

Nondeterminism and Alternating Finite Automata

Gustav Grabolle

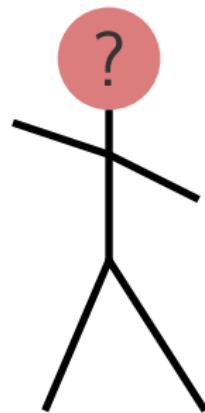


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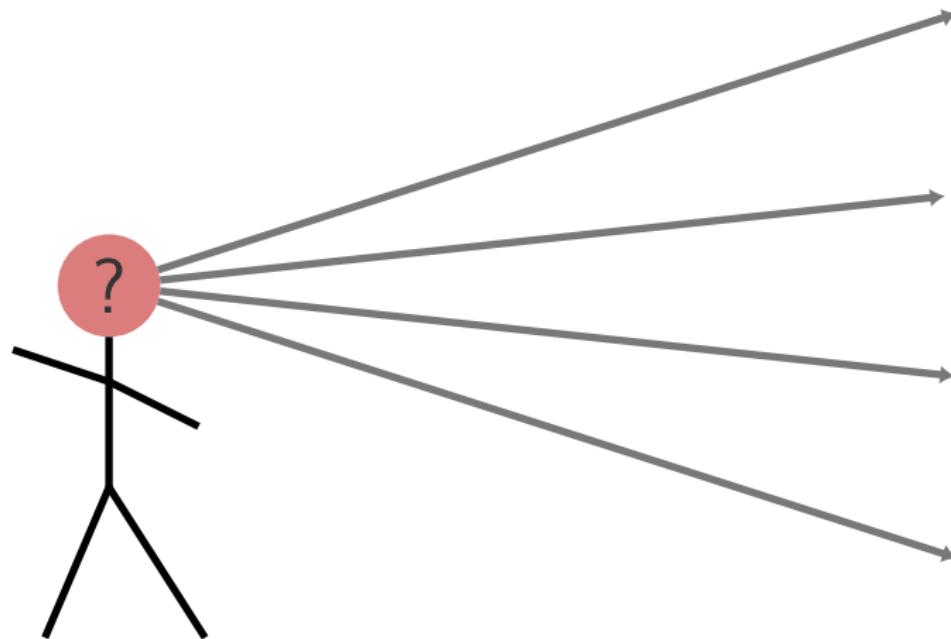
Thessaloniki
Juli 4, 2019

IDEAS

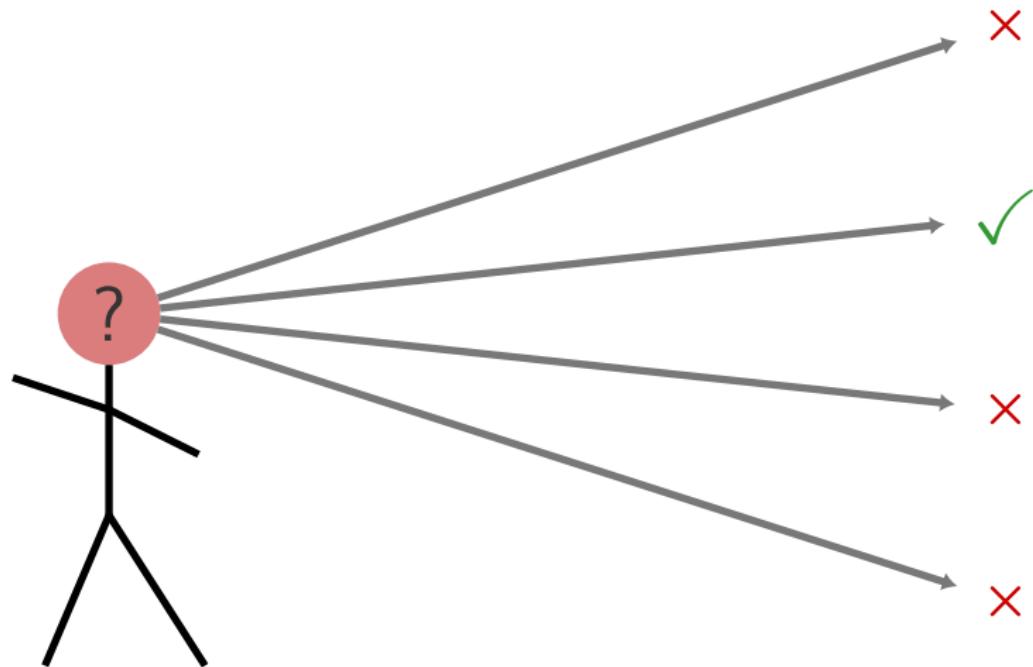
Nondeterminism



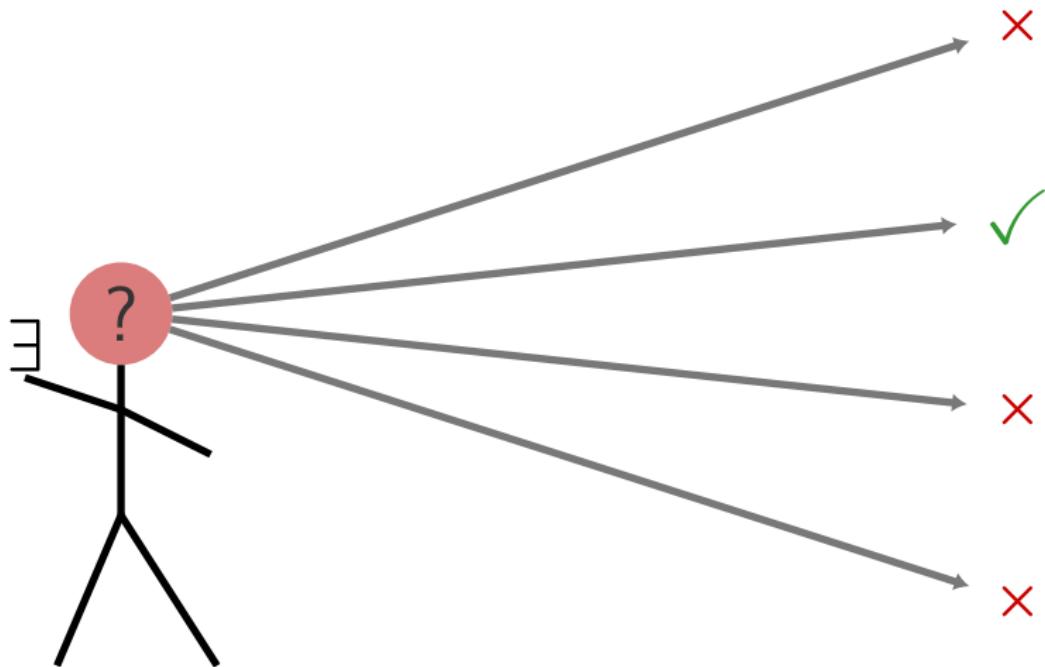
Nondeterminism



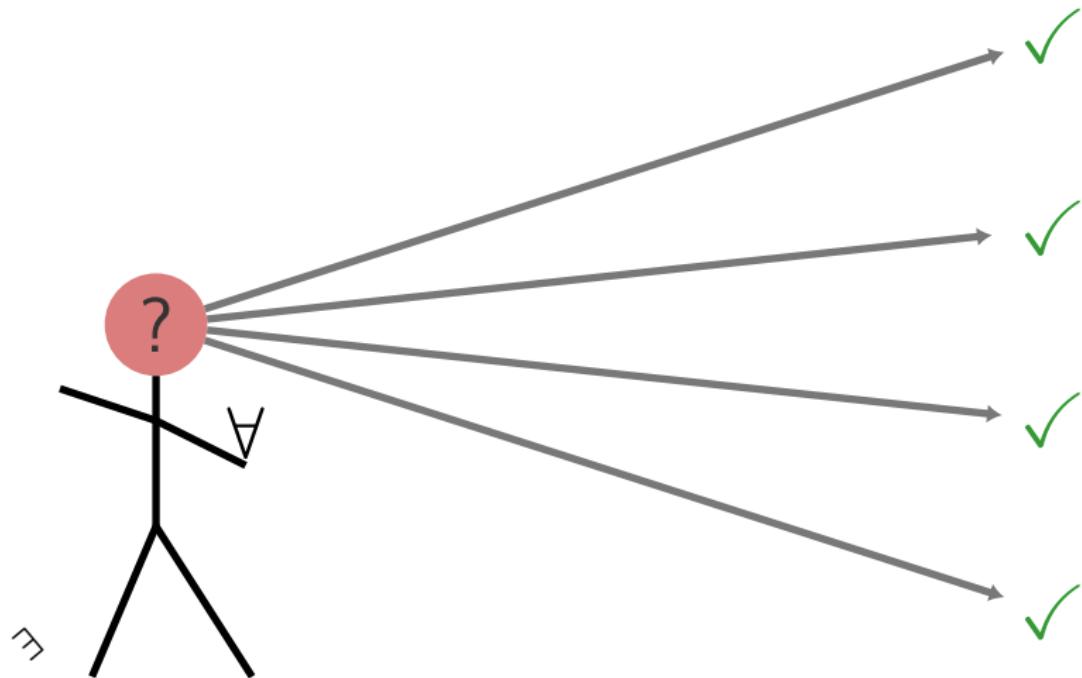
Nondeterminism



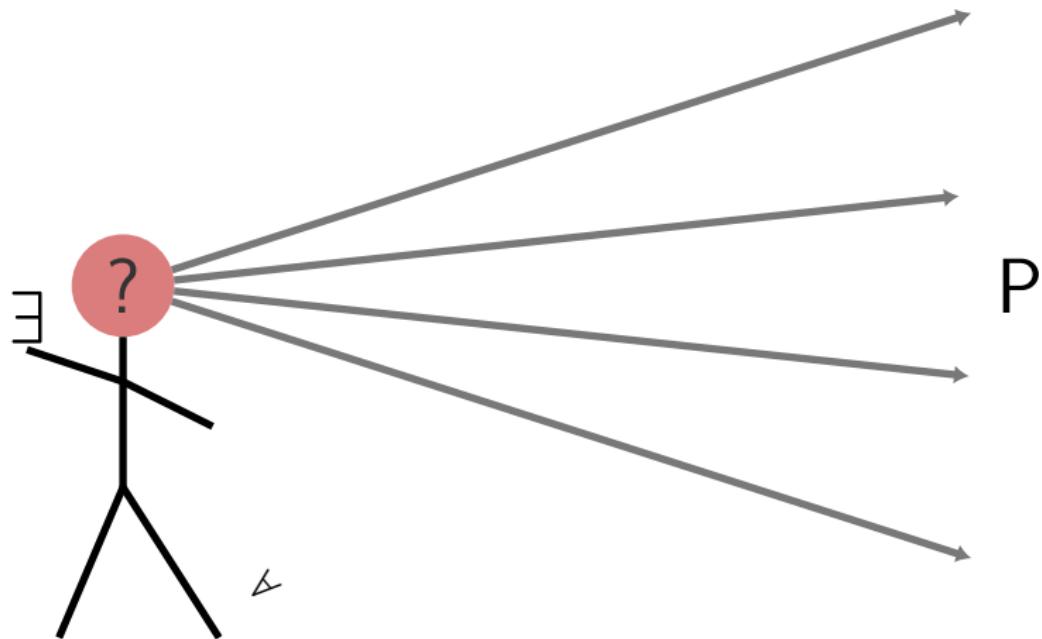
Nondeterminism



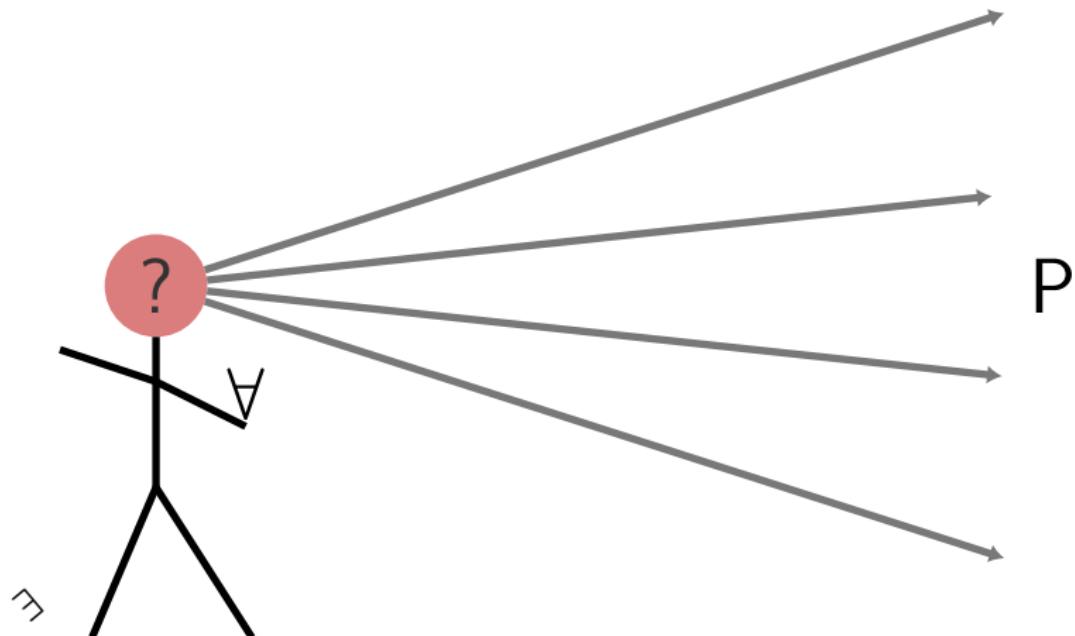
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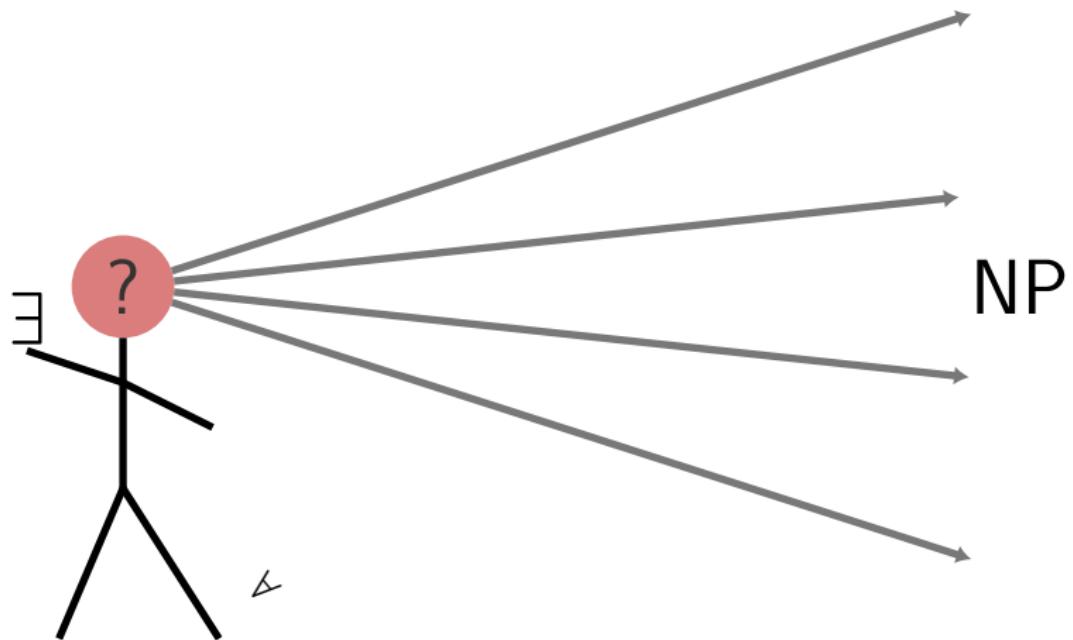
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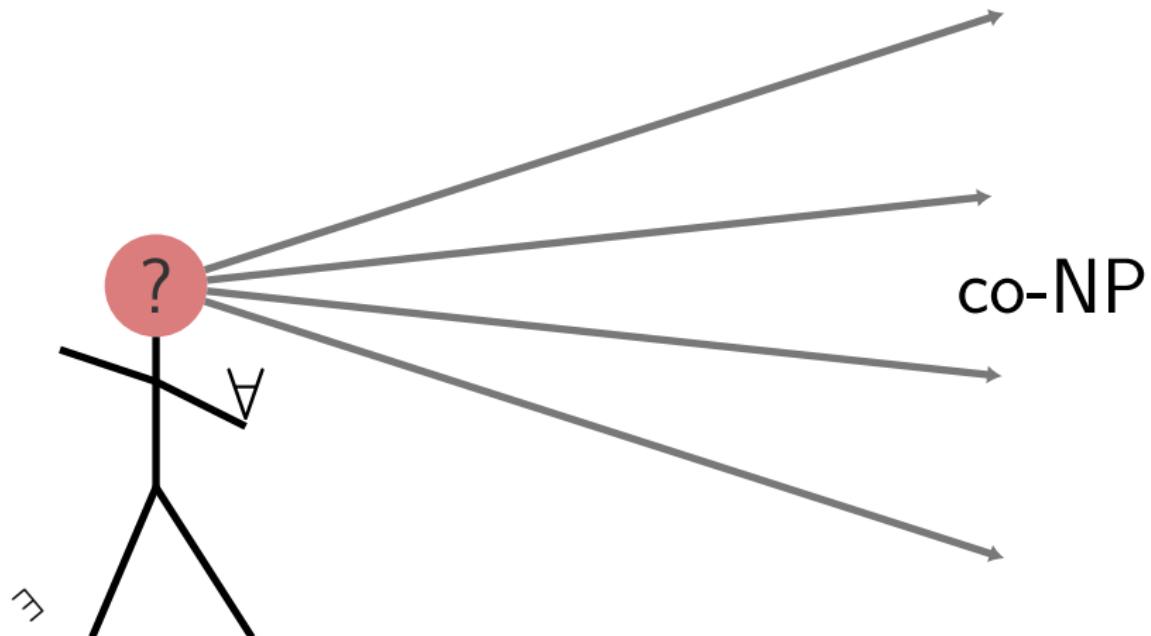
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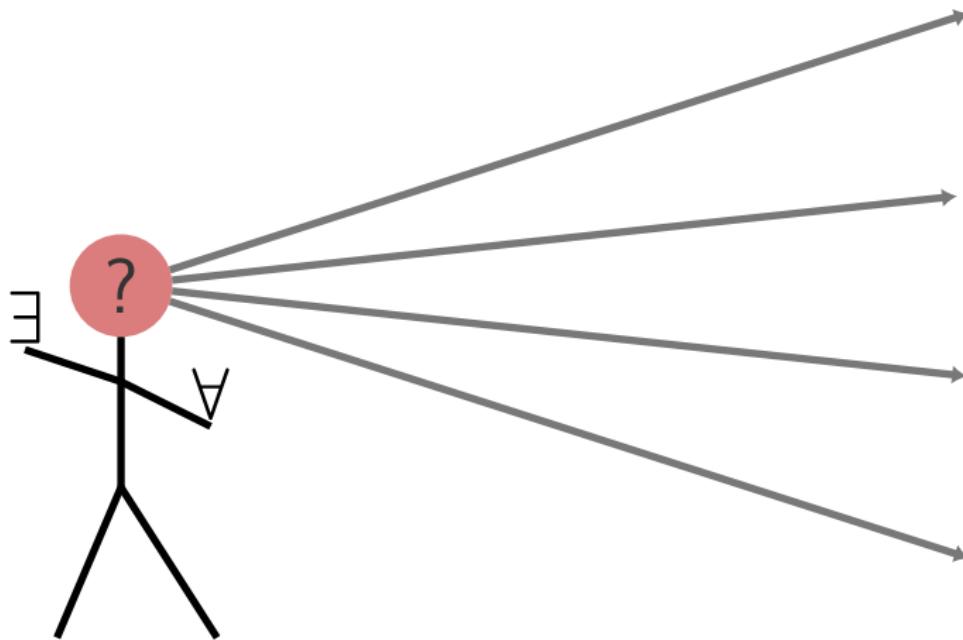
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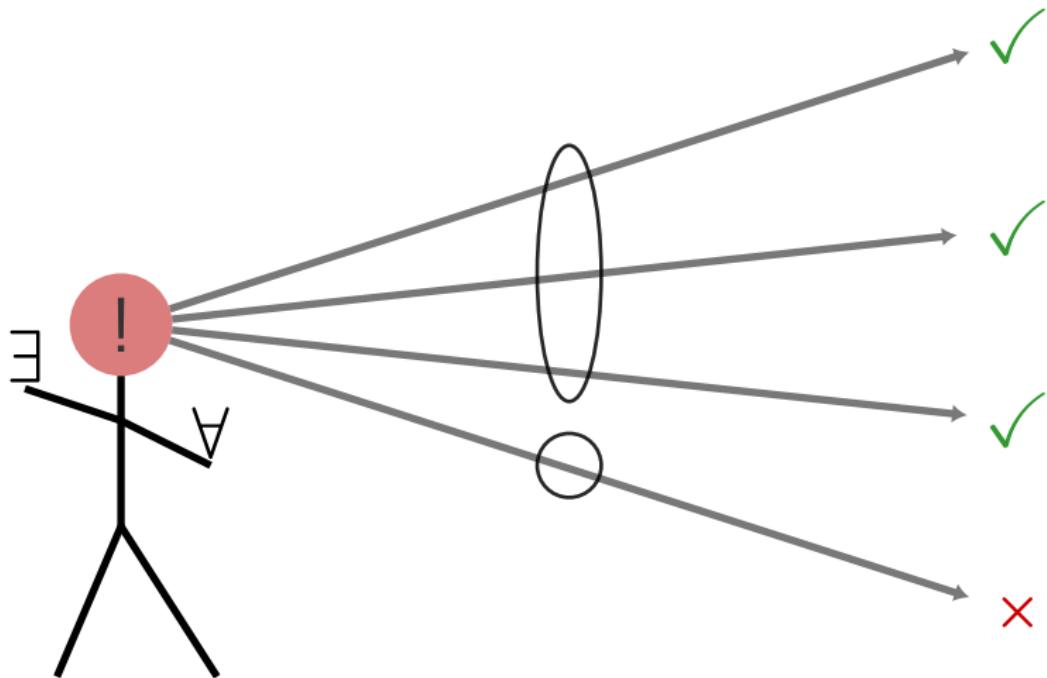
Nondeterminism



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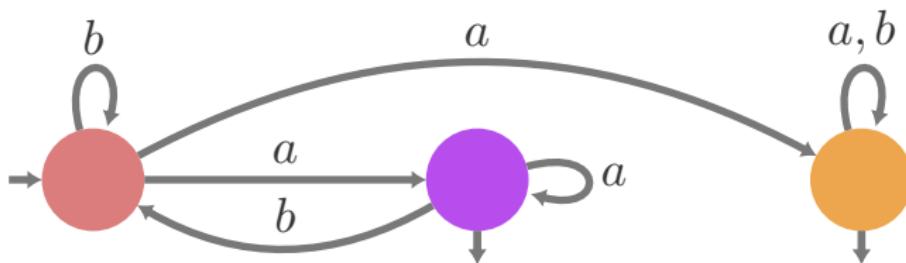


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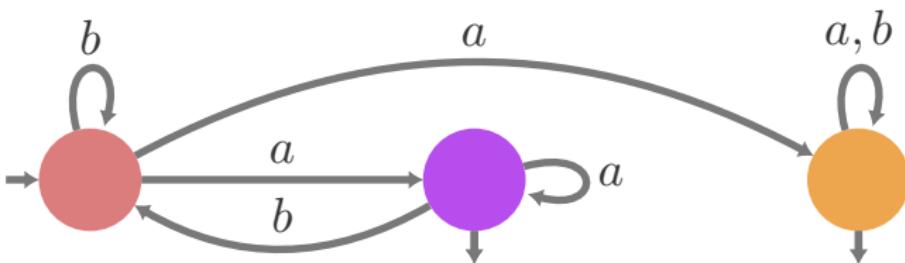


WARM UP

Finite automata



Finite automata



$$\mathcal{A} = (Q, \Sigma, I, \Delta, F)$$

$$\Sigma = \{a, b\}$$

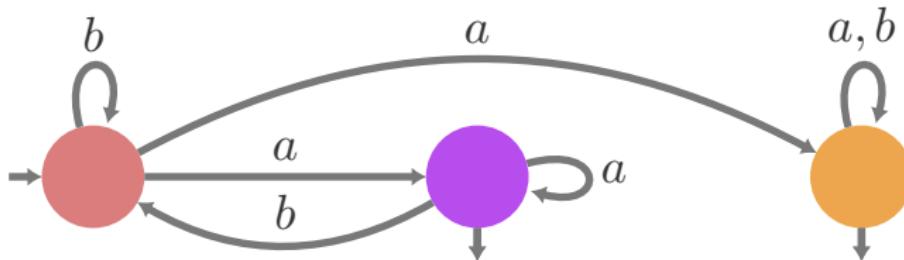
$$Q = \{\bullet, \circ, \bullet\}$$

$$\Delta = \rightarrow$$

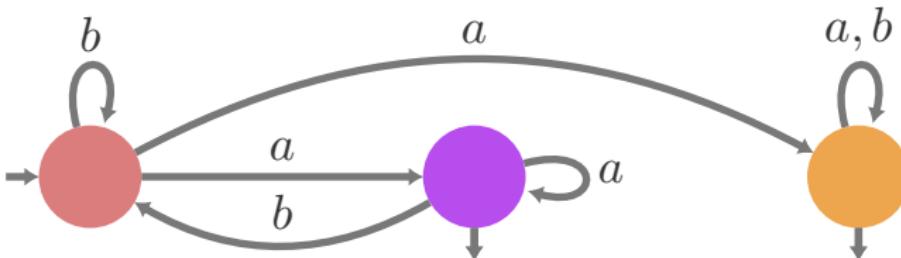
$$I = \{\bullet\}$$

$$F = \{\circ, \bullet\}$$

Existential acceptance condition

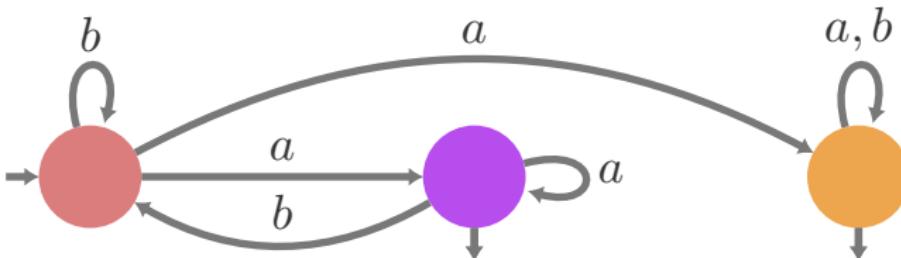


Existential acceptance condition



$$L^{\exists}(\mathcal{A}) = \{w \in \Sigma^* \mid \text{accepting run on } w \text{ exists}\}$$

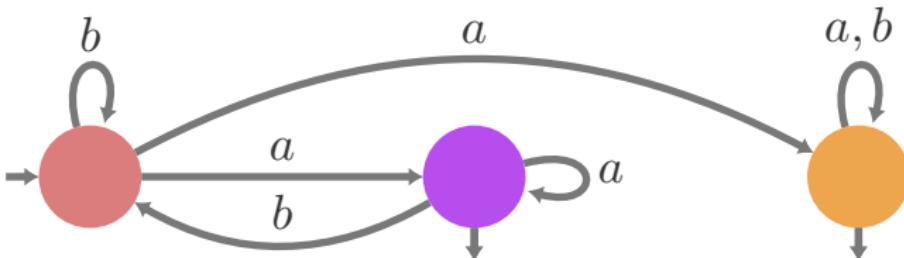
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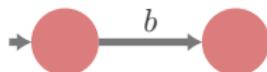
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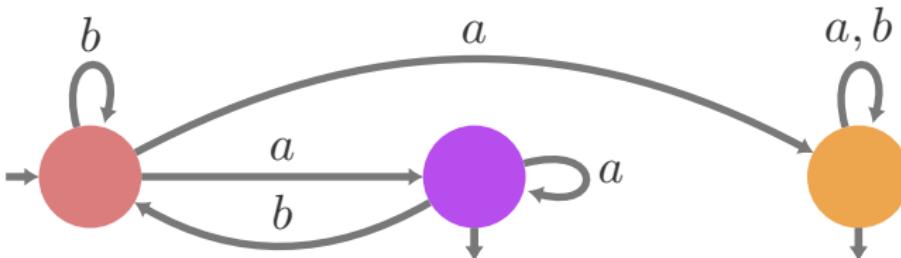
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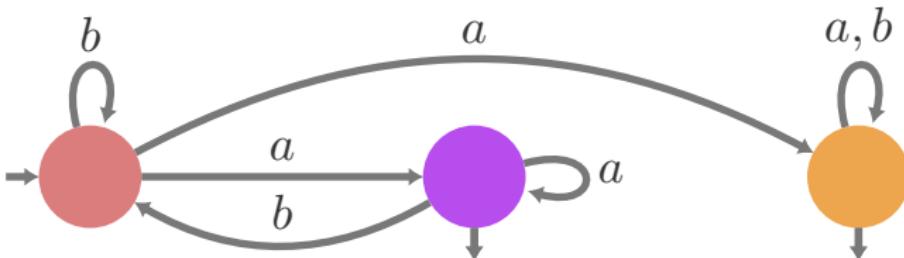
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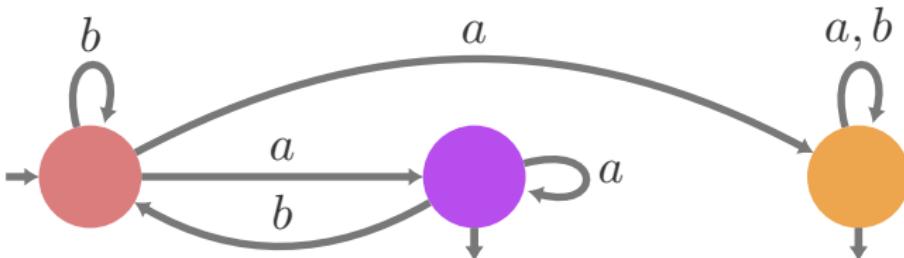
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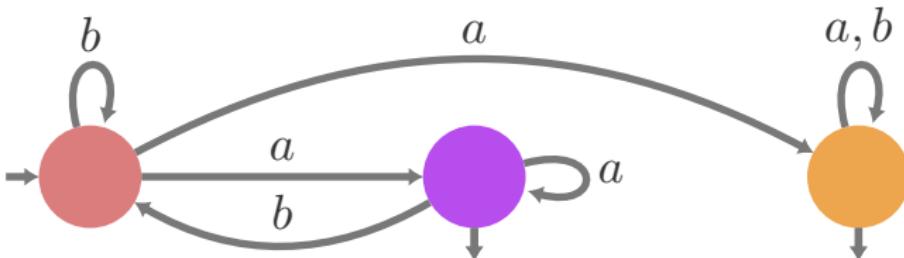
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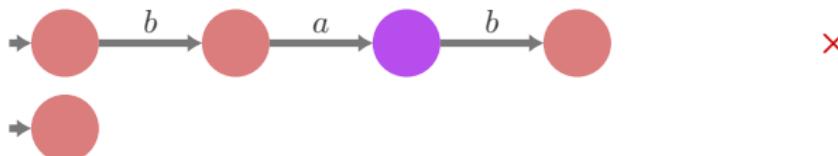
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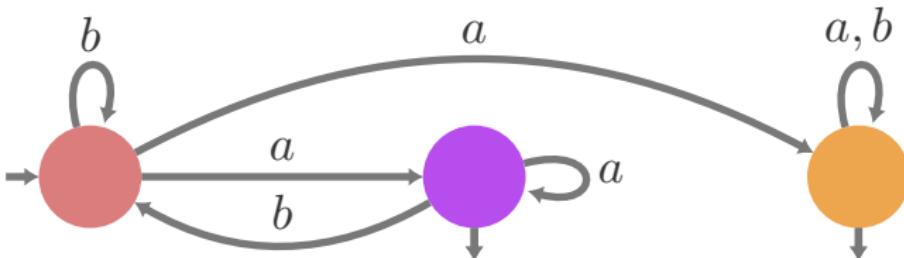
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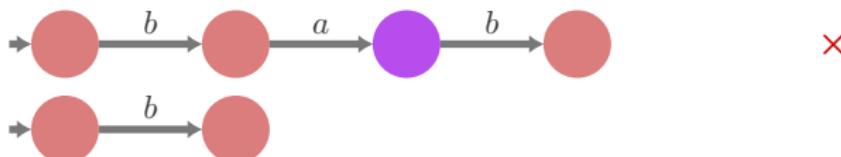
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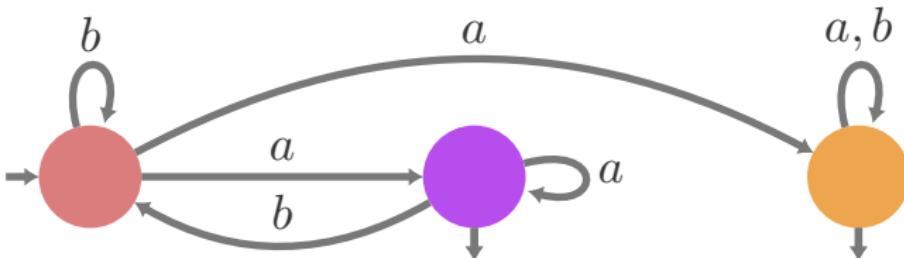
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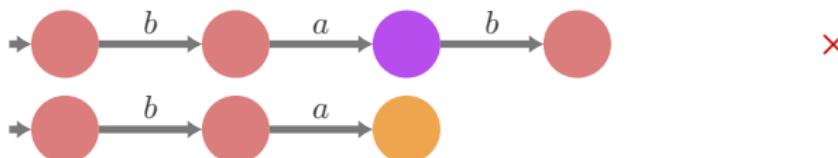
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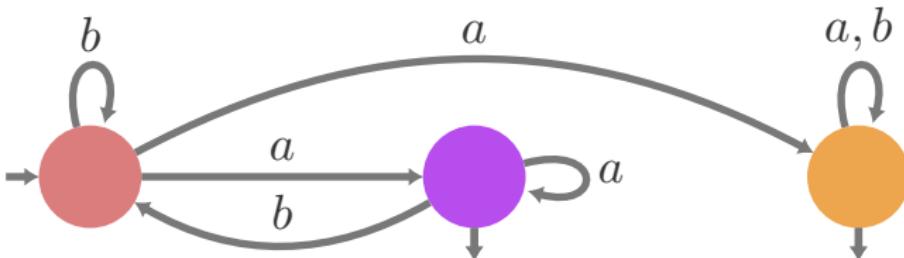
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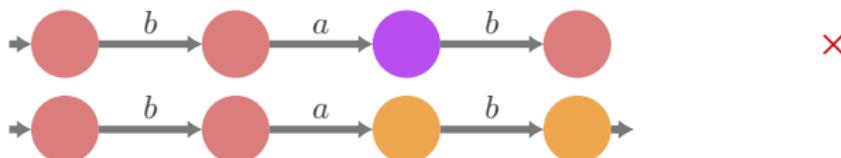
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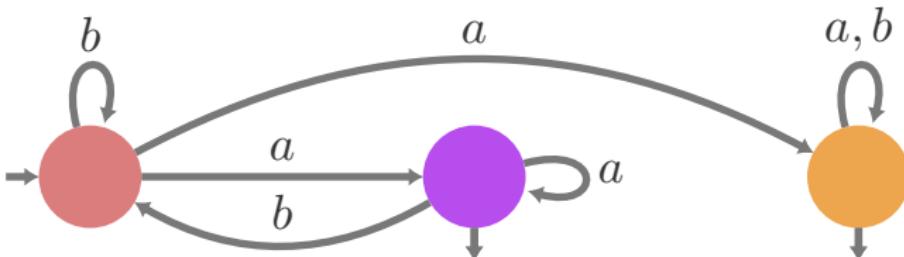
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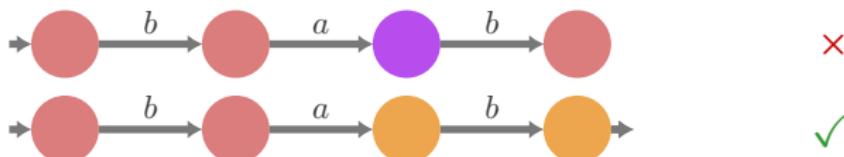
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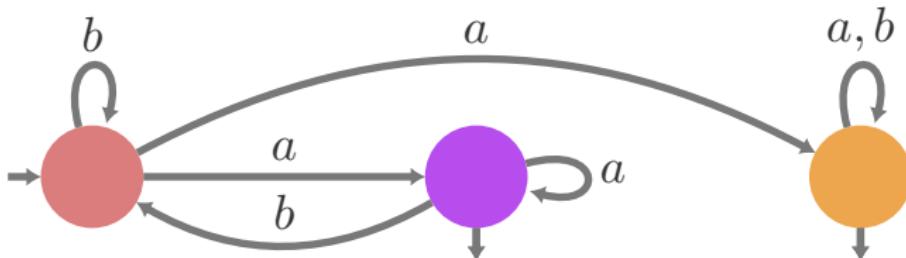
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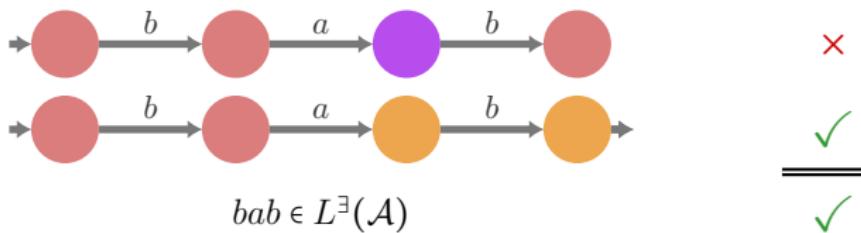
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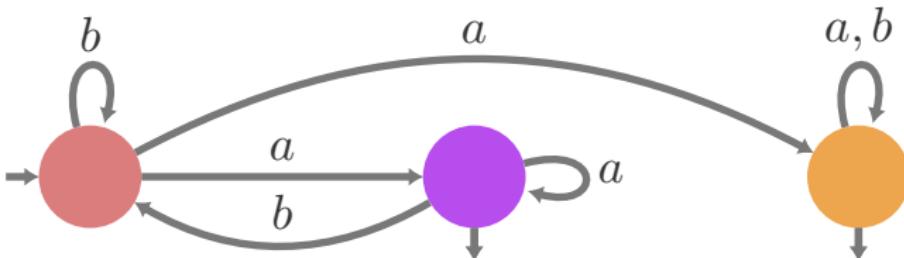
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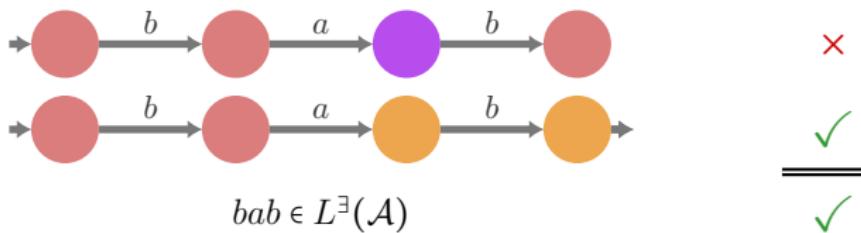


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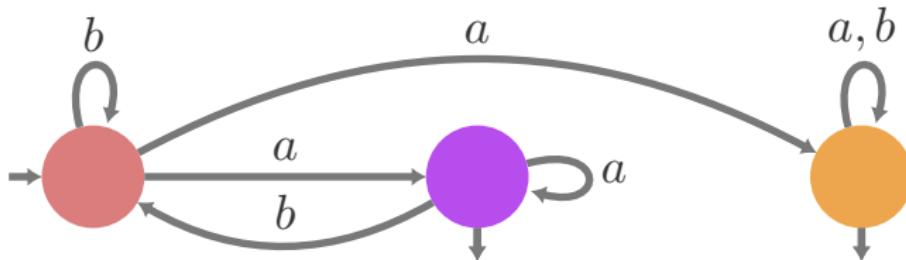


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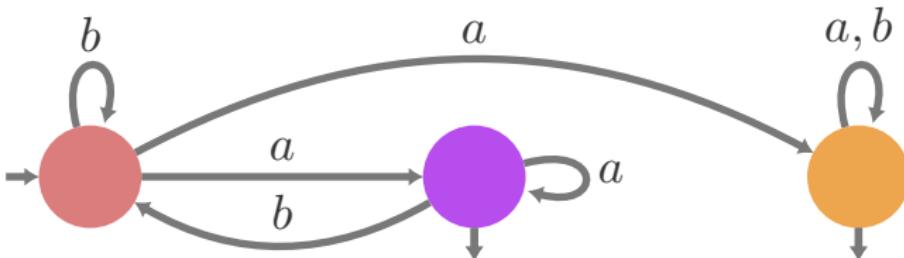
$\Sigma^* \cdot a \cdot \Sigma^*$



Universal acceptance condition

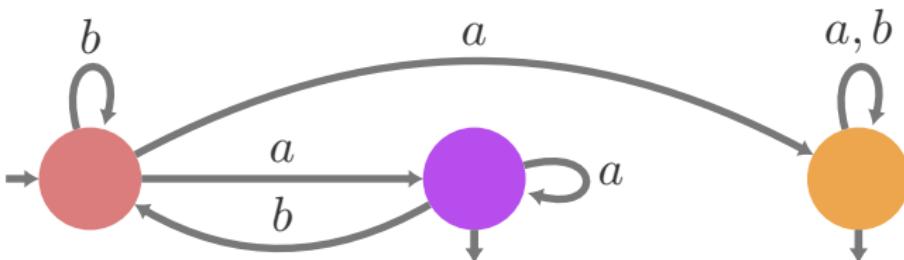


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$$L^{\forall}(\mathcal{A}) = \{w \in \Sigma^* \mid \text{all runs on } w \text{ are accepting}\}$$

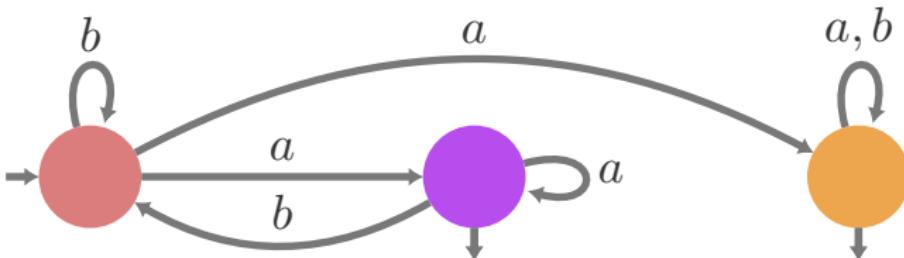
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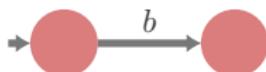
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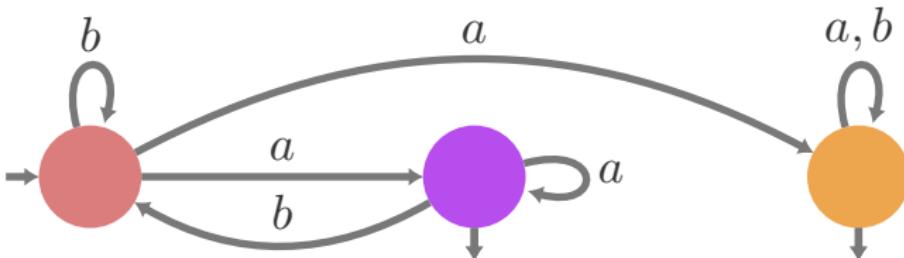
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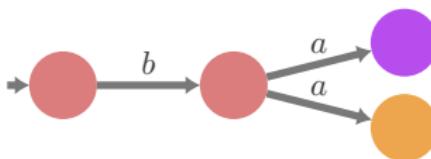
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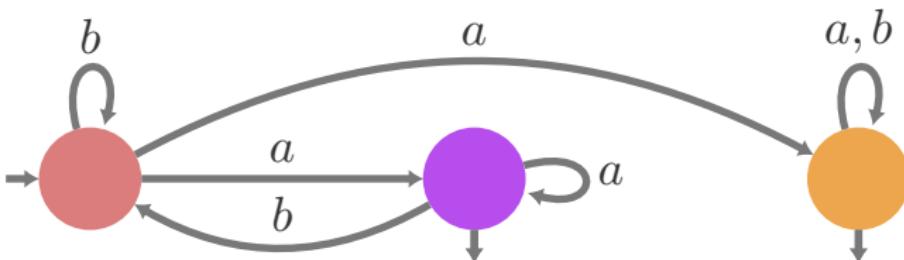
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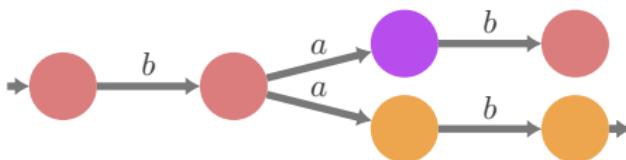
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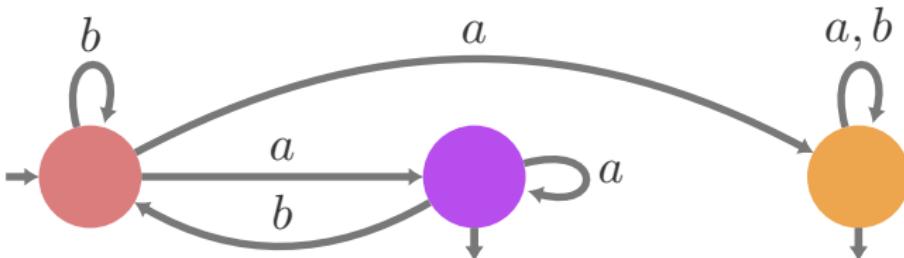
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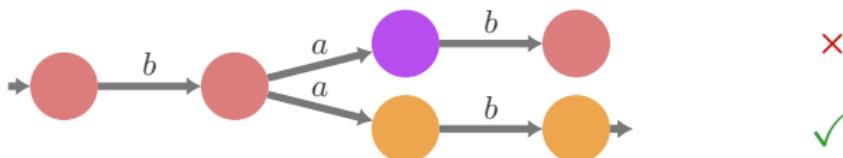
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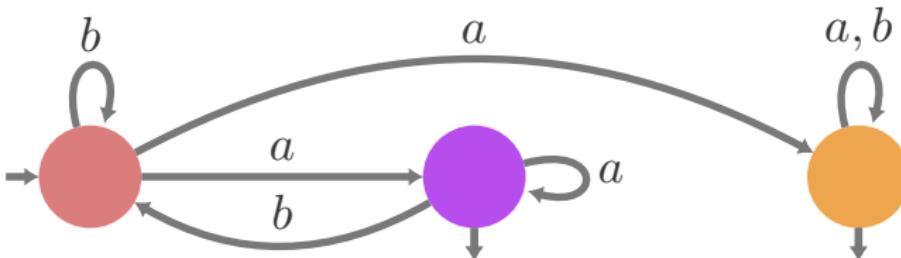
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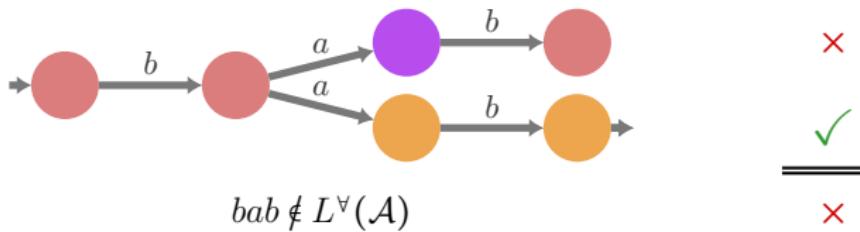
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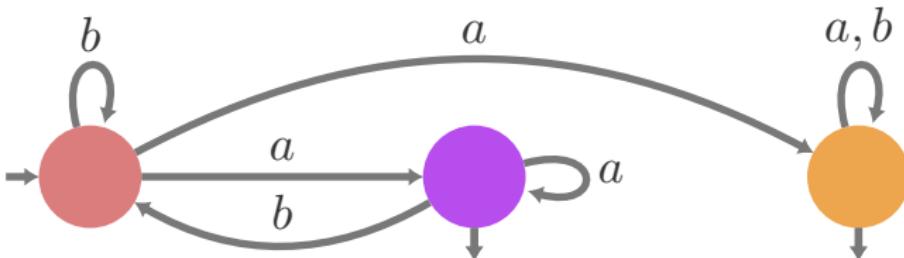
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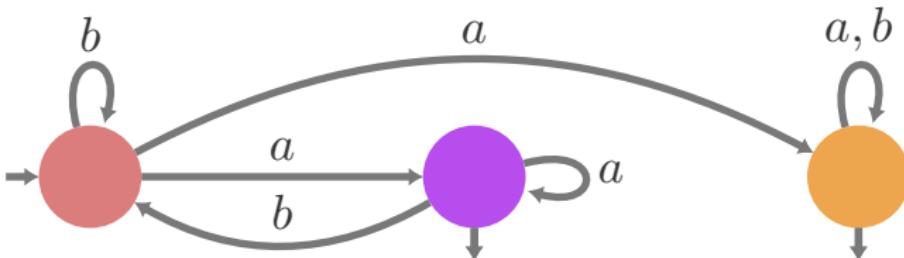


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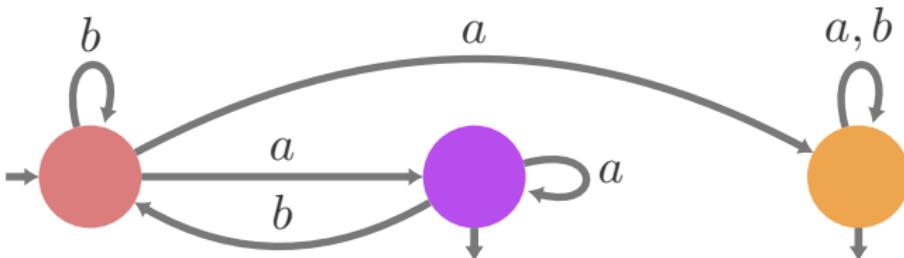
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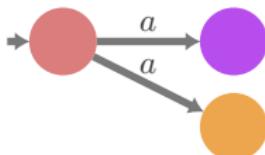
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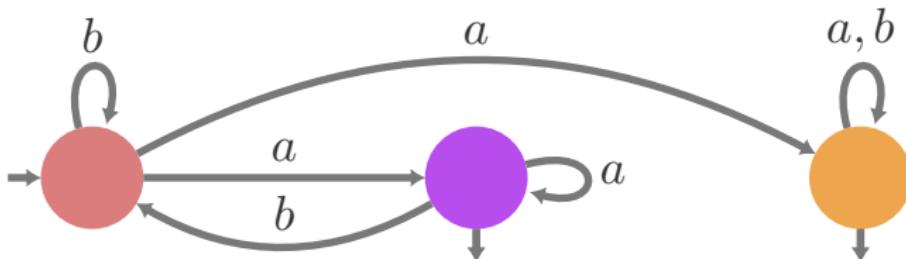
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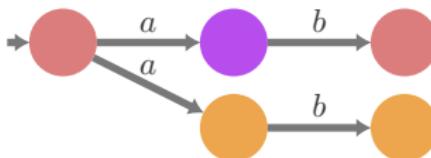
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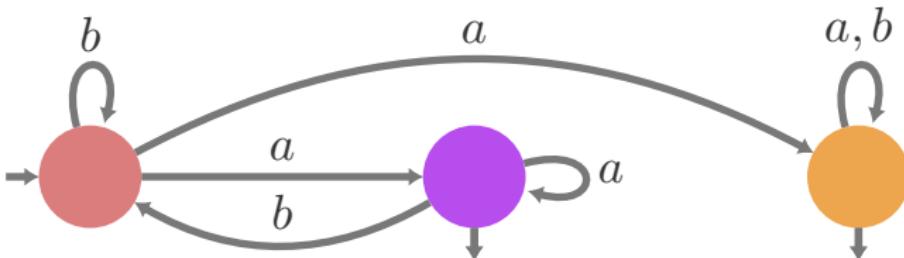
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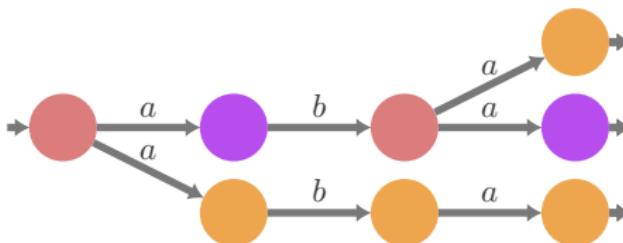
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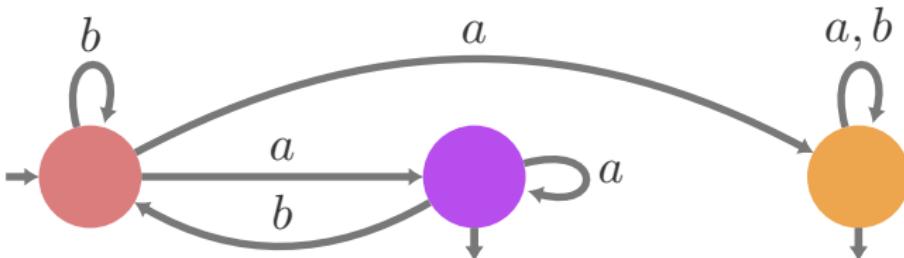
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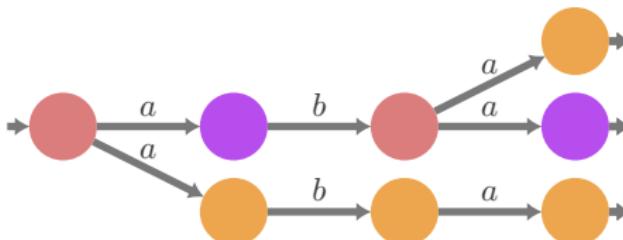
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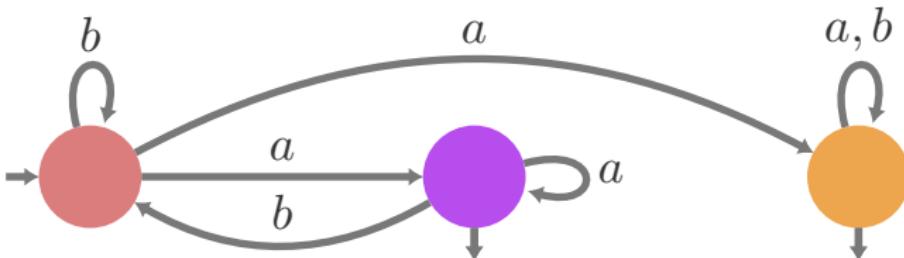


✓

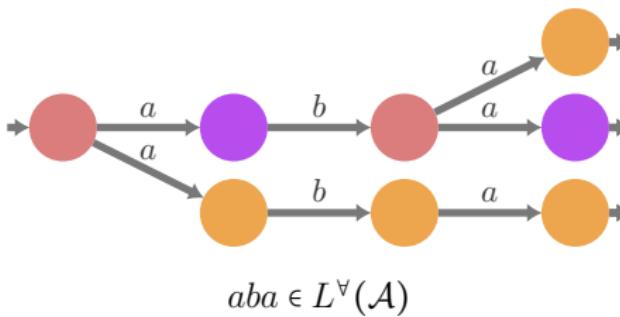
✓

✓

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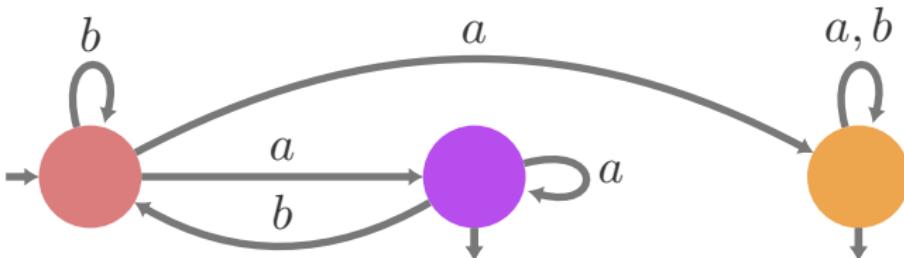


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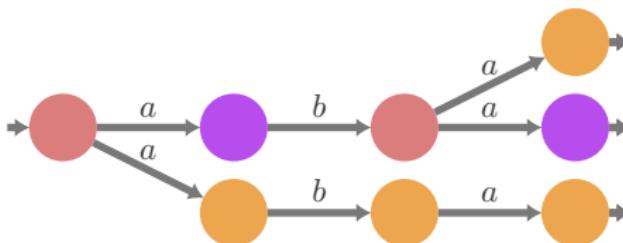
✓
✓
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=====✓

Universal acceptance condition



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$\Sigma^* \cdot a$

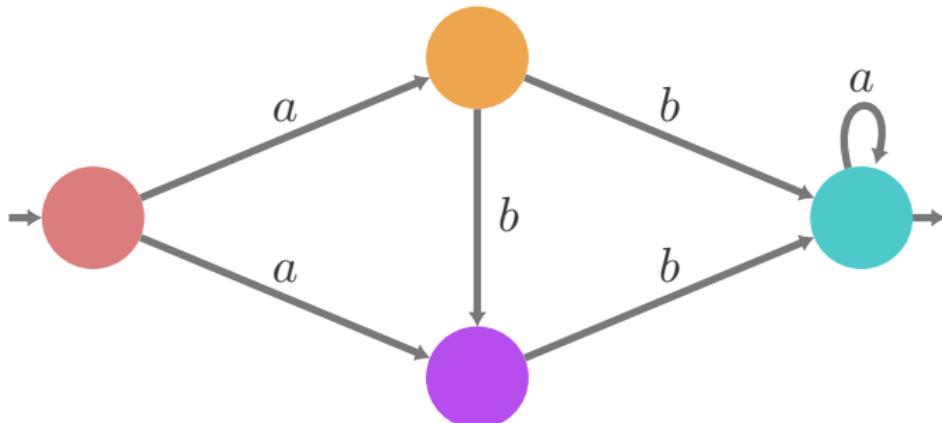


$aba \in L^{\forall}(\mathcal{A})$

✓
✓
✓
=====✓

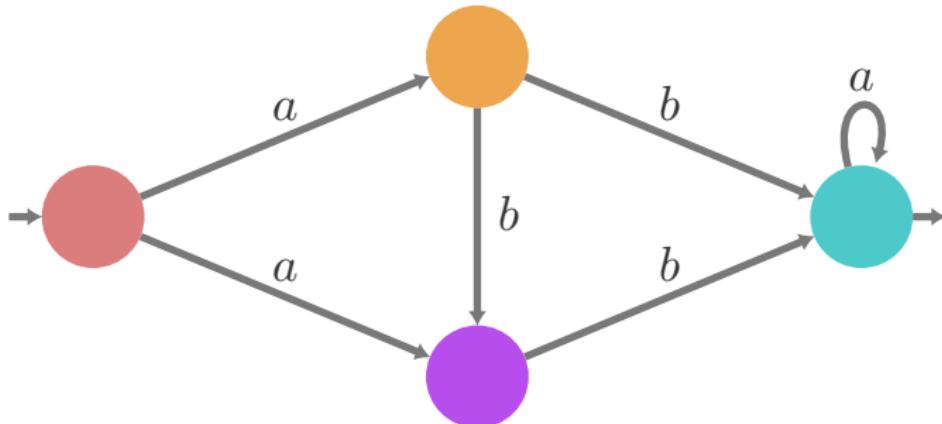
Universal acceptance condition II

$$L^{\forall}(\mathcal{A}) = \{w \in \Sigma^* \mid \text{all runs on } w \text{ are accepting}\}$$



Universal acceptance condition II

$$L^{\forall}(\mathcal{A}) = \{w \in \Sigma^* \mid \text{all runs on } w \text{ are accepting}\}$$



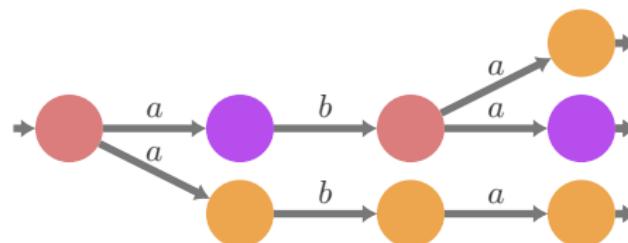
$$L^{\forall}(\mathcal{A}) = \emptyset$$

Determinization of universal finite automata

$$L = L^{\forall}(\mathcal{A}) \Rightarrow L = L^{\forall}(\mathcal{A}') \wedge \mathcal{A}' \text{ deterministic}$$

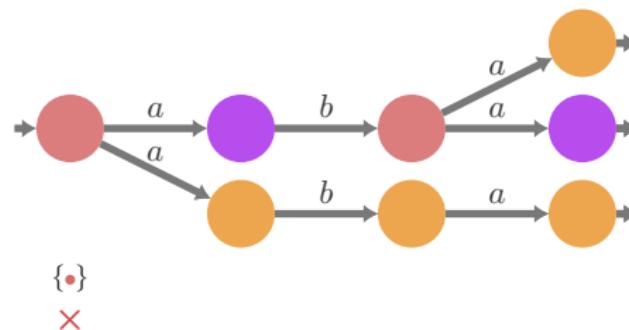
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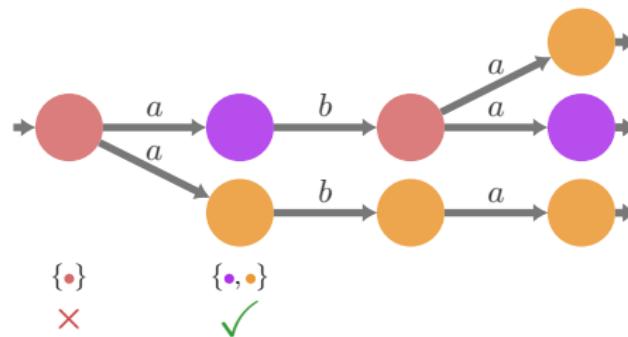
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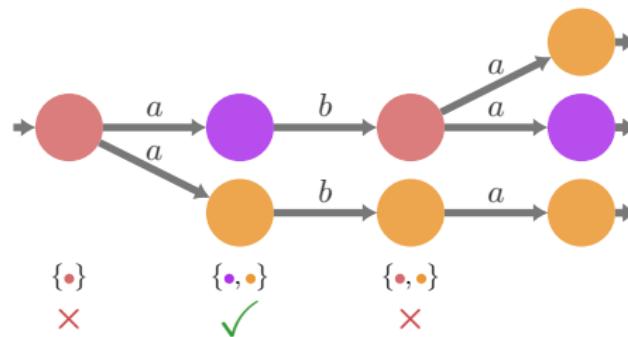
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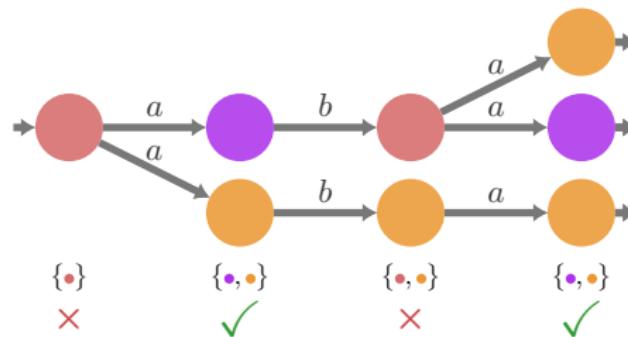
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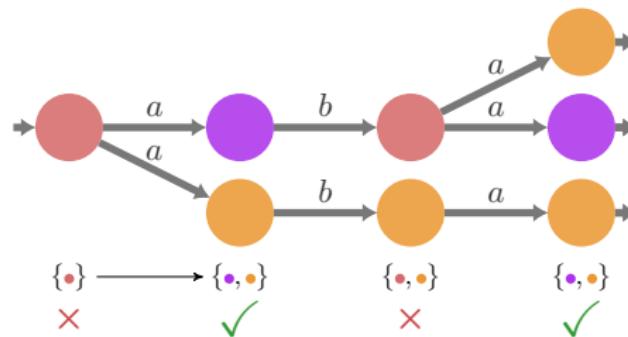
Determinization of universal finite automata

$$L = L^{\forall}(\mathcal{A}) \Rightarrow L = L^{\forall}(\mathcal{A}') \wedge \mathcal{A}' \text{ deterministic}$$



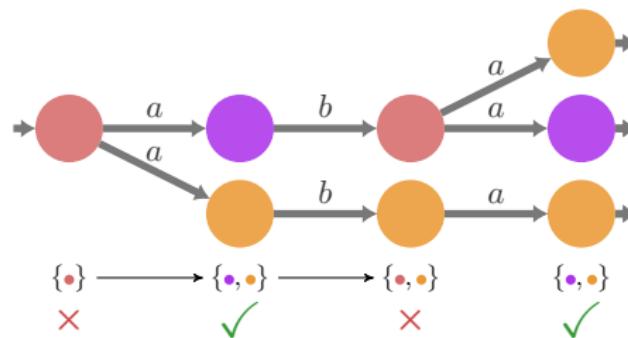
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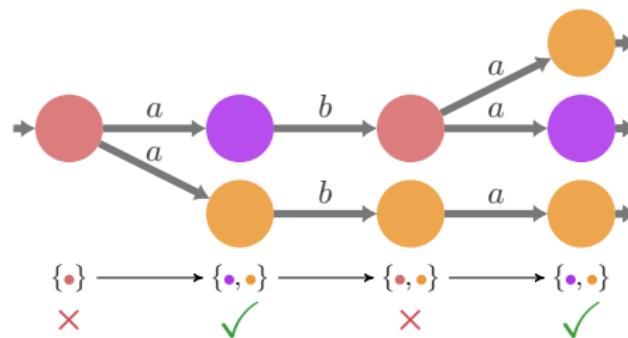
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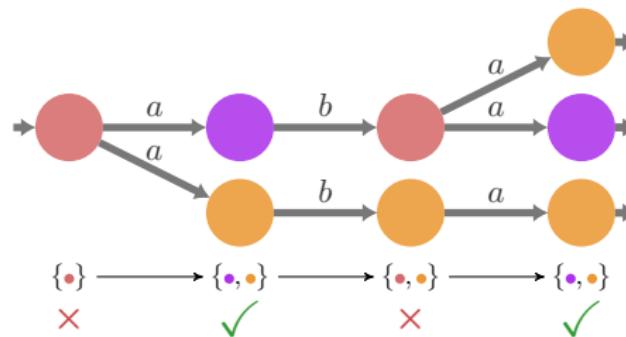
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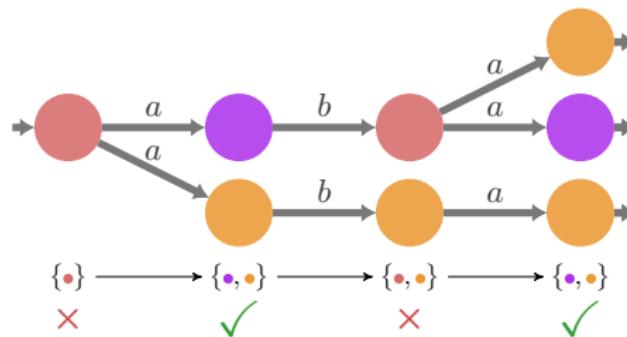
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$$\mathcal{A}' = (Q', \Sigma, I', \Delta', F')$$

Determinization of universal finite automata

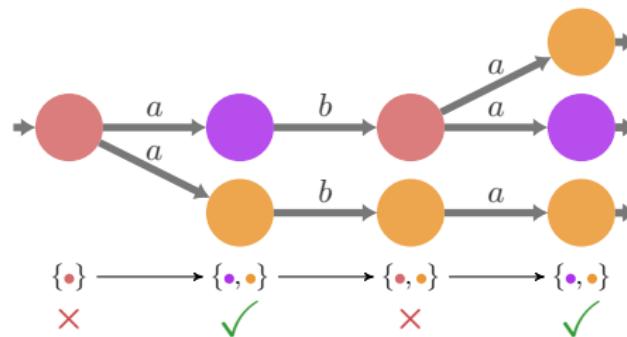
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$$\mathcal{A}' = (Q', \Sigma, I', \Delta', F') \quad Q' = \{S \subseteq Q\}$$

Determinization of universal finite automata

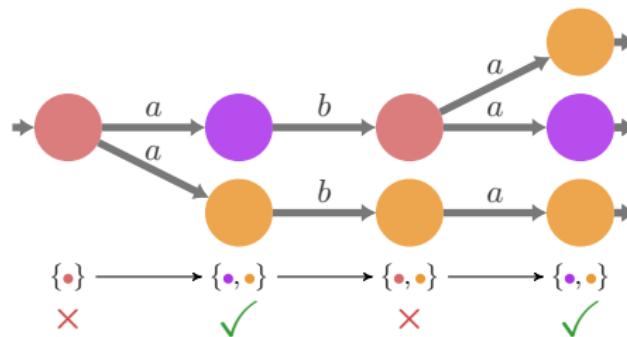
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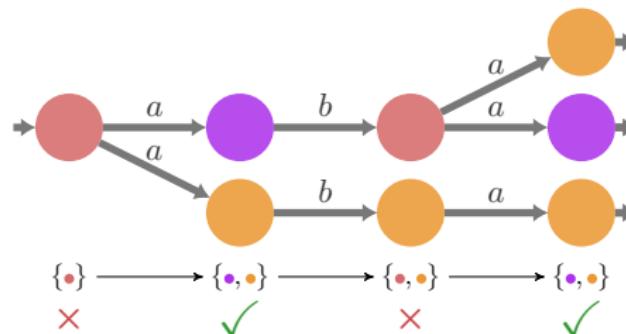


$$\begin{array}{ll} \mathcal{A}' &= (Q', \Sigma, I', \Delta', F') \\ I' &= \{I\} \end{array}$$

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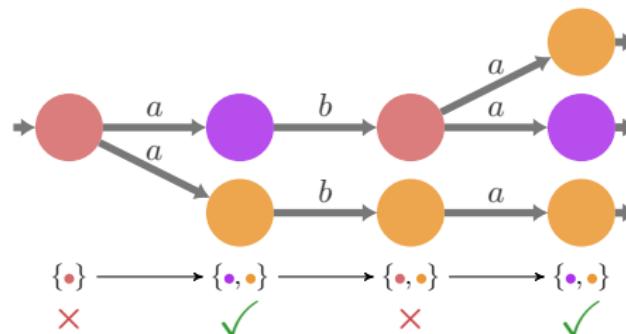
$$\begin{aligned}\mathcal{A}' &= (Q', \Sigma, I', \Delta', F') \\ I' &= \{I\}\end{aligned}$$

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Determinization of universal finite automata

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$$\begin{aligned}\mathcal{A}' &= (Q', \Sigma, I', \Delta', F') \\ I' &= \{I\}\end{aligned}$$

$$\Delta' = \{(S, a, T) \mid \textcolor{teal}{T} = \{p \mid \exists q \in S : (q, a, p) \in \Delta\}\}$$

$$\begin{aligned}Q' &= \{S \subseteq Q\} \\ F' &= \{S \subseteq F\}\end{aligned}$$

Comparing existential and universal acceptance

$\{L^{\exists}(\mathcal{A}) \mid \mathcal{A} \text{ finite automaton}\} \subsetneq \{L^{\forall}(\mathcal{A}) \mid \mathcal{A} \text{ finite automaton}\}$

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Comparing existential and universal acceptance

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$$L = L^{\exists}(\mathcal{A}) \iff L = L^{\exists}(\mathcal{A}') \wedge \mathcal{A}' \text{ deterministic}$$

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$$\begin{aligned} L = L^{\exists}(\mathcal{A}) &\Leftrightarrow L = L^{\exists}(\mathcal{A}') \wedge \mathcal{A}' \text{ deterministic} \\ &\Leftrightarrow L = L^{\forall}(\mathcal{A}') \wedge \mathcal{A}' \text{ deterministic} \end{aligned}$$

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$$\begin{aligned} L = L^{\exists}(\mathcal{A}) &\iff L = L^{\exists}(\mathcal{A}') \wedge \mathcal{A}' \text{ deterministic} \\ &\iff L = L^{\forall}(\mathcal{A}') \wedge \mathcal{A}' \text{ deterministic} \\ &\iff L = L^{\forall}(\mathcal{A}'') \end{aligned}$$

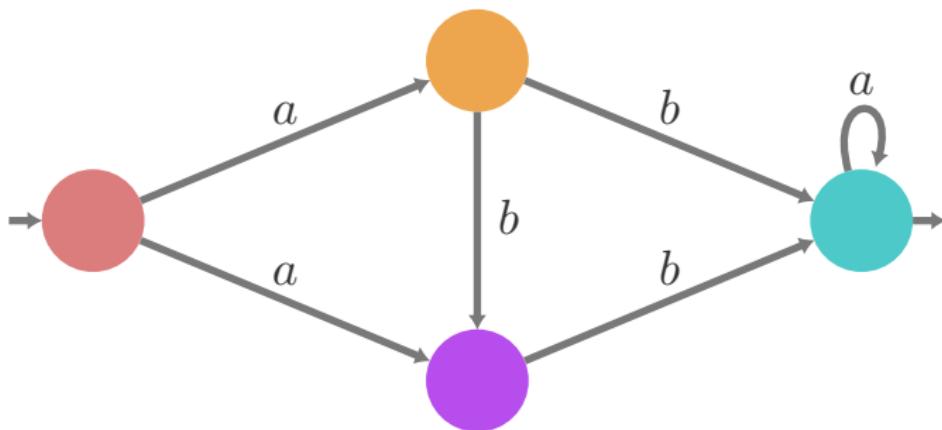
Comparing existential and universal acceptance

$$\{L^{\exists}(\mathcal{A}) \mid \mathcal{A} \text{ finite automaton}\} = \{L^{\forall}(\mathcal{A}) \mid \mathcal{A} \text{ finite automaton}\}$$

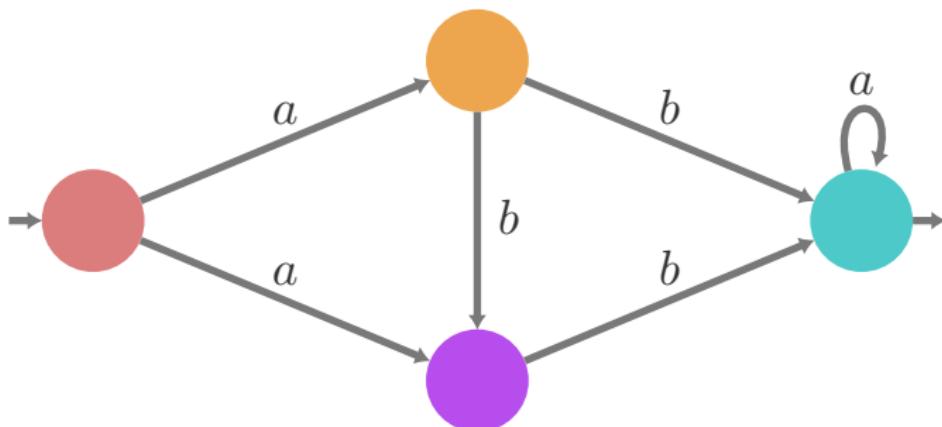
$$\begin{aligned} L = L^{\exists}(\mathcal{A}) &\Leftrightarrow L = L^{\exists}(\mathcal{A}') \wedge \mathcal{A}' \text{ deterministic} \\ &\Leftrightarrow L = L^{\forall}(\mathcal{A}') \wedge \mathcal{A}' \text{ deterministic} \\ &\Leftrightarrow L = L^{\forall}(\mathcal{A}'') \end{aligned}$$

ALTERNATING AUTOMATA

Alternation in finite automata

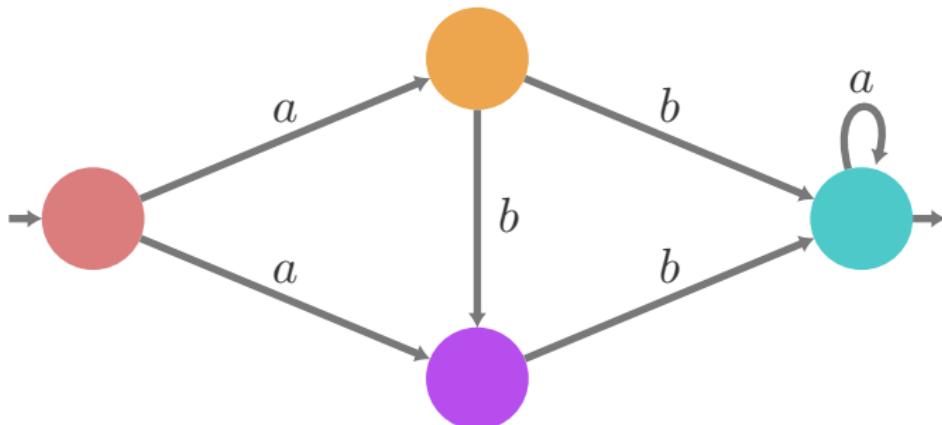


Alternation in finite automata



$$L_{\bullet} = a^*$$

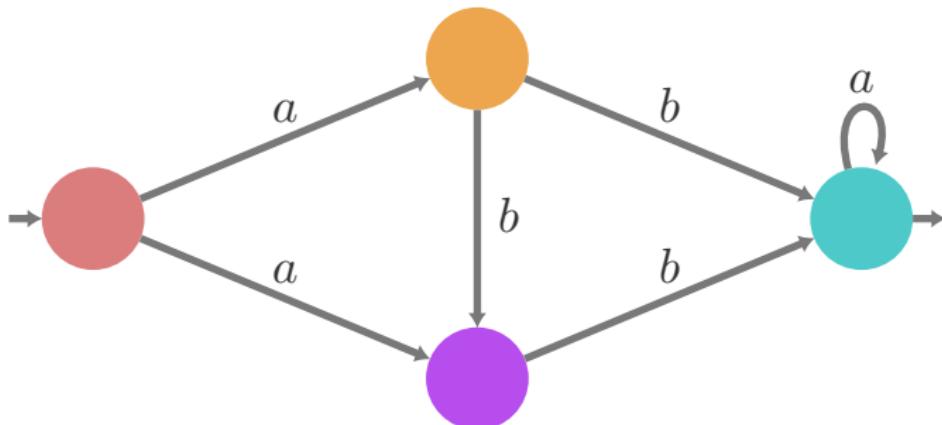
Alternation in finite automata



$$L_{\text{cyan}} = a^*$$

$$L_{\text{purple}} = b \cdot L_{\text{cyan}}$$

Alternation in finite automata

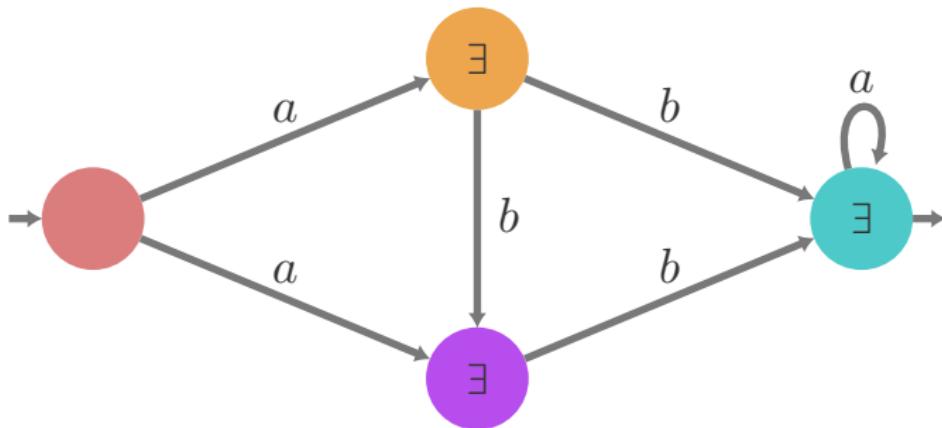


$$L_{\text{cyan}} = a^*$$

$$L_{\text{purple}} = b \cdot L_{\text{cyan}}$$

$$L_{\text{orange}} = b \cdot L_{\text{cyan}} \cup b \cdot L_{\text{purple}}$$

Alternation in finite automata



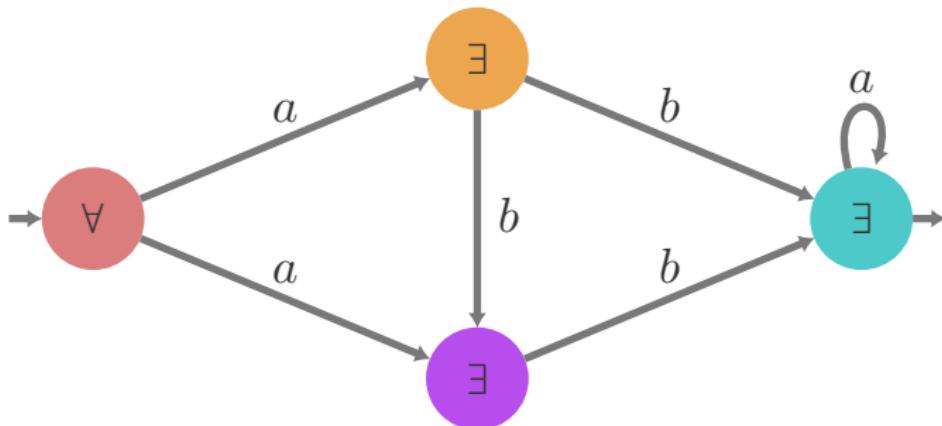
$$L_{\text{cyan}} = a^*$$

$$L_{\text{purple}} = b \cdot L_{\text{cyan}}$$

$$L_{\text{yellow}} = b \cdot L_{\text{cyan}} \cup b \cdot L_{\text{purple}}$$

$$\exists : \quad L_q = \bigcup_{\sigma \in \Sigma} \sigma \cdot \left(\bigcup_{p \in T_\sigma(q)} L_p \right)$$

Alternation in finite automata



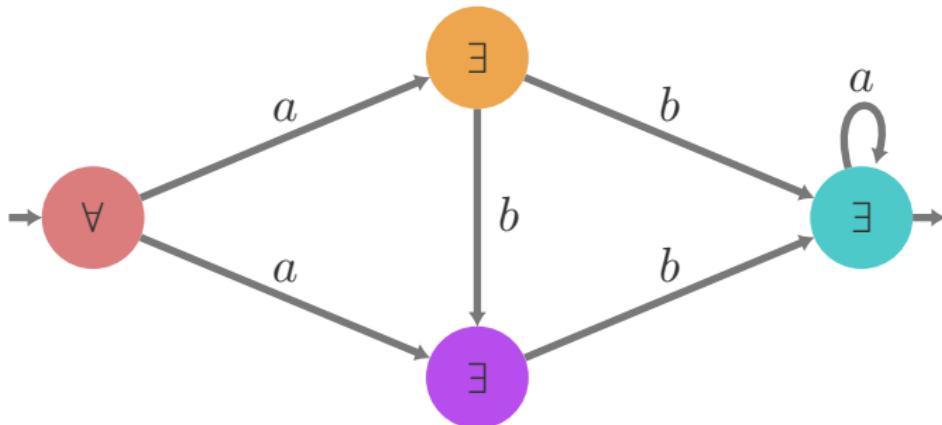
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Alternation in finite automata



$$L_{\text{teal}} = a^*$$

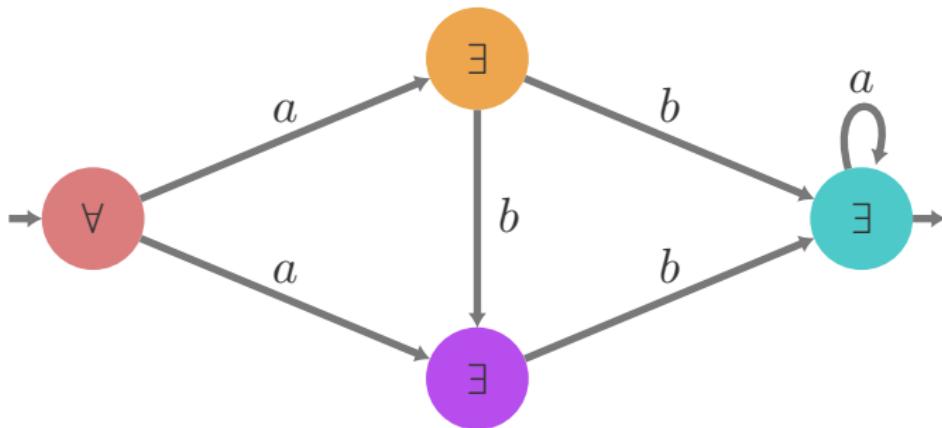
$$L_{\text{purple}} = b \cdot L_{\text{teal}}$$

$$L_{\text{orange}} = b \cdot L_{\text{teal}} \cup b \cdot L_{\text{purple}}$$

$$L_{\text{red}} = a \cdot L_{\text{orange}} \cap a \cdot L_{\text{purple}}$$

$$\exists : \quad L_q = \bigcup_{\sigma \in \Sigma} \sigma \cdot \left(\bigcup_{p \in T_\sigma(q)} L_p \right)$$

Alternation in finite automata



$$L_{\text{teal}} = a^*$$

$$L_{\text{purple}} = b \cdot L_{\text{teal}}$$

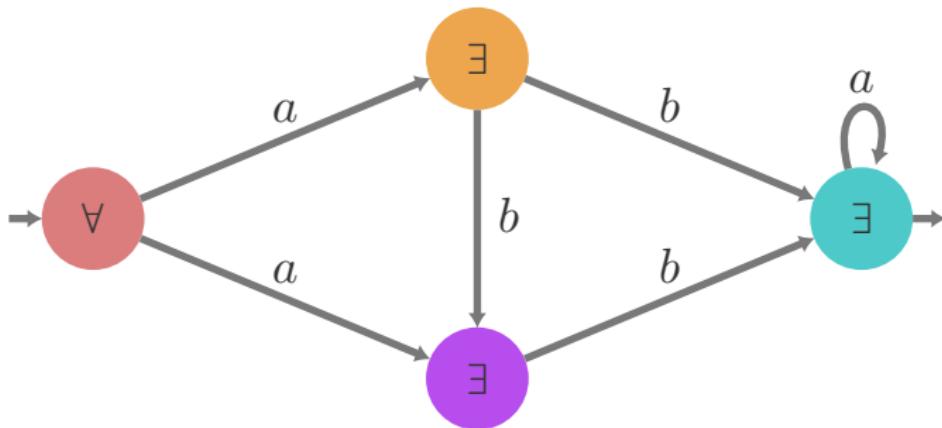
$$L_{\text{orange}} = b \cdot L_{\text{teal}} \cup b \cdot L_{\text{purple}}$$

$$L_{\text{red}} = a \cdot L_{\text{orange}} \cap a \cdot L_{\text{purple}}$$

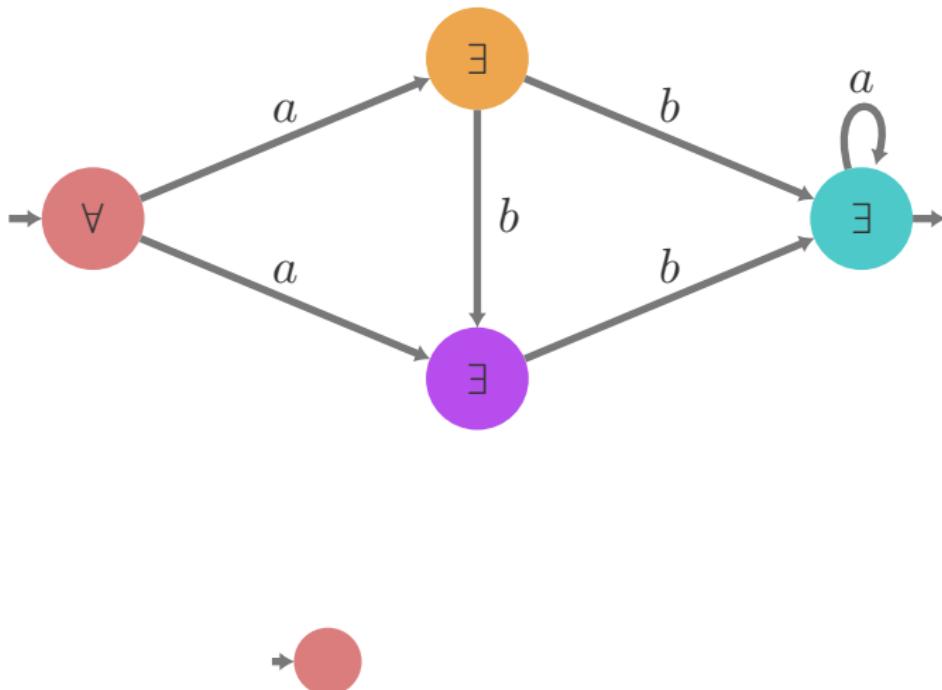
$$\exists : \quad L_q = \bigcup_{\sigma \in \Sigma} \sigma \cdot \left(\bigcup_{p \in T_\sigma(q)} L_p \right)$$

$$\forall : \quad L_q = \bigcup_{\sigma \in \Sigma} \sigma \cdot \left(\bigcap_{p \in T_\sigma(q)} L_p \right)$$

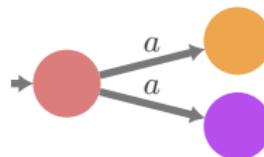
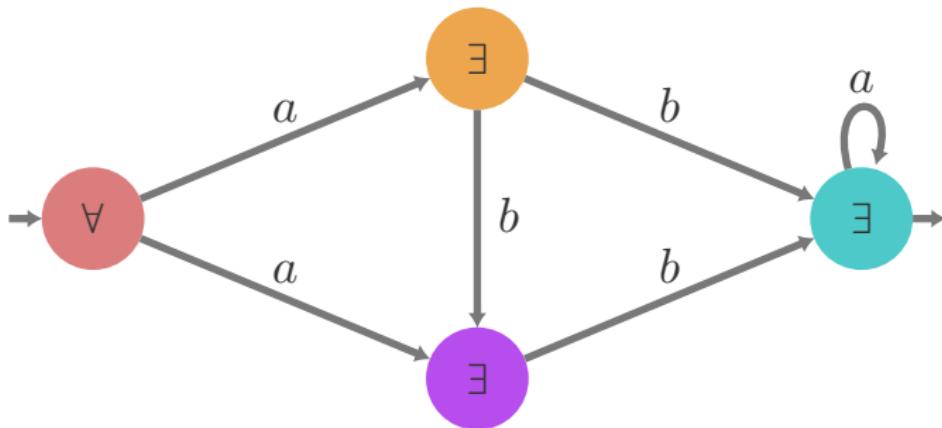
Alternation in finite automata



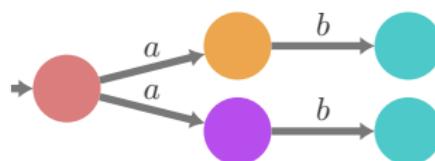
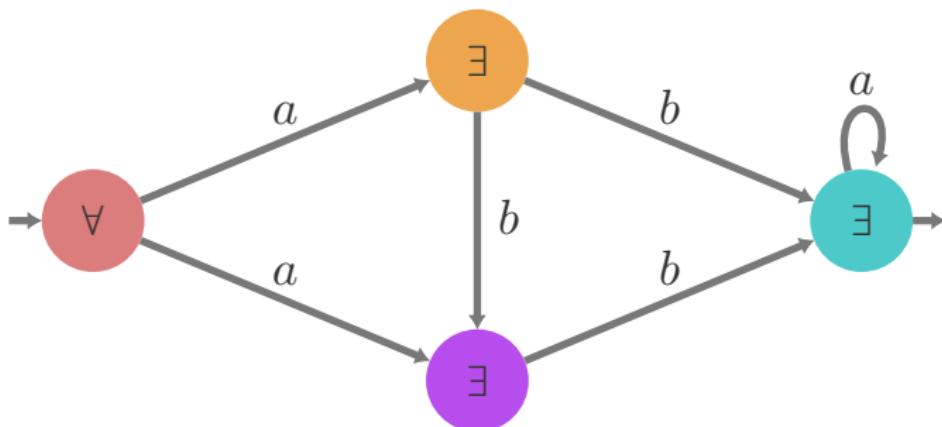
Alternation in finite automata



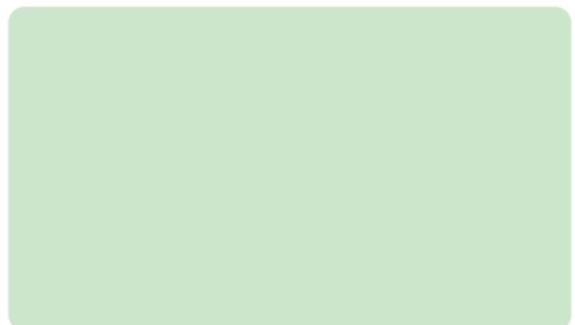
Alternation in finite automata



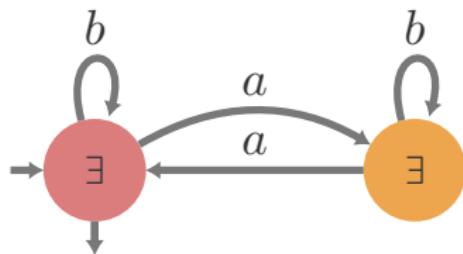
Alternation in finite automata



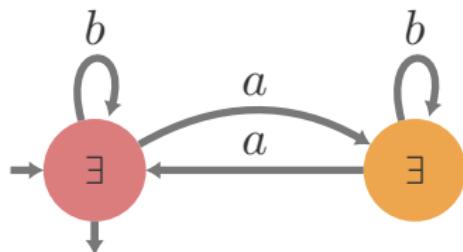
Cool properties



Cool properties

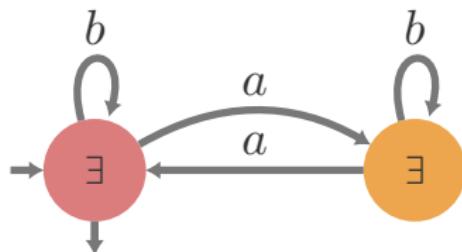


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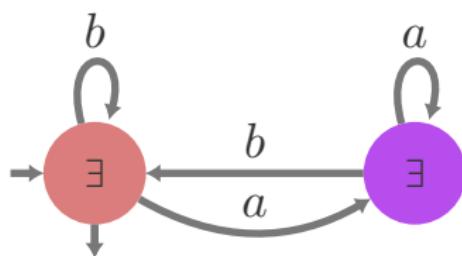


$$L_1 = \{w \in \Sigma^* \mid |w|_a \in 2\mathbb{N}\}$$

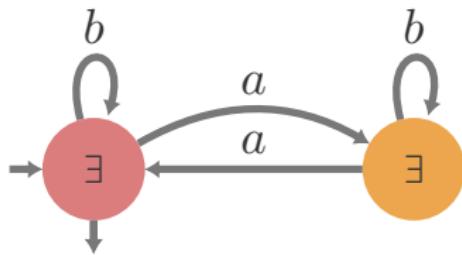
Cool properties



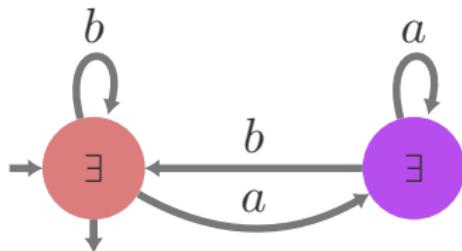
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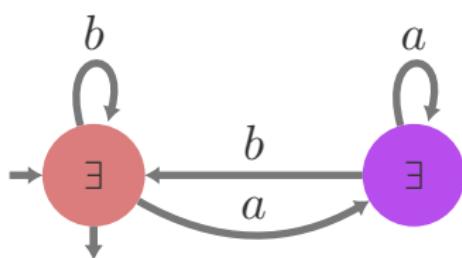
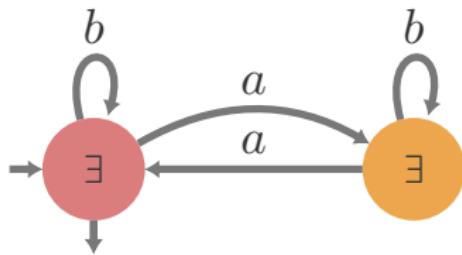
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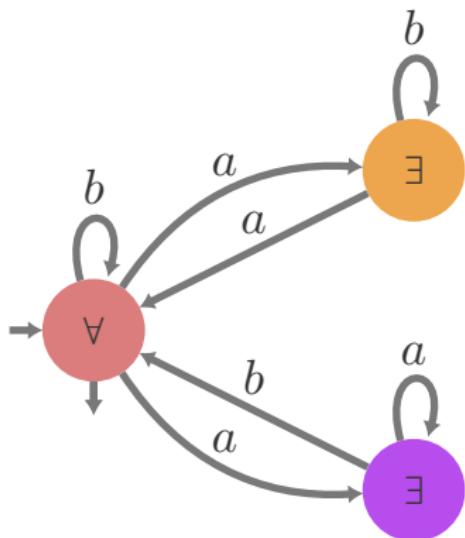
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$$L_3 = L_1 \cap L_2$$

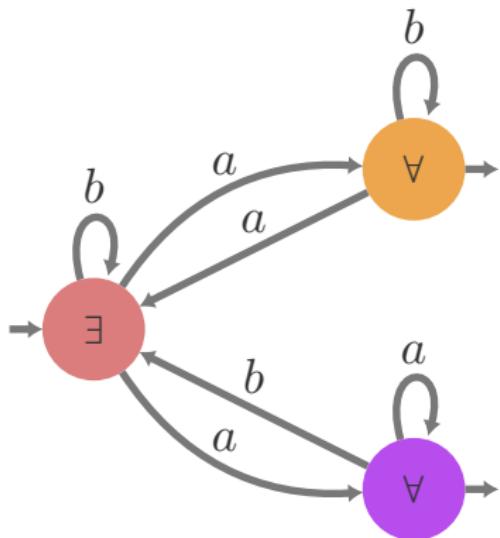
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Cool properties

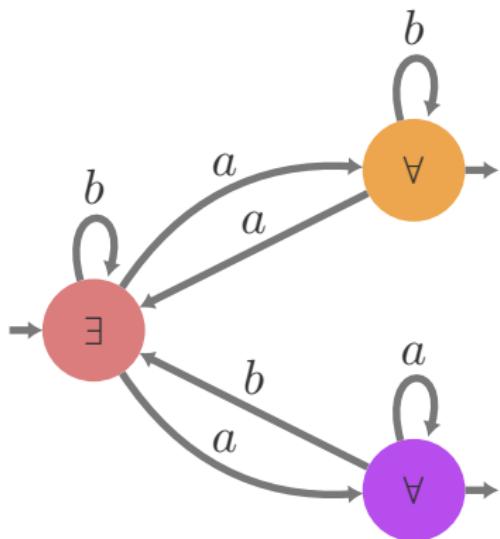


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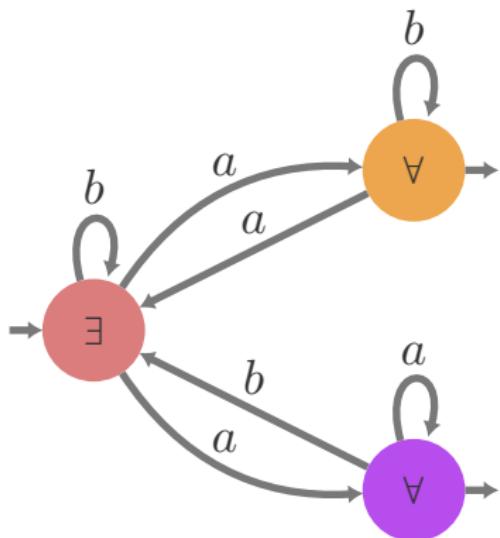
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$$L_4 = L_3^C$$

Cool properties



$$L_1 = \{w \in \Sigma^* \mid |w|_a \in 2\mathbb{N}\}$$

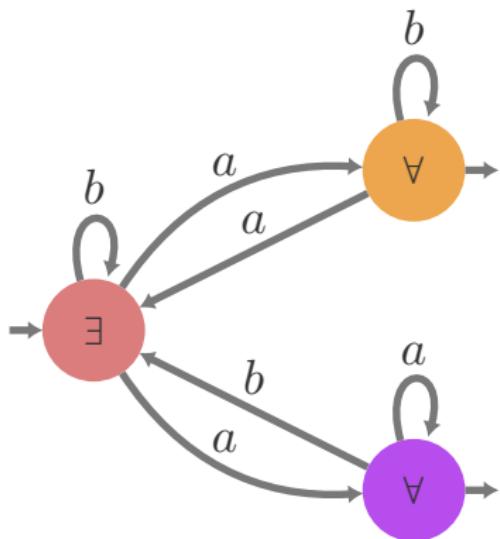
$$L_2 = \Sigma^* \cdot b \cup \{\varepsilon\}$$

$$L_3 = L_1 \cap L_2$$

$$L_4 = L_3^C$$

$\{L^{\exists}(\mathcal{A}) \mid \mathcal{A} \text{ finite automaton}\} ? \{L(\mathcal{A}) \mid \mathcal{A} \text{ alternating}\}$

Cool properties



$$\begin{aligned}L_1 &= \{w \in \Sigma^* \mid |w|_a \in 2\mathbb{N}\} \\L_2 &= \Sigma^* \cdot b \cup \{\varepsilon\}\end{aligned}$$

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$$\{L^{\exists}(\mathcal{A}) \mid \mathcal{A} \text{ finite automaton}\} = \{L(\mathcal{A}) \mid \mathcal{A} \text{ alternating}\}$$

Non-cool properties

AFA \hookrightarrow DFA $\in \mathbf{DOUBLEEXP}$

Non-cool properties

AFA \hookrightarrow DFA $\in \textbf{DOUBLEEXP}$

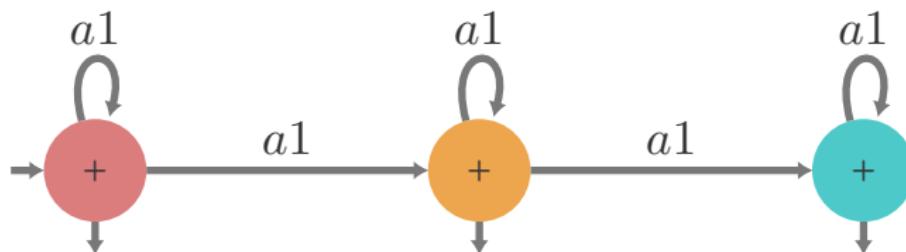
EMPTINESS is **PSPACE** complete

INPUT: AFA \mathcal{A}

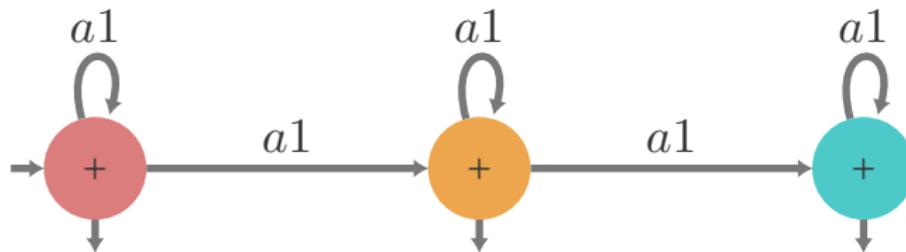
OUTPUT: yes if $L(\mathcal{A}) = \emptyset$, no otherwise

WEIGHTED ALTERNATION

Alternation in weighted finite automata

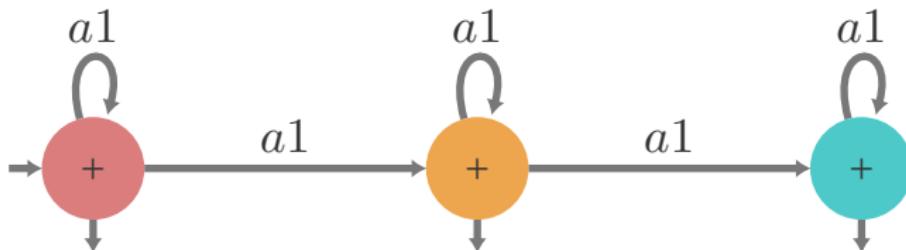


Alternation in weighted finite automata



$$\llbracket \mathcal{A} \rrbracket_{\bullet}(a^{n+1}) = 1$$

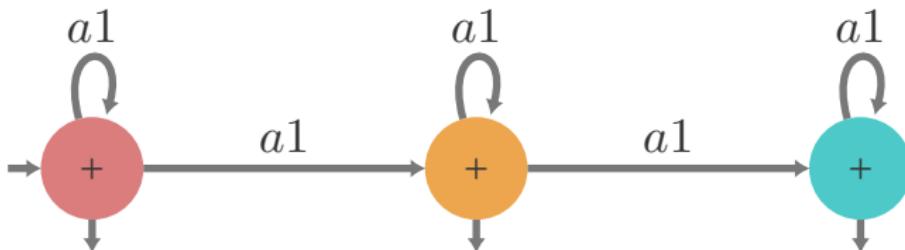
Alternation in weighted finite automata



$$\llbracket \mathcal{A} \rrbracket_{\textcolor{teal}{\bullet}}(a^{n+1}) = 1$$

$$\llbracket \mathcal{A} \rrbracket_{\textcolor{orange}{\bullet}}(a^{n+1}) = \llbracket \mathcal{A} \rrbracket_{\textcolor{orange}{\bullet}}(a^n) + \llbracket \mathcal{A} \rrbracket_{\textcolor{teal}{\bullet}}(a^n) = n + 2$$

Alternation in weighted finite automata

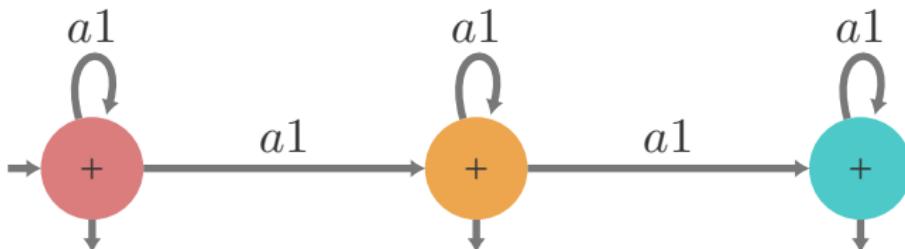


$$\llbracket \mathcal{A} \rrbracket_{\textcolor{teal}{\bullet}}(a^{n+1}) = 1$$

$$\llbracket \mathcal{A} \rrbracket_{\textcolor{orange}{\bullet}}(a^{n+1}) = \llbracket \mathcal{A} \rrbracket_{\textcolor{orange}{\bullet}}(a^n) + \llbracket \mathcal{A} \rrbracket_{\textcolor{teal}{\bullet}}(a^n) = n + 2$$

$$\llbracket \mathcal{A} \rrbracket_{\textcolor{red}{\bullet}}(a^{n+1}) = \llbracket \mathcal{A} \rrbracket_{\textcolor{red}{\bullet}}(a^n) + \llbracket \mathcal{A} \rrbracket_{\textcolor{orange}{\bullet}}(a^n) = 1 + \sum_{i=1}^{n+1} i$$

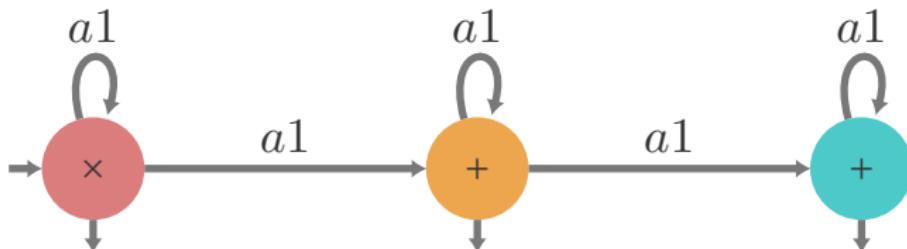
Alternation in weighted finite automata



$$\begin{aligned} \llbracket \mathcal{A} \rrbracket_{\textcolor{teal}{\bullet}}(a^{n+1}) &= 1 \\ \llbracket \mathcal{A} \rrbracket_{\textcolor{orange}{\bullet}}(a^{n+1}) &= \llbracket \mathcal{A} \rrbracket_{\textcolor{orange}{\bullet}}(a^n) + \llbracket \mathcal{A} \rrbracket_{\textcolor{teal}{\bullet}}(a^n) = n + 2 \\ \llbracket \mathcal{A} \rrbracket_{\textcolor{red}{\bullet}}(a^{n+1}) &= \llbracket \mathcal{A} \rrbracket_{\textcolor{red}{\bullet}}(a^n) + \llbracket \mathcal{A} \rrbracket_{\textcolor{orange}{\bullet}}(a^n) = 1 + \sum_{i=1}^{n+1} i \end{aligned}$$

$$+ : \quad \llbracket A \rrbracket_q(\sigma w) = \sum_{p \in T_a(q)} \text{wt}(q, a, p) \cdot \llbracket A \rrbracket_p(w)$$

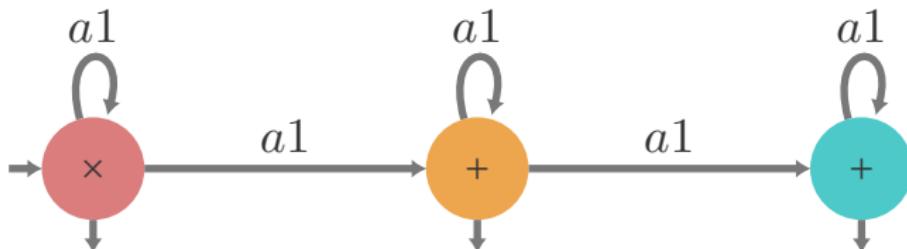
Alternation in weighted finite automata



$$\begin{aligned}\llbracket \mathcal{A} \rrbracket_{\textcolor{teal}{\bullet}}(a^{n+1}) &= 1 \\ \llbracket \mathcal{A} \rrbracket_{\textcolor{orange}{\bullet}}(a^{n+1}) &= \llbracket \mathcal{A} \rrbracket_{\textcolor{orange}{\bullet}}(a^n) + \llbracket \mathcal{A} \rrbracket_{\textcolor{teal}{\bullet}}(a^n) = n + 2\end{aligned}$$

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Alternation in weighted finite automata



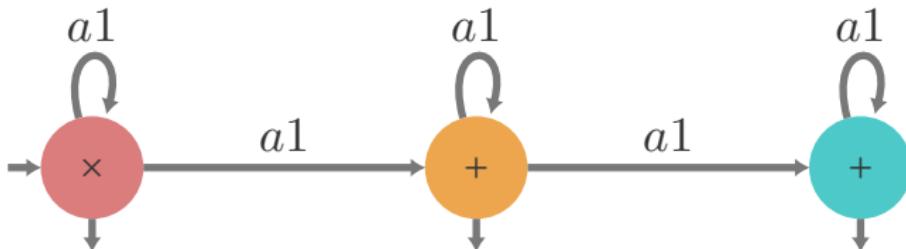
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$$+ : \quad \llbracket A \rrbracket_q(\sigma w) = \sum_{p \in T_a(q)} \text{wt}(q, a, p) \cdot \llbracket A \rrbracket_p(w)$$

Alternation in weighted finite automata



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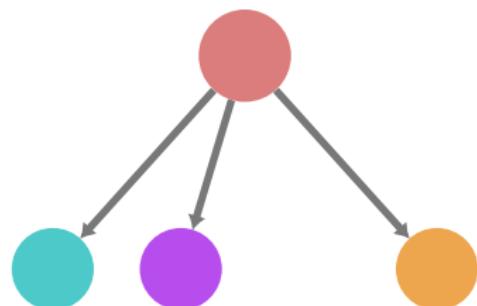
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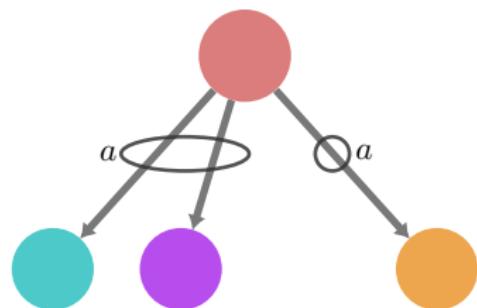
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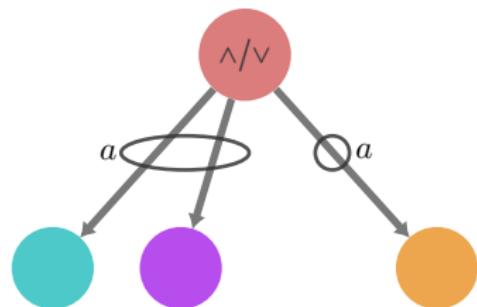
Mixed state alternation



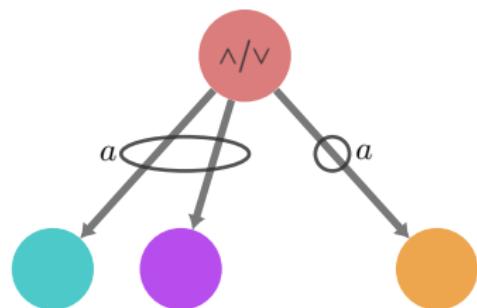
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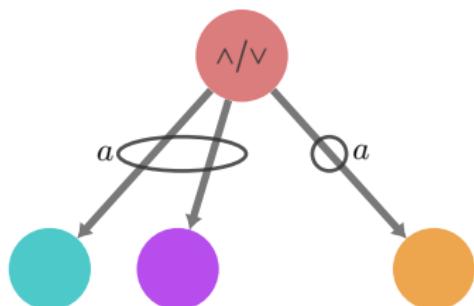


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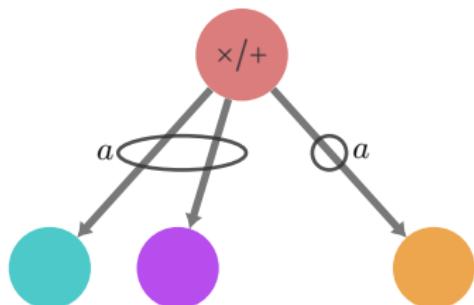
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Modified transition function

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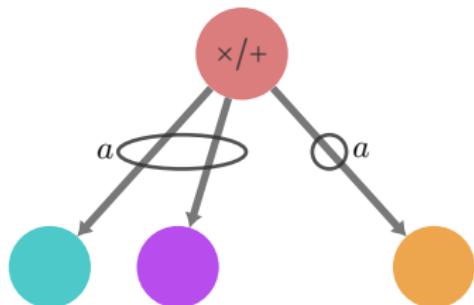
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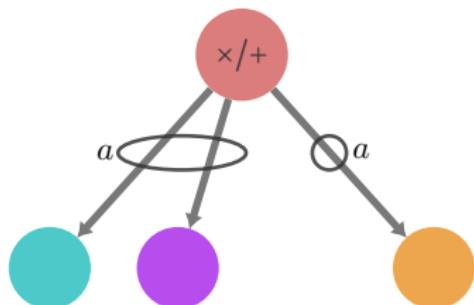
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$$\delta(q_1, a) = q_1 q_2 + q_3^2$$

Weighted alternating finite automata

WAFA [Kostolányi, Mišún '18]

- S commutative semiring

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- $\delta : Q \times \Sigma \rightarrow S[q_1, \dots, q_n]_{\text{const}=0}$ transition function
- $\tau : Q \rightarrow S$ final weight function

State behavior and behavior

State behavior for $q \in Q$

$$\begin{aligned} \llbracket \mathcal{A} \rrbracket_q : \Sigma^* &\rightarrow S[q_1, \dots, q_n]_{\text{const}=0} : \\ w &\mapsto \begin{cases} q & \text{if } w = \varepsilon \\ \delta(q, a)(\llbracket \mathcal{A} \rrbracket_{q_1}(v), \dots, \llbracket \mathcal{A} \rrbracket_{q_n}(v)) & \text{if } w = av \end{cases} \end{aligned}$$

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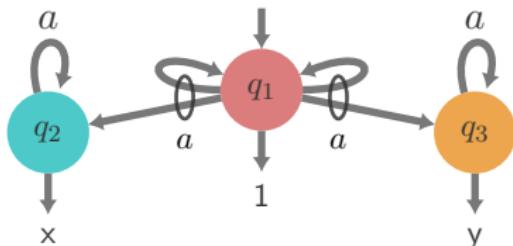
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$$\mathcal{A} = (\{q_1, q_2, q_3\}, \{a\}, q_1, \delta, \tau)$$

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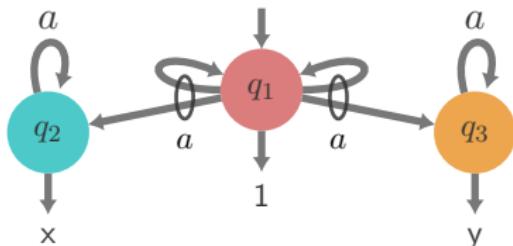


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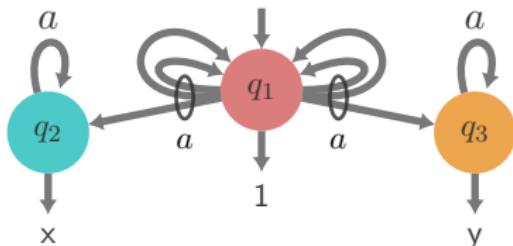


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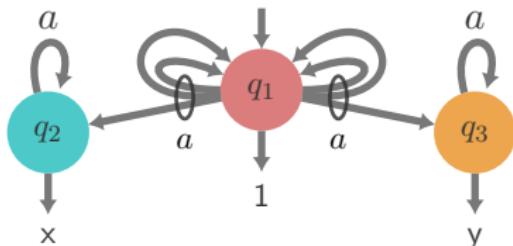


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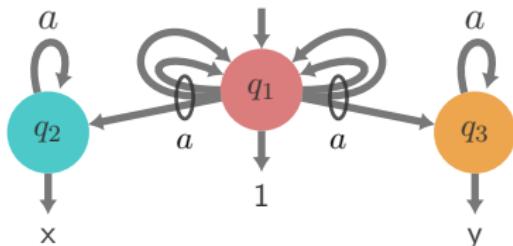


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Expressive power of WAFA and WFA

$\text{WAFA}(S, \Sigma) = \text{WFA}(S, \Sigma) \Leftrightarrow S$ locally finite

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Simplified WAFA

Without loss of generality:

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$$s \in \text{WAFA}(S, \Sigma) \not\Rightarrow s \circ h^{-1} \in \text{WAFA}(S, \Sigma)$$

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$$s : \{a, c, d, \#\} \rightarrow \mathbb{B}[y] : \quad w \mapsto \begin{cases} y^{i,j} & \text{if } w = a^i \# c^j d^k \\ 0 & \text{otherwise} \end{cases}$$

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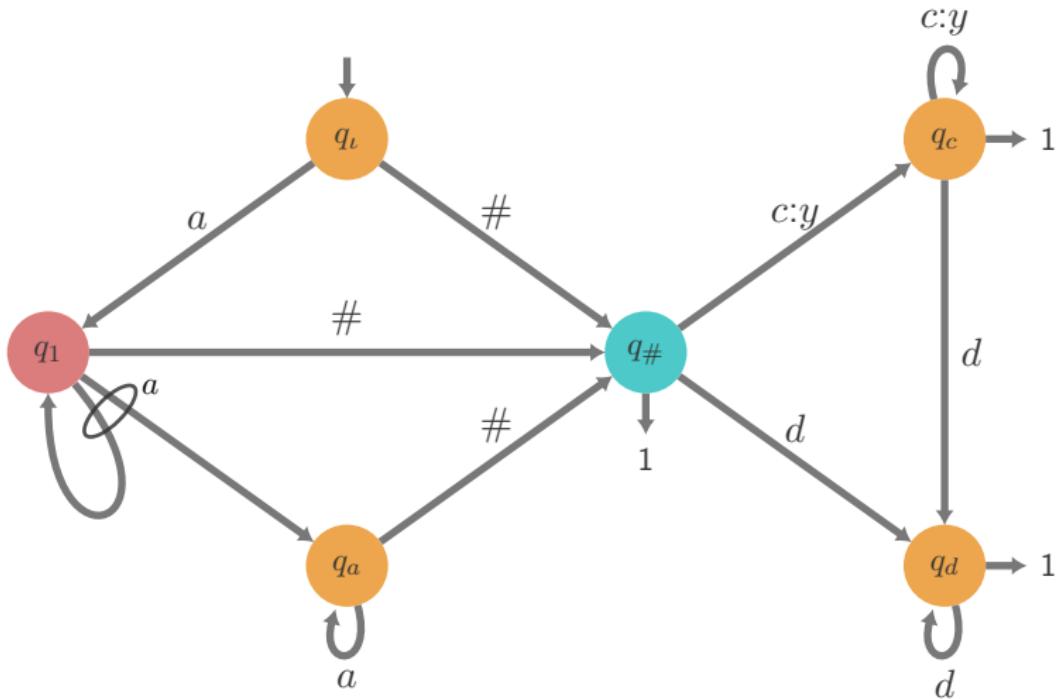
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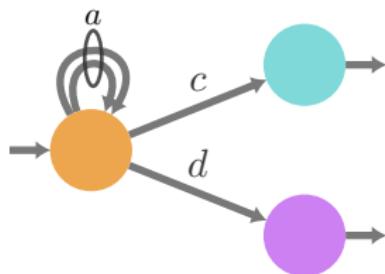
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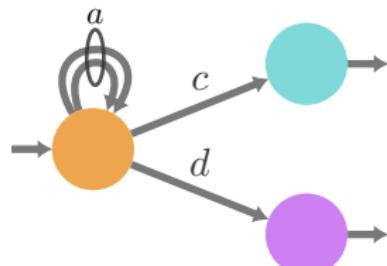
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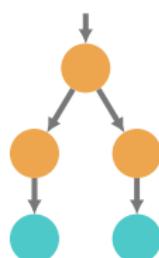
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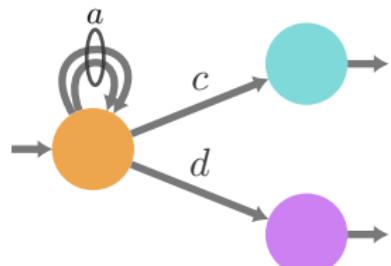
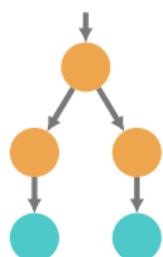
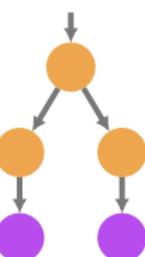
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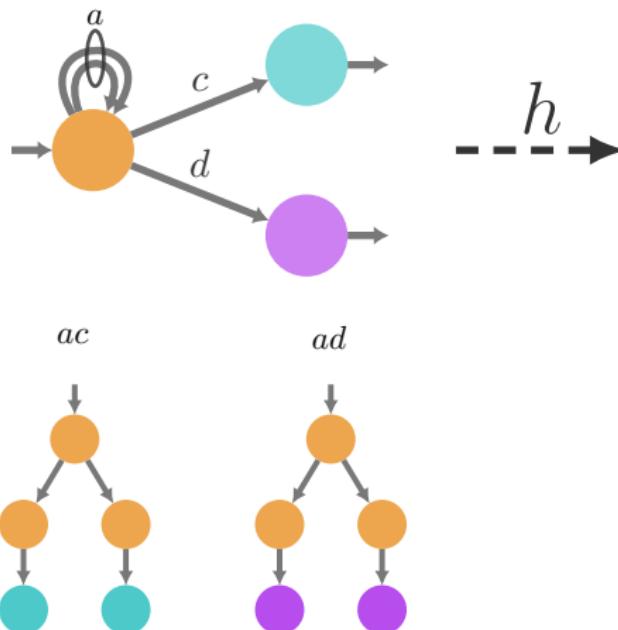
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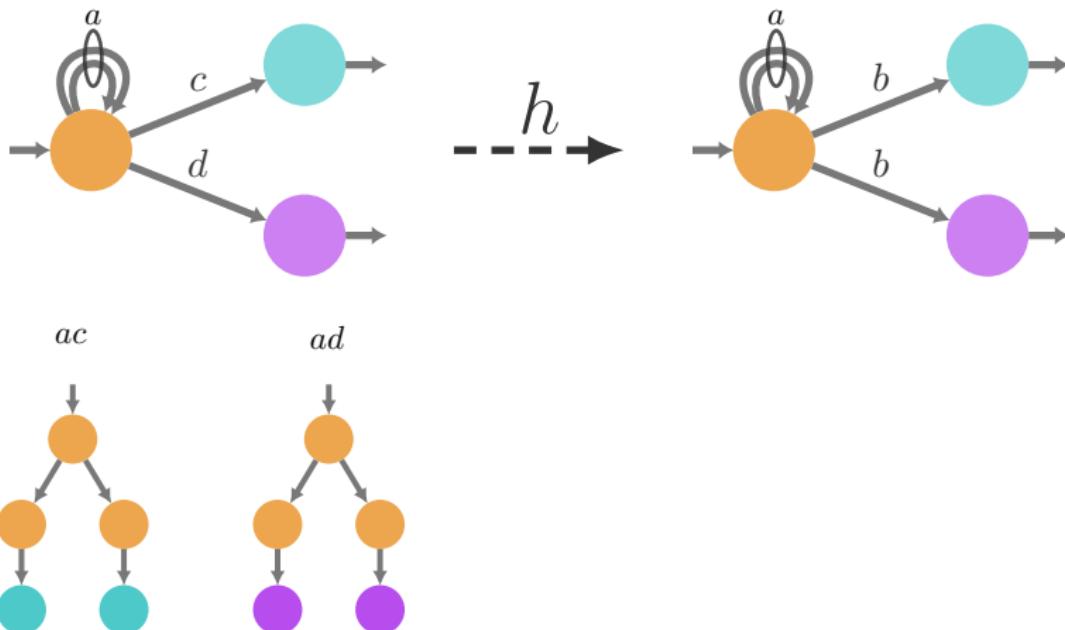
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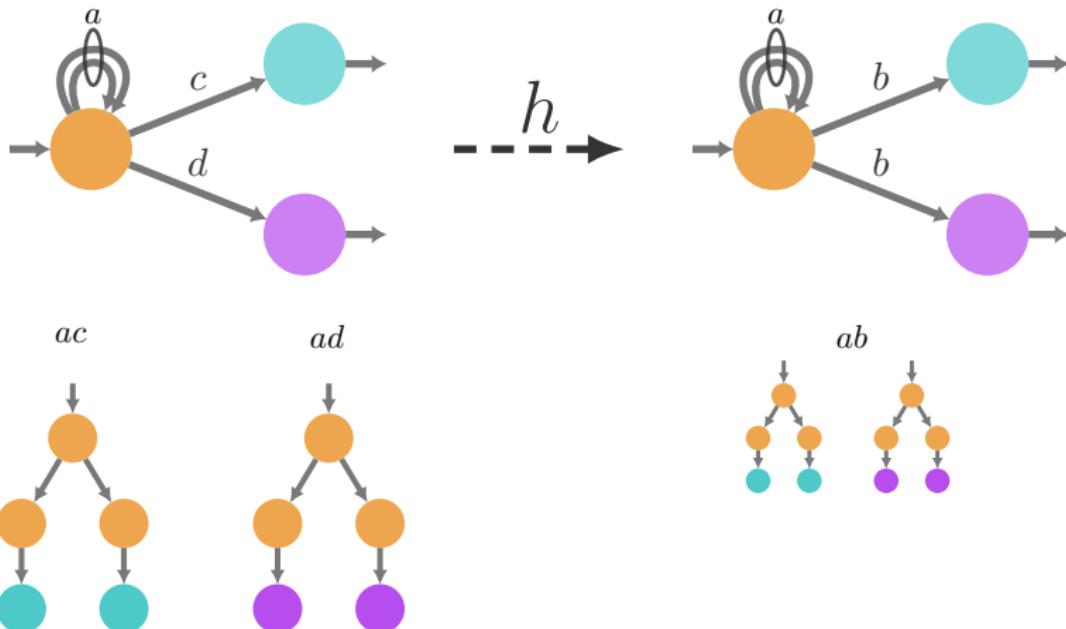
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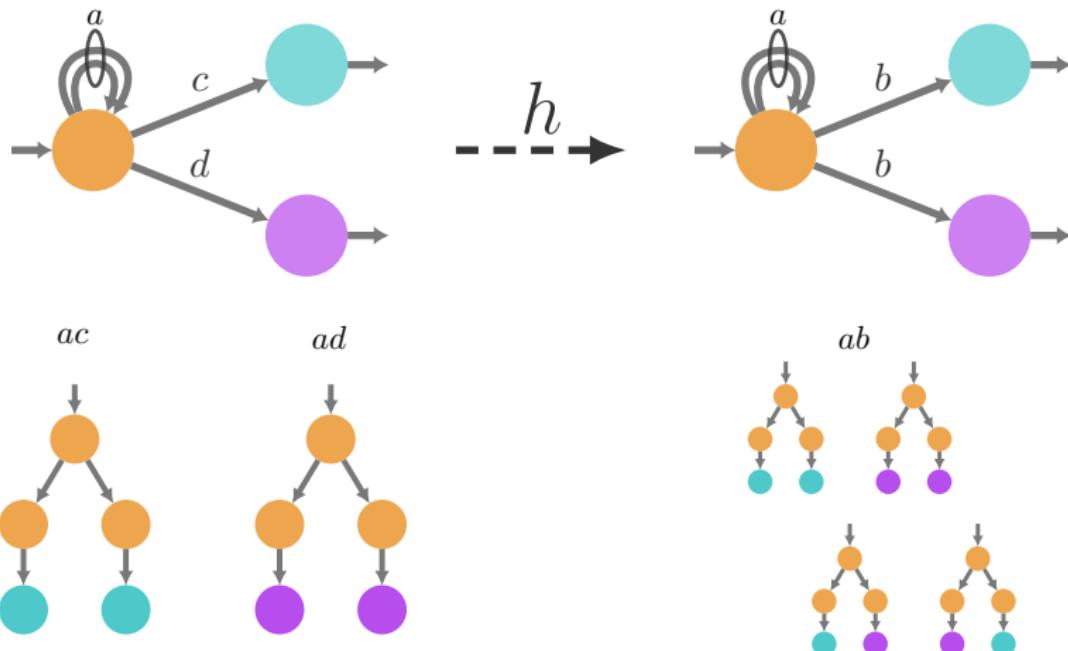
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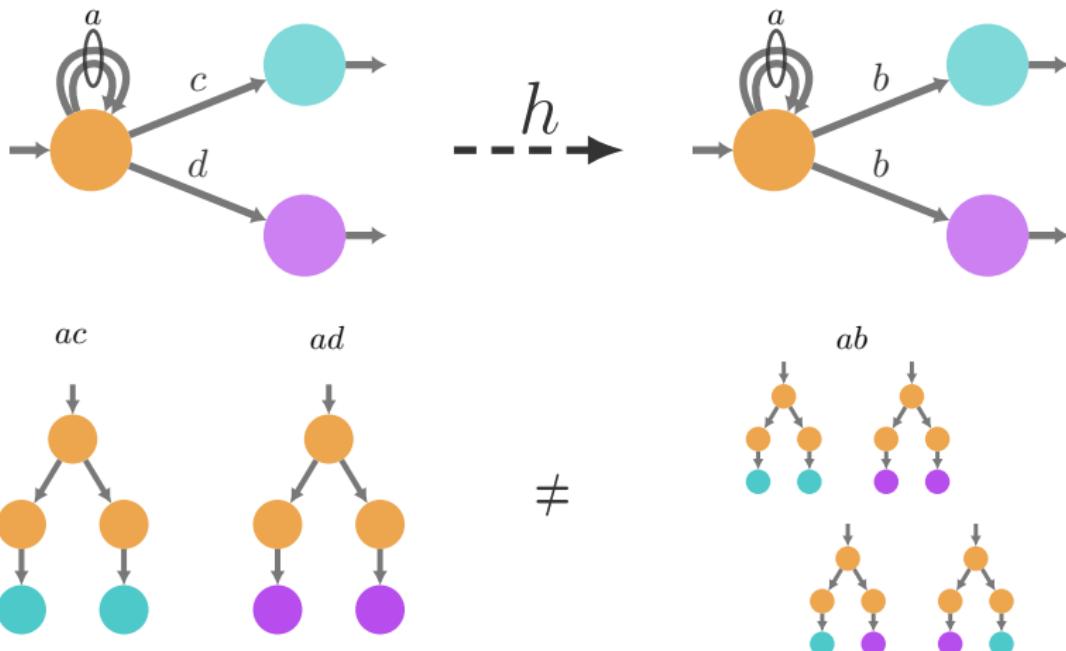
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WAFA and WFTA

Characterization by WFTA (G.)

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Characterization of WFTA closed under Inverses of Homomorphisms

$$\text{WFTA}(S) \circ \text{HOM} = \text{WFTA}(S) \Leftrightarrow S \text{ locally finite}$$

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