On the enumeration of lattice points in polytopes

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Let $P \subset \mathbb{R}^n$ be a lattice polytope, i. e.the convex hull of finitely many points of the integral lattice $\mathbb{Z}^n$. A classical object of enumerative combinatorics is the function

$$E(P,k) = \#(kP \cap \mathbb{Z}^n),$$

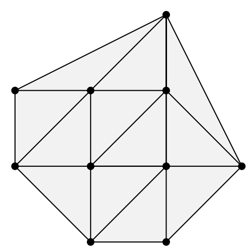
whose study was pioneered by the French mathematician Ehrhart. The generating function

$$E_P(t) = \sum_{k=0}^{\infty} E(P,k)t^k$$

turns out to be (the power series expansion of) a rational function of negative degree:

$$E_P(t) = \frac{1 + h_1 t + \cdots + h_d t^d}{(1-t)^{\dim(P)+1}}, \qquad d \le \dim(P).$$

This equivalent to the fact that $E(P,k)$ is given by a polynomial for all $k \ge 0$, the Ehrhart polynomial of P. We are interested in the so-called h-vector $(h_0 = 1, h_1, \dots, h_d)$, $h_d \neq 0$; it reflects several properties of P.



$$E_P(t) = \frac{1 + 7t + 4t^2}{(1-t)^3}$$

A class of special interest is the one in which h is palindromic, i. e. $h_i = h_{d-i}$ for all i. This is the case when a multiple kP contains a lattice point x that has lattice distance 1 to all facets of kP. For a certain polytope P in this class Stanley conjectured that the h-vector of P is unimodal, i. e. satisfies the inequalities $h_0 \le h_1 \le \cdots \le h_{[d/2]}$. Stanley's conjecture was recently proved by Athanasiadis. We extend his theorem to the class of all polytopes (with palindromic h-vector) that have a regular unimodular triangulation, i. e. a triangulation into lattice simplices of smallest possible volume that can be realized as the subdivision of P into the domains of linearity of a piecewise affine, concave function on P. The proof is reduction to the g-theorem, conjectured by McMullen and proved by Stanley. (Joint work with Tim Römer.)