Hopf Algebras and Quantum Groups

C. Daskaloyannis, AUTH

Notions discussed:

- TENSOR PRODUCT of vector spaces $U \otimes_k V$,
  TENSOR ALGEBRA of V: $T(V) = k \oplus V \oplus (V \otimes V) \oplus \cdots$,

- LIE ALGEBRA g and UNIVERSAL ENVELOPING ALGEBRA of g: $U(g) \cong T(g)/I$ where $I =< [x, y] - (x \otimes y - y \otimes x) >$

- Definition of an ALGEBRA USING COMMUTATIVE DIAGRAMS (A with product $m : A \otimes A \to A$ and unity $\eta : k \to A$)

- Definition of a CO-ALGEBRA USING DUAL DIAGRAMS (A with co-product $\Delta : A \to A \otimes A$ and co-unity $\epsilon A \to k$)

- BI-ALGEBRA = algebra+co-algebra, with co-product and co-unity, which are algebraic morphisms $\Delta(xy) = \Delta(x)\Delta(y), \ \Delta(1) = 1 \otimes 1, \ \epsilon(xy) = \epsilon(x)\epsilon(y)$

- HOPF ALGEBRA $(A, m, \eta, \Delta, \epsilon, S)$ where $S : A \to A$ is the antipode(co-inverse)

- Examples of Hopf algebras:
  - $U(g)$ is a Hopf algebra,
  - The group of transformations on the plane $SL(2)$. THE FUNCTIONS ON SL(2), $O(SL(2)) \cong k[a, b, c, d]/(ab - cd - 1)$ is a Hopf algebra with

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

  - The QUANTIZATION of the plane and the $SL_q(2)$
    The MANIN PLANE (noncommutative plane), $k_q\{x, y\} = k\{x, y\}/I$ where $I =< yx - qxy >$.
    The transformations on the Manin plane $SL_q(2) = k_q\{a, b, c, d\}/(ad - qbc - 1)$ where $ab = qba$, $ac = qca$, $bd = qbd$, $cd = qdc$, $bc = cb$ and $ad - da = (q - q^{-1})c$ is a Hopf algebra. $SL_q(2)$ is a QUANTUM GROUP corresponding to $SL(2)$.
  - The QUANTIZATION OF THE LIE ALGEBRA $sl(2)$: $U(sl(2))$ and the QUANTUM LIE ALGEBRA $U_q(sl(2))$. Finite dim representations of .
  - $U_q(g)$ DRINFELD-JIMBO QUANTIZATION of a Lie algebra g

Bibliography:

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