.2}$$

as

$$\begin{aligned}
&\varinjlim_{Y\to Y'} \mathrm{Hom}_{\mathcal{C}}(X,Y') \times \varinjlim_{Z\to Z'} \mathrm{Hom}_{\mathcal{C}}(Y,Z') \\
&\qquad\qquad \simeq \varinjlim_{Y\to Y'} \big(\mathrm{Hom}_{\mathcal{C}}(X,Y') \times \varinjlim_{Z\to Z'} \mathrm{Hom}_{\mathcal{C}}(Y,Z')\big) \\
&\qquad\qquad \xleftarrow{\sim} \varinjlim_{Y\to Y'} \big(\mathrm{Hom}_{\mathcal{C}}(X,Y') \times \varinjlim_{Z\to Z'} \mathrm{Hom}_{\mathcal{C}}(Y',Z')\big) \\
&\qquad\qquad \to \varinjlim_{Y\to Y'} \varinjlim_{Z\to Z'} \mathrm{Hom}_{\mathcal{C}}(X,Z') \\
&\qquad\qquad \simeq \varinjlim_{Z\to Z'} \mathrm{Hom}_{\mathcal{C}}(X,Z')
\end{aligned}$$

Lemma 6.1.11. *The composition* (6.2) *is associative.*

The verification is left to the reader.

Hence we get a category $\mathcal{C}^r_{\mathcal{S}}$ whose objects are those of $\mathcal{C}$ and morphisms are given by Definition 6.1.9.

Let us denote by $Q_{\mathcal{S}} : \mathcal{C} \to \mathcal{C}^r_{\mathcal{S}}$ the natural functor associated with

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \varinjlim_{(Y\to Y')\in\mathcal{S}^Y} \mathrm{Hom}_{\mathcal{C}}(X,Y').$$

If there is no risk of confusion, we denote this functor simply by Q.

Lemma 6.1.12. *If $s : X \to Y$ belongs to $\mathcal{S}$, then $Q(s)$ is invertible.*

Proof. For any $Z \in \mathcal{C}^r_{\mathcal{S}}$, the map $\mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X,Z)$ is bijective by Lemma 6.1.10. q.e.d.

A morphism $f : X \to Y$ in $\mathcal{C}^r_{\mathcal{S}}$ is thus given by an equivalence class of triplets (Y', t, f') with $t : Y \to Y', t \in \mathcal{S}$ and $f' : X \to Y'$, that is:

$$X \xrightarrow[f']{} Y' \xleftarrow[t]{} Y$$

the equivalence relation being defined as follows: $(Y', t, f') \sim (Y'', t', f'')$ if there exists (Y''', t'', f''') $(t, t', t'' \in S)$ and a commutative diagram:

$$\begin{tikzcd}
& Y' \arrow[d, dotted] & \\
X \arrow[ur, "f'"] \arrow[r, dotted, "f'''"] \arrow[dr, "f''"'] & Y''' & Y \arrow[ul, "t"'] \arrow[l, dotted, "t''"'] \arrow[dl, "t'"] \\
& Y'' \arrow[u, dotted] &
\end{tikzcd} \tag{6.3}$$

Note that the morphism (Y', t, f') in $\mathcal{C}^r_{\mathcal{S}}$ is $Q(t)^{-1} \circ Q(f')$, that is,

$$f = Q(t)^{-1} \circ Q(f'). \tag{6.4}$$

For two parallel arrows $f, g : X \rightrightarrows Y$ in $\mathcal{C}$ we have the equivalence

$$Q(f) = Q(g) \in \mathcal{C}^r_{\mathcal{S}} \iff \text{there exits } s : Y \to Y', s \in \mathcal{S} \text{ with } s \circ f = s \circ g. \tag{6.5}$$

The composition of two morphisms $(Y', t, f') : X \to Y$ and $(Z', s, g') : Y \to Z$ is defined by the diagram below in which $t, s, s' \in \mathcal{S}$:

$$\begin{tikzcd}
 & & & W & & & \\
X \arrow[r, "f'"'] & Y' \arrow[urr, dotted, "h"] & & Y \arrow[ll, "t"] \arrow[r, "g'"'] & Z' \arrow[ul, dotted, "s'"'] & & Z \arrow[ll, "s"]
\end{tikzcd}$$

Theorem 6.1.13. *Assume that $\mathcal{S}$ is a right multiplicative system.*

(i) *The category $\mathcal{C}^r_{\mathcal{S}}$ and the functor Q define a localization of $\mathcal{C}$ by $\mathcal{S}$.*

(ii) *For a morphism $f : X \to Y$, $Q(f)$ is an isomorphism in $\mathcal{C}^r_{\mathcal{S}}$ if and only if there exist $g : Y \to Z$ and $h : Z \to W$ such that $g \circ f \in \mathcal{S}$ and $h \circ g \in \mathcal{S}$.*

Corollary 6.1.14. *If $\mathcal{S}$ is saturated, a morphism f in $\mathcal{C}$ belongs to $\mathcal{S}$ if and only if $Q(f)$ is an isomorphism.*

Notation 6.1.15. From now on, we shall write $\mathcal{C}_{\mathcal{S}}$ instead of $\mathcal{C}^r_{\mathcal{S}}$. This is justified by Theorem 6.1.13.

Remark 6.1.16. (i) In the above construction, we have used the property of $\mathcal{S}$ of being a right multiplicative system. If $\mathcal{S}$ is a left multiplicative system, one sets

$$\mathrm{Hom}_{\mathcal{C}^l_{\mathcal{S}}}(X, Y) = \varinjlim_{(X' \to X) \in \mathcal{S}_X} \mathrm{Hom}_{\mathcal{C}}(X', Y).$$

By Proposition 6.1.2 (i), the two constructions give equivalent categories.
(ii) If $\mathcal{S}$ is both a right and left multiplicative system,

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{S}}}(X, Y) \simeq \varinjlim_{(X' \to X) \in \mathcal{S}_X, (Y \to Y') \in \mathcal{S}^Y} \mathrm{Hom}_{\mathcal{C}}(X', Y').$$

6.2 Localization of subcategories

Proposition 6.2.1. *Let $\mathcal{C}$ be a category, $\mathcal{I}$ a full subcategory, $\mathcal{S}$ a right multiplicative system in $\mathcal{C}$, $\mathcal{T}$ the family of morphisms in $\mathcal{I}$ which belong to $\mathcal{S}$.*

(i) *Assume that $\mathcal{T}$ is a right multiplicative system in $\mathcal{I}$. Then $\mathcal{I}_{\mathcal{T}} \to \mathcal{C}_{\mathcal{S}}$ is well-defined.*

(ii) *Assume that for every $f : Y \to X$, $f \in \mathcal{S}$, $Y \in \mathcal{I}$, there exists $g : X \to W$, $W \in \mathcal{I}$, with $g \circ f \in \mathcal{S}$. Then $\mathcal{T}$ is a right multilplicative system and $\mathcal{I}_{\mathcal{T}} \to \mathcal{C}_{\mathcal{S}}$ is fully faithful.*

Proof. (i) is obvious.
(ii) It is left to the reader to check that $\mathcal{T}$ is a right multpiplicative system. For $X \in \mathcal{I}$, $\mathcal{T}^X$ is the full subcategory of $\mathcal{S}^X$ whose objects are the morphisms $s : X \to Y$ with $Y \in \mathcal{I}$. By Proposition 6.1.8 and the hypothesis, the functor $\mathcal{T}^X \to \mathcal{S}^X$ is cofinal, and the result follows from Definition 6.1.9. q.e.d.

Corollary 6.2.2. *Let $\mathcal{C}$ be a category, $\mathcal{I}$ a full subcategory, $\mathcal{S}$ a right multiplicative system in $\mathcal{C}$, $\mathcal{T}$ the family of morphisms in $\mathcal{I}$ which belong to $\mathcal{S}$. Assume that for any $X \in \mathcal{C}$ there exists $s : X \to W$ with $W \in \mathcal{I}$ and $s \in \mathcal{S}$.*

Then $\mathcal{T}$ is a right multpiplicative system and $\mathcal{I}_{\mathcal{T}}$ is equivalent to $\mathcal{C}_{\mathcal{S}}$.

Proof. The natural functor $\mathcal{I}_{\mathcal{T}} \to \mathcal{C}_{\mathcal{S}}$ is fully faithful by Proposition 6.2.1 and is essentially surjective by the assumption. q.e.d.

6.3 Localization of functors

Let $\mathcal{C}$ be a category, $\mathcal{S}$ a right multiplicative system in $\mathcal{C}$ and $F : \mathcal{C} \to \mathcal{A}$ a functor. In general, F does not send morphisms in $\mathcal{S}$ to isomorphisms in $\mathcal{A}$. In other words, F does not factorize through $\mathcal{C}_{\mathcal{S}}$. It is however possible in some cases to define a localization of F as follows.

Definition 6.3.1. A right localization of F (if it exists) is a functor $F_{\mathcal{S}} : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ and a morphism of functors $\tau : F \to F_{\mathcal{S}} \circ Q$ such that for any functor $G : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ the map

$$\mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C}_{\mathcal{S}},\mathcal{A})}(F_{\mathcal{S}}, G) \to \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C},\mathcal{A})}(F, G \circ Q) \tag{6.6}$$

is bijective. (This map is obtained as the composition $\mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C}_{\mathcal{S}},\mathcal{A})}(F_{\mathcal{S}}, G) \to \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C},\mathcal{A})}(F_{\mathcal{S}} \circ Q, G \circ Q) \xrightarrow{\tau} \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C},\mathcal{A})}(F, G \circ Q)$.)

We shall say that F is right localizable if it admits a right localization.

One defines similarly the left localization. Since we mainly consider right localization, we shall sometimes omit the word "right" as far as there is no risk of confusion.

If $(\tau, F_{\mathcal{S}})$ exists, it is unique up to unique isomorphisms. Indeed, $F_{\mathcal{S}}$ is a representative of the functor

$$G \mapsto \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C},\mathcal{A})}(F, G \circ Q).$$

(This last functor is defined on the category $\mathrm{Fct}(\mathcal{C}_{\mathcal{S}}, \mathcal{A})$ with values in **Set**.)

Proposition 6.3.2. *Let $\mathcal{C}$ be a category, $\mathcal{I}$ a full subcategory, $\mathcal{S}$ a right multiplicative system in $\mathcal{C}$, $\mathcal{T}$ the family of morphisms in $\mathcal{I}$ which belong to $\mathcal{S}$. Let $F : \mathcal{C} \to \mathcal{A}$ be a functor. Assume that*

(i) *for any $X \in \mathcal{C}$ there exists $s : X \to W$ with $W \in \mathcal{I}$ and $s \in \mathcal{S}$,*

(ii) *for any $t \in \mathcal{T}$, $F(t)$ is an isomorphism.*

Then F is right localizable.

Proof. We shall apply Corollary (6.2.2).

Denote by $\iota : \mathcal{I} \to \mathcal{C}$ the natural functor. By the hypothesis, the localization $F_{\mathcal{T}}$ of $F \circ \iota$ exists. Consider the diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "Q_{\mathcal{S}}"] & & \mathcal{C}_{\mathcal{S}} \arrow[dd, dotted, "F_{\mathcal{S}}"] \\
\mathcal{I} \arrow[u, "\iota"] \arrow[r, "Q_{\mathcal{T}}"] \arrow[drr, "F\circ\iota"'] & \mathcal{I}_{\mathcal{T}} \arrow[ur, "\sim", "\iota_Q"'] \arrow[dr, "F_{\mathcal{T}}"] & \\
& & \mathcal{A}
\end{tikzcd}$$

Denote by ι_Q^{-1} a quasi-inverse of ι_Q and set $F_{\mathcal{S}} := F_{\mathcal{T}} \circ \iota_Q^{-1}$. Let us show that $F_{\mathcal{S}}$ is the localization of F. Let $G : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ be a functor. We have the chain of morphisms:

$$\begin{aligned}
\mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C},\mathcal{A})}(F, G \circ Q_{\mathcal{S}}) &\xrightarrow{\lambda} \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{I},\mathcal{A})}(F \circ \iota, G \circ Q_{\mathcal{S}} \circ \iota) \\
&\simeq \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{I},\mathcal{A})}(F_{\mathcal{T}} \circ Q_{\mathcal{T}}, G \circ \iota_Q \circ Q_{\mathcal{T}}) \\
&\simeq \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{I}_{\mathcal{T}},\mathcal{A})}(F_{\mathcal{T}}, G \circ \iota_Q) \\
&\simeq \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C}_{\mathcal{S}},\mathcal{A})}(F_{\mathcal{T}} \circ \iota_Q^{-1}, G) \\
&\simeq \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C}_{\mathcal{S}},\mathcal{A})}(F_{\mathcal{S}}, G).
\end{aligned}$$

The first isomomorphism above follows from the fact that $Q_{\mathcal{T}}$ satisfies the hypothesis (c) of Definition 6.1.1 and the other isomorphisms are obvious. It remains to check that λ is an isomorphism. This is left to the reader. q.e.d.

Remark 6.3.3. Let $\mathcal{C}$ (resp. $\mathcal{C}'$) be a category and $\mathcal{S}$ (resp. $\mathcal{S}'$) a right multiplicative system in $\mathcal{C}$ (resp. $\mathcal{C}'$). One checks immediately that $\mathcal{S} \times \mathcal{S}'$ is a right multiplicative system in the category $\mathcal{C} \times \mathcal{C}'$ and $(\mathcal{C} \times \mathcal{C}')_{\mathcal{S}\times\mathcal{S}'}$ is equivalent to $\mathcal{C}_{\mathcal{S}} \times \mathcal{C}'_{\mathcal{S}'}$. Since a bifunctor is a functor on the product $\mathcal{C} \times \mathcal{C}'$, we may apply the preceding results to the case of bifunctors. In the sequel, we shall write $F_{\mathcal{S}\mathcal{S}'}$ instead of $F_{\mathcal{S}\times\mathcal{S}'}$.

Exercises to Chapter 6

Exercise 6.1. Let $\mathcal{C}$ be a category, $\mathcal{S}$ a right multiplicative system. Let $\mathcal{T}$ be the set of morphisms $f : X \to Y$ in $\mathcal{C}$ such that there exist $g : Y \to Z$ and $h : Z \to W$, with $h \circ g$ and $g \circ f$ in $\mathcal{S}$.

Prove that $\mathcal{T}$ is a right saturated multiplicative system and that the natural functor $\mathcal{C}_{\mathcal{S}} \to \mathcal{C}_{\mathcal{T}}$ is an equivalence.

Exercise 6.2. Let $\mathcal{C}$ be a category, $\mathcal{S}$ a right and left multiplicative system. Prove that $\mathcal{S}$ is saturated if and only if for any $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$, $h \circ g \in \mathcal{S}$ and $g \circ f \in \mathcal{S}$ imply $g \in \mathcal{S}$.

Exercise 6.3. Let $\mathcal{C}$ be a category with a zero object 0, $\mathcal{S}$ a right and left saturated multiplicative system.
(i) Show that $\mathcal{C}_{\mathcal{S}}$ has a zero object (still denoted by 0).
(ii) Prove that $Q(X) \simeq 0$ if and only if the zero morphism $0 : X \to X$ belongs to $\mathcal{S}$.

Exercise 6.4. Let $\mathcal{C}$ be a category, $\mathcal{S}$ a right multiplicative system. Consider morphisms $f : X \to Y$ and $f' : X' \to Y'$ in $\mathcal{C}$ and morphism $\alpha : X \to X'$ and $\beta : Y \to Y'$ in $\mathcal{C}_{\mathcal{S}}$, and assume that $f' \circ \alpha = \beta \circ f$ (in $\mathcal{C}_{\mathcal{S}}$). Prove that there exists a commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "\alpha'"] \arrow[d, "f"'] & X_1 \arrow[d] & X' \arrow[l, "s"'] \arrow[d, "f'"] \\ Y \arrow[r, "\beta'"] & Y_1 & Y' \arrow[l, "t"'] \end{tikzcd}$$

with s and t in $\mathcal{S}$, $\alpha = Q(s)^{-1} \circ Q(\alpha')$ and $\beta = Q(t)^{-1} \circ Q(\beta')$.

Exercise 6.5. Let $F : \mathcal{C} \to \mathcal{A}$ be a functor and assume that $\mathcal{C}$ admits fimite inductive limits and F is right exact. Let $\mathcal{S}$ denote the set of morphisms s in $\mathcal{C}$ such that $F(s)$ is an isomorphism.
(i) Prove that $\mathcal{S}$ is a right saturated multiplicative system.
(ii) Prove that the localized functor $F_{\mathcal{S}} : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ is faithful.

Exercise 6.6. Let A be a commutative ring, $S \subset A$ a multiplicative subset (i.e. $1 \in S$ and $s, t \in S$ implies $s \cdot t \in S$). Let $S^{-1}A$ denote the localization of the ring A and if M is an A-module, denote by $S^{-1}M$ its localization, $S^{-1}M = S^{-1}A \otimes M$. Note that the functor $M \mapsto S^{-1}M$ is exact. Let $\mathcal{S}$ denote the family of morphisms in $\mathrm{Mod}(A)$ defined by: $f : M \to N \in \mathcal{S}$ if and only if f induces an isomorphism $S^{-1}M \to S^{-1}N$.
(i) Prove that $\mathcal{S}$ is a right and left multiplicative system.
(ii) Construct the natural functor $(\mathrm{Mod}(A))_{\mathcal{S}} \to \mathrm{Mod}(S^{-1}A)$.
(iii) Prove that this functor is an equivalence.

Chapter 7

Triangulated categories

Triangulated categories play an increasing role in mathematics and this subject might deserve a whole book. However, we have restricted ourselves to describe their main properties with the construction of derived categories in mind.
Some references: [6], [11], [12], [15], [16], [17].

7.1 Triangulated categories

Let $\mathcal{D}$ be an additive category endowed with an automorphism T (i.e., an invertible functor $T : \mathcal{D} \to \mathcal{D}$).

Definition 7.1.1. Let $\mathcal{D}$ be an additive category endowed with an automorphism T. A triangle in $\mathcal{D}$ is a sequence of morphisms:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X). \tag{7.1}$$

A morphism of triangles is a commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & T(X) \arrow[d, "T(\alpha)"'] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X').
\end{tikzcd}$$

Example 7.1.2. The triangle $X \xrightarrow{f} Y \xrightarrow{-g} Z \xrightarrow{-h} T(X)$ is isomorphic to the triangle (7.1), but the triangle $X \xrightarrow{-f} Y \xrightarrow{-g} Z \xrightarrow{-h} T(X)$ is not isomorphic to the triangle (7.1) in general.

Definition 7.1.3. A triangulated category is an additive category $\mathcal{D}$ endowed with an automorphism T and a family of triangles called distinguished triangles (d.t. for short), this family satisfying axioms TR0 - TR5 below.

TR0 A triangle isomorphic to a d.t. is a d.t.

TR1 The triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to T(X)$ is a d.t.

TR2 For all $f : X \to Y$ there exists a d.t. $X \xrightarrow{f} Y \to Z \to T(X)$.

TR3 A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ is a d.t. if and only if $Y \xrightarrow{g} Z \xrightarrow{h} T(X) \xrightarrow{-T(f)} T(Y)$ is a d.t.

TR4 Given two d.t. $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} T(X')$ and morphisms $\alpha : X \to X'$ and $\beta : Y \to Y'$ with $f' \circ \alpha = \beta \circ f$, there exists a morphism $\gamma : Z \to Z'$ giving rise to a morphism of d.t.:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, dotted, "\gamma"'] & T(X) \arrow[d, "T(\alpha)"'] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X'),
\end{tikzcd}$$

TR5 (Octahedral axiom) Given three d.t.

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \to T(X),$$
$$Y \xrightarrow{g} Z \xrightarrow{k} X' \to T(Y),$$
$$X \xrightarrow{g \circ f} Z \xrightarrow{l} Y' \to T(X),$$

there exists a distinguished triangle $Z' \xrightarrow{\varphi} Y' \xrightarrow{\psi} X' \to T(Z')$ making the diagram below commutative:

(7.2)
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & Y \arrow[r, "h"] \arrow[d, "g"'] & Z' \arrow[r] \arrow[d, dotted, "\varphi"] & T(X) \arrow[d, "\mathrm{id}"'] \\
X \arrow[r, "g\circ f"] \arrow[d, "f"'] & Z \arrow[r, "l"] \arrow[d, "\mathrm{id}"'] & Y' \arrow[r] \arrow[d, dotted, "\psi"] & T(X) \arrow[d, "T(f)"'] \\
Y \arrow[r, "g"] \arrow[d, "h"'] & Z \arrow[r, "k"] \arrow[d, "l"'] & X' \arrow[r] \arrow[d, "\mathrm{id}"'] & T(Y) \arrow[d, "T(h)"] \\
Z' \arrow[r, dotted, "\varphi"] & Y' \arrow[r, dotted, "\psi"] & X' \arrow[r, dotted] & T(Z')
\end{tikzcd}$$

Diagram (7.2) is often called the octahedron diagram. Indeed, it can be written using the vertexes of an octahedron.

$$\begin{tikzcd}
& Y' \arrow[ddl, dashed] \arrow[dr, dotted, "\psi"] & \\
Z' \arrow[ur, dotted, "\varphi"] \arrow[d] & & X' \arrow[ll] \arrow[ddl] \\
X \arrow[rr, dashed] \arrow[dr, "f"'] & & Z \arrow[u] \arrow[uul, dashed] \\
& Y \arrow[uul] \arrow[ur, "g"'] &
\end{tikzcd}$$

Remark 7.1.4. The morphism γ in TR 4 is not unique and this is the origin of many troubles.

Remark 7.1.5. The category $\mathcal{D}^{\mathrm{op}}$ endowed with the image by the contravariant functor op : $\mathcal{D} \to \mathcal{D}^{\mathrm{op}}$ of the family of the d.t. in $\mathcal{D}$, is a triangulated category.

Definition 7.1.6. (i) A triangulated functor of triangulated categories $F : (\mathcal{D}, T) \to (\mathcal{D}', T')$ is an additive functor which satisfies $F \circ T \simeq T' \circ F$ and which sends distinguished triangles to distinguished triangles.

(ii) A triangulated subcategory $\mathcal{D}'$ of $\mathcal{D}$ is a subcategory $\mathcal{D}'$ of $\mathcal{D}$ which is triangulated and such that the functor $\mathcal{D}' \to \mathcal{D}$ is triangulated.

(iii) Let $(\mathcal{D}, T)$ be a triangulated category, $\mathcal{C}$ an abelian category, $F : \mathcal{D} \to \mathcal{C}$ an additive functor. One says that F is a cohomological functor if for any d.t. $X \to Y \to Z \to T(X)$ in $\mathcal{D}$, the sequence $F(X) \to F(Y) \to F(Z)$ is exact in $\mathcal{C}$.

Remark 7.1.7. By TR3, a cohomological functor gives rise to a long exact sequence:

$$\cdots \to F(X) \to F(Y) \to F(Z) \to F(T(X)) \to \cdots \tag{7.3}$$

Proposition 7.1.8. (i) *If* $X \xrightarrow{f} Y \xrightarrow{g} Z \to T(X)$ *is a d.t. then* $g \circ f = 0$.

(ii) *For any* $W \in \mathcal{D}$, *the functors* $\mathrm{Hom}_{\mathcal{D}}(W, \cdot)$ *and* $\mathrm{Hom}_{\mathcal{D}}(\cdot, W)$ *are cohomological.*

Note that (ii) means that if $\varphi : W \to Y$ (resp. $\varphi : Y \to W$) satisfies $g \circ \varphi = 0$ (resp. $\varphi \circ f = 0$), then φ factorizes through f (resp. through g).

Proof. (i) Applying TR1 and TR4 we get a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & X \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & T(X) \arrow[d, "\mathrm{id}"'] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & T(X). \end{tikzcd}$$

Then $g \circ f$ factorizes through 0.

(ii) Let $X \to Y \to Z \to T(X)$ be a d.t. and let $W \in \mathcal{D}$. We want to show that

$$\mathrm{Hom}\,(W, X) \xrightarrow{f\circ} \mathrm{Hom}\,(W, Y) \xrightarrow{g\circ} \mathrm{Hom}\,(W, Z)$$

is exact, i.e., : for all $\varphi : W \to Y$ such that $g \circ \varphi = 0$, there exists $\psi : W \to X$ such that $\varphi = f \circ \psi$. This means that the dotted arrow below may be completed, and this follows from the axioms TR4 and TR3.

$$\begin{tikzcd} W \arrow[r, "\mathrm{id}"] \arrow[d, dotted] & W \arrow[r] \arrow[d, "\varphi"'] & 0 \arrow[r] \arrow[d] & T(W) \arrow[d, dotted] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & T(X). \end{tikzcd}$$

The proof for $\mathrm{Hom}\,(\cdot, W)$ is similar. q.e.d.

Proposition 7.1.9. *Consider a morphism of d.t.:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & T(X) \arrow[d, "T(\alpha)"'] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X'). \end{tikzcd}$$

If α and β are isomorphisms, then so is γ.

Proof. Apply $\mathrm{Hom}\,(W, \cdot)$ to this diagram and write $\tilde{X}$ instead of $\mathrm{Hom}\,(W, X)$, $\tilde{\alpha}$ instead of $\mathrm{Hom}\,(W, \alpha)$, etc. We get the commutative diagram:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\alpha}"'] & \tilde{Y} \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{\beta}"'] & \tilde{Z} \arrow[r, "\tilde{h}"] \arrow[d, "\tilde{\gamma}"'] & \widetilde{T(X)} \arrow[d, "\widetilde{T(\alpha)}"'] \\ \tilde{X}' \arrow[r, "\tilde{f}'"] & \tilde{Y}' \arrow[r, "\tilde{g}'"] & \tilde{Z}' \arrow[r, "\tilde{h}'"] & \widetilde{T(X')}. \end{tikzcd}$$

The rows are exact in view of the preceding proposition, and $\tilde{\alpha}$, $\tilde{\beta}$, $\widetilde{T(\alpha)}$, $\widetilde{T(\beta)}$ are isomorphisms. Therefore $\tilde{\gamma} = \mathrm{Hom}\,(W, \gamma) : \mathrm{Hom}\,(W, Z) \to \mathrm{Hom}\,(W, Z')$ is an isomorphism. This implies that γ is an isomorphism by the Yoneda lemma. q.e.d.

Corollary 7.1.10. *Let $\mathcal{D}'$ be a full triangulated category of $\mathcal{D}$.*

(i) *Consider a triangle $X \xrightarrow{f} Y \to Z \to T(X)$ in $\mathcal{D}'$ and assume that this triangle is distinguished in $\mathcal{D}$. Then it is distinguished in $\mathcal{D}'$.*

(ii) *Consider a d.t. $X \to Y \to Z \to T(X)$ in $\mathcal{D}$, with X and Y in $\mathcal{D}'$. Then there exists $Z' \in \mathcal{D}'$ and an isomorphism $Z \simeq Z'$.*

Proof. (i) There exists a d.t. $X \xrightarrow{f} Y \to Z' \to T(X)$ in $\mathcal{D}'$. Then Z' is isomorphic to Z by TR4 and Proposition 7.1.9.
(ii) Apply TR2 to the morphism $X \to Y$ in $\mathcal{D}'$. q.e.d.

Remark 7.1.11. The proof of Proposition 7.1.9 does not make use of axiom TR 5, and this proposition implies that TR 5 is equivalent to the axiom:
TR5': given $f : X \to Y$ and $g : Y \to Z$, there exists a commutative diagram (7.2) such that all rows are d.t.

By Proposition 7.1.9, one gets that the object Z given in TR4 is unique up to isomorphism. However, this isomorphism is not unique, and this is the source of many difficulties (e.g., glueing problems in sheaf theory).

7.2 The homotopy category $K(\mathcal{C})$

Let $\mathcal{C}$ be an additive category.

Starting with $C(\mathcal{C})$, we shall construct a new category by deciding that a morphism of complexes homotopic to zero is isomorphic to the zero morphism. Set:

$$Ht(X, Y) = \{f : X \to Y; f \text{ is homotopic to } 0\}.$$

If $f : X \to Y$ and $g : Y \to Z$ are two morphisms in $C(\mathcal{C})$ and if f or g is homotopic to zero, then $g \circ f$ is homotopic to zero. This allows us to state:

Definition 7.2.1. The homotopy category $K(\mathcal{C})$ is defined by:

$$\begin{aligned} \mathrm{Ob}(K(\mathcal{C})) &= \mathrm{Ob}(C(\mathcal{C})) \\ \mathrm{Hom}_{K(\mathcal{C})}(X, Y) &= \mathrm{Hom}_{C(\mathcal{C})}(X, Y)/Ht(X, Y) \end{aligned}$$

In other words, a morphism homotopic to zero in $C(\mathcal{C})$ becomes the zero morphism in $K(\mathcal{C})$ and a homotopy equivalence becomes an isomorphism.

One defines similarly $K^*(\mathcal{C})$, $(* = b, +, -)$. They are clearly additive categories, endowed with an automorphism, the shift functor $[1] : X \mapsto X[1]$.

Recall that if $f : X \to Y$ is a morphism in $C(\mathcal{C})$, one defines its mapping cone $\mathrm{Mc}(f)$, an object of $C(\mathcal{C})$, and there is a natural triangle

$$Y \xrightarrow{\alpha(f)} \mathrm{Mc}(f) \xrightarrow{\beta(f)} X[1] \xrightarrow{f[1]} Y[1]. \tag{7.4}$$

Such a triangle is called a mapping cone triangle.

Definition 7.2.2. A distinguished triangle (d.t. for short) in $K(\mathcal{C})$ is a triangle isomorphic in $K(\mathcal{C})$ to a mapping cone triangle.

Theorem 7.2.3. *The category $K(\mathcal{C})$ endowed with the shift functor $[1]$ and the family of d.t. is a triangulated category.*

We shall not give the proof of this fundamental result here.

Notation 7.2.4. For short, we shall sometimes write $X \to Y \to Z \xrightarrow{+1}$ instead of $X \to Y \to Z \to X[1]$ to denote a d.t. in $K(\mathcal{C})$.

The complex $\mathrm{Hom}^\bullet$

Let $X \in C^-(\mathcal{C})$ and $Y \in C^+(\mathcal{C})$. Recall that

$$\mathrm{Hom}^\bullet_{\mathcal{C}}(X,Y) = \mathrm{tot}(\mathrm{Hom}^{\bullet,\bullet}_{\mathcal{C}}(X,Y)). \tag{7.5}$$

Hence, $\mathrm{Hom}_{\mathcal{C}}(X,Y)^n = \oplus_k \mathrm{Hom}_{\mathcal{C}}(X^k, Y^{n+k})$ and

$$d^n \; : \; \mathrm{Hom}_{\mathcal{C}}(X,Y)^n \to \mathrm{Hom}_{\mathcal{C}}(X,Y)^{n+1}$$

is defined as follows. To $f = \{f^k\}_k \in \oplus_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X^k, Y^{n+k})$ one associates

$$d^n f = \{g^k\}_k \in \oplus_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X^k, Y^{n+k+1}),$$

with

$$g^k = d'^{n+k,-k} f^k + (-)^{k+n+1} d''^{k+n+1,-k-1} f^{k+1}$$

In other words, the components of df in $\mathrm{Hom}_{\mathcal{C}}(X,Y)^{n+1}$ will be

$$(d^n f)^k = d_Y^{k+n} \circ f^k + (-)^n f^{k+1} \circ d_X^k. \tag{7.6}$$

Proposition 7.2.5. *Let $\mathcal{C}$ be an additive category and let $X, Y \in C(\mathcal{C})$. There are isomorphisms:*

$$\begin{aligned} Z^0(\mathrm{Hom}^\bullet_{\mathcal{C}}(X,Y)) = \mathrm{Ker}\, d^0 &\simeq \mathrm{Hom}_{C(\mathcal{C})}(X,Y), \\ B^0(\mathrm{Hom}^\bullet_{\mathcal{C}}(X,Y)) = \mathrm{Im}\, d^{-1} &\simeq Ht(X,Y), \\ H^0(\mathrm{Hom}^\bullet_{\mathcal{C}}(X,Y)) = (\mathrm{Ker}\, d^0)/(\mathrm{Im}\, d^{-1}) &\simeq \mathrm{Hom}_{K(\mathcal{C})}(X,Y). \end{aligned}$$

Proof. (i) Let us calculate $Z^0(\mathrm{Hom}^\bullet_{\mathcal{C}}(X,Y))$. By (7.6), the component of $d^0\{f^k\}_k$ in $\mathrm{Hom}_{\mathcal{C}}(X^k, Y^{k+1})$ will be zero if and only if $d^k_Y \circ f^k = f^{k+1} \circ d^k_X$, that is, if the family $\{f^k\}_k$ defines a morphism of complexes.
(ii) Let us calculate $B^0(\mathrm{Hom}^\bullet_{\mathcal{C}}(X,Y))$. An element $f^k \in \mathrm{Hom}_{\mathcal{C}}(X^k, Y^k)$ will be in the image of d^{-1} if it is in the sum of the image of $\mathrm{Hom}_{\mathcal{C}}(X^k, Y^{k-1})$ by d^{k-1}_Y and the image of $\mathrm{Hom}_{\mathcal{C}}(X^{k+1}, Y^k)$ by d^k_X. Hence, if it can be written as $f^k = d^{k-1}_Y \circ s^k + s^{k+1} \circ d^k_X$. q.e.d.

7.3 Localization of triangulated categories

Definition 7.3.1. Let $\mathcal{D}$ be a category and let $\mathcal{N} \subset \mathrm{Ob}(\mathcal{D})$. One says that $\mathcal{N}$ is a null system if it satisfies:

N1 $0 \in \mathcal{N}$,

N2 $X \in \mathcal{N}$ if and only if $T(X) \in \mathcal{N}$,

N3 if $X \to Y \to Z \to T(X)$ is a d.t. in $\mathcal{D}$ and $X, Y \in \mathcal{N}$ then $Z \in \mathcal{N}$.

To a null system one associates a multiplicative system as follows. Define:

$$\mathcal{S} = \{f : X \to Y, \text{ there exists a d.t. } X \to Y \to Z \to T(X) \text{ with } Z \in \mathcal{N}\}.$$

Theorem 7.3.2. (i) *$\mathcal{S}$ is a right and left multiplicative system.*

(ii) *Denote as usual by $\mathcal{D}_{\mathcal{S}}$ the localization of $\mathcal{D}$ by $\mathcal{S}$ and by Q the localization functor. Then $\mathcal{D}_{\mathcal{S}}$ is an additive category endowed with an automorphism (the image of T, still denoted by T).*

(iii) *Define a d.t. in $\mathcal{D}_{\mathcal{S}}$ as being isomorphic to the image by Q of a d.t. in $\mathcal{D}$. Then $\mathcal{D}_{\mathcal{S}}$ is a triangulated category.*

(iv) *If $X \in \mathcal{N}$ then $Q(X) \simeq 0$.*

(v) *Let $F : \mathcal{D} \to \mathcal{D}'$ be a functor of triangulated categories such that $F(X) \simeq 0$ for any $X \in \mathcal{N}$. Then F factors uniquely through Q.*

The proof is tedious and will not be given here.

Notation 7.3.3. We will write $\mathcal{D}/\mathcal{N}$ instead of $\mathcal{D}_{\mathcal{S}}$.

Let $\mathcal{N}$ be a null system and let $X \in \mathcal{D}$.

$$\mathrm{Ob}(\mathcal{S}^X) = \{s : X \to X';\ \text{there exists a d.t. } X \xrightarrow{s} X' \to Z \to T(X) \text{ with } Z \in \mathcal{N}\}$$
$$\mathrm{Hom}_{\mathcal{S}^X}((s : X \to X'), (s : X \to X'')) = \{h : X' \to X''; h \circ s = s'\}$$

and similarly for $\mathcal{S}_X$. Recall that the categories $\mathcal{S}_X^{\mathrm{op}}$ and $\mathcal{S}^X$ are filtrant.

Now consider a full triangulated subcategory $\mathcal{I}$ of $\mathcal{D}$. We shall write $\mathcal{N} \cap \mathcal{I}$ instead of $\mathcal{N} \cap \mathrm{Ob}(\mathcal{I})$. This is clearly a null system in $\mathcal{I}$.

Proposition 7.3.4. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{N}$ a null system, $\mathcal{I}$ a full triangulated category of $\mathcal{D}$. Assume condition* (i) *or* (ii) *below*

(i) *any morphism $Y \to Z$ with $Y \in \mathcal{I}$ and $Z \in \mathcal{N}$, factorizes as $Y \to Z' \to Z$ with $Z' \in \mathcal{N} \cap \mathcal{I}$,*

(ii) *any morphism $Z \to Y$ with $Y \in \mathcal{I}$ and $Z \in \mathcal{N}$, factorizes as $Z \to Z' \to Y$ with $Z' \in \mathcal{N} \cap \mathcal{I}$.*

Then $\mathcal{I}/(\mathcal{N} \cap \mathcal{I}) \to \mathcal{D}/\mathcal{N}$ is fully faithful.

Proof. We shall apply Proposition 6.2.1. We may assume (ii), the case (i) being deduced by considering $\mathcal{D}^{\mathrm{op}}$. Let $f : Y \to X$ is a morphism in $\mathcal{S}$ with $Y \in \mathcal{I}$. We shall show that there exists $g : X \to W$ with $W \in I$ and $g \circ f \in \mathcal{S}$. The morphism f is embedded in a d.t. $Y \to X \to Z \to T(Y)$, with $Z \in \mathcal{N}$. By the hypothesis, the morphism $Z \to T(Y)$ factorizes through an object $Z' \in \mathcal{N} \cap \mathcal{I}$. We may embed $Z' \to T(Y)$ into a d.t. and obtain a commutative diagram of d.t.:

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & X \arrow[r] \arrow[d, dotted, "g"] & Z \arrow[r] \arrow[d] & T(Y) \arrow[d, "\mathrm{id}"] \\
Y \arrow[r] & W \arrow[r] & Z' \arrow[r] & T(Y)
\end{tikzcd}$$

By TR4, the dotted arrow g may be completed, and Z' belonging to $\mathcal{N}$, this implies that $g \circ f \in \mathcal{S}$. q.e.d.

Proposition 7.3.5. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{N}$ a null system, $\mathcal{I}$ a full triangulated subcategory of $\mathcal{D}$, and assume conditions* (i) *or* (ii) *below:*

(i) *for any $X \in \mathcal{D}$, there exists a d.t. $X \to Y \to Z \to T(X)$ with $Z \in \mathcal{N}$ and $Y \in \mathcal{I}$,*

(ii) *for any $X \in \mathcal{D}$, there exists a d.t. $Y \to X \to Z \to T(X)$ with $Z \in \mathcal{N}$ and $Y \in \mathcal{I}$.*

Then $\mathcal{I}/\mathcal{N} \cap \mathcal{I} \to \mathcal{D}/\mathcal{N}$ is an equivalence of categories.

Proof. Apply Corollary 6.2.2. q.e.d.

Localization of triangulated functors

Let $F : \mathcal{D} \to \mathcal{D}'$ be a functor of triangulated categories, $\mathcal{N}$ a null system in $\mathcal{D}$. One defines the localization of F similarly as in the usual case, replacing all categories and functors by triangulated ones. Applying Proposition 6.3.2, we get:

Proposition 7.3.6. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{N}$ a null system, $\mathcal{I}$ a full triangulated category of $\mathcal{D}$. Let $F : \mathcal{D} \to \mathcal{D}'$ be a triangulated functor, and assume*

(i) *for any $X \in \mathcal{D}$, there exists a d.t. $X \to Y \to Z \to T(X)$ with $Z \in \mathcal{N}$ and $Y \in \mathcal{I}$,*

(ii) *for any $Y \in \mathcal{N} \cap \mathcal{I}$, $F(Y) \simeq 0$.*

Then F is right localizable.

One can define $F_{\mathcal{N}}$ by the diagram:

$$\begin{tikzcd}
\mathcal{D} \arrow[rr] & & \mathcal{D}/\mathcal{N} \arrow[dd, dotted, "F_{\mathcal{N}}"] \\
\mathcal{I} \arrow[u] \arrow[r] \arrow[drr] & \mathcal{I}/\mathcal{I}\cap\mathcal{N} \arrow[ur, "\sim"] \arrow[dr] & \\
& & \mathcal{D}'
\end{tikzcd}$$

If one replace condition (i) in Proposition 7.3.6 by the condition

(i)' for any $X \in \mathcal{D}$, there exists a d.t. $Y \to X \to Z \to T(X)$ with $Z \in \mathcal{N}$ and $Y \in \mathcal{I}$,

one gets that F is left localizable.

Finally, let us consider triangulated bifunctors, i.e., bifunctors which are additive and triangulated with respect to each of their arguments.

Proposition 7.3.7. *Let $\mathcal{D}, \mathcal{N}, \mathcal{I}$ and $\mathcal{D}', \mathcal{N}', \mathcal{I}'$ be as in Proposition 7.3.6. Let $F : \mathcal{D} \times \mathcal{D}' \to \mathcal{D}''$ be a triangulated bifunctor. Assume:*

(i) *for any $X \in \mathcal{D}$, there exists a d.t. $X \to Y \to Z \to T(X)$ with $Z \in \mathcal{N}$ and $Y \in \mathcal{I}$*

(ii) *for any $X' \in \mathcal{D}'$, there exists a d.t. $X' \to Y' \to Z' \to T(X')$ with $Z' \in \mathcal{N}'$ and $Y' \in \mathcal{I}'$*

(iii) *for any* $Y \in \mathcal{I}$ *and* $Y' \in \mathcal{I}' \cap \mathcal{N}'$, $F(Y, Y') \simeq 0$,

(iv) *for any* $Y \in \mathcal{I} \cap \mathcal{N}$ *and* $Y' \in \mathcal{I}'$, $F(Y, Y') \simeq 0$.

Then F *is right localizable.*

One denotes by $F_{\mathcal{N}\mathcal{N}'}$ its localization.

Of course, there exists a similar result for left localizable functors by reversing the arrows in the hypotheses (i) and (ii) above.

Exercises to Chapter 7

Exercise 7.1. Let $\mathcal{D}$ be a triangulated category and consider a commutative diagram in $\mathcal{D}$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & T(X) \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X),
\end{tikzcd}$$

Assume that $T(f) \circ h' = 0$ and the first row is a d.t. Prove that the second row is also a d.t. under one of the hypotheses:
(i) for any $P \in \mathcal{D}$, the sequence below is exact:

$$\mathrm{Hom}\,(P, X) \to \mathrm{Hom}\,(P, Y) \to \mathrm{Hom}\,(P, Z') \to \mathrm{Hom}\,(P, T(X)),$$

(ii) for any $P \in \mathcal{D}$, the sequence below is exact:

$$\mathrm{Hom}\,(T(Y), P) \to \mathrm{Hom}\,(T(X), P) \to \mathrm{Hom}\,(Z', P) \to \mathrm{Hom}\,(Y, P).$$

Exercise 7.2. Let $\mathcal{D}$ be a triangulated category and let $X_1 \to Y_1 \to Z_1 \to T(X_1)$ and $X_2 \to Y_2 \to Z_2 \to T(X_2)$ be two d.t. Show that $X_1 \oplus X_2 \to Y_1 \oplus Y_2 \to Z_1 \oplus Z_2 \to T(X_1) \oplus T(X_2)$ is a d.t.
In particular, $X \to X \oplus Y \to Y \xrightarrow{0} T(X)$ is a d.t.
(Hint: Consider a d.t. $X_1 \oplus X_2 \to Y_1 \oplus Y_2 \to H \to T(X_1) \oplus T(X_2)$ and construct the morphisms $H \to Z_1 \oplus Z_2$, then apply the result of Exercise 7.1.)

Exercise 7.3. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ be a d.t. in a triangulated category.
(i) Prove that if $h = 0$, this d.t. is isomorphic to $X \to X \oplus Z \to Z \xrightarrow{0} T(X)$.
(ii) Prove the same result by assuming now that there exists $k : Y \to X$ with $k \circ f = \mathrm{id}_X$.
(Hint: to prove (i), construct the morphism $Y \to X \oplus Z$ by TR4, then use Proposition 7.1.9.)

Exercise 7.4. Let $X \xrightarrow{f} Y \to Z \to T(X)$ be a d.t. in a triangulated category. Prove that f is an isomorphism if and only if Z is isomorphic to 0.

Exercise 7.5. Let $\mathcal{D}$ be a triangulated category, $\mathcal{N}$ a null system, and let Y be an object of $\mathcal{D}$ such that $\mathrm{Hom}_{\mathcal{D}}(Z, Y) \simeq 0$ for all $Z \in \mathcal{N}$. Prove that $\mathrm{Hom}_{\mathcal{D}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}/\mathcal{N}}(X, Y)$.

Chapter 8

Derived categories

In this chapter we construct the derived category of an abelian category $\mathcal{C}$ and the right derived functor RF of a left exact functor $F : \mathcal{C} \to \mathcal{C}'$ of abelian categories.
Some references: [6], [10], [11], [12], [15], [16], [17].

8.1 Derived categories

In all this chapter, $\mathcal{C}$ will denote an abelian category.

Recall that if $f : X \to Y$ is a morphism in $C(\mathcal{C})$, one says that f is a quasi-isomorphism (a qis, for short) if $H^k(f) : H^k(X) \to H^k(Y)$ is an isomorphism for all k. One extends this definition to morphisms in $K(\mathcal{C})$.

If one embeds f into a d.t. $X \xrightarrow{f} Y \to Z \xrightarrow{+1}$, then f is a qis iff $H^k(Z) \simeq 0$ for all $k \in \mathbb{Z}$, that is, if Z is qis to 0.

Proposition 8.1.1. *Let $\mathcal{C}$ be an abelian category. Then the functor $H^0 \colon K(\mathcal{C}) \to \mathcal{C}$ is a cohomological functor.*

Proof. Let $X \xrightarrow{f} Y \to Z \xrightarrow{+1}$ be a d.t. Then it is isomorphic to $X \to Y \xrightarrow{\alpha(f)} \mathrm{Mc}(f) \xrightarrow{\beta(f)} X[1] \xrightarrow{+1}$. Since the sequence in $C(\mathcal{C})$:

$$0 \to Y \to \mathrm{Mc}(f) \to X[1] \to 0$$

is exact, it follows from Theorem 5.2.6 that the sequence

$$H^k(Y) \to H^k(\mathrm{Mc}(f)) \to H^{k+1}(X)$$

is exact. Therefore, $H^k(Y) \to H^k(Z) \to H^{k+1}(X)$ is exact. q.e.d.

Corollary 8.1.2. *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence in $C(\mathcal{C})$ and define $\varphi\colon \mathrm{Mc}(f) \to Z$ as $\varphi^n = (0, g^n)$. Then φ is a qis.*

Proof. Consider the exact sequence in $C(\mathcal{C})$:

$$0 \to M(\mathrm{id}_X) \xrightarrow{\gamma} \mathrm{Mc}(f) \xrightarrow{\varphi} Z \to 0$$

where $\gamma^n\colon (X^{n+1} \oplus X^n) \to X^{n+1} \oplus Y^n$ is defined by: $\gamma^n = \begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 \\ 0 & f^n \end{pmatrix}$.
Since $H^k(\mathrm{Mc}(\mathrm{id}_X)) \simeq 0$ for all k, we get the result. q.e.d.

We shall localize $K(\mathcal{C})$ with respect to the family of objects qis to zero (see Section 7.3). Define:

$$N(\mathcal{C}) = \{X \in K(\mathcal{C}); H^k(X) \simeq 0 \text{ for all } k\}.$$

One also defines $N^*(\mathcal{C}) = N(\mathcal{C}) \cap K^*(\mathcal{C})$ for $^* = b, +, -$.
Clearly, $N^*(\mathcal{C})$ is a null system in $K^*(\mathcal{C})$.

Definition 8.1.3. One defines the derived categories $D^*(\mathcal{C})$ as $K^*(\mathcal{C})/N^*(\mathcal{C})$, where $* = \emptyset, b, +, -$. One denotes by Q the localization functor $K^*(\mathcal{C}) \to D^*(\mathcal{C})$.

By Theorem 7.3.2, these are triangulated categories.
Hence, *a quasi-isomorphism in $K(\mathcal{C})$ becomes an isomorphism in $D(\mathcal{C})$.*
The functors below are well defined:

$$\begin{aligned} H^j(\cdot) : D(\mathcal{C}) &\to \mathcal{C} \\ \tau^{\leq n} : D(\mathcal{C}) &\to D^-(\mathcal{C}) \\ \tau^{\geq n} : D(\mathcal{C}) &\to D^+(\mathcal{C}) \end{aligned}$$

and $H^j(\cdot)$ is a cohomological functor on $D^*(\mathcal{C})$. In fact, if $X \in N(\mathcal{C})$, then $H^j(X) \simeq 0$ in $\mathcal{C}$, and if $f : X \to Y$ is a qis in $K(\mathcal{C})$, then $\tau^{\leq n}(f)$ and $\tau^{\geq n}(f)$ are qis.

In particular, if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$ is a d.t. in $D(\mathcal{C})$, we get a long exact sequence:

$$\cdots \to H^k(X) \to H^k(Y) \to H^k(Z) \to H^{k+1}(X) \to \cdots \tag{8.1}$$

Let $X \in K(\mathcal{C})$, with $H^j(X) = 0$ for $j > n$. Then the morphism $\tau^{\leq n}X \to X$ in $K(\mathcal{C})$ is a qis, hence an isomorphism in $D(\mathcal{C})$.

It follows from Proposition 7.3.4 that $D^+(\mathcal{C})$ is equivalent to the full subcategory of $D(\mathcal{C})$ consisting of objects X satisfying $H^j(X) \simeq 0$ for $j << 0$, and similarly for $D^-(\mathcal{C}), D^b(\mathcal{C})$. Moreover, $\mathcal{C}$ is equivalent to the full subcategory of $D(\mathcal{C})$ consisting of objects X satisfying $H^j(X) \simeq 0$ for $j \neq 0$.

Definition 8.1.4. Let X, Y be objects of $\mathcal{C}$. One sets

$$\mathrm{Ext}^k_{\mathcal{C}}(X,Y) = \mathrm{Hom}_{D(\mathcal{C})}(X, Y[k]).$$

We shall see in Theorem 8.4.5 below that if $\mathcal{C}$ has enough injectives, this definition is compatible with (5.13).

Notation 8.1.5. Let A be a ring. We shall write for short $D^*(A)$ instead of $D^*(\mathrm{Mod}(A))$, for $^* = \emptyset, b, +, -$.

Remark 8.1.6. (i) Let $X \in K(\mathcal{C})$, and let $Q(X)$ denote its image in $D(\mathcal{C})$. One can prove that:

$$Q(X) \simeq 0 \Leftrightarrow X \text{ is qis to } 0 \text{ in } K(\mathcal{C}).$$

(ii) Let $f : X \to Y$ be a morphism in $C(\mathcal{C})$. Then $f \simeq 0$ in $D(\mathcal{C})$ iff there exists X' and a qis $g : X' \to X$ such that $f \circ g$ is homotopic to 0, or else iff there exists Y' and a qis $h : Y \to Y'$ such that $h \circ f$ is homotopic to 0.

Remark 8.1.7. Consider the morphism $\gamma : Z \to X[1]$ in $D(\mathcal{C})$. If X, Y, Z belong to $\mathcal{C}$ (i.e. are concentrated in degree 0), the morphism $H^k(\gamma) : H^k(Z) \to H^{k+1}(X)$ is 0 for all $k \in \mathbb{Z}$. However, γ is *not* the zero morphism in $D(\mathcal{C})$ in general (this happens if the short exact sequence splits). In fact, let us apply the cohomological functor $\mathrm{Hom}_{\mathcal{C}}(W, \cdot)$ to the d.t. above. It gives rise to the long exact sequence:

$$\cdots \to \mathrm{Hom}_{\mathcal{C}}(W,Y) \to \mathrm{Hom}_{\mathcal{C}}(W,Z) \xrightarrow{\tilde{\gamma}} \mathrm{Hom}_{\mathcal{C}}(W,X[1]) \to \cdots$$

where $\tilde{\gamma} = \mathrm{Hom}_{\mathcal{C}}(W,\gamma)$. Since $\mathrm{Hom}_{\mathcal{C}}(W,Y) \to \mathrm{Hom}_{\mathcal{C}}(W,Z)$ is not an epimorphism in general, $\tilde{\gamma}$ is not zero. Therefore γ is not zero in general. The morphism γ may be described as follows.

$$\begin{tikzcd}
Z := & 0 \arrow[r] & 0 \arrow[r] & Z \arrow[r] & 0 \\
\mathrm{Mc}(f) := \arrow[u, "\varphi"] \arrow[d, "\beta(f)"'] & 0 \arrow[r] & X \arrow[r, "f"] \arrow[u] \arrow[d, "\mathrm{id}"'] & Y \arrow[r] \arrow[u] \arrow[d] & 0 \\
X[1] := & 0 \arrow[r] & X \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

Proposition 8.1.8. *Let $X \in D(\mathcal{C})$.*

(i) *There are d.t. in $D(\mathcal{C})$:*

$$\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \xrightarrow{+1} \tag{8.2}$$

$$\tau^{\leq n-1}X \to \tau^{\leq n}X \to H^n(X)[-n] \xrightarrow{+1} \tag{8.3}$$

$$H^n(X)[-n] \to \tau^{\geq n}X \to \tau^{\geq n+1}X \xrightarrow{+1} \tag{8.4}$$

(ii) *Moreover,* $H^n(X)[-n] \simeq \tau^{\leq n}\tau^{\geq n}X \simeq \tau^{\geq n}\tau^{\leq n}X$.

Corollary 8.1.9. *Let $\mathcal{C}$ be an abelian category and assume that for any $X, Y \in \mathcal{C}$, $\mathrm{Ext}^k(X,Y) = 0$ for $k \geq 2$. Let $X \in D^b(\mathcal{C})$. Then:*

$$X \simeq \oplus_j H^j(X)\,[-j].$$

Proof. Call "amplitude of X" the smallest integer k such that $H^j(X) = 0$ for j not belonging to some interval of length k. If $k = 0$, this means that there exists some i with $H^j(X) = 0$ for $j \neq i$, hence $X \simeq H^i(X)\,[-i]$. Now we argue by induction on the amplitude. Consider the d.t. (8.3):

$$\tau^{\leq n-1}X \to \tau^{\leq n}X \to H^n(X)\,[-n] \xrightarrow{+1}$$

and assume $\tau^{\leq n-1}X \simeq \oplus_{j<n}H^j(X)\,[-j]$. By the result of Exercise 7.3, it it enough to show that $\mathrm{Hom}_{D^b(\mathcal{C})}(H^n(X)[-n], H^j(X)\,[-j+1]) = 0$ for $j < n$. Since $n + 1 - j \geq 2$, the result follows. q.e.d.

Example 8.1.10. (i) If a ring A is a principal ideal domain (such as a field, or $\mathbb{Z}$, or $k[x]$ for k a field), then the category $\mathrm{Mod}(A)$ satisfies the hypotheses of Corollary 8.1.9.
(ii) See Example 8.4.8 to see an object which does not split.

8.2 Resolutions

Lemma 8.2.1. *Let $\mathcal{J}$ be an additive subcategory of $\mathcal{C}$, and assume that $\mathcal{J}$ is cogenerating. Let $X^\bullet \in C^+(\mathcal{C})$.*

Then there exists $Y^\bullet \in K^+(\mathcal{J})$ and a qis $X^\bullet \to Y^\bullet$.

Proof. The proof is of the same kind of those in Section 5.5 and is left to the reader. q.e.d.

We set $N^+(\mathcal{J}) := N(\mathcal{C}) \cap K^+(\mathcal{J})$. It is clear that $N^+(\mathcal{J})$ is a null system in $K^+(\mathcal{J})$.

Proposition 8.2.2. *Assume $\mathcal{J}$ is cogenerating in $\mathcal{C}$. Then the natural functor $\theta : K^+(\mathcal{J})/N^+(\mathcal{J}) \to D^+(\mathcal{C})$ is an equivalence of categories.*

Proof. Apply Lemma 8.2.1 and Proposition 7.3.4. q.e.d.

Let us apply the preceding proposition to the category $\mathcal{I}_\mathcal{C}$ of injective objects of $\mathcal{C}$.

Corollary 8.2.3. *Assume that $\mathcal{C}$ admits enough injectives. Then $K^+(\mathcal{I}_\mathcal{C}) \to D^+(\mathcal{C})$ is an equivalence of categories.*

Proof. Recall that if $X^\bullet \in C^+(\mathcal{I}_\mathcal{C})$ is qis to 0, then $X^\bullet$ is homotopic to 0. q.e.d.

8.3 Derived functors

In this section, $\mathcal{C}$ and $\mathcal{C}'$ will denote abelian categories. Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor. It defines naturally a functor

$$K^+F : K^+(\mathcal{C}) \to K^+(\mathcal{C}').$$

For short, one often writes F instead of K^+F. Applying the results of Chapter **??**, we shall construct (under suitable hypotheses) the right localization of F. Recall Definition 5.6.5. By Lemma 5.6.8, $K^+(F)$ sends $N^+(\mathcal{J})$ to $N^+(\mathcal{C}')$.

Definition 8.3.1. If the functor $K^+(F)\colon K^+(\mathcal{C}) \to D^+(\mathcal{C}')$ admits a right localization (with respect to the qis in $K^+(\mathcal{C})$), one says that F admits a right derived functor and one denotes by $RF\colon D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ the right localization of F.

Theorem 8.3.2. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor of abelian categories, and let $\mathcal{J} \subset \mathcal{C}$ be a full additive subcategory. Assume that $\mathcal{J}$ is F-injective. Then F admits a right derived functor $RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}')$.*

Proof. This follows immediately from Lemma 8.2.1 and Proposition 7.3.6 applied to $K^+(F) : K^+(\mathcal{C}) \to D^+(\mathcal{C}')$. q.e.d.

It is vizualised by the diagram

$$\begin{tikzcd}
K^+(\mathcal{J}) \arrow[r, "K^+(F)"] \arrow[d, "Q"'] & K^+(\mathcal{C}') \arrow[dd, "Q"] \\
K^+(\mathcal{J})/N^+(\mathcal{J}) \arrow[d, "\sim"'] \arrow[dr, "K^+(F)_{N(\mathcal{J})}"] & \\
D^+(\mathcal{C}) \arrow[r, dotted, "RF"'] & D^+(\mathcal{C}').
\end{tikzcd}$$

Note that if $\mathcal{C}$ admits enough injectives, then

$$R^kF = H^k \circ RF. \tag{8.5}$$

Recall that the derived functor RF is triangulated, and does not depend on the category $\mathcal{J}$. Hence, if $X' \to X \to X'' \xrightarrow{+1}$ is a d.t. in $D^+(\mathcal{C})$, then $RF(X') \to RF(X) \to RF(X'') \xrightarrow{+1}$ is a d.t. in $D^+(\mathcal{C}')$. (Recall that an exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ gives rise to a d.t. in $D(\mathcal{C})$.)

Applying the cohomological functor H^0, we get the long exact sequence in $\mathcal{C}'$:

$$\cdots \to R^kF(X') \to R^kF(X) \to R^kF(X'') \to R^{k+1}F(X') \to \cdots$$

By considering the category $\mathcal{C}^{\mathrm{op}}$, one defines the notion of left derived functor of a right exact functor F.

We shall study the derived functor of a composition.

Let $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}''$ be left exact functors of abelian categories. Then $G \circ F : \mathcal{C} \to \mathcal{C}''$ is left exact. Using the universal property of the localization, one shows that if F, G and $G \circ F$ are right derivable, then there exists a natural morphism of functors

$$R(G \circ F) \to RG \circ RF. \tag{8.6}$$

Proposition 8.3.3. *Assume that there exist full additive subcategories $\mathcal{J} \subset \mathcal{C}$ and $\mathcal{J}' \subset \mathcal{C}'$ such that $\mathcal{J}$ is F-injective, $\mathcal{J}'$ is G-injective and $F(\mathcal{J}) \subset \mathcal{J}'$. Then $\mathcal{J}$ is $(G \circ F)$-injective and the morphism in (8.6) is an isomorphism:*

$$R(G \circ F) \simeq RG \circ RF.$$

Proof. The fact that $\mathcal{J}$ is $(G \circ F)$ injective follows immediately from the definition. Let $X \in K^+(\mathcal{C})$ and let $Y \in K^+(\mathcal{J})$ with a qis $X \to Y$. Then $RF(X)$ is represented by the complex $F(Y)$ which belongs to $K^+(\mathcal{J}')$. Hence $RG(RF(X))$ is represented by $G(F(Y)) = (G \circ F)(Y)$, and this last complex also represents $R(G \circ F)(Y)$ since $Y \in \mathcal{J}$ and $\mathcal{J}$ is $G \circ F$ injective. q.e.d.

Note that in general F does not send injective objects of $\mathcal{C}$ to injective objects of $\mathcal{C}'$, and that is why we had to introduce the notion of "F-injective" category.

8.4 Bifunctors

Now consider three abelian categories $\mathcal{C}, \mathcal{C}', \mathcal{C}''$ and an *additive* bifunctor:

$$F : \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''.$$

We shall assume that F is left exact with respect to each of its arguments.

Let $X \in K^+(\mathcal{C}), X' \in K^+(\mathcal{C}')$ and assume X (or X') is homotopic to 0. Then one checks easily that $\mathrm{tot}(F(X, X'))$ is homotopic to zero. Hence one can naturally define:

$$K^+(F) : K^+(\mathcal{C}) \times K^+(\mathcal{C}') \to K^+(\mathcal{C}'')$$

by setting:

$$K^+(F)(X, X') = \mathrm{tot}(F(X, X')).$$

If there is no risk of confusion, we shall sometimes write F instead of K^+F.

Definition 8.4.1. One says $(\mathcal{J}, \mathcal{J}')$ is F-injective if:

(i) for all $X \in \mathcal{J}$, $\mathcal{J}'$ is $F(X, \cdot)$-injective.

(ii) for all $X' \in \mathcal{J}'$, $\mathcal{J}$ is $F(\cdot, X')$-injective.

Lemma 8.4.2. *Let $X \in K^+(\mathcal{J})$, $X' \in K^+(\mathcal{J}')$. If X or X' is qis to 0, then $F(X, X')$ is qis to zero.*

Proof. The double complex $F(X, Y)$ will satisfy the hypothesis of Proposition 5.2.11. q.e.d.

Using Lemma 8.4.2 and Proposition 7.3.7 one gets that F admits a right derived functor,

$$RF : D^+(\mathcal{C}) \times D^+(\mathcal{C}') \to D^+(\mathcal{C}'').$$

Example 8.4.3. Assume $\mathcal{C}$ has enough injectives. Then

$$\mathrm{RHom}_{\mathcal{C}} : D^-(\mathcal{C})^{\mathrm{op}} \times D^+(\mathcal{C}) \to D^+(\mathbf{Ab})$$

exists and may be calculated as follows. Let $X \in D^-(\mathcal{C})$, $Y \in D^+(\mathcal{C})$. There exists a qis in $K^+(\mathcal{C})$, $Y \to I$, the I^j's being injective. Then:

$$\mathrm{RHom}_{\mathcal{C}}(X, Y) \simeq \mathrm{Hom}^{\bullet}_{\mathcal{C}}(X, I).$$

If $\mathcal{C}$ has enough projectives, and $P \to X$ is a qis in $K^-(\mathcal{C})$, the P^j's being projective, one also has:

$$\mathrm{RHom}_{\mathcal{C}}(X, Y) \simeq \mathrm{Hom}^{\bullet}_{\mathcal{C}}(P, Y).$$

These isomorphisms hold in $D^+(\mathbf{Ab})$.

Example 8.4.4. Let A be a ring. The functor

$$\cdot \otimes^L_A \cdot : D^-(\mathrm{Mod}(A^{\mathrm{op}})) \times D^-(\mathrm{Mod}(A)) \to D^-(\mathbf{Ab})$$

is well defined.

$$\begin{aligned} N \otimes^L_A M &\simeq s(N \otimes_A P) \\ &\simeq s(Q \otimes_A M) \end{aligned}$$

where P (resp. Q) is a complex of projective A-modules qis to M (resp. N).

In the preceding situation, one has:

$$\mathrm{Tor}^A_{-k}(N, M) = H^k(N \otimes^L_A M).$$

The following result relies the derived functor of $\mathrm{Hom}_{\mathcal{C}}$ and $\mathrm{Hom}_{D(\mathcal{C})}$.

Theorem 8.4.5. *Let $\mathcal{C}$ be an abelian category with enough injectives. Then for $X \in D^-(\mathcal{C})$ and $Y \in D^+(\mathcal{C})$*

$$H^0\mathrm{RHom}_{\mathcal{C}}(X, Y) \simeq \mathrm{Hom}_{D(\mathcal{C})}(X, Y).$$

Proof. By Proposition **??**, there exists $I_Y \in C^+(\mathcal{I})$ and a qis $Y \to I_Y$. Then we have the isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{D(\mathcal{C})}(X, Y[k]) &\simeq \mathrm{Hom}_{K(\mathcal{C})}(X, I_Y[k]) \\
&\simeq H^0(\mathrm{Hom}^\bullet_{\mathcal{C}}(X, I_Y[k])) \\
&\simeq R^k\mathrm{Hom}_{\mathcal{C}}(X, Y),
\end{aligned}$$

where the second isomorphism follows from Proposition 7.2.5. q.e.d.

Theorem 8.4.5 implies the isomorphism

$$\mathrm{Ext}^k_{\mathcal{C}}(X, Y) \simeq H^k\mathrm{RHom}_{\mathcal{C}}(X, Y).$$

Example 8.4.6. Let W be the Weyl algebra in one variable over a field k: $W = k[x, \partial]$ with the relation $[x, \partial] = -1$.

Let $\mathcal{O} = W/W \cdot \partial$, $\quad \Omega = W/\partial \cdot W$ and let us calculate $\Omega \otimes^L_W \mathcal{O}$. We have an exact sequence:

$$0 \to W \xrightarrow{\partial\cdot} W \to \Omega \to 0$$

hence Ω is qis to the complex

$$0 \to W^{-1} \xrightarrow{\partial\cdot} W^0 \to 0$$

where $W^{-1} = W^0 = W$ and W^0 is in degree 0. Then $\Omega \otimes^L_W \mathcal{O}$ is qis to the complex

$$0 \to \mathcal{O}^{-1} \xrightarrow{\partial\cdot} \mathcal{O}^0 \to 0,$$

where $\mathcal{O}^{-1} = \mathcal{O}^0 = \mathcal{O}$ and $\mathcal{O}^0$ is in degree 0. Since $\partial : \mathcal{O} \to \mathcal{O}$ is surjective and has k as kernel, we obtain:

$$\Omega \otimes^L_W \mathcal{O} \simeq k[1].$$

Example 8.4.7. Let k be a field and let $A = k[x_1, \dots, x_n]$. This is a commutative noetherian ring and it is known (Hilbert) that any finitely generated A-module M admits a finite free presentation of length at most n, i.e. M is qis to a complex:

$$L := \quad 0 \to L^{-n} \to \cdots \xrightarrow{P_0} L^0 \to 0$$

where the L^j's are free of finite rank. Consider the functor

$$\mathrm{Hom}_A(\cdot, A) : \mathrm{Mod}(A) \to \mathrm{Mod}(A).$$

It is contravariant and left exact.

Since free A-modules are projective, we find that $\mathrm{RHom}_A(M, A)$ is isomorphic in $D^b(A)$ to the complex

$$L^* := \quad 0 \leftarrow L^{-n*} \leftarrow \cdots \xleftarrow{P_0} L^{0*} \leftarrow 0$$

where $L^{j*} = \mathrm{Hom}_A(L^j, A)$. Set for short $^* = \mathrm{RHom}_A(\cdot, A)$ Using (8.6), we find a natural morphism of functors

$$\mathrm{id} \to {}^{**}.$$

Applying $\mathrm{RHom}_A(\cdot, A)$ to the object $\mathrm{RHom}_A(M, A)$ we find:

$$\begin{aligned} \mathrm{RHom}_A(\mathrm{RHom}_A(M, A), A) &\simeq \mathrm{RHom}_A(L^*, A) \\ &\simeq L \\ &\simeq M. \end{aligned}$$

In other words, we have proved the isomorphism in $D^b(A)$: $M \simeq M^{**}$.

Assume now $n = 1$, i.e. $A = k[x]$ and consider the natural morphism in $\mathrm{Mod}(A)$: $f : A \to A/Ax$. Applying the functor $* = \mathrm{RHom}_A(\cdot, A)$, we get the morphism in $D^b(A)$:

$$f^* : \mathrm{RHom}_A(A/Ax, A) \to A.$$

Remember that $\mathrm{RHom}_A(A/Ax, A) \simeq A/xA[-1]$. Hence $H^j(f^*) = 0$ for all $j \in \mathbb{Z}$, although $f^* \neq 0$ since $f^{**} = f$.

Let us give an example of an object of a derived category which is not isomorphic to the direct sum of its cohomology objects (hence, a situation in which Corollary 8.1.9 does not apply).

Example 8.4.8. Let k be a field and let $A = k[x_1, x_2]$. Define the A-modules $M' = A/(Ax_1 + Ax_2)$, $M = A/(Ax_1^2 + Ax_1x_2)$ and $M'' = A/Ax_1$. There is an exact sequence

$$0 \to M' \to M \to M'' \to 0 \tag{8.7}$$

and this exact sequence does not split since x_1 kills M' and M'' but not M. For N an A-module, set $N^* = \mathrm{RHom}_A(N, A)$, an object of $D^b(A)$ (see Example 8.4.7). We have $M'^* \simeq H^2(M'^*)[-2]$ and $M''^* \simeq H^1(M'^*)[-1]$, and the functor $^* = \mathrm{RHom}_A(\cdot, A)$ applied to the exact sequence (8.7) gives rise to the long exact sequence

$$0 \to H^1(M''^*) \to H^1(M^*) \to 0 \to 0 \to H^2(M^*) \to H^2(M'^*) \to 0.$$

Hence $H^1(M^*)[-1] \simeq H^1(M''^*)[-1] \simeq M''^*$ and $H^2(M^*)[-2] \simeq H^2(M'^*)[-2] \simeq M'^*$. Assume for a while $M^* \simeq \oplus_j H^j(M^*)[-j]$. This implies $M^* \simeq M'^* \oplus M''^*$ hence (by applying again the functor *), $M \simeq M' \oplus M''$, which is a contradiction.

Exercises to Chapter 8

Exercise 8.1. Let $\mathcal{C}$ be an abelian category with enough injectives. Prove that the two conditions below are equivalent.
(i) For all X and Y in $\mathcal{C}$, $Ext^j(X, Y) \simeq 0$ for all $j > n$.
(ii) For all X in $\mathcal{C}$, there exists an exact sequence $0 \to X \to X^0 \to \cdots \to X^n \to 0$, with the X^j's injective.
In such a situation, one says that $\mathcal{C}$ has homological dimension $\leq n$ and one writes $dh(\mathcal{C}) \leq n$.
(iii) Assume moreover that $\mathcal{C}$ has enough projectives. Prove that (i) is equivalent to: for all X in $\mathcal{C}$, there exists an exact sequence $0 \to X^n \to \cdots \to X^0 \to X \to 0$, with the X^j's projective.

Exercise 8.2. Let $\mathcal{C}$ be an abelian category with enough injective and such that $dh(\mathcal{C}) \leq 1$. Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor and let $X \in D^+(\mathcal{C})$.
(i)Construct an isomorphism $H^k(RF(X)) \simeq F(H^k(X)) \oplus R^1F(H^{k-1}(X))$.
(ii) Recall that $dh(\mathrm{Mod}(\mathbb{Z})) = 1$. Let $X \in D^-(\mathrm{Mod}(\mathbb{Z}))$, and let $M \in \mathrm{Mod}(\mathbb{Z})$. Deduce the isomorphism $H^k(X \otimes^L M) \simeq (H^k(X) \otimes M) \oplus Tor_1(H^{k+1}(X), M)$.

Exercise 8.3. Let $\mathcal{C}$ be an abelian category with enough injectives and let $0 \to X' \to X \to X'' \to 0$ be an exact sequence in $\mathcal{C}$. Assuming that $Ext^1(X'', X') \simeq 0$, prove that the sequence splits.

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Institut de Mathématiques, Analyse Algébrique
Université Pierre et Marie Curie, Case 82
4, place Jussieu F-75252, Paris Cedex 05, France
email: `schapira@math.jussieu.fr`
Homepage: `www.math.jussieu.fr/~schapira`