Weighted Automata and Networks

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Seminar on Theoretical Computer Science and Discrete Mathematics
Aristotle University of Thessaloniki, Greece
October 30, 2018

I

Weighted and Multi-Valued Automata

⋄ Weights and Truth values ⋄

- ***weighted automata*** – classical nondeterministic automata in which the transitions carry weights
- weights are also assigned to states – ***initial*** and ***final weights***
- formally $\mathcal{A} = (A, \sigma, \tau, \{\delta_x\}_{x\in X})$

  A – set of states with $|A| = n$

  X – fixed input alphabet

  $\sigma, \tau : A \to K$ – ***initial*** and ***final vectors*** *with entries in* K

  $\delta_x : A \times A \to K$ – ***transition matrices*** *with entries in* K

- $\delta_x(a, b)$ – weight of a transition from a to b imposed by x

  $\sigma(a)$ / $\tau(a)$ – measure how much a is an initial / final state

Weights

- ⋆ model certain ***quantitative properties***
- ⋆ amount of resources needed for the execution of a transition
- ⋆ time needed for the execution
- ⋆ cost of the execution
- ⋆ probability of successful execution of a transition
- ⋆ reliability of successful execution ...

- ⋆ ***operations on weights***: ***multiplication*** and ***addition*** – accumulation of weights
- ⋆ ***structures of weights***: ***semirings, bimonoids*** (distributivity is not neccessary), ...

Truth values

- ***ordered algebraic structures* – *truth values***
- ***multi-valued logic* – *graded truth*** or ***intermediate truth***
- *subtle nuances in reasoning* – modeling of ***uncertainty***
- representation of imprecise aspects of human knowledge

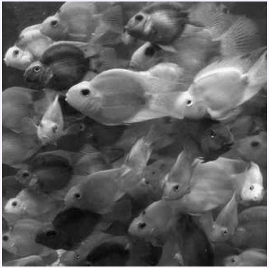
- ***ordering*** is essential – comparison of truth values
- ***operations on truth values*** – logic conjectives
- $[0,1]$ (real unit interval), max, min – ***Gödel structure***
- $[0,1]$, t-norm, t-conorm – ***Łukasiewicz*** and ***product structure***

- *linearity of the ordering is not essential*
- truth values – ***lattices, ordered algebraic structures***

Multi-valued logics

Classical Boolean logic

two-element Boolean algebra



Multi-valued logics with linearly ordered structures of truth values

structures on $[0, 1]$ *determined by* t*-norms*

Multi-valued logics with general structures of truth values (not necessarily linearly ordered)

lattices, residuated latices, etc.

- ***lattices***: finite infimum and supremum –
  conjunction and ***disjunction*** (***intersection*** and ***union***)
- ***complete lattices***: infinite infimum and supremum –
  universal and ***existential quantifiers***

- *infimum does not necessarily distribute over supremum*
  (except in distributive lattices)
- new operation: ***multiplication*** $\otimes$ – *distributes over suprema*
  strong conjunction

- ***lattice ordered monoid***
  monoid + *partial order* (*compatible* w.r.t. *multiplication* $\otimes$)
  lattice w.r.t. this partial order
  $\otimes$ *distributes* over (finite) suprema
  $(\vee, \otimes)$-reduct – ***semiring reduct***

- ***How to model the implication?***

- ***Residuated function***
- $(P, \leqslant)$, $(Q, \leqslant)$ – partially ordered sets, $f : P \to Q$
- f is ***residuated*** if there is $g : Q \to P$ satisfying

$$f(x) \leqslant y \quad \Leftrightarrow \quad x \leqslant g(y)$$

- ***residuation property***
- if exists, such g is unique
- it is called the ***residual*** of f and denoted by $f^\sharp$

Theorem on residuated functions

The following conditions for $f : P \to Q$ are equivalent:

(i) *f is residuated;*

(ii) *f is isotone and there is an isotone $g : Q \to P$ such that*

$$I_P \leqslant f \circ g, \qquad g \circ f \leqslant I_Q;$$

(iii) *the inverse image under f of every principal down-set of Q is a principal down-set of P;*

(iv) *f is isotone and the set $\{x \in P \mid f(x) \leqslant y\}$ has the greatest element, for every $y \in Q$.*

- ★ principal down-set: $a\downarrow = \{x \in P \mid x \leqslant a\}$
- ★ lattice-theoretical counterpart of a ***continuous function***
- ★ $f^\sharp(y) = \top\{x \in P \mid f(x) \leqslant y\}$ ($\top H$ – greatest element of H)

- $(S, \otimes, \leqslant)$ – ***ordered semigroup***
- $\leqslant$ is compatible w.r.t. $\otimes$
- for $a \in S$, functions $\lambda_a, \varrho_a : S \to S$ are defined by

$$\lambda_a(x) = a \otimes x, \qquad \varrho_a(x) = x \otimes a$$

- λ_a – ***inner left translation*** w.r.t. a
  ϱ_a – ***inner right translation*** w.r.t. a
- ***residuated semigroup*** – λ_a and ϱ_a are ***residuated functions***
- $a \backslash b = \lambda_a^\sharp(b) = \top\{x \in S \mid a \otimes x \leqslant b\}$ – *right residual* of b by a
- $b / a = \varrho_a^\sharp(b) = \top\{x \in S \mid x \otimes a \leqslant b\}$ – *left residual* of b by a
- ***residuation property***

$$a \otimes b \leqslant c \quad \Leftrightarrow \quad b \leqslant a \backslash c \quad \Leftrightarrow \quad a \leqslant c / b$$

★ ***Quantale***

ordered semigroup $(S, \otimes, \leqslant)$

complete lattice w.r.t. $\leqslant$

$\otimes$ *distributes* over ***arbitrary suprema*** (finite and infinite)

★ $\otimes$ is *not necessarily commutative*

★ ***inifinite distributivity*** $\Rightarrow$ existence of residuals

$$a \backslash b = \bigvee \{x \in S \mid a \otimes x \leqslant b\} = \top \{x \in S \mid a \otimes x \leqslant b\}$$

$$b / a = \bigvee \{x \in S \mid x \otimes a \leqslant b\} = \top \{x \in S \mid x \otimes a \leqslant b\}$$

★ ***unital quantale*** – with a *multiplicative unit* e

★ $(S, \vee, \otimes, 0, e)$ – semiring (semiring reduct)

★ ***integral quantale*** – e is the greatest element ($e = 1$)

- general meaning: *lattice-ordered residuated semigroup*
- *not necessarily commutative* (left and right residuals)
- *not necessarily complete, not necessarily bounded*
- ***multi-valued logic*** – requires *commutativity* and *completeness*

- ***Residuated lattice*** – algebra $\mathbb{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$

  $(L, \vee, \wedge, 0, 1)$ – *bounded lattice* with the least element 0 and the greatest element 1

  $(L, \otimes, 1)$ – *commutative monoid* with the unit 1

  $\otimes$ and $\to$ satisfy the ***residuation property***

$$x \otimes y \leqslant z \quad \Leftrightarrow \quad x \leqslant y \to z$$

- ***Complete residuated lattice*** – the lattice reduct is complete

  commutative integral quantale

- ⋆ *only one residual* (left and right residuals coincide)
- ⋆ operation $\to$: ***residuum*** or ***residual implication***

  models the implication
- ⋆ *residuation property* $x \otimes y \leqslant z \quad \Leftrightarrow \quad x \leqslant y \to z$

  modus ponens rule

  deduction theorem
- ⋆ ***biresiduum*** or ***residual equivalence***:

$$x \leftrightarrow y = (x \to y) \wedge (y \to x)$$

  models the equivalence
- ⋆ ***negation***: $\neg x = x \to 0$

on $[0,1]$ with $x \wedge y = \min(x,y)$ and $x \vee y = \max(x,y)$

- ***Göedel structure***

$$x \otimes y = \min(x,y), \qquad x \to y = \begin{cases} 1 & \text{if } x \leqslant y \\ y & \text{otherwise} \end{cases}$$

- ***Product structure*** or ***Goguen structure***

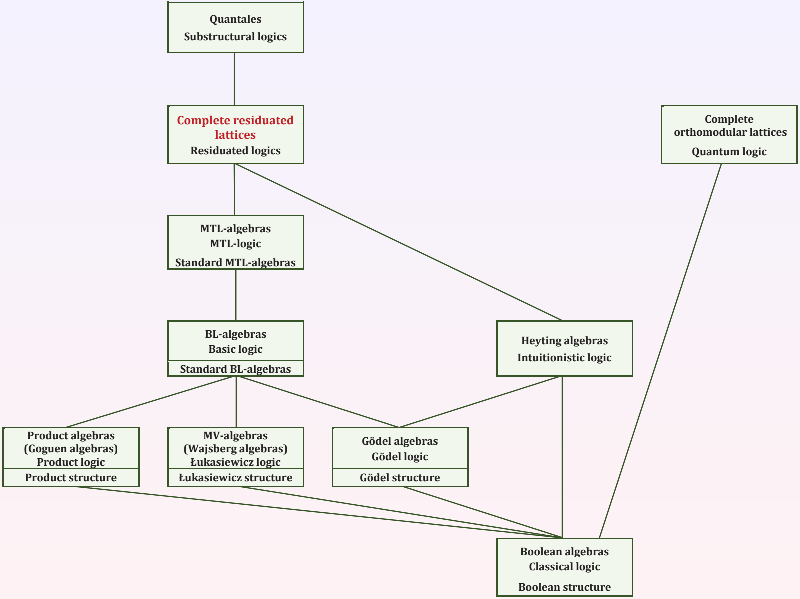
$$x \otimes y = x \cdot y, \qquad x \to y = \begin{cases} 1 & \text{if } x \leqslant y \\ \frac{y}{x} & \text{otherwise} \end{cases}$$

- ***Łukasiewicz structure***

$$x \otimes y = \max(x+y-1,0), \qquad x \to y = \min(1-x+y,1)$$

- Göedel and Łukasiewicz structure on finite chains in $[0,1]$
- ***Heyting algebra***: $\mathbb{L}$ with $\otimes = \wedge$ (bounded ***Brouwer lattice***)

Multi-valued structures and logics



- ⋆ ***Dedekind*** (1894) – quantales of ideals of rings
- ⋆ ***Schröder***, Algebra und Logik der Relative (Leipzig, 1895)

  quantales of binary relations
- ⋆ ***Brouwer*** (1920s) – relative pseudo complementation
- ⋆ ***Heyting*** (1930) – Heyting algebras
- ⋆ ***Ward, Dilworth*** (1930s) – (noncommutative) residuated lattices, arithmetical applications
- ⋆ ***Mulvey*** (1986) – quantale of closed linear subspaces of a non-commutative $C^\star$-algebra

  applications in functional analysis, topology

  Gelfand, *von Neumann*, and *Hilbert quantales*

II

Weighted Automata

◇ State Reduction ◇

- ***behaviour of*** a WFA $\mathcal{A}$ or ***language recognized by*** $\mathcal{A}$

  $[\![\mathcal{A}]\!] : X^* \to S$ given by $[\![\mathcal{A}]\!](u) = \sigma \cdot \delta_u \cdot \tau$

  $\delta_u = \delta_{x_1} \cdot \ldots \cdot \delta_{x_s}$, if $u = x_1 \cdots x_s$, $x_1, \ldots, x_s \in X$

- $\mathcal{A} = (A, X, \sigma^A, \tau^A, \{\delta_x^A\}_{x\in X})$ and $\mathcal{B} = (B, X, \sigma^B, \tau^B, \{\delta_x^B\}_{x\in X})$ are ***equivalent WFAs*** if $[\![\mathcal{A}]\!] = [\![\mathcal{B}]\!]$

- ***State Reduction Problem*:**

  Provide efficient methods for constructing a reasonably small WFA equivalent to $\mathcal{A}$ (not necessarily minimal)

- ***Equivalence Problem*:**

  Provide efficient methods for testing whether two WFAs $\mathcal{A}$ and $\mathcal{B}$ are equivalent

- *Mimimization of NFA* – ***computationally hard*** problem
- the same goes for weighted and multi-valued automata
- more practical – ***state reduction problem***

  *give an **efficient construction** of a **reasonably small** automaton (not necessarily minimal) **equivalent** to the given automaton*

- ***How to make a state reduction?***

- our main ideas came from algebra – ***quotient algebra***
- ***congruences*** – compatible equivalence relations
- elements of the quotient algebra – ***equivalence classes***
- ***rows*** or ***columns*** in the correspodning Boolean matrix

- $(S, +, \cdot, 0, e)$ – semiring with the unit e
- $\mathcal{A} = (A, \sigma, \tau, \{\delta_x\}_{x \in X})$ – weighted finite automaton over S
- $\pi \in S^{A \times A}$ – given matrix

- ***Our idea***: *Construct an WFA whose states would be rows of* π

- $a\pi$ – a-row of π – $\pi(a, \cdot)$, πb – b-column of π – $\pi(\cdot, b)$
- $\overline{A}$ – the set of all different rows of π
- define $\overline{\sigma}, \overline{\tau} : \overline{A} \to S$, $\overline{\delta}_x : \overline{A} \times \overline{A} \to S$, $x \in X$ by

$$\overline{\sigma}(a\pi) = (\sigma \cdot \pi)(a) = \sigma \cdot (\pi a)$$
$$\overline{\tau}(a\pi) = (\pi \cdot \tau)(a) = (a\pi) \cdot \tau$$
$$\overline{\delta}_x(a\pi, b\pi) = (\pi \cdot \delta_x \cdot \pi)(a, b) = (a\pi) \cdot \delta_x \cdot (\pi b)$$

- ***Question***: *Are these definitions good?*
- ***Answer***: *Not necessarily.*

  If $a\pi = a'\pi$ and $b\pi = b'\pi$, it does not have to be

  $\sigma \cdot (\pi a) = \sigma \cdot (\pi a')$ or $(a\pi) \cdot \delta_x \cdot (\pi b) = (a'\pi) \cdot \delta_x \cdot (\pi b')$

- ***Question***: *Under what conditions the definitions are good?*

- we need a ***partial order*** $\leqslant$ on S (not necessarily compatible)
- a square matrix $\pi : A \times A \to S$ is

  reflexive if $e \leqslant \pi(a, a)$, for all $a \in A$

  transitive if $\pi(a, b) \cdot \pi(b, c) \leqslant \pi(a, c)$, for all $a, b, c \in A$

- ***Quasi-order matrix*** – reflexive and transitive matrix

Theorem 1

Let $\pi \in S^{A\times A}$ be a quasi-order matrix and $a, b \in A$. Then the following conditions are equivalent:

(i) $\pi(a,b) = \pi(b,a) = e$

(ii) $a\pi = b\pi$

(iii) $\pi a = \pi b$

Theorem 2

Let $\pi \in S^{A\times A}$ be a quasi-order matrix.
Then $\overline{\sigma}$, $\overline{\tau}$ and $\overline{\delta}_x$ are well-defined and $\overline{\mathcal{A}} = (\overline{A}, \overline{\sigma}, \overline{\tau}, \{\overline{\delta}_x\}_{x\in X})$ is an WFA satisfying $|\overline{\mathcal{A}}| \leqslant |\mathcal{A}|$.

$\overline{\mathcal{A}}$ – ***row automaton*** – isomorphic to ***column automaton***

- S – semiring with unit e, $\leqslant$ – partial order on S
- partial order on $S^{A\times A}$ is defined entrywise

$$\mu \leqslant \eta \quad\Leftrightarrow\quad \mu(a,b) \leqslant \eta(a,b), \text{ for all } a, b \in S$$

- *if the ordering on S is compatible, then the ordering of matrices is also compatible*

- for a matrix π with entries in a lattice-ordered monoid

$$\pi \text{ is reflexive} \quad\Leftrightarrow\quad \Delta \leqslant \pi \quad\Rightarrow\quad \pi \leqslant \pi^2$$

$$\pi \text{ is transitive} \quad\Leftrightarrow\quad \pi^2 \leqslant \pi$$

$$\pi \text{ is a quasi-order matrix} \quad\Rightarrow\quad \pi^2 = \pi$$

- $\Delta(a,a) = e$ (the unit), $\Delta(a,b) = 0$, for $a \neq b$ – *unit matrix*

- ***Question:*** *Under what conditions this holds for matrices with entries in a semiring?*

- S – *positively ordered semiring* – compatible partial order $\leqslant$ and 0 is the least element

$$\pi \text{ is reflexive} \quad \Rightarrow \quad \Delta \leqslant \pi \quad \Rightarrow \quad \pi \leqslant \pi^2$$

- to prove

$$\pi \text{ is transitive} \quad \Rightarrow \quad \pi^2 \leqslant \pi$$

we need something like

$$a_1, \ldots, a_s \leqslant a \quad \Rightarrow \quad a_1 + \ldots + a_s \leqslant a$$

the addition behaves somehow like supremum

- ***Question***: *In which class of semirings all of this is true?*

- ***Answer***: *Additively idempotent semirings*

- $a + a = a$, for every $a \in S$ (equivalently $1 + 1 = 1$)
- positively partially ordered
- partial ordering: $a \leqslant b \iff a + b = b$
- supremum coincides with addition
- *every quasi-order matrix π satisfies $\pi^2 = \pi$*
- ***Question**: Why $\pi^2 = \pi$ is so important?*
- ***behaviour of the row automaton $\overline{\mathcal{A}}$*** (the general case)

  $[\![\overline{\mathcal{A}}]\!](\varepsilon) = \sigma \cdot \pi^2 \cdot \tau$

  $[\![\overline{\mathcal{A}}]\!](x_1 x_2 \cdots x_k) = \sigma \cdot \pi^2 \cdot \delta_{x_1} \cdot \pi^2 \cdot \delta_{x_2} \cdot \pi^2 \cdot \ldots \cdot \pi^2 \cdot \delta_{x_k} \cdot \pi^2 \cdot \tau$
- with $\pi^2 = \pi$ we avoid squares

Additively idempotent semirings – importance

- the basic concept of ***idempotent analysis***

  the usual arithmetic operations $(+, \cdot)$ *are replaced by a new set of basic operations – semiring operations* $(\max, +)$, $(\min, +)$, *etc.*

  some problems that are non-linear in the traditional analysis turn out to be linear over a suitable semiring

  tropical mathematics, tropical geometry ... (tropical semiring)

- ***algebraic path problems*** (generalization of the shortest path problem in graphs)
- ***optimization problems*** (including ***dynamic programming***)
- ***discrete-event systems***
- ***automata and formal language theory***

- ***Question*:** *Is $\overline{\mathcal{A}}$ equivalent to $\mathcal{A}$?* ***Answer*:** *Not necessarily.*
- ***Question*:** *Under what conditions they are equivalent?*

$$[\![\mathcal{A}]\!](\varepsilon) = \sigma \cdot \tau$$
$$[\![\mathcal{A}]\!](x_1x_2\cdots x_k) = \sigma \cdot \delta_{x_1} \cdot \delta_{x_2} \cdot \ldots \cdot \delta_{x_k} \cdot \tau$$

$$[\![\overline{\mathcal{A}}]\!](\varepsilon) = \sigma \cdot \pi \cdot \tau$$
$$[\![\overline{\mathcal{A}}]\!](x_1x_2\cdots x_k) = \sigma \cdot \pi \cdot \delta_{x_1} \cdot \pi \cdot \delta_{x_2} \cdot \pi \cdot \ldots \cdot \pi \cdot \delta_{x_k} \cdot \pi \cdot \tau$$

- π has to be a solution of ***the general system***

$$\sigma \cdot \tau = \sigma \cdot \pi \cdot \tau$$
$$\sigma \cdot \delta_{x_1} \cdot \delta_{x_2} \cdot \ldots \cdot \delta_{x_k} \cdot \tau = \sigma \cdot \pi \cdot \delta_{x_1} \cdot \pi \cdot \delta_{x_2} \cdot \pi \cdot \ldots \cdot \pi \cdot \delta_{x_k} \cdot \pi \cdot \tau$$

- ⋆ it may consist of infinitely many equations
- ⋆ can not be solved efficiently
- ⋆ we have to find as possible greater solutions (greater solutions provide better reductions)
- ⋆ in the general case, there is no the greatest solution

- ⋆ ***instances of the general system***
- ⋆ systems whose any solution is a solution to the general system
- ⋆ *we need instances with finitely many equations or inequations which have the greatest solution and can be solved efficiently*

III

Weakly linear systems

◇ The General Results ◇

Weakly linear systems (The general form)

- $I = I_1 \cup I_2 \cup I_3 \cup I_4$ – nonempty set
- $A, \; B$ – nonempty sets
- $\{\alpha_i\}_{i\in I} \in S^{A\times A}, \; \{\beta_i\}_{i\in I} \in S^{B\times B}$ – given families of matrices
- μ – unknown taking values in $S^{A\times B}$
- ***weakly linear system***

$$
\begin{aligned}
\alpha_i \cdot \mu &\leqslant \mu \cdot \beta_i, && i \in I_1,\\
\mu \cdot \beta_i &\leqslant \alpha_i \cdot \mu, && i \in I_2,\\
\mu^\top \cdot \alpha_i &\leqslant \beta_i \cdot \mu^\top, && i \in I_3,\\
\beta_i \cdot \mu^\top &\leqslant \mu^\top \cdot \alpha_i, && i \in I_4,
\end{aligned}
$$

- ***homogeneous system*** – $A = B$ and $\alpha_i = \beta_i$, for all $i \in I$
- otherwise – ***heterogeneous system***

Theorem 1

*For an arbitrary $\gamma_0 \in S^{A\times B}$ the exists the **greatest solution** of the WLS which is less than or equal to γ_0.*

- ⋆ in the case of an homogeneeous WLS we have

Theorem 2

*Let γ_0 be a quasi-order matrix, and γ the greatest solution of the WLS such that $\gamma \leqslant \gamma_0$. **Then γ is also a quasi-order matrix**.*

- ⋆ ***How to compute the greatest solutions?***

$$
\begin{aligned}
\alpha_i \cdot \mu &\leqslant \mu \cdot \beta_i, & i &\in I_1, \\
\mu \cdot \beta_i &\leqslant \alpha_i \cdot \mu, & i &\in I_2, \\
\mu^\top \cdot \alpha_i &\leqslant \beta_i \cdot \mu^\top, & i &\in I_3, \\
\beta_i \cdot \mu^\top &\leqslant \mu^\top \cdot \alpha_i, & i &\in I_4,
\end{aligned}
$$

Definition

A function $\phi : S^{A\times B} \to S^{A\times B}$ is defined as follows

$$
\phi(\gamma) = \Big(\bigwedge_{i\in I_1} \alpha_i\backslash(\gamma\cdot\beta_i)\Big) \wedge \Big(\bigwedge_{i\in I_2} (\alpha_i\cdot\gamma)/\beta_i)\Big)
\wedge \Big(\bigwedge_{i\in I_3}\big[(\beta_i\cdot\gamma^\top)/\alpha_i)\big]^\top\Big) \wedge \Big(\bigwedge_{i\in I_4}\big[\beta_i\backslash(\gamma^\top\cdot\alpha_i)\big]^\top\Big),
$$

The function ϕ (cont.)

Theorem 3

- ϕ *is an isotone function on the complete lattice* $S^{A\times B}$
- *the considered WLS is equivalent to the inequation*

$$\mu \leqslant \phi(\mu)$$

- $\gamma \leqslant \phi(\gamma)$ – γ is a ***post-fixed point*** of ϕ
- $\gamma = \phi(\gamma)$ – γ is a ***fixed point*** of ϕ

- ***solving the WLS*** $\equiv$ ***computing post-fixed points*** of ϕ

Knaster-Tarski Fixed Point Theorem

Let L be a complete lattice, $\phi : L \to L$ an isotone function, and $a_0 \in L$

- ★ *there exists the greatest post-fixed point a of ϕ satisfying $a \leqslant a_0$*
- ★ *a is also the greatest fixed point of ϕ satisfying $a \leqslant a_0$*

- ★ Knaster-Tarski theorem provides the existence of the greatest solution of WLS which is less or equal to a given $\gamma_0 \in S^{A\times B}$
- ★ it does not provide a way to compute this solution

- ★ ***Problem***: *How to compute the greatest solutions?*

Kleene Fixed Point Theorem

Let L be a complete lattice, $\phi : L \to L$ an isotone function, and $a_0 \in L$

Define Kleene's descending chain $\{a_k\}_{k\in\mathbb{N}}$ of ϕ by

$$a_1 = a_0, \qquad a_{k+1} = a_k \wedge \phi(a_k)$$

- ★ *If the chain stabilizes at some a_k (i.e. $a_{k+1} = a_k$), then a_k is the greatest fixed point of ϕ less than or equal to a_0*
- ★ *if ϕ is Scott-continuous (i.e., it preserves lower-directed infima), then the greatest fixed point of ϕ less than or equal to a_0 is*

$$a = \bigwedge_{k\in\mathbb{N}} a_k$$

The greatest solutions of WLS

- ★ for $\gamma_0 \in S^{A\times B}$ we consider Kleene's descending chain $\{\gamma_k\}_{k\in\mathbb{N}}$ given by
$$\gamma_1 = \gamma_0, \qquad \gamma_{k+1} = \gamma_k \wedge \phi(\gamma_k)$$
- ★ if the subalgebra of S generated by all entries of matrices α_i, β_i and γ_0 satisfies DCC, *the chain stabilizes at some* γ_k
- ★ then γ_k *is the greatest solution of the WLS*
- ★ special cases: S *satisfies DCC* or *is locally finite*
- ★ special case: S is the ***max-plus quantale***
  - ϕ is Scott-continuous (i.e., ω-continuous)
  - the greatest solution which is less than or equal to γ_0 is
$$\gamma = \bigwedge_{k\in\mathbb{N}} \gamma_k$$

- $\mathbb{R}_\infty = \mathbb{R} \cup \{-\infty, +\infty\}$
- usual ordering
- ***multiplication***

$$a \otimes b = \begin{cases} a + b & \text{if } a, b \in \mathbb{R}, \\ -\infty & \text{if } a = -\infty \ \text{ or } \ b = -\infty, \\ +\infty & \text{if } a = +\infty,\ b \neq -\infty \ \text{ or } \ a \neq -\infty,\ b = +\infty, \end{cases}$$

- *commutative unital quantale*
- ***residuation***

$$a \to b = \begin{cases} b - a & \text{if } a, b \in \mathbb{R}, \\ -\infty & \text{if } b = -\infty, a \neq -\infty, \\ +\infty & \text{if } b = +\infty \ \text{ or } \ a = b = -\infty. \end{cases}$$

Related quantales

- $\mathbb{Q}_\infty = \mathbb{Q} \cup \{-\infty, +\infty\}$, $\quad \mathbb{Z}_\infty = \mathbb{Z} \cup \{-\infty, +\infty\}$
- subquantales of $\mathbb{R}_\infty$
- $\mathbb{R}_{\geqslant 0} \cup \{+\infty\}$, with multiplication and residuation

$$a \otimes b = \begin{cases} a + b & \text{if } a, b \in \mathbb{R}_{\geqslant 0}, \\ +\infty & \text{if } a = +\infty \ \text{ or } \ b = +\infty, \end{cases}$$

$$a \to b = \begin{cases} b - a & \text{if } a, b \in \mathbb{R}_{\geqslant 0} \ \text{ and } \ a \leqslant b, \\ \ 0 & \text{if } b \in \mathbb{R}_{\geqslant 0} \ \text{ and } \ a > b, \\ +\infty & \text{if } b = +\infty. \end{cases}$$

- subquantales $\mathbb{Q}_{\geqslant 0} \cup \{+\infty\}$ and $\mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$,

Related quantales (cont.)

- ★ $\mathbb{R}_{\geqslant 0} \cup \{+\infty\}$, with

$$a \otimes b = \begin{cases} a \cdot b & \text{if } a, b \in \mathbb{R}_{\geqslant 0}, \\ 0 & \text{if } a = 0 \ \text{ or } \ b = 0, \\ +\infty & \text{if } a = +\infty,\ b \neq 0, \ \text{ or } \ b = +\infty,\ a \neq 0 \end{cases}$$

$$a \to b = \begin{cases} b/a & \text{if } a \in \mathbb{R}_{>0},\ \ b \in \mathbb{R}_{\geqslant 0}, \\ 0 & \text{if } a = +\infty,\ \ b \in \mathbb{R}_{\geqslant 0}, \\ +\infty & \text{if } a = 0 \ \text{ or } \ b = +\infty. \end{cases}$$

- ★ ***max-min quantale*** or ***fuzzy algebra*** – $\mathbb{R} \cup \{-\infty, +\infty\}$, with

$$a \otimes b = a \wedge b, \qquad a \to b = \begin{cases} b & \text{if } a > b, \\ +\infty & \text{if } a \leqslant b. \end{cases}$$

- ⋆ ***Max-plus semiring*** – carrier $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$
- ⋆ ***Open problem*: *How to avoid* $+\infty$ *as an entry in matrices* γ_k*?***
- ⋆ if $\tau(a) = -\infty$, for some $a \in A$, then

$$\gamma_0(a,a) = (\tau/\tau)(a,a) = \tau(a) \to \tau(a) = -\infty \to -\infty = +\infty$$

- ⋆ *To replace* $+\infty$ *on the diagonal od* γ_0 *by an enough big real number?*
- ⋆ the final result should be a *quasi-order matrix*
- ⋆ the new starting matrix should also be *quasi-order matrix*

IV

Weighted Automata

⋄ Back to State Reduction ⋄

Right and left invariant matrices

- ⋆ $\mathcal{A} = (A, \sigma, \tau, \{\delta_x\}_{x\in X})$ – weighted automaton over a quantale S
- ⋆ ***right invariant matrices*** – solutions of

$$\mu \cdot \delta_x \leqslant \delta_x \cdot \mu, \qquad x \in X,$$
$$\mu \cdot \tau \leqslant \tau$$

- ⋆ ***left invariant matrices*** – solutions of

$$\delta_x \cdot \mu \leqslant \mu \cdot \delta_x, \qquad x \in X,$$
$$\sigma \cdot \mu \leqslant \sigma$$

- ⋆ $\mu \cdot \tau \leqslant \tau \iff \mu \leqslant \tau/\tau$ and $\sigma \cdot \mu \leqslant \sigma \iff \mu \leqslant \sigma\backslash\sigma$
- ⋆ $\tau/\tau, \sigma\backslash\sigma \in S^{A\times A}$ are ***quasi-order matrices*** given by

$$(\tau/\tau)(a,b) = \tau(a)/\tau(b), \qquad (\sigma\backslash\sigma)(a,b) = \sigma(a)\backslash\sigma(b)$$

- *How to compute the greatest right and left invariant matrices?*
- ***they can be computed as the greatest solutions of the corresponding WLS that are less than or equal to*** τ/τ ***or*** $\sigma\backslash\sigma$
- *Why we need right and left invariant matrices?*
- ***right and left invariant matrices are solutions of the general system***
- ***right and left invariant matrices provide state reductions that may be efficiently realised***

- ⋆ $\mathcal{A}$ – WFA, π – the greatest left invariant q-o.m. on $\mathcal{A}$

  when $\mathcal{A}$ is reduced by means of π, the resulting row automaton $\mathcal{A}/\pi$ *can not be reduced by means of right invariant q-o.m.*

  however, *it could be reduced by means of left invariant q-o.m.*

- ⋆ ***Alternate reductions***

  we alternately make a series of reductions by means of the greatest right and left invariant q-o.m., or vice versa

  this procedure will be interrupted when we get an automaton that can not be reduced by means of alternate reductions

V

Weighted Automata

◇ Equivalence ◇

◇ Simulation and Bisimulation ◇

- ***Equivalence Problem**: Provide efficient methods for testing whether two WFAs $\mathcal{A}$ and $\mathcal{B}$ are equivalent*

- *Equivalence Problem is computationally hard*
- equivalence of WFAs *can not be expressed through matrices*, as some kind of *relationship between states*

- ***simulation**: $A \times B$-matrix which provides that $\mathcal{B}$ **simulates** $\mathcal{A}$*
- ***bisimulation**: $A \times B$-matrix which, together with its transpose, provides that $\mathcal{B}$ and $\mathcal{A}$ **simulate each other***
- *existence of a bisimulation implies equivalence of $\mathcal{A}$ and $\mathcal{B}$*

  bisimulations provide ***approximations of equivalence***

- ***simulations*** and ***bisimulations*** – defined as solutions of particular *systems of matrix inequations*

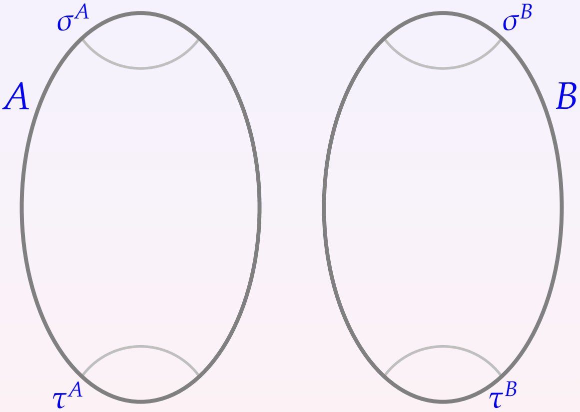
- ⋆ S – ***unital quantale***
- ⋆ WFAs $\mathcal{A} = (A, X, \sigma^A, \tau^A, \{\delta_x^A\}_{x\in X})$, $\mathcal{B} = (B, X, \sigma^B, \tau^B, \{\delta_x^B\}_{x\in X})$
- ⋆ ***forward simulations*** – solutions of

$$\mu^\top \cdot \delta_x^A \leqslant \delta_x^B \cdot \mu^\top, \qquad x \in X,$$
$$\mu^\top \cdot \tau^A \leqslant \tau^B \qquad \text{(equivalently } \mu \leqslant \gamma_0 = (\tau^B/\tau^A)^\top\text{)}$$
$$\sigma^A \leqslant \sigma^B \cdot \mu^\top$$

- ⋆ ***backward simulations*** – solutions of

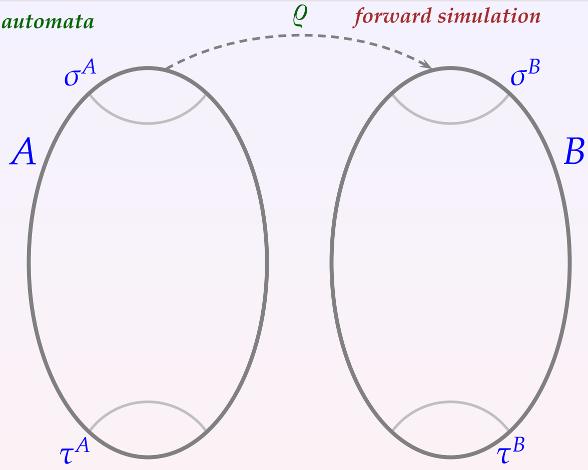
$$\delta_x^A \cdot \mu \leqslant \mu \cdot \delta_x^B, \qquad x \in X,$$
$$\sigma^A \cdot \mu \leqslant \sigma^B \qquad \text{(equivalently } \mu \leqslant \gamma_0 = \sigma^A \backslash \sigma^B\text{)}$$
$$\tau^A \leqslant \mu \cdot \tau^B$$

nondeterministic automata

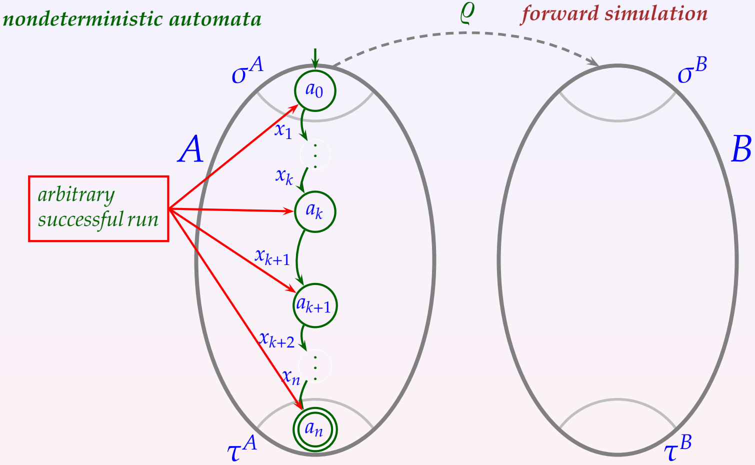


Forward and backward simulations

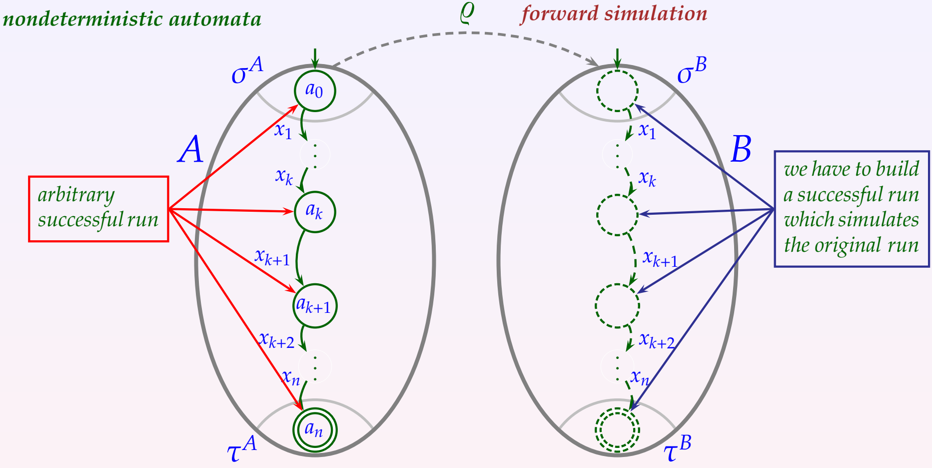
nondeterministic automata



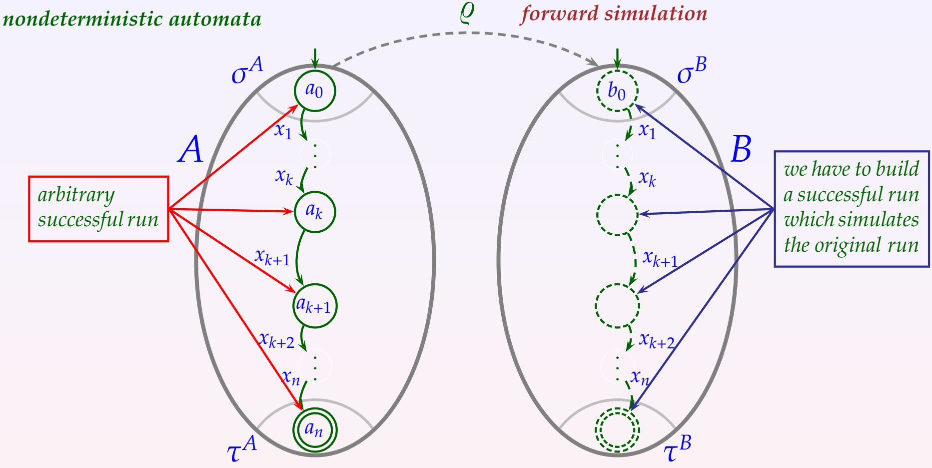
Forward and backward simulations



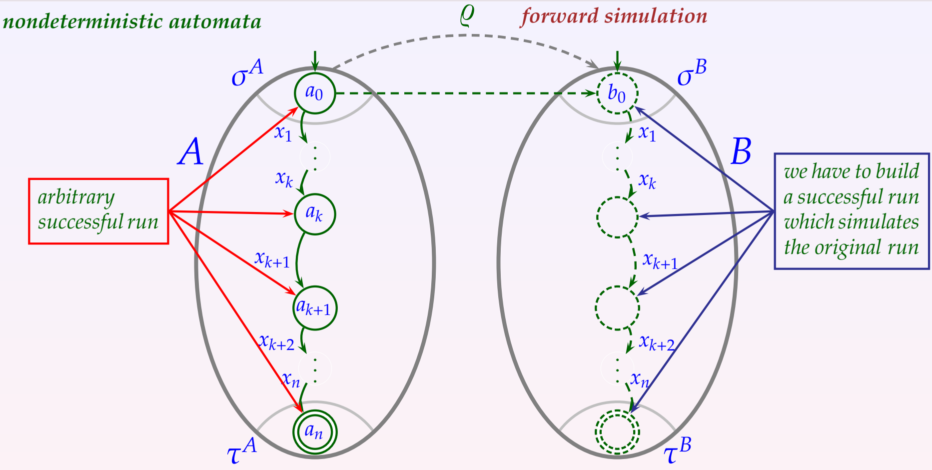
Forward and backward simulations

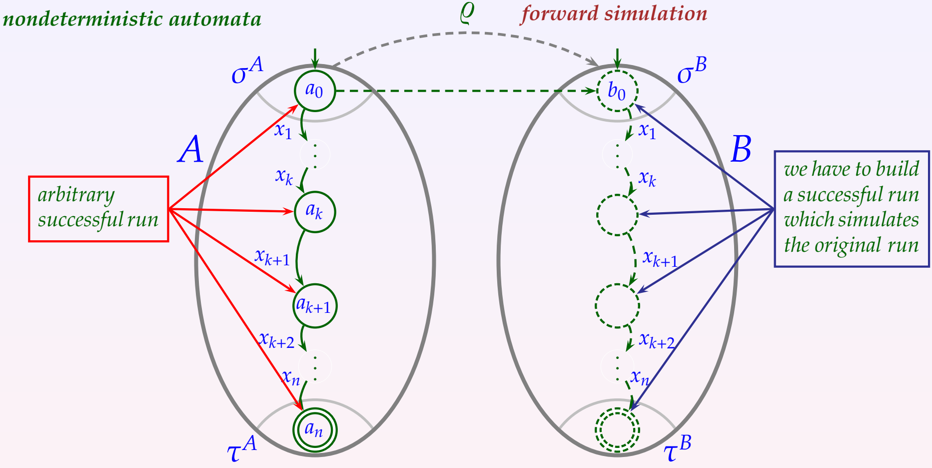


Forward and backward simulations



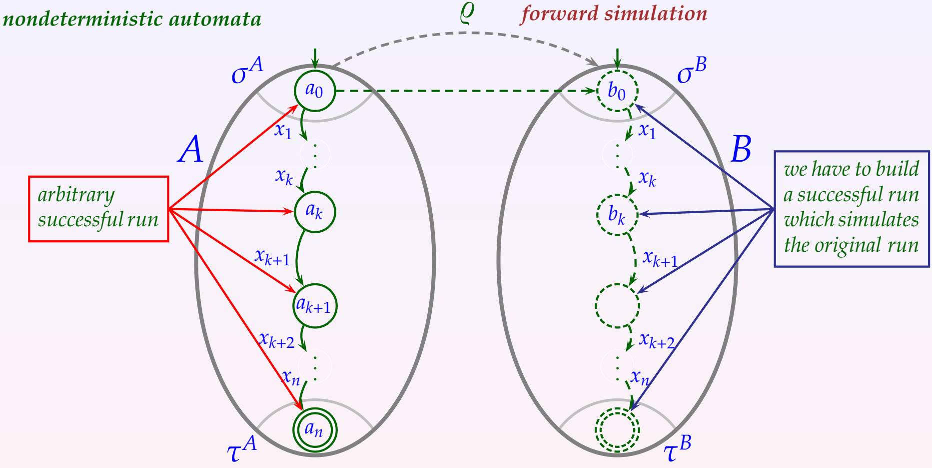
Forward and backward simulations



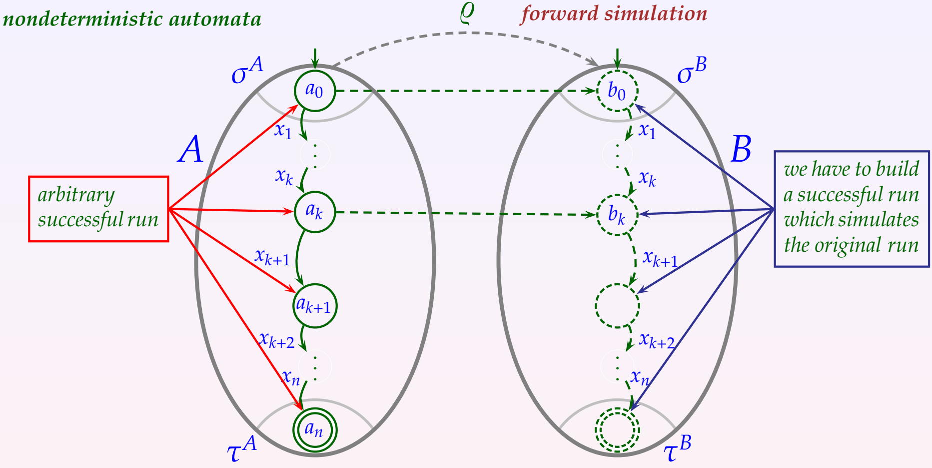


$$(1)\quad \sigma^A \leqslant \sigma^B \cdot \varrho^\top$$

Forward and backward simulations

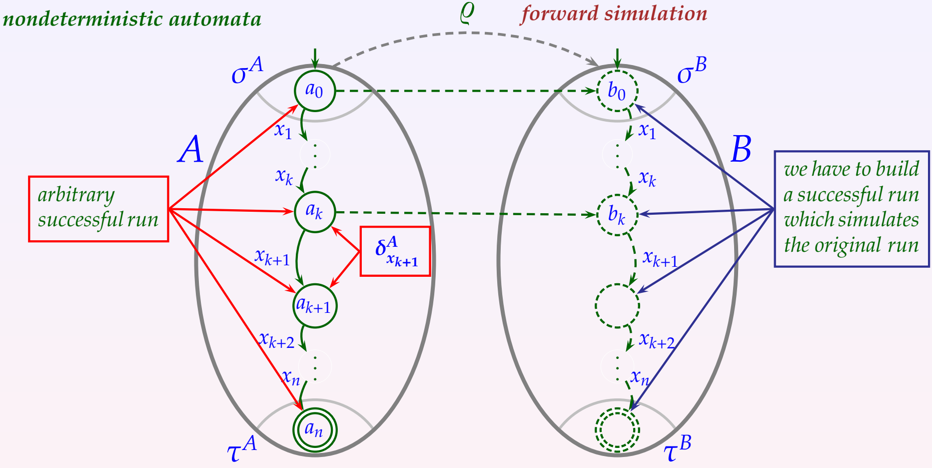


(1) $\sigma^A \leqslant \sigma^B \cdot \varrho^\top$



(1) $\sigma^A \leqslant \sigma^B \cdot \varrho^\top$

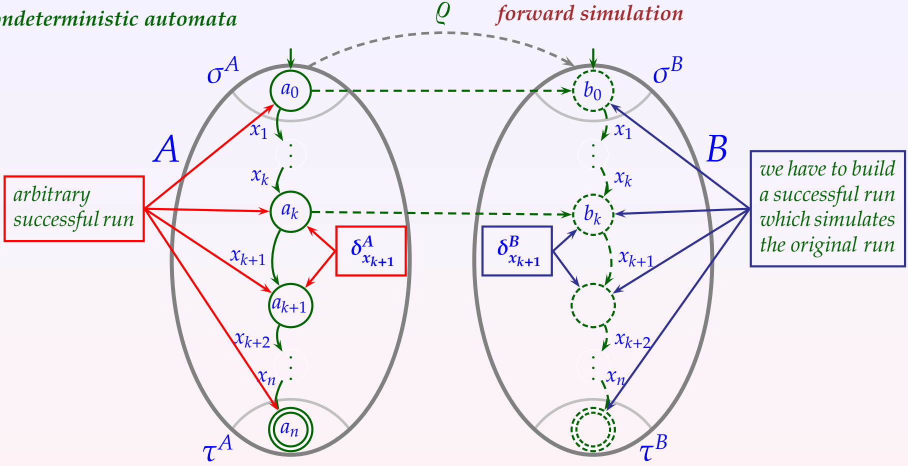
Forward and backward simulations



(1) $\sigma^A \leqslant \sigma^B \cdot \varrho^\top$

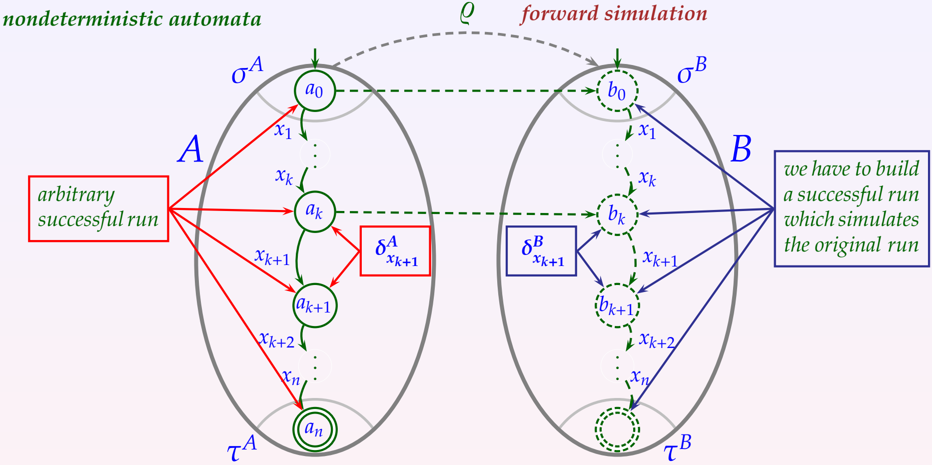
Forward and backward simulations

nondeterministic automata



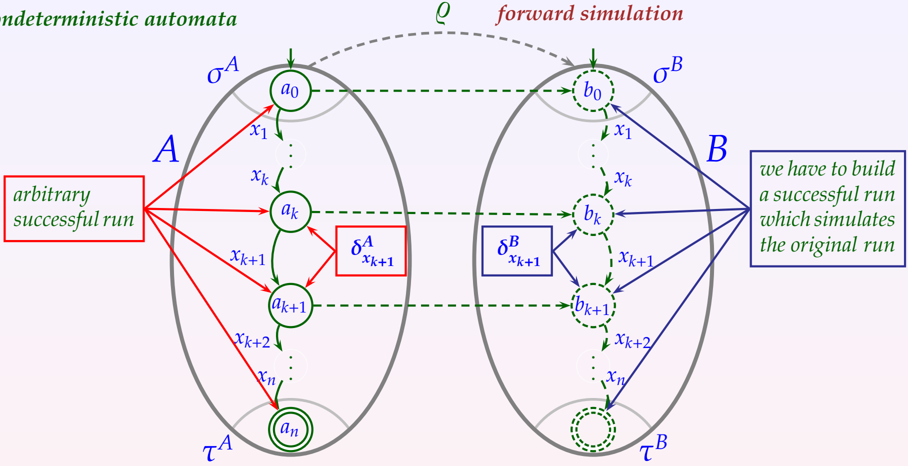
(1) $\sigma^A \leqslant \sigma^B \cdot \varrho^\top$

Forward and backward simulations



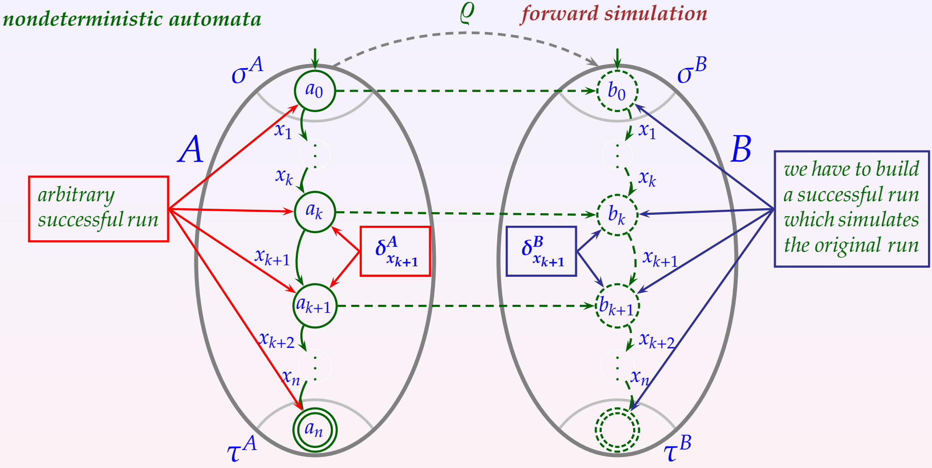
(1) $\sigma^A \leqslant \sigma^B \cdot \varrho^\top$

nondeterministic automata



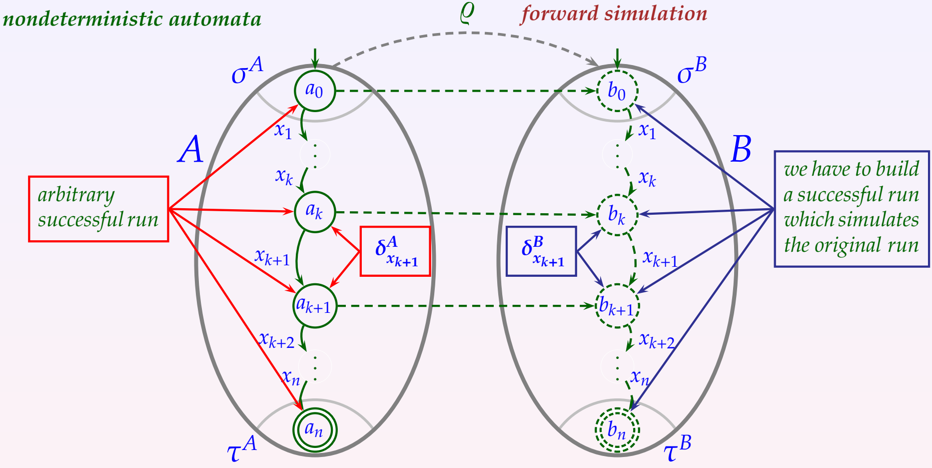
(1) $\sigma^A \leqslant \sigma^B \cdot \varrho^\top$

Forward and backward simulations



(1) $\sigma^A \leqslant \sigma^B \cdot \varrho^\top$ (2) $\varrho^\top \cdot \delta^A_x \leqslant \delta^B_x \cdot \varrho^\top$

Forward and backward simulations



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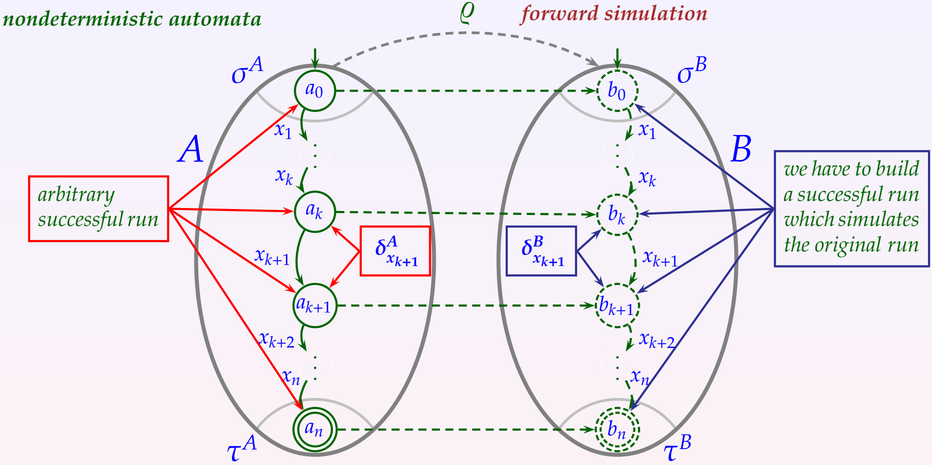
Forward and backward simulations



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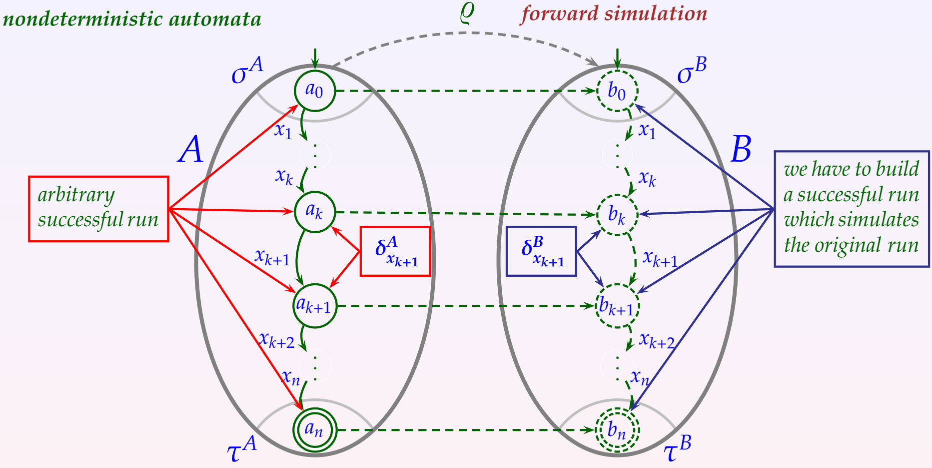
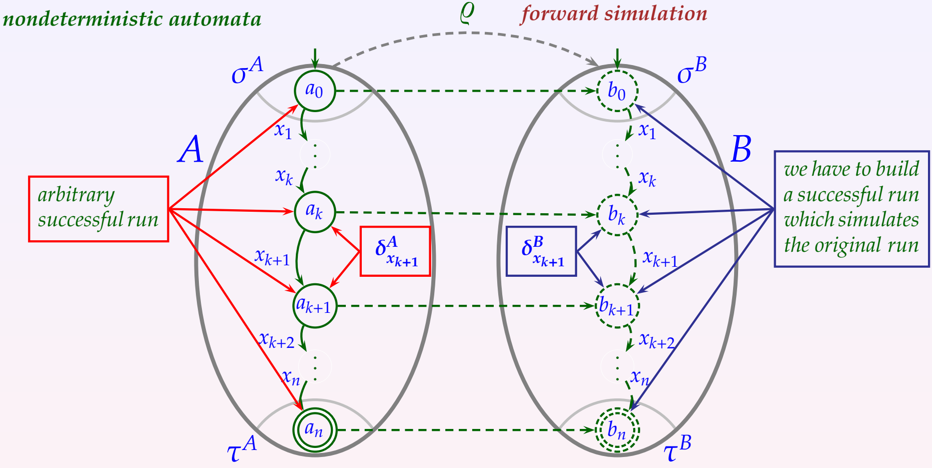


(1) $\sigma^A \leqslant \sigma^B \cdot \varrho^\top$ (2) $\varrho^\top \cdot \delta^A_x \leqslant \delta^B_x \cdot \varrho^\top$ (3) $\varrho^\top \cdot \tau^A \leqslant \tau^B$

Forward and backward simulations



forward simulation – the sequence $b_0, \dots, b_n$ is built **starting from** b_0 and **ending with** b_n

backward simulation – the sequence $b_0, \dots, b_n$ is built **starting from** b_n and **ending with** b_0

type of bisimulations	ϱ	$\varrho^\top$
forward	*forward*	*forward*
backward	*backward*	*backward*
forward-backward	*forward*	*backward*
backward-forward	*backward*	*forward*

- ⋆ there are also ***two-way simulations***

  $\mathcal{B}$ simulates $\mathcal{A}$ and $\mathcal{A}$ simulates $\mathcal{B}$, but two simulations are independent

★ ***Forward bisimulation***

$$\mu^\top \cdot \delta_x^A \leqslant \delta_x^B \cdot \mu^\top, \qquad \mu \cdot \delta_x^B \leqslant \delta_x^A \cdot \mu, \qquad x \in X$$
$$\mu^\top \cdot \tau^A \leqslant \tau^B, \qquad \mu \cdot \tau^B \leqslant \tau^A$$
$$\sigma^A \leqslant \sigma^B \cdot \mu^\top, \qquad \sigma^B \leqslant \sigma^A \cdot \mu$$

★ first row – WLS with $I_1 = I_4 = \emptyset$, $I = I_2 \cup I_3$

★ second row – equivalent to $\mu \leqslant \gamma_0 = (\tau^B/\tau^A)^\top \wedge (\tau^A/\tau^B)$

Theorem (Test of existence for forward bisimulations)

Let γ be the greatest solution of the above WLS such that $\gamma \leqslant \gamma_0$.

★ *If $\sigma^A \leqslant \sigma^B \cdot \gamma^\top$ and $\sigma^B \leqslant \sigma^A \cdot \gamma$, then γ is the* ***greatest forward bisimulation*** *between $\mathcal{A}$ and $\mathcal{B}$.*

★ *If γ does not satisfy this condition, then* ***there is no any forward bisimulation*** *between $\mathcal{A}$ and $\mathcal{B}$.*

- ★ S – ***additively idempotent semiring***
  - ▹ $\leqslant$ – *natural partial ordering* on S and its *extension to matrices*
- ★ ***we can define all types of simulations and bisimulations for WFAs over*** S
- ★ ***Problem: How to test the existence and compute the greatest ones?***
- ★ *in the general case,* ***there is no residuation*** *for matrices over* S
  - ▹ for $\alpha \in S^{A\times B}$ and $\gamma \in S^{A\times C}$ inequation $\alpha \cdot \mu \leqslant \gamma$ ***may not have the greatest solution in*** $S^{B\times C}$ (similarly for $\mu \cdot \beta \leqslant \gamma$)
- ★ ***Problem: Can this inequation have the greatest solution in some*** $M \subseteq S^{B\times C}$***?***

Relative residuals

- ⋆ $\alpha \in S^{A\times B}$, $\beta \in S^{B\times C}$, $\gamma \in S^{A\times C}$
- ⋆ $M \subseteq S^{B\times C}$, $N \subseteq S^{A\times B}$
- ⋆ ***relative right residual*** of γ by α ***w.r.t.*** M – greatest solution of $\alpha \cdot \mu \leqslant \gamma$ in M, if it exists
- ⋆ ***relative left residual*** of γ by β ***w.r.t.*** N – greatest solution of $\mu \cdot \beta \leqslant \gamma$ in N, if it exists
- ⋆ when $(M, +, \mathbf{0})$ and $(N, +, \mathbf{0})$ are ***finite submonoids*** of $(S^{B\times C}, +, \mathbf{0})$ and $(S^{A\times B}, +, \mathbf{0})$, ***relative residuals always exist***
- ⋆ ***Problem***: *How to compute them?*
- ⋆ *We solved the problem for* $M = \mathbf{2}^{B\times C}$, $N = \mathbf{2}^{A\times B}$ *– Boolean matrices*

- ⋆ *relative right residual* of γ by α *w.r.t.* $\mathbf{2}^{B\times C}$ *exists*
  - ▹ we call it the ***Boolean right residual***, and denote it by $\alpha \nwarrow \gamma$
- ⋆ *relative left residual* of γ by β *w.r.t.* $\mathbf{2}^{A\times B}$ *exists*
  - ▹ we call it the ***Boolean left residual***, and denote it by $\gamma \nearrow \beta$
- ⋆ for any assertion Ψ of a classical Boolean logic, $\lceil \Psi \rceil$ denotes its ***truth value***

Theorem (Boolean residuals)

$$(\alpha \nwarrow \gamma)(b,c) = \lceil \alpha(\cdot,b) \leqslant \gamma(\cdot,c) \rceil, \qquad (\gamma \nearrow \beta)(a,b) = \lceil \beta(b,\cdot) \leqslant \gamma(a,\cdot) \rceil$$

For all $\xi \in \mathbf{2}^{B\times C}$ *and* $\eta \in \mathbf{2}^{A\times B}$ *we have*

$$\alpha \cdot \xi \leqslant \gamma \;\Leftrightarrow\; \xi \leqslant \alpha \nwarrow \gamma, \qquad \eta \cdot \beta \leqslant \gamma \;\Leftrightarrow\; \eta \leqslant \gamma \nearrow \beta$$

- ★ ***Boolean sumulations and bisimulations***
  solutions of systems which define simulations and bisimulations in the class of ***Boolean matrices***

- ★ ***Test of existence*****:** *similar as for simulations and bisimulations for automata over a unital quantale*

- ★ ***Difference*****:** *we compute a sequence* $\{\gamma_k\}_{k\in\mathbb{N}}$ *of Boolean matrices*

- ▹ since $\mathbf{2}^{A\times B}$ is finite, *the sequence stabilizes* at some γ_k

- □ *if* γ_k *passes the test, then there is a Boolean simulation (bisimulation) and* γ_k *is the greatest one*

- □ *if* γ_k *does not pass the test, then there is no any Boolean simulation (bisimulation)*

- ***Past results:***
- state reduction, simulation and bisimulation for

  fuzzy automata over a complete residuated lattice

  nondeterministic automata

  weighted automata over an additively idempotent semiring

  relative residuation – Boolean residuation

- ***Further work:***

  weighted automata over a max-plus semiring

V

Weighted Networks

◇ Positional Analysis ◇

- mathematical methods for the study of social structures
- they can also be applied to many other types of networks that arise in computer science, physics, biology, etc.

- a ***social network*** is made up of
  - a set of social ***actors*** (individuals or organizations)
  - ***ties*** or ***social interactions*** between actors

- most often, ties are modeled by *Boolean-valued relations* or *Boolean matrices*
- in many real-world networks, not all ties have the same *strength*, *intensity*, *duration*, or some other *quantitative property*
- in these cases, it is natural to assign ***weights*** to ties, to model these quantitative properties

- S – *unital quantale*
- ***Weighted network*** (*one-mode network*): $\mathcal{N} = (A, \{\varrho_i\}_{i\in I})$

  A – set of ***actors***

  $\{\varrho_i\}_{i\in I} \subseteq S^{A\times A}$ – family of matrices

  *matrices represent **social relations** between actors*

  sometimes we consider $\mathcal{N} = (A, \{\varrho_i\}_{i\in I}, \{\sigma_j\}_{j\in J})$

  $\{\sigma_j\}_{j\in J} \subseteq S^{A}$ – family of vectors

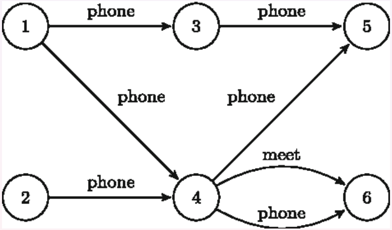
  *vectors represent certain **individual properties** of actors*

- most often, *Boolean matrices* have been taken in account
- another name: *valued networks* (usually integer weights)

Positional analysis

★ *identify the **position** or **role** of actors in a network on the basis of mutual relationships*

★ example: ***terroristic group*** or ***criminal group*** *- identify roles (leaders, etc.) on the basis on communication between the group members (without insight into the content of the conversation)*



J. Brynielsson, L. Kaati, P. Svenson, *Social positions and simulation relations*, Soc. Netw. Anal. Min. 2 (2012) 39–52

- closely related: ***blockmodeling*** – data reduction method

  large and complex social networks are mapped into simpler structures – ***blockmodel images***

  blockmodel image – *structural summary of the original network*

- common idea

  to cluster actors who have substantially similar patterns of relationships with others

  to interpret the pattern of relationships among the clusters

- *behavior of individuals is often determined by the affiliation of the group*
- *such influence of the group to the behavior of an individual can be very important*

- ***Structural equivalences*** (Lorrain and White, 1971)

  two actors are considered to be structurally equivalent if they have identical neighborhoods

  in our terminology – greatest solutions of the system

$$\mu \cdot \varrho_i \leqslant \varrho_i, \qquad \mu^\top \cdot \varrho_i \leqslant \varrho_i, \qquad \varrho_i \cdot \mu \leqslant \varrho_i, \qquad \varrho_i \cdot \mu^\top \leqslant \varrho_i$$

  this concept has shown oneself to be too strong

- ***Regular equivalences*** (White and Reitz, 1983)

  less restrictive than structural equivalences and more appropriate for modeling social positions

  two actors are considered to be regularly equivalent if they are equally related to equivalent others

⋆ ***Regular matrix*** – solution of the system

$$\varrho_i \cdot \mu = \mu \cdot \varrho_i, \qquad \varrho_i \cdot \mu^\top = \mu^\top \cdot \varrho_i, \qquad\qquad i \in I$$

weakly linear system

greatest solution – ***regular equivalence matrix***

it can be computed using the previously described method

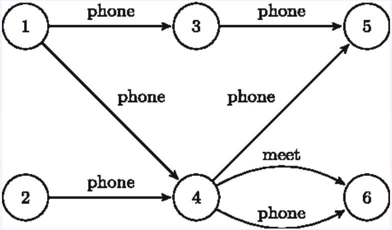
⋆ we can also use any of the following three systems

$$\begin{aligned} \varrho_i \cdot \mu &= \mu \cdot \varrho_i, & i \in I \\ \varrho_i \cdot \mu &\leqslant \mu \cdot \varrho_i, & i \in I \\ \mu \cdot \varrho_i &\leqslant \varrho_i \cdot \mu, & i \in I \end{aligned}$$

the greatest solutions – *quasi-order matrices*

if we need the greatest solutions which are *equivalence matrices* – we have to add inequations with $\mu^\top$

Example (Brynielsson, Kaati, and Svenson)



- ⋆ ***regular equivalence*** classes – $\{1\}, \{2\}, \{3\}, \{4\}, \{5,6\}$

- ⋆ ***simulation equivalence*** classes – $\{1,2\}, \{3\}, \{4\}, \{5,6\}$

  simulation equivalence ≡ natural equivalence of *simulation quasi-order* (its symmetric opening)

  simulation quasi-orders – (greatest) solutions of $\mu \cdot \varrho_i \leqslant \varrho_i \cdot \mu$

- ⋆ *regular equivalences can not identify group leaders* – 1 *and* 2

- ***identify similar positions in different networks***
- networks: $\mathcal{N} = (A, \{\varrho_i\}_{i\in I}, \{\sigma_j\}_{j\in J}), \quad \mathcal{N}' = (A', \{\varrho'_i\}_{i\in I}, \{\sigma'_j\}_{j\in J})$
- ***regular bisimulations*** – solutions of the system

$$\begin{aligned} \sigma_j &= \mu \cdot \sigma'_j, && j \in J \\ \sigma'_j &= \mu^\top \cdot \sigma_j, && j \in J \\ \varrho_i \cdot \mu &= \mu \cdot \varrho'_i, && i \in I \\ \mu^\top \cdot \varrho_i &= \varrho'_i \cdot \mu^\top, && i \in I \end{aligned}$$

μ – unknown taking values in $S^{A\times B}$

- *algorithm for testing the existence of a regular bisimulation and computing the greatest one*
- other types of simulations and bisimulations (unpublished)

- ***Two-mode network*** – $\mathcal{T} = (A, B, \{\varrho_i\}_{i\in I})$

  A, B – two sets of entities

  $\{\varrho_i\}_{i\in I} \subseteq S^{A\times B}$ – family of matrices (represent relationships)

  affiliation or ***bipartite networks***

- examples: *people attending events, organizations employing people, authors and articles*, etc.

- ***Positional analysis*** – *identify positions in both modes of the network*
- *Indirect approach* – reduction to one-mode networks

  single one-mode network on $A \cup B$

  two one-mode networks $(A, \{\varrho_i \cdot \varrho_i^\top\}_{i\in I})$, $(B, \{\varrho_i^\top \cdot \varrho_i\}_{i\in I})$

Our direct approach

★ ***Two-mode systems***

$\alpha \cdot \varrho_i = \varrho_i \cdot \beta,$		$i \in I$
$\alpha \cdot \varrho_i \leqslant \varrho_i \cdot \beta,$		$i \in I$
$\varrho_i \cdot \beta \leqslant \alpha \cdot \varrho_i,$		$i \in I$
$\alpha \cdot \varrho_i = \varrho_i \cdot \beta,$	$\alpha^\top \cdot \varrho_i = \varrho_i \cdot \beta^\top,$	$i \in I$
$\alpha \cdot \varrho_i \leqslant \varrho_i \cdot \alpha,$	$\alpha^\top \cdot \varrho_i \leqslant \varrho_i \cdot \beta^\top$	$i \in I$
$\varrho_i \cdot \beta \leqslant \alpha \cdot \varrho_i,$	$\varrho_i \cdot \beta^\top \leqslant \alpha^\top \cdot \varrho_i$	$i \in I$

$\{\varrho_i\}_{i\in I} \subseteq S^{A\times B}$ – given family of matrices

α and β – unknowns taking values in $S^{A\times A}$ and $S^{B\times B}$

★ ***solutions*** – *pairs of matrices* (ordered coordinatewise)

★ ***algorithms for computing the greatest solutions for all two-mode systems***

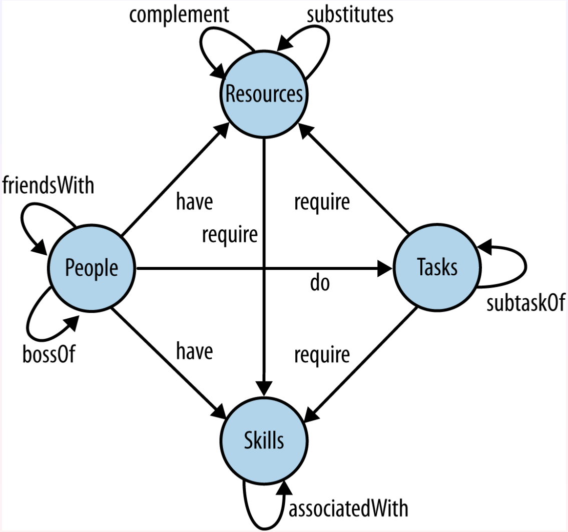
- ***Multi-mode network*** – $\mathcal{T} = (A_1, \ldots, A_n, \mathcal{R})$
- $A_1, \ldots, A_n$ – multiple non-empty sets
- $\mathscr{R}$ – *system of* $A_j \times A_k$*-matrices defined for some pairs* (j,k)
- formally: $J \subseteq [1,n] \times [1,n]$ satisfying

  $$(\forall j \in [1,n])(\exists k \in [1,n])\ (j,k) \in J \text{ or } (k,j) \in J$$

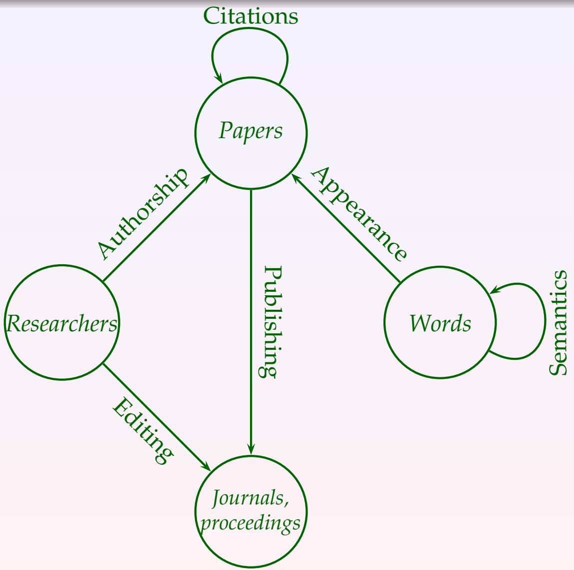
  $\{I_{j,k}\}_{(j,k)\in J}$ – collection of non-empty sets

  $\mathcal{R} = \{\varrho_i^{j,k} \mid (j,k) \in J,\ i \in I_{j,k}\}, \quad \varrho_i^{j,k} \in S^{A_j \times A_k}$, for all $(j,k) \in J$, $i \in I_{j,k}$

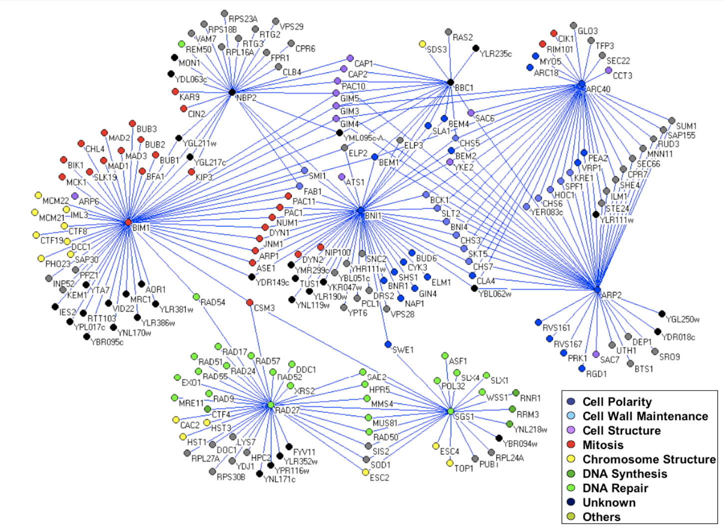
- *complex synthesis of one-mode and two-mode networks*
- ***Positional analysis*** – *identify positions in all modes of the network*



Simple organization network



Network of academic publications



Genetic regulatory (interaction) network

Multi-mode systems

$$\alpha_j \cdot \varrho_i^{j,k} = \varrho_i^{j,k} \cdot \alpha_k, \qquad (j,k) \in J,\ i \in I_{j,k},$$
$$\alpha_j \cdot \varrho_i^{j,k} \leqslant \varrho_i^{j,k} \cdot \alpha_k, \qquad (j,k) \in J,\ i \in I_{j,k},$$
$$\alpha_j \cdot \varrho_i^{j,k} \geqslant \varrho_i^{j,k} \cdot \alpha_k, \qquad (j,k) \in J,\ i \in I_{j,k},$$

- ⋆ $\alpha_1, \dots, \alpha_n$ – *unknowns*, α_j takes values in $S^{A_j \times A_j}$
- ⋆ *solutions* – n-tuples of matrices from $S^{A_1 \times A_1} \times \cdots \times S^{A_n \times A_n}$
- ⋆ n-tuples are ordered coordinatewise

- ⋆ greatest solutions – *n-tuples of quasi-order matrices* (to get *equivalence matrices* we add (in)equations with $\alpha_j^\top$ and $\alpha_k^\top$)
- ⋆ ***algorithms for computing the greatest solutions for all multi-mode systems***

Grouping employees and jobs

- ⋆ two-mode network $\mathscr{T} = (A, B, \{\varrho_i\}_{i\in I})$
- ⋆ A – the set of all ***employees*** of some company
- ⋆ B – the set of all ***jobs*** which this company performs for other companies
- ⋆ I set of these other companies
- ⋆ the jobs for the company $i \in I$ are allotted to employees by a relation $\varrho_i \subseteq A \times B$
- ⋆ **Task:** *group employees into* ***teams*** *and jobs into* ***groups of jobs*** so that
  - ▷ *the teams and groups of jobs are as wide as possible*
  - ▷ *for any company, a group of jobs γ is assigned to a team θ if and only if*
    - ∗ *for every employee from θ there is a job from γ which he has already performed for that company*
    - ∗ *for every job from γ there is an employee from θ who has already preformed that job for that company*
- ⋆ *this can be done using the greatest solution of* $\alpha \cdot \varrho_i = \varrho_i \cdot \beta, \quad \alpha^\top \cdot \varrho_i = \varrho_i \cdot \beta^\top$

Example 2

The modified problem

- the teams and the groups of jobs have ***wider*** and ***narrower parts***
- the ***narrower parts*** – the ***cores*** of the teams and groups of jobs
- for any company, a group of jobs γ is assigned to a team θ if and only if
  - *for every employee **from the core** of θ there is a job from γ which he has already performed for that company*
  - *for every job **from the core** of γ there is an employee from θ who has already preformed that job for that company*
- *this can be done using the greatest solution of* $\alpha \cdot \varrho_i = \varrho_i \cdot \beta$
- the *wider teams and groups of jobs* are the *rows* of α and the *columns* of β
- the *narrower teams and groups of jobs* are *equivalence classes* of the natural equivalences of α and β
- *the core of the team performs the main part of the assigned jobs, and the rest of the team assists the core in the jobs that they have not previously performed and in other cases when they need help*
- *the core of the group of jobs assigned to the team are main jobs they have to perform, while the rest of this group are those jobs for which the members of the team could be engaged to assist*

Adding a third mode

- third mode: ***skills***, e.g., knowledge of specific ***software packages***, if the considered company is a software company
- ***groups of software packages*** assigned to teams
- for any company, a group of software packages π is assigned to a team θ if and only if
  - *for every employee from θ there is a software package from π for which the employee is qualified*
  - *for every software package from π there is an employee from θ who is qualified for that software package*
- *such grouping can be done using the **triples of equivalences*** that are solutions of our three-mode systems
- version with cores – ***triples of quasi-orders*** that are solutions of our three-mode systems

Thank you
for your attention!