Periodic orbits of trans-Neptunian objects at the 2/3 and 3/4 resonances

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1. Introduction

The interest on studying the dynamics of exterior resonances is continuously growing after the discover of objects in the Edgeworth-Kuiper belt. Useful information on the structure of the phase space can be obtained by studying families of 2-D and 3-D periodic orbits in the circular restricted three-body problem and study their stability. This is so, because the periodic orbits and their stability character define the structure of the phase space and in this way they put an order in the study of all the orbits of the system.

We compute families of 2-D and 3-D resonant periodic orbits, and their stability, for the exterior 2/3 and 3/4 resonances with Neptune.

2. Families of periodic orbits and stability

We assume that the orbit of Neptune is circular and we consider a rotating frame of reference $Oxyz$ whose x-axis is the line joining the Sun, S, with Neptune, N, the positive direction being from S to N, its origin is at their centre of mass, the y-axis is in the orbital plane of Neptune and the z-axis is perpendicular to the xy plane. In the usual normalized units where the distance $(SN) = 1$, the gravitational constant is $G = 1$ and the total mass $m_S + m_N = 1$, the differential equations of motion are the equations of the restricted circular 3-body problem (Roy, 1982). The mass of Neptune was taken equal to $\mu = 5.178 \times 10^{-5}$. In the above mentioned rotating frame there exist planar families of symmetric periodic orbits of the small object, in the Oxy plane, which "lie" outside the orbit of Neptune and are nearly circular or nearly elliptic. In figure 1a we have drawn the families of periodic orbits at the exterior resonances 2/3 and 3/4 in a diagram $x_0 - Jacobi\ constant$. Families $I_{2/3}$ and $I_{3/4}$ are unstable, but families $II_{2/3}$ and $II_{3/4}$ are stable. Along the families $II_{2/3}$ and $II_{3/4}$ gaps appear at $e = 0.80$ and $e = 0.60$, respectively due to difficulties in the

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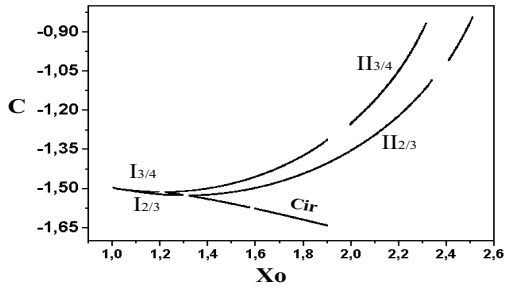

Figure 1. Families of simple periodic orbits at the resonances 2/3 and 3/4. The symbol *cir* denotes the circular orbits for $\mu > 0$.

numerical computations, because we are close to a collision orbit. At this point we have an abrupt change of the stability and also of the multiplicity of the orbits, (number of intersections with the x-axis). Along the above planar families of periodic orbits we have computed the vertical stability, i.e. the stability with respect to perturbations perpendicular to the plane of motion. Several vertical critical orbits have been found, where there is a change from stability to instability, or vice versa. At these orbits we have a bifurcation of families of 3-D periodic orbits. For the computation of the vertical stability we used the method developed by Hénon (1973).

There are two types of three-dimensional symmetric periodic orbits:
(a) The third body starts perpendicularly from the xz-plane and crosses again perpendicularly the xz-plane after half period. The nonzero initial conditions of a periodic orbit are $(x_0,\ z_0,\ \dot{y}_0)$.
(b) The third body starts perpendicularly from the x-axis and crosses again perpendicularly the x-axis after half period. The nonzero initial conditions of a periodic orbit are $(x_0,\ \dot{y}_0,\ \dot{z}_0)$.
The eccentricity and the inclination vary along the families, but the semimajor axis is almost constant, $a_{N2/3} = 1.31037$ and $a_{3/4} = 1.21141$.
The 2/3 Resonance
Three vertical critical orbits along the family $II_{N2/3}$ of the planar problem were found from which bifurcate the 3-D families of periodic orbits. These points correspond to the eccentricities $e_1 = 0.42125$, $e_2 = 0.45004$ and $e_3 = 0.967767$. From the first point there bifurcates the family A_1 and from the second and third points there bifurcate two families, which in fact coincide, so we have one more family of 3-D periodic orbits, A_{23}. In figures 2a-2b we show these families. Family A_1 is of type (a) and family A_{23} is of type (b) mentioned above. In the type (b) orbits we presented the families in the space (x_0, *eccentricity*, *inclination*), instead of the initial conditions.

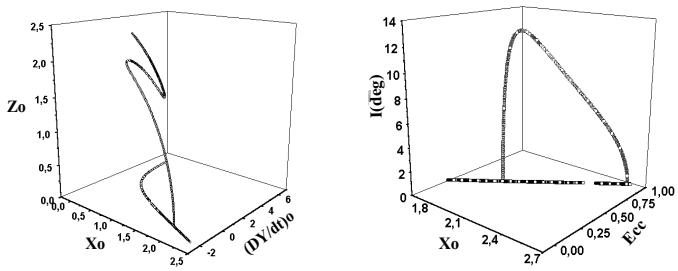

Figure 2. (2a): Family A_1 of the N2/3 resonance. There is a maximum at $z_1 = 2.0330960676$ and a minimum at $z_2 = 1.727231054$. (2b): Family A_{23} of the N2/3 resonance. The maximum is at $i = 13^o$.

The 3/4 Resonance

At this case five vertical critical orbits along the family $II_{3/4}$ of the planar problem were found, corresponding to the eccentricities $e_1 = 0.291335$, $e_2 = 0.306772$, $e_3 = 0.662853$, $e_4 = 0.752902$ and $e_5 = 0.767551$. In Figure 3a we show the family B_1, which bifurcates from the first point mentioned above and is of type (a). It starts with direct orbits (revolving, in the inertial frame, in the same direction as Neptune), with multiplicity 2, and terminates at a point corresponding to a retrograde periodic orbit with multiplicity 7, with almost zero eccentricity, belonging to a family of retrograde planar circular periodic orbits. In Figure 3b we show the family B_{25}, which bifurcates from the second point and terminates to the fifth point and is of type (b). It starts with orbits of multiplicity 2 and terminates with multiplicity 4. In Figure 4a we show the family B_3, which bifurcates from the third point and is of type (a). It starts with direct orbits of multiplicity 4, and terminates at a point corresponding to a retrograde planar periodic orbit with eccentricity $e = 0.35754$ and multiplicity 7, belonging to a family of retrograde planar elliptic periodic orbits. Finally in Figure 4b we show the family B_4, which bifurcates from the fourth point and is of type (a). It starts with direct orbits of multiplicity 4 and ends to a collision orbit with eccentricity almost equal to 1.

The linear isoenergetic stability of the three-dimensional periodic orbits can be studied by the method of surface of section of Poincaré (e.g. Siegel and Moser, 1971) and is similar to the one developed by Hadjidemetriou (1975) for the case of the planar general three-body problem (for details see Michalodimitrakis, 1979).

Family A_1: stable up to $z_{crit} = 1.8941766379$ (after the maximum),then unstable. There is a small unstable region between $z_3 = 1.952$ and $z_4 = 1.994$.

Family A_{23}: unstable until $x_1 = 2.5264204785$, then stable.

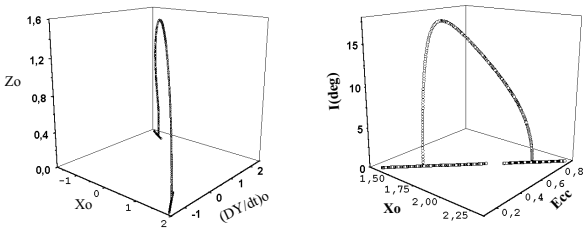

Figure 3. (3a): Family B_1 at the 3/4 resonance. The maximum is at $z_0 = 1.5688723869$. (3b): Family B_{25} at the 3/4 resonance. The maximum is at $i = 18^o$.

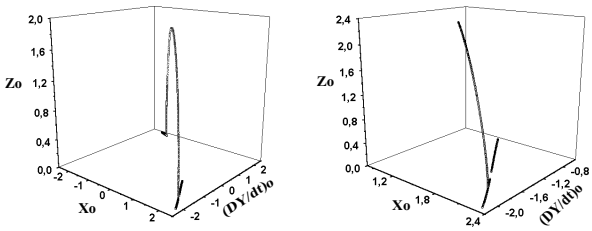

Figure 4. (4a): Family B_3 at the 3/4 resonance. The maximum is at $z = 1.8525582916$ (4b): Family B_4 at the 3/4 resonance.

Family B_1: stable up to $x = 0.9199809211$, unstable until $x = -1.1798368940$, and then stable again.
Family B_{25}: unstable up to $x = 1.6950556580$, stable up to $x = 1.86462425$, unstable until $x = 2.1161860449$ and finally again stable.
Family B_3: unstable up to $x = 1.9929097838$, stable up to $x = 1.96887977$, unstable up to $x = 0.9510947$ and again stable.
Family B_4: unstable until $x = 2.1071284826$ and then stable.

Keywords: TNOs - periodic orbits - bifurcation - resonances

Acknowledgement: Th. A. Kotoulas is financially supported by the Hellenic Scholarship Foundation (I.K.Y.).

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