ARISTOTLE UNIVERSITY OF THESSALONIKI
DEPARTMENT OF MATHEMATICS
POSTGRADUATE PROGRAM
“THEORETICAL COMPUTER SCIENCE AND THEORY OF CONTROL & SYSTEMS”

Weighted tree automata with discounting

MASTER THESIS

Eleni G. Mandrali

Supervisor: George Rahonis
Assistant Professor
Aristotle University of Thessaloniki

Thessaloniki, December 2008

ABSTRACT

We consider weighted top-down tree automata and weighted bottom-up tree automata with discounting over commutative semirings, and we show that the two models accept the same class of formal tree series. For this class of tree series we establish a Kleene theorem and an MSO logical characterization. Then, we introduce weighted Muller tree automata with discounting over the max-plus and the min-plus semirings, and we show their expressive equivalence with two fragments of weighted MSO sentences.

Key words: Semirings, discounting, weighted tree automata with discounting, rational tree series operations with discounting, weighted Muller tree automata, weighted MSO logic over finite and infinite trees.

CONTENTS

1 Introduction

Weighted tree automata over finite trees have been considered by many researchers (see [1, 3, 5, 7, 8, 9, 10, 21, 24, 25, 28, 31, 40, 42, 43]) and have been contributed in important areas of Computer Science like code selection [6, 29] and monadic second order evaluations on graphs [48]. Weighted tree automata models are obtained by classical tree automata, top-down or bottom-up, whose transitions are equipped with weights mainly from a semiring. The weights might model resources used for the execution of transitions, the time needed or reliability. For an excellent survey on weighted tree automata we refer the reader to [32] (see also [5]).

If we require that weighted tree automata can work also on infinite trees, then clearly the underlying semiring should admit infinite sums and products satisfying special axioms (see [47]). *Discounting* is a common strategy to face problems arising on systems with non-terminating behavior, in particular in economic mathematics, in Markov decision processes, and in game theory (see [16, 30, 49]). This method was incorporated for weighted automata over infinite words by Droste and Kuske in [20]. More precisely, the authors considered weighted automata over the max-plus and min-plus semirings, acting on infinite words, and employed a discounting parameter which permitted the summation of infinitely many values. In this way, they achieved a Kleene theorem for the infinitary series obtained as the behaviors of their automata. They also considered weighted automata with discounting over finite words, and they showed a Kleene-Schützenberger theorem for the series accepted by these automata. In [12, 13] further properties of weighted automata with discounting over infinite words were investigated. In [22] a weighted MSO logic with discounting has been introduced and a Büchi-type characterization of infinitary recognizable series with discounting has been established. Kuich [41] proved Kleene theorems for weighted automata with discounting acting on finite and infinite words over Conway semirings. Recently, in [23] the authors investigated weighted automata with discounting over semirings and finitely generated graded monoids.

It is the first goal of this thesis, to introduce weighted tree automata with discounting, acting on finite trees, and achieve a Kleene theorem for the tree series obtained as the behaviors of these automata. Furthermore, we consider a weighted MSO logic on trees with discounting, and in our second main result we provide a logical characterization for the family of the behaviors of our tree automata. Our MSO logic is a slight modification of the MSO logic in [24, 32] and goes back to the pioneering work of Droste and Gastin [18, 19] in which weighted logics over semirings were first time considered. As in [32], our logic has a purely syntactic description whenever the underlying semiring is additively locally finite. Very recently, in [25] the authors achieved a purely syntactic MSO logical description for weighted tree automata over arbitrary semirings. The discounting method for stochastic tree automata has been also used in [45, 46].

Infinite trees play a crucial role in practical applications, namely in program optimization [36], and in proving termination of nondeterministic or concurrent programs under any reasonable notion of fairness [38]. Furthermore, tree automata over infinite trees contribute in program synthesis in model checking [53]. All these applications are based on the fundamental fact that every program can be described by an infinite tree (see [15, 36, 54]). Weighted Muller tree automata were investigated in [47] but for the underlying semirings special completeness axioms were required. It is our third goal to provide a weighted Muller tree automaton model with discounting, over the max-plus and the min-plus semirings, and to show an equivalent MSO logical characterization. This part of our work, is motivated by the following fact. Currently, several tools for model checking are built in a weighted setting, in particular over De Morgan algebras (see [11, 14, 37]). Therefore, taking into account the important role of tree automata in program synthesis [53], we wish to study the extension of these models in the weighted setting, provided that our underlying semirings (like max-plus and min-plus) are already used in practical applications and do not require any completeness axioms.

In the sequel, we briefly describe the contents of the thesis. In Section 2, we recall notions on trees, semirings, tree series, and define the discounting over ranked alphabets and semirings.

In Section 3, we deal with weighted top-down and bottom-up tree automata with discounting. We show that these models are equivalent, and we investigate properties of their behaviors.

Then in Section 4, we define rational tree series operations with discounting and we show our Kleene theorem. For this, we use similar arguments as in [21], but now in our constructions we employ the discounting.

In Section 5, we introduce a weighted MSO logic with discounting on trees, which is a slight modification of the logics in [24, 32]. We show that sentences from two fragments of this logic are expressively equivalent with the recognizable tree series with discounting. The one fragment called *almost existential* has a purely syntactic definition.

In Section 6, we investigate weighted Muller tree automata with discounting, over the max-plus and the min-plus semirings. Due to our discounting, we do not require any completeness axioms for summation.

Then in Section 7, we extend the syntax of our weighted MSO logic to represent also infinite trees. In fact we use the same syntax as in [47]. We show that the infinitary tree series obtained as the behaviors of our weighted Muller tree automata with discounting are definable by two fragments in this logic, the *restricted* and the *incomplete universal*. The latter has a purely syntactic definition.

Finally in Section 8, we apply our theory to monadic ranked alphabets and reobtain the Kleene-Schützenberger theorem for words series with discounting which proved in [20].

2 Preliminaries

Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. The *prefix relation* $\leq$ over $\mathbb{N}^*$ is a partial order defined in the usual way: for every $w, v \in \mathbb{N}^*$, $w \leq v$ iff there exists $u \in \mathbb{N}^*$ such that $wu = v$. A set $A \subseteq \mathbb{N}^*$ is called *prefix-closed* if $v \in A$ implies $w \in A$ for every $w \leq v$.

A *ranked alphabet* Σ is a pair (Σ, rk_Σ) (simply denoted by Σ) where Σ is a finite set and $rk_\Sigma : \Sigma \to \mathbb{N}$. As usually, we set $\Sigma_k = \{\sigma \in \Sigma \mid rk_\Sigma(\sigma) = k\}$ for every $k \geq 0$, and $\deg(\Sigma) = \max\{k \in \mathbb{N} \mid \Sigma_k \neq \emptyset\}$.

A *tree t over* Σ is a partial mapping $t : \mathbb{N}_+^* \to \Sigma$ such that the domain $dom(t)$ of t is a non-empty prefix-closed set, and for every $w \in dom(t)$ if $t(w) \in \Sigma_k$ $k \geq 0$, then for $i \in \mathbb{N}_+$, $wi \in dom(t)$ iff $1 \leq i \leq k$. The elements of $dom(t)$ are called the *nodes of t*. We let $lv(t) = \{w \in dom(t) \mid t(w) \in \Sigma_0\}$ to be the set of *leaves of t*. A tree t is called *finite* (resp. *infinite)* if its domain is finite (resp. infinite). As usually, we shall denote by T_Σ (resp. T_Σ^ω) the set of all finite (resp. infinite) trees over Σ. Moreover, for every finite set A, we shall write $T_\Sigma(A)$ for the set of finite trees over the ranked alphabet Σ', where $\Sigma'_0 = \Sigma_0 \cup A$ and $\Sigma'_k = \Sigma_k$ for every $k > 0$.

The set T_Σ of all finite trees over Σ can be also inductively defined as the smallest set

$T \subseteq (\Sigma \cup \{(,),\ ,\})^*$ such that (i) $\Sigma_0 \subseteq T$ and (ii) if $k \geq 1, \sigma \in \Sigma_k$, and $t_1, \ldots, t_k \in T$, then $\sigma(t_1, \ldots, t_k) \in T$. The *height* $ht(t)$ of every finite tree $t \in T_\Sigma$ is defined by: (i) if $t \in \Sigma_0$, then $ht(t) = 0$ and (ii) if $t = \sigma(t_1, \ldots, t_k)$ with $\sigma \in \Sigma_k$ $(k \geq 1)$ and $t_1, \ldots, t_k \in T_\Sigma$, then $ht(t) = 1 + \max\{ht(t_i) \mid 1 \leq i \leq k\}$.

Let $t, t_1, \ldots, t_m \in T_\Sigma$ and $w_1, \ldots, w_m \in dom(t)$ be nodes of t which are pairwise incomparable according to the prefix relation. We denote by $t\,[w_1 \leftarrow t_1, \ldots, w_m \leftarrow t_m]$ the tree obtained by substituting in t the tree t_i at w_i for every $1 \leq i \leq m$. Formally, $dom\,(t\,[w_1 \leftarrow t_1, \ldots, w_m \leftarrow t_m])$ $= \{w \in dom(t) \mid w < w_i$ for every $1 \leq i \leq m\} \cup \{w \in dom(t) \mid w$ is incomparable with w_i for every $1 \leq i \leq m\} \cup \{w_i u \mid u \in dom\,(t_i)$ for every $1 \leq i \leq m\}$. Moreover, for every $w \in dom\,(t\,[w_1 \leftarrow t_1, \ldots, w_m \leftarrow t_m])$, we set

$$t\,[w_1 \leftarrow t_1, \ldots, w_m \leftarrow t_m]\,(w) = \begin{cases} t(w) & \text{if } w < w_j \text{ for some } 1 \leq j \leq m, \text{ or } w \text{ is} \\ & \text{incomparable with } w_i \text{ for every } 1 \leq i \leq m \\ t_j(u) & \text{if } w = w_j u \text{ for some } 1 \leq j \leq m,\ u \in dom\,(t_j)\,. \end{cases}$$

For every $t \in T_\Sigma$ and $w \in dom(t)$, the *subtree* $t|_w$ *of* t *at* w is defined as follows: $dom\,(t|_w) = \{u \in \mathbb{N}_+^* \mid wu \in dom(t)\}$, and $t|_w(u) = t(wu)$ for every $u \in dom\,(t|_w)$.

A ranked alphabet Σ is called *a-monadic* if $\Sigma_0 = \{a\}$ and $\Sigma_k = \emptyset$ for every $k > 1$.

Let Σ, Γ be two ranked alphabets. A *relabeling* from Σ to Γ is a mapping $h : \Sigma \to \mathcal{P}(\Gamma)$ such that $h(\sigma) \subseteq \Gamma_k$ for every $\sigma \in \Sigma_k$, $k \geq 0$. Then h is extended to a mapping $h : T_\Sigma \to \mathcal{P}(T_\Gamma)$ in the following way:

$$h(t) = \{s \in T_\Gamma \mid dom(s) = dom(t) \text{ and } s(w) \in h(t(w)),\ \forall w \in dom(s)\}$$

for every $t \in T_\Sigma$.

Later on in our constructions, we employ only surjective relabelings $h : \Sigma \to \mathcal{P}(\Gamma)$ with $|h(\sigma)| = 1$ for every $\sigma \in \Sigma$. Therefore, we make the following convention for the rest of the paper: *a relabeling from* Σ *to* Γ *is a surjective mapping* $h : \Sigma \to \Gamma$ *such that* $h(\sigma) \in \Gamma_k$ *for every* $\sigma \in \Sigma_k$, $k \geq 0$. Obviously, this requires that $\deg(\Sigma) = \deg(\Gamma)$.

A *(nondeterministic) bottom-up tree automaton (buta* for short) is a tuple $\mathcal{M} = (Q, \Sigma, \Delta, F)$ with *finite state set* Q, *input ranked alphabet* Σ, *set of transitions* $\Delta \subseteq \bigcup_{k \geq 0} Q^k \times \Sigma_k \times Q$, and *final states* $F \subseteq Q$.

Let $t \in T_\Sigma$. A *run of* $\mathcal{M}$ *over* t is a mapping $r_t : dom(t) \to Q$ such that $r_t(\varepsilon) \in F$, and for every $w \in dom(t)$ it holds that $((r_t(w1), \ldots, r_t(w.rk_\Sigma(t(w)))), t(w), r_t(w)) \in \Delta$. We say that a tree $t \in T_\Sigma$ is *recognized* (or *accepted*) by $\mathcal{M}$ if there is a run r_t of $\mathcal{M}$ over t. The *tree language of* $\mathcal{M}$ consists of all trees accepted by $\mathcal{M}$ and is denoted by $|\mathcal{M}|$.

A tree language $L \subseteq T_\Sigma$ is called *recognizable* if there is a buta $\mathcal{M} = (Q, \Sigma, \Delta, F)$ such that $|\mathcal{M}| = L$. It is well-known that the class of recognizable tree languages is closed under the Boolean operations, relabelings, and inverse relabelings [33, 34].

A buta $\mathcal{M}$ is called *deterministic* if the transition set Δ is replaced by a family $\delta = (\delta_\sigma)_{\sigma \in \Sigma}$ of mappings $\delta_\sigma : Q^k \to Q$, $\sigma \in \Sigma_k, k \geq 0$. In this case, for every input tree $t \in T_\Sigma$ there is at most one run r_t of $\mathcal{M}$ over t. The collection of all tree languages accepted by deterministic buta coincides with the class of recognizable tree languages [26, 33, 34].

Next we turn to semirings. A *semiring* $(K, +, \cdot, 0, 1)$ consists of a set K, two binary operations $+$ and $\cdot$, and two constant elements 0 and 1 such that

(i) $(K, +, 0)$ is a commutative monoid,

(ii) $(K, \cdot, 1)$ is a monoid,

(iii) the distributivity laws $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ hold for every a, b, $c \in K$, and

(iv) $0 \cdot a = a \cdot 0 = 0$ for every $a \in K$.

If the operations and the constant elements are understood, then the semiring is simply denoted by K.

A semiring K is called *commutative* if $a \cdot b = b \cdot a$ for every $a, b \in K$, and *idempotent* if $1 + 1 = 1$. If K is idempotent, then the relation $\leq$ defined for every $a, b \in K$ by $a \leq b$ iff $a + b = b$ is a partial order on K (see Proposition 20.19 in [35]). Then, clearly $a \leq a + b$ for every $a, b \in K$.

Example 1 *The following structures are commutative semirings:*

- *the semiring* $(\mathbb{N} \cup \{\infty\}, +, \cdot, 0, 1)$ *of* extended natural numbers*,*
- *the* arctic semiring *or* max-plus semiring $\mathbb{R}_{\max} = (\mathbb{R}_+ \cup \{-\infty\}, \max, +, -\infty, 0)$ *where* $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$ *and* $-\infty + x = -\infty$ *for every* $x \in \mathbb{R}_+$*,*
- *the* tropical *or* min-plus semiring $(\mathbb{R}_+ \cup \{\infty\}, \min, +, \infty, 0)$*,*
- *the* subset semiring $(P(A), \cup, \cap, \emptyset, A)$ *for every non-empty set* A*,*
- every bounded distributive lattice (see [2]) with the operations supremum and infimum, in particular the *fuzzy semiring* $([0, 1], \vee, \wedge, 0, 1)$ and the *Boolean semiring* $\mathbb{B} = (\{0, 1\}, \vee, \wedge, 0, 1)$.

The second main result of our paper will apply to commutative semirings K which are *additively locally finite*, i.e. such that every finitely generated submonoid of $(K, +, 0)$ is finite. Important examples of additively locally finite semirings include:

- all idempotent semirings K, in particular the arctic and the tropical semirings and every bounded distributive lattice,
- all fields of characteristic p, for every prime p,
- all products $K_1 \times \ldots \times K_n$ (with operations defined pointwise) of additively locally finite semirings K_i $(1 \leq i \leq n)$,
- the semiring of polynomials $(K[X], +, \cdot, 0, 1)$ over a variable X and an additively locally finite semiring K.

Consider two semirings K_1 and K_2. A mapping $f : K_1 \to K_2$ is called a *semiring homomorphism* (or simply an *homomorphism*) if $f(a+b) = f(a)+f(b)$ and $f(a \cdot b) = f(a) \cdot f(b)$ for every $a, b \in K_1$, and $f(0) = 0$ and $f(1) = 1$. An homomorphism $f : K \to K$ is an *endomorphism* of K. The set $End(K)$ of all endomorphisms of K is a monoid with operation the usual composition mapping $\circ$ and unit element the identity mapping id on K. If no confusion arises, we shall simply denote the operation $\cdot$ of K and the composition operation $\circ$ of $End(K)$ by concatenation.

Example 2 *Let $K = \mathbb{R}_{\max}$, the max-plus semiring. For every $p \in \mathbb{R}_+$ we put $p \cdot (-\infty) = -\infty$. Then the mapping $\overline{p} : \mathbb{R}_{\max} \to \mathbb{R}_{\max}$ given by $x \longmapsto p \cdot x$ is an endomorphism of $\mathbb{R}_{\max}$ which can be considered as a* discounting *of $\mathbb{R}_{\max}$. Conversely, every endomorphism of $\mathbb{R}_{\max}$ is of this form (see [20], Lemma 15).*

Let Σ be a ranked alphabet and K be a semiring. A Φ*-discounting over* Σ *and* K is a family $\Phi = (\Phi_k)_{k \geq 0}$ of mappings $\Phi_k : \Sigma_k \to (End(K))^k$ for $k \geq 1$, and $\Phi_0 : \Sigma_0 \to End(K)$. For every $\sigma \in \Sigma_k$ $(k \geq 1)$ we shall write $\left(\Phi_\sigma^1, \ldots, \Phi_\sigma^k\right)$ for the k-tuple $\Phi_k(\sigma)$. If no confusion arises with the rank of σ, then we simply denote $\Phi_k(\sigma)$ by Φ_σ. For every $t \in T_\Sigma$ and every $w \in dom(t)$, we define the endomorphism Φ_w^t of K as follows:

$$\Phi_w^t = \begin{cases} id & \text{if } w = \varepsilon \\ \Phi_{t(\varepsilon)}^{i_1} \circ \Phi_{t(i_1)}^{i_2} \circ \ldots \circ \Phi_{t(i_1 \ldots i_{n-1})}^{i_n} & \text{if } w = i_1 \ldots i_n \text{ with } i_1, \ldots, i_n \in \mathbb{N}_+, n > 0 \end{cases}$$

where id is the identity endomorphism of K.

Next, we turn to formal series over trees. Given a ranked alphabet Σ and a semiring K a *formal tree series (*or *tree series* for short) *over* Σ *and* K is a mapping $S : T_\Sigma \to K$. As usual we denote by (S,t) the *coefficient* $S(t)$. The *support* of S is the tree language $supp(S) = \{t \in T_\Sigma \mid (S,t) \neq 0\}$. The class of all tree series over Σ and K is denoted by $K\langle\langle T_\Sigma \rangle\rangle$, and the class of *polynomials* (i.e. tree series with finite support) is denoted by $K\langle T_\Sigma \rangle$.

For every tree language $L \subseteq T_\Sigma$, the *characteristic series* $1_L : T_\Sigma \to K$ *of* L is determined by

$$(1_L, t) = \begin{cases} 1 & \text{if } t \in L \\ 0 & \text{otherwise} \end{cases}$$

for every $t \in T_\Sigma$.

Let $S, T \in K\langle\langle T_\Sigma \rangle\rangle$ and $k \in K$. The sum $S + T$, the scalar products kS and Sk as well as the Hadamard product $S \odot T$ are defined elementwise

$$\begin{gathered} (S+T, t) = (S,t) + (T,t), \\ (kS, t) = k \cdot (S,t), \quad (Sk, t) = (S,t) \cdot k, \\ (S \odot T, t) = (S,t) \cdot (T,t) \end{gathered}$$

for every $t \in T_\Sigma$.

Then, it is easily seen that $(K\langle\langle T_\Sigma \rangle\rangle, +, \odot, \mathbf{0}, \mathbf{1})$ and $(K\langle T_\Sigma \rangle, +, \odot, \mathbf{0}, \mathbf{1})$ are semirings, where $\mathbf{0}$ is the tree series (over Σ and K) with all its coefficients being 0, and $\mathbf{1}$ is the tree series (over Σ and K) with all its coefficients being 1. If K is commutative, then $K\langle\langle T_\Sigma \rangle\rangle$ and $K\langle T_\Sigma \rangle$ are commutative. If K is idempotent, then $K\langle\langle T_\Sigma \rangle\rangle$ and $K\langle T_\Sigma \rangle$ are also idempotent; furthermore they are partially ordered with $\leq$ determined for $S, T \in K\langle\langle T_\Sigma \rangle\rangle$ (resp. $S, T \in K\langle T_\Sigma \rangle$) by $S \leq T$ iff $(S,t) \leq (T,t)$ for every $t \in T_\Sigma$.

Let now Σ, Γ be ranked alphabets and $h : \Sigma \to \Gamma$ be a relabeling. For every tree series $S \in K\langle\langle T_\Sigma \rangle\rangle$ the tree series $h(S) \in K\langle\langle T_\Gamma \rangle\rangle$ is well-defined by

$$(h(S), s) = \sum_{t \in h^{-1}(s)} (S,t)$$

for $s \in T_\Gamma$.

Similarly, if $T \in K\langle\langle T_\Gamma \rangle\rangle$, then the tree series $h^{-1}(T) \in K\langle\langle T_\Sigma \rangle\rangle$ is determined by

$$(h^{-1}(T), t) = (T, h(t))$$

for every $t \in T_\Sigma$.

For the rest of Sections 3, 4, and 5, we fix a ranked alphabet Σ, a commutative semiring K, and a Φ-discounting over Σ and K.

3 Weighted tree automata with Φ-discounting

In this section we introduce weighted top-down tree automata (with and without terminal weights). The behaviors of these models are discounted by Φ. This means that for every input tree t the weight of every node of t is discounted according to the distance of the node from the root of t; the longer this distance is the greater is the grade of discounting; nodes of the same level get a weight with the same grade of discounting. We show that the classes of the Φ-behaviors of such tree automata with and without terminal weights, coincide.

Next, we consider weighted bottom-up tree automata (with and without initial weights). Again for every input tree, the nodes with a longer distance from the root get a weight discounted by a greater grade. But now, as usual, the process of an input tree is done from its leaves to its root. We show that weighted bottom-up tree automata (with and without initial weights), and weighed top-down tree automata (with and without terminal weights), accept the same class of tree series. Here, we mainly work with weighted bottom-up models since we want to take advantage of the determinization property of unweighted bottom-up tree automata. Therefore, we shall develop our theory for weighted bottom-up tree automata, but all our main results hold also for weighted top-down tree automata.

3.1 Weighted top-down tree automata with Φ-discounting

Definition 3 *(i) A* weighted top-down tree automaton with terminal weights *(*wtdta-t *for short)* over Σ and K *is a quadruple* $\mathcal{M} = (Q, in, wt, ter)$, *where* Q *is the* finite state set, $in : Q \to K$ *is the* initial distribution, $wt : \bigcup_{k \geq 0} Q \times \Sigma_k \times Q^k \to K$ *is the* mapping assigning weights to the transitions *of the automaton, and* $ter : Q \to K$ *is the* final distribution.

(ii) A weighted top-down tree automaton *(*wtdta *for short)* over Σ and K *is a wtdta-t* $\mathcal{M} = (Q, in, wt, ter)$ *such that* $ter(q) = 1$ *for every* $q \in Q$. *In this case, we simply write* $\mathcal{M} = (Q, in, wt)$.

Let $t \in T_\Sigma$. A *run of* $\mathcal{M}$ *over* t is a mapping $r_t : dom(t) \to Q$. The *weight of* r_t *at* $w \in dom(t)$ is the value

$$wt(r_t, w) = wt\left(r_t(w), t(w), (r_t(w1), \ldots, r_t(w.rk_\Sigma(t(w))))\right).$$

The Φ-*weight* (or simply *weight*) *of* r_t, which is denoted by $weight(r_t)$ (or by $weight_{\mathcal{M}}(r_t)$ whenever we want to notify the tree automaton $\mathcal{M}$), is defined by

$$weight(r_t) = in(r_t(\varepsilon)) \cdot \prod_{w \in dom(t)} \Phi_w^t\left(wt(r_t, w)\right) \cdot \prod_{w \in lv(t)} \Phi_w^t \circ \Phi_{t(w)}\left(ter(r_t(w))\right).$$

If $\mathcal{M}$ is a wtdta then the $weight(r_t)$ is simply computed by

$$weight(r_t) = in(r_t(\varepsilon)) \cdot \prod_{w \in dom(t)} \Phi_w^t \left(wt(r_t, w) \right).$$

For every $t \in T_\Sigma$, and every $q \in Q$, we denote by $R_{\mathcal{M}}(t, q)$ the set of all runs r_t of $\mathcal{M}$ over t with $r_t(\varepsilon) = q$. Furthermore, we set $R_{\mathcal{M}}(t) = \bigcup_{q \in Q} R_{\mathcal{M}}(t, q)$ for the set of all runs r_t of $\mathcal{M}$ over t.

The Φ-*behavior* (or simply *behavior*) of $\mathcal{M}$ is the formal tree series

$$\|\mathcal{M}\| : T_\Sigma \to K$$

whose coefficients are given by

$$(\|\mathcal{M}\|, t) = \sum_{r_t \in R_{\mathcal{M}}(t)} weight(r_t)$$

for every $t \in T_\Sigma$.

We shall denote by $K^{\Phi-td-t} \langle\langle T_\Sigma \rangle\rangle$ (resp. by $K^{\Phi-td} \langle\langle T_\Sigma \rangle\rangle$) the collection of the Φ-behaviors of all wtdta-t (resp. wtdta) over Σ and K. Trivially $K^{\Phi-td} \langle\langle T_\Sigma \rangle\rangle \subseteq K^{\Phi-td-t} \langle\langle T_\Sigma \rangle\rangle$. In fact, the following stronger result holds.

Proposition 4 $K^{\Phi-td} \langle\langle T_\Sigma \rangle\rangle = K^{\Phi-td-t} \langle\langle T_\Sigma \rangle\rangle$.

Proof. Let $\mathcal{M} = (Q, in, wt, ter)$ be a wtdta-t over Σ and K. We consider the wtdta $\mathcal{M}' = (Q, in, wt')$ over Σ and K with

- $wt'(q, \sigma, (q_1, \ldots, q_k)) = wt(q, \sigma, (q_1, \ldots, q_k))$ for every $q, q_1, \ldots, q_k \in Q, \sigma \in \Sigma_k, k > 0$
- $wt'(q, a) = wt(q, a) \cdot \Phi_a(ter(q))$ for every $q \in Q, a \in \Sigma_0$.

Let $t \in T_\Sigma$. Then, every run $r_t \in R_{\mathcal{M}}(t)$ is also a run of $\mathcal{M}'$ over t, and vice versa. Moreover, we compute

$$\begin{aligned}
weight_{\mathcal{M}}(r_t) &= in(r_t(\varepsilon)) \cdot \prod_{w \in dom(t)} \Phi_w^t \left(wt(r_t, w) \right) \cdot \prod_{w \in lv(t)} \Phi_w^t \circ \Phi_{t(w)} \left(ter(r_t(w)) \right) \\
&= in(r_t(\varepsilon)) \cdot \prod_{w \in dom(t) \setminus lv(t)} \Phi_w^t \left(wt(r_t, w) \right) \cdot \prod_{w \in lv(t)} \Phi_w^t \left(wt(r_t, w) \right) \\
&\qquad \cdot \prod_{w \in lv(t)} \Phi_w^t \circ \Phi_{t(w)} \left(ter(r_t(w)) \right) \\
&= in(r_t(\varepsilon)) \cdot \prod_{w \in dom(t) \setminus lv(t)} \Phi_w^t \left(wt(r_t, w) \right) \\
&\qquad \cdot \prod_{w \in lv(t)} \Phi_w^t \left(wt(r_t, w) \right) \cdot \left(\Phi_w^t \circ \Phi_{t(w)} \left(ter(r_t(w)) \right) \right)
\end{aligned}$$

which equals

$$
\begin{aligned}
& in(r_t(\varepsilon)) \cdot \prod_{w \in dom(t) \setminus lv(t)} \Phi_w^t \left(wt(r_t, w) \right) \\
& \qquad \cdot \prod_{w \in lv(t)} \Phi_w^t \left(wt(r_t, w) \cdot \Phi_{t(w)} \left(ter(r_t(w)) \right) \right) \\
& = in(r_t(\varepsilon)) \cdot \prod_{w \in dom(t) \setminus lv(t)} \Phi_w^t \left(wt'(r_t, w) \right) \cdot \prod_{w \in lv(t)} \Phi_w^t \left(wt'(r_t, w) \right) \\
& = in(r_t(\varepsilon)) \cdot \prod_{w \in dom(t)} \Phi_w^t \left(wt'(r_t, w) \right) \\
& = weight_{\mathcal{M}'}(r_t).
\end{aligned}
$$

Therefore, $(\|\mathcal{M}'\|, t) = (\|\mathcal{M}\|, t)$ for every $t \in T_\Sigma$, and thus $\|\mathcal{M}'\| = \|\mathcal{M}\|$. ■

3.2 Weighted bottom-up tree automata with Φ-discounting

In this subsection, we introduce the bottom-up version of our weighted tree automata, with and without initial weights. We show that the Φ-behaviors of the two models constitute the same class of tree series which moreover coincides with the class of Φ-behaviors of wtdta.

Definition 5 *(i) A* weighted bottom-up tree automaton with initial weights *(*wbuta-i *for short) over* Σ *and* K *is a quadruple* $\mathcal{M} = (Q, in, wt, ter)$*, where* Q *is the* finite state set*,* $in : Q \to K$ *is the* initial distribution*,* $wt : \bigcup_{k \geq 0} Q^k \times \Sigma_k \times Q \to K$ *is the* mapping assigning weights to the transitions *of the automaton, and* $ter : Q \to K$ *is the* final distribution*.*

(ii) A weighted bottom-up tree automaton *(*wbuta *for short) over* Σ *and* K *is a wbuta-i* $\mathcal{M} = (Q, in, wt, ter)$ *such that* $in(q) = 1$ *for every* $q \in Q$*. In this case, we simply write* $\mathcal{M} = (Q, wt, ter)$*.*

Let $t \in T_\Sigma$. A *run of* $\mathcal{M}$ *over* t is a mapping $r_t : dom(t) \to Q$. The *weight of* r_t *at* $w \in dom(t)$ is the value

$$wt(r_t, w) = wt\left((r_t(w1), \ldots, r_t(w.rk_\Sigma(t(w)))), t(w), r_t(w) \right).$$

The Φ-*weight* (or simply *weight*) *of* r_t, which is denoted by $weight(r_t)$ (or by $weight_{\mathcal{M}}(r_t)$ whenever we want to notify the tree automaton $\mathcal{M}$), is defined by

$$weight(r_t) = \prod_{w \in lv(t)} \Phi_w^t \circ \Phi_{t(w)} \left(in(r_t(w)) \right) \cdot \prod_{w \in dom(t)} \Phi_w^t \left(wt(r_t, w) \right) \cdot ter(r_t(\varepsilon)).$$

If $\mathcal{M}$ is a wbuta then the Φ-weight of r_t is simply given by

$$weight(r_t) = \prod_{w \in dom(t)} \Phi_w^t \left(wt(r_t, w) \right) \cdot ter(r_t(\varepsilon)).$$

For every $t \in T_\Sigma$, and every $q \in Q$, we denote by $R_{\mathcal{M}}(t,q)$ the set of all runs r_t of $\mathcal{M}$ over t with $r_t(\varepsilon) = q$. Furthermore, we set $R_{\mathcal{M}}(t) = \bigcup_{q \in Q} R_{\mathcal{M}}(t,q)$.

Then the Φ-*behavior* (or simply *behavior*) of $\mathcal{M}$ is the formal tree series

$$\|\mathcal{M}\| : T_\Sigma \to K$$

whose coefficients are given by

$$(\|\mathcal{M}\|, t) = \sum_{r_t \in R_{\mathcal{M}}(t)} weight(r_t)$$

for every $t \in T_\Sigma$.

The collection of the Φ-behaviors of all wbuta-i (resp. wbuta) over Σ and K is denoted by $K^{\Phi-bu-i}\langle\langle T_\Sigma \rangle\rangle$ (resp. by $K^{\Phi-bu}\langle\langle T_\Sigma \rangle\rangle$). The proof of the following proposition employs the same arguments with the ones in the proof of Proposition 4.

Proposition 6 $K^{\Phi-bu}\langle\langle T_\Sigma \rangle\rangle = K^{\Phi-bu-i}\langle\langle T_\Sigma \rangle\rangle$.

It is well-known (see [5]) that the behaviors of weighted bottom-up tree automata (without initial weights) coincide with the behaviors of weighted top-down tree automata (without terminal weights). This result is generalized in the subsequent theorem which refers to the Φ-behaviors of weighted tree automata.

Theorem 7 *Let Σ be a ranked alphabet, K be a commutative semiring, and Φ be a discounting over Σ and K. Then*

$$K^{\Phi-td}\langle\langle T_\Sigma \rangle\rangle = K^{\Phi-td-t}\langle\langle T_\Sigma \rangle\rangle = K^{\Phi-bu}\langle\langle T_\Sigma \rangle\rangle = K^{\Phi-bu-i}\langle\langle T_\Sigma \rangle\rangle.$$

Proof. Let $\mathcal{M} = (Q, wt, ter)$ be a wbuta over Σ and K. We consider the wtdta $\mathcal{M}' = (Q, in, wt')$ with $in(q) = ter(q)$ for every $q \in Q$, and $wt'(q, \sigma, (q_1, \ldots, q_k)) = wt((q_1, \ldots, q_k), \sigma, q)$ for every $q, q_1, \ldots, q_k \in Q, \sigma \in \Sigma_k, k \geq 0$.

Let $t \in T_\Sigma$. Obviously, every run r_t of $\mathcal{M}$ over t is also a run of $\mathcal{M}'$ over t, and vice versa. Moreover, for every $w \in dom(t)$ we have $wt(r_t, w) = wt'(r_t, w)$. Now, we compute

$$\begin{aligned} weight_{\mathcal{M}}(r_t) &= \prod_{w \in dom(t)} \Phi_w^t\left(wt(r_t, w)\right) \cdot ter(r_t(\varepsilon)) \\ &= in(r_t(\varepsilon)) \cdot \prod_{w \in dom(t)} \Phi_w^t\left(wt'(r_t, w)\right) = weight_{\mathcal{M}'}(r_t) \end{aligned}$$

which in turn implies that $(\|\mathcal{M}\|, t) = (\|\mathcal{M}'\|, t)$. Thus, $K^{\Phi-bu}\langle\langle T_\Sigma \rangle\rangle \subseteq K^{\Phi-td}\langle\langle T_\Sigma \rangle\rangle$.

The converse inclusion is proved by similar arguments. ■

We shall denote by $K^{\Phi-rec}\langle\langle T_\Sigma \rangle\rangle$ the class of tree series derived in the above theorem. The elements of $K^{\Phi-rec}\langle\langle T_\Sigma \rangle\rangle$ will be called Φ-*recognizable tree series over* Σ *and* K. It should be clear that if our Φ-discounting employs only the identity mapping, i.e, $\Phi_\sigma = (id, \ldots, id)$ for every $\sigma \in \Sigma_k$ $(k \geq 1)$ and $\Phi_0 = id$ for every $\sigma \in \Sigma_0$, then $K^{\Phi-rec}\langle\langle T_\Sigma \rangle\rangle = K^{rec}\langle\langle T_\Sigma \rangle\rangle$ where $K^{rec}\langle\langle T_\Sigma \rangle\rangle$ as usual denotes the class of recognizable formal tree series over Σ and K (see [3, 5]). Next, we give an example of a wbuta over $\mathbb{R}_{\max}$ which is *deterministic* (see [5]).

Example 8 *Let Σ be a ranked alphabet with $\Sigma_0 = \{a\}, \Sigma_2 = \{\sigma, \gamma\}$, and $\Sigma_3 = \{\delta\}$. We consider the wbuta $\mathcal{M} = (\{q\}, wt, ter)$ over Σ and $\mathbb{R}_{\max}$ with its weight assigning mapping defined by $wt((q,q),\sigma,q) = 1$ and $wt(a,q) = wt((q,q),\gamma,q) = wt((q,q,q),\delta,q) = 0$. The final distribution is given by $ter(q) = 0$. We define a Φ-discounting over Σ and $\mathbb{R}_{\max}$ specified by $\Phi_a = \overline{0}$, $\Phi_\sigma = (\overline{1}, \overline{1})$, $\Phi_\gamma = (\overline{0}, \overline{0})$, and $\Phi_\delta = (\overline{0}, \overline{0}, \overline{0})$ (see Example 2).*

Then for every $t \in T_\Sigma$ the coefficient $(\|\mathcal{M}\|, t)$ equals the number of occurrences of σ in the greatest initial σ-subtree of t. One can easily show that there is no deterministic wbuta without discounting over Σ and $\mathbb{R}_{\max}$ with the same behavior.

For our constructions in Sections 4 we shall need normalized forms of wbuta. First, we show that every wbuta can be normalized according to its final distribution in the following sense.

A wbuta $\mathcal{M} = (Q, wt, ter)$ is called *final weight normalized* if there is one state $q_f \in Q$ such that

- $ter(q_f) = 1$, and for every other $q \in Q$ with $q \neq q_f$, $ter(q) = 0$,
- for every $k > 0, \sigma \in \Sigma_k, q_1, \ldots, q_k, q \in Q$, if there is an $1 \leq i \leq k$ with $q_i = q_f$, then $wt((q_1, \ldots, q_k), \sigma, q) = 0$.

Then we write $\mathcal{M} = (Q, wt, q_f)$.

Lemma 9 *For every wbuta $\mathcal{M}$ there is an equivalent final weight normalized wbuta $\mathcal{M}'$. Moreover, $\mathcal{M}'$ can be chosen to have one more state than $\mathcal{M}$.*

Proof. Let $\mathcal{M} = (Q, wt, ter)$ be a wbuta over Σ and K. We construct the final weight normalized wbuta $\mathcal{M}' = (Q', wt', q_f)$ over Σ and K with $Q' = Q \cup \{q_f\}$ where q_f is a new state, and wt' is determined by

$$wt'((q_1,\ldots,q_k),\sigma,q) = \begin{cases} wt((q_1,\ldots,q_k),\sigma,q) & \text{if } q_1,\ldots,q_k,q \in Q \\ \sum\limits_{\substack{p \in Q \\ ter(p) \neq 0}} wt((q_1,\ldots,q_k),\sigma,p) \cdot ter(p) & \text{if } q_1,\ldots,q_k \in Q, q = q_f \\ 0 & \text{otherwise} \end{cases}$$

for every $k \geq 0, \sigma \in \Sigma_k, q_1, \ldots, q_k, q \in Q'$.

Let now $t \in T_\Sigma$ and r'_t be a run of $\mathcal{M}'$ over t with non-zero weight. Then $r'_t(\varepsilon) = q_f$. We let $R_{r'_t}$ to be the set of all runs r_t of $\mathcal{M}$ over t, which differ with r'_t only at their roots. Clearly, $weight_{\mathcal{M}'}(r'_t) = \sum_{r_t \in R_{r'_t}} weight_{\mathcal{M}}(r_t)$. Conversely, let r_t be a run of $\mathcal{M}$ over t with $weight_{\mathcal{M}}(r_t) \neq 0$. Then, there exists a run r'_t of $\mathcal{M}'$ over t (which differs with r_t only at its root node) with $weight_{\mathcal{M}'}(r'_t) \neq 0$. Furthermore, if R is the set of all runs of $\mathcal{M}$ over t which differ with r_t only at their roots, then $R = R_{r'_t}$. We conclude

$$\begin{aligned}
(\|\mathcal{M}'\|, t) &= \sum_{r'_t \in R_{\mathcal{M}'}(t)} weight_{\mathcal{M}'}(r'_t) \\
&= \sum_{r'_t \in R_{\mathcal{M}'}(t)} \sum_{r_t \in R_{r'_t}} weight_{\mathcal{M}}(r_t) \\
&= \sum_{r_t \in R_{\mathcal{M}}(t)} weight_{\mathcal{M}}(r_t) \\
&= (\|\mathcal{M}\|, t).
\end{aligned}$$

■

Let $a \in \Sigma_0$. A tree series $S \in K\langle\langle T_\Sigma \rangle\rangle$ is called *a-proper* if $(S, a) = 0$. We shall denote by $K^a\langle\langle T_\Sigma \rangle\rangle$ the collection of all a-proper tree series over Σ and K. Now let $\mathcal{M} = (Q, wt, ter)$ be a wbuta over Σ and K. We set $I_a = \{q \in Q \mid wt(a, q) \neq 0\}$, and we call I_a the set of *initial a-states* of $\mathcal{M}$. Then $\mathcal{M}$ is called *initial a-state normalized* if there is a state $q_a \in Q$ such that $I_a = \{q_a\}$, $wt(a, q_a) = 1$, and $wt((q_1, \ldots, q_k), \sigma, q_a) = 0$ for every $\sigma \in \Sigma_k \setminus \{a\}, k \geq 0$.

Lemma 10 *Let $a \in \Sigma_0$ and $\mathcal{M} = (Q, wt, ter)$ be a wbuta with $\|\mathcal{M}\| \in K^a\langle\langle T_\Sigma \rangle\rangle$. Then there is an initial a-state normalized wbuta $\mathcal{M}'$ such that $\|\mathcal{M}'\| = \|\mathcal{M}\|$. Moreover, $\mathcal{M}$ can be chosen to have one more state than $\mathcal{M}$.*

Proof. We construct the wbuta $\mathcal{M}' = (Q', wt', ter')$ with $Q' = Q \cup \{q_a\}$ and

$$ter'(q) = \begin{cases} ter(q) & \text{if } q \in Q \\ 0 & \text{if } q = q_a \end{cases}$$

for every $q \in Q'$. The weight assigning mapping wt' is determined as follows:

- $wt'((q_1, \ldots, q_k), \sigma, q) = \sum \left\{ \begin{array}{c} \left(\prod\limits_{\substack{1 \leq i \leq k \\ q_i = q_a}} \Phi^i_\sigma(wt(a, p_i)) \right) \cdot wt((p_1, \ldots, p_k), \sigma, q) \mid \\ p_1, \ldots, p_k \in Q, \text{ and } p_i = q_i \text{ if } q_i \in Q \end{array} \right\}$

  for every $k \geq 0, \sigma \in \Sigma_k \setminus \{a\}, q_1, \ldots, q_k \in Q'$, and $q \in Q$
- $wt'((q_1, \ldots, q_k), \sigma, q_a) = 0$ for every $k \geq 0, \sigma \in \Sigma_k \setminus \{a\}$, and $q_1, \ldots, q_k \in Q'$
- $wt'(a, q) = 0$ for every $q \in Q$
- $wt'(a, q_a) = 1$.

Obviously, $\mathcal{M}'$ is initial a-state normalized and q_a is the initial a-state. Note that for every tree $t \in T_\Sigma$, every run r'_t of $\mathcal{M}'$ over t, and every node $w \in dom(t)$, if $t(w) \neq a$ and $r_t(w) = q_a$, then $weight_{\mathcal{M}'}(r'_t) = 0$. We will show that $\|\mathcal{M}'\| = \|\mathcal{M}\|$. For $t = a$ we have $(\|\mathcal{M}'\|, a) = (\|\mathcal{M}\|, a) = 0$. Now let $t \neq a$. For every $q \in Q$, we define the mapping $v : R_{\mathcal{M}}(t, q) \to R_{\mathcal{M}'}(t, q)$ as follows. For every run $r_t \in R_{\mathcal{M}}(t, q)$ we set

$$(v(r_t))(w) = \begin{cases} r_t(w) & \text{if } t(w) \neq a \\ q_a & \text{otherwise} \end{cases}$$

for every $w \in dom(t)$.

We let $dom_a(t) = \{w \in dom(t) \mid t(w) = a\}$ and $pre_a(t) = \{w \in dom(t) \mid$ there exists an $i \in \mathbb{N}_+$ such that $wi \in dom_a(t)\}$, i.e., the set of all nodes of t which are predecessors of the a-labeled nodes. Let $pre_a(t) = \{w_1, \ldots, w_m\}$. Then for every $1 \leq j \leq m$, we set $dom_{a,j}(t) = \{i \mid t(w_j i) = a\} = \left\{i_{j1}, \ldots, i_{jk_j}\right\}$ (with $i_{j1} < \ldots < i_{jk_j}$) which indicates the set of all a-labeled nodes following w_j. Clearly, $dom_a(t) = \bigcup_{1 \leq j \leq m} \{w_j i \mid i \in dom_{a,j}(t)\}$. Finally we define the *a-surrounding* of t to be the set $sur_a(t) = pre_a(t) \cup dom_a(t)$. Let $q \in Q$ and $r'_t \in R_{\mathcal{M}'}(t, q)$ with $r'_t(w) = q_a$ for every $w \in dom_a(t)$. Then we calculate

$$
\begin{aligned}
&\sum_{r_t \in v^{-1}(r'_t)} \prod_{w \in sur_a(t)} \Phi^t_w (wt(r_t, w)) \\
&= \sum_{r_t \in v^{-1}(r'_t)} \prod_{1 \leq j \leq m} \left(\prod_{i \in dom_{a,j}(t)} \Phi^t_{w_j i} (wt(r_t, w_j i)) \right) \cdot \Phi^t_{w_j} (wt(r_t, w_j)) \\
&= \sum_{r_t \in v^{-1}(r'_t)} \prod_{1 \leq j \leq m} \left(\prod_{1 \leq l \leq k_j} \Phi^t_{w_j i_{jl}} (wt(a, r_t(w_j i_{jl}))) \right) \\
&\cdot \Phi^t_{w_j} \left(wt \left(\begin{array}{c} \left(\begin{array}{c} r_t(w_j 1), \ldots, r_t(w_j(i_{j1} - 1)), \\ r_t(w_j i_{j1}), r_t(w_j(i_{j1} + 1)), \ldots, r_t(w_j(i_{j2} - 1)), r_t(w_j i_{j2}), \\ \ldots, r_t\left(w_j\left(i_{j(k_j-1)} + 1\right)\right), \ldots, r_t\left(w_j\left(i_{jk_j} - 1\right)\right), r_t\left(w_j i_{jk_j}\right), \\ r_t\left(w_j\left(i_{jk_j} + 1\right)\right), \ldots, r_t(w_j \rho_j) \end{array} \right), \\ t(w_j), r_t(w_j) \end{array} \right) \right) \\
&= \prod_{1 \leq j \leq m} \sum \left\{ \begin{array}{c} \left(\prod_{1 \leq l \leq k_j} \Phi^t_{w_j i_{jl}} (wt(a, p_{jl})) \right) \cdot \\ \Phi^t_{w_j} \left(wt \left(\begin{array}{c} \left(\begin{array}{c} r'_t(w_j 1), \ldots, r'_t(w_j(i_{j1} - 1)), p_{j1}, \\ r'_t(w_j(i_{j1} + 1)), \ldots, r'_t(w_j(i_{j2} - 1)), p_{j2}, \\ \ldots, p_{jk_j}, \ldots, r'_t(w_j \varrho_j) \end{array} \right), \\ t(w_j), r'_t(w_j) \end{array} \right) \right) \\ \mid p_{j1}, \ldots, p_{jk_j} \in Q \end{array} \right\}
\end{aligned}
$$

where for every $1 \leq j \leq m$, we assume that $t(w_j) \in \Sigma_{\rho_j}$.
On the other side

$$
\begin{aligned}
\prod_{w \in sur_a(t)} \Phi^t_w (wt'(r'_t, w)) &= \prod_{1 \leq j \leq m} \left(\Phi^t_{w_j} (wt'(r'_t, w_j)) \cdot \prod_{i \in dom_{a,j}(t)} \Phi^t_{w_j i} (wt'(r'_t, w_j i)) \right) \\
&= \prod_{1 \leq j \leq m} \Phi^t_{w_j} (wt'(r'_t, w_j)) \\
&= \prod_{1 \leq j \leq m} \Phi^t_{w_j} \left(wt'\left(\left(r'_t(w_j 1), \ldots, r'_t(w_j \rho_j)\right), t(w_j), r'_t(w_j) \right) \right)
\end{aligned}
$$

which equals

$$\prod_{1\leq j\leq m}\Phi^{t}_{w_j}\left(\sum\left\{\begin{array}{c}\left(\prod\limits_{1\leq l\leq k_j}\Phi^{i_{jl}}_{t(w_j)}\left(wt\left(a,p_{jl}\right)\right)\right)\\ \cdot wt\left(\begin{array}{c}\left(\begin{array}{c}r'_t\left(w_j1\right),\ldots,r'_t\left(w_j\left(i_{j1}-1\right)\right),p_{j1},\\ r'_t\left(w_j\left(i_{j1}+1\right)\right),\ldots,r'_t\left(w_j\left(i_{j2}-1\right)\right),p_{j2},\\ \ldots,p_{jk_j},\ldots,r'_t\left(w_j\varrho_j\right)\end{array}\right),\\ t\left(w_j\right),r'_t\left(w_j\right)\end{array}\right)\\ \mid p_{j1},\ldots,p_{jk_j}\in Q\end{array}\right\}\right)$$

$$=\prod_{1\leq j\leq m}\sum\left\{\begin{array}{c}\left(\prod\limits_{1\leq l\leq k_j}\Phi^{t}_{w_ji_{j_l}}\left(wt\left(a,p_{jl}\right)\right)\right)\cdot\\ \Phi^{t}_{w_j}\left(wt\left(\begin{array}{c}\left(\begin{array}{c}r'_t\left(w_j1\right),\ldots,r'_t\left(w_j\left(i_{j1}-1\right)\right),p_{j1},\\ \ldots,r'_t\left(w_j\left(i_{j1}+1\right)\right),\ldots,r'_t\left(w_j\left(i_{j2}-1\right)\right),p_{j2},\\ \ldots,p_{jk_j},\ldots,r'_t\left(w_j\varrho_j\right)\end{array}\right),\\ t(w_j),r'_t\left(w_j\right)\end{array}\right)\right)\\ \mid p_{j1},\ldots,p_{jk_j}\in Q\end{array}\right\}.$$

Now, for every $t\in T_\Sigma\setminus\{a\}$ we get

$$\begin{aligned}(\|\mathcal{M}\|,t)&=\sum_{r_t\in R_{\mathcal{M}}(t)}weight_{\mathcal{M}}\left(r_t\right)\\
&=\sum_{\substack{q\in Q\\ ter(q)\neq 0}}\sum_{r_t\in R_{\mathcal{M}}(t,q)}\prod_{w\in dom(t)}\Phi^t_w\left(wt\left(r_t,w\right)\right)\cdot ter\left(q\right)\\
&=\sum_{\substack{q\in Q\\ ter(q)\neq 0}}\sum_{r_t\in R_{\mathcal{M}}(t,q)}\left(\begin{array}{c}\prod\limits_{w\in dom(t)\setminus sur_a(t)}\Phi^t_w\left(wt\left(r_t,w\right)\right)\\ \cdot\prod\limits_{w\in sur_a(t)}\Phi^t_w\left(wt\left(r_t,w\right)\right)\end{array}\right)\cdot ter\left(q\right)\\
&\overset{(*)}{=}\sum_{\substack{q\in Q\\ ter(q)\neq 0}}\sum_{\substack{r'_t\in R_{\mathcal{M}'}(t,q)\\ \forall w\in dom_a(t):r'_t(w)=q_a}}\left(\sum_{r_t\in v^{-1}(r'_t)}\left(\begin{array}{c}\prod\limits_{w\in dom(t)\setminus sur_a(t)}\Phi^t_w\left(wt\left(r_t,w\right)\right)\\ \cdot\prod\limits_{w\in sur_a(t)}\Phi^t_w\left(wt\left(r_t,w\right)\right)\end{array}\right)\right)\cdot ter\left(q\right)\\
&\overset{(**)}{=}\sum_{\substack{q\in Q\\ ter'(q)\neq 0}}\sum_{\substack{r'_t\in R_{\mathcal{M}'}(t,q)\\ \forall w\in dom_a(t):r'_t(w)=q_a}}\left(\begin{array}{c}\prod\limits_{w\in dom(t)\setminus sur_a(t)}\Phi^t_w\left(wt'\left(r'_t,w\right)\right)\\ \cdot\left(\sum\limits_{r_t\in v^{-1}(r'_t)}\prod\limits_{w\in sur_a(t)}\Phi^t_w\left(wt\left(r_t,w\right)\right)\right)\end{array}\right)\cdot ter'\left(q\right)\end{aligned}$$

which equals

$$\sum_{\substack{q\in Q \\ ter'(q)\neq 0}} \sum_{\substack{r'_t\in R_{\mathcal{M}'}(t,q) \\ \forall w\in dom_a(t): r'_t(w)=q_a}} \left(\begin{array}{c} \prod\limits_{w\in dom(t)\setminus sur_a(t)} \Phi^t_w\left(wt'\left(r'_t,w\right)\right) \\ \cdot \prod\limits_{w\in sur_a(t)} \Phi^t_w\left(wt'\left(r'_t,w\right)\right) \end{array} \right) \cdot ter'(q)$$

$$= \sum_{\substack{q\in Q \\ ter'(q)\neq 0}} \sum_{\substack{r'_t\in R_{\mathcal{M}'}(t,q) \\ \forall w\in dom_a(t): r'_t(w)=q_a}} \prod_{w\in dom(t)} \Phi^t_w\left(wt'\left(r'_t,w\right)\right) \cdot ter'(q)$$

$$= \sum_{\substack{q\in Q \\ ter'(q)\neq 0}} \sum_{r'_t\in R_{\mathcal{M}'}(t,q)} \prod_{w\in dom(t)} \Phi^t_w\left(wt'\left(r'_t,w\right)\right) \cdot ter'(q)$$

$$= \sum_{r'_t\in R_{\mathcal{M}'}(t)} weight_{\mathcal{M}'}\left(r'_t\right)$$

$$= \left(\left\|\mathcal{M}'\right\|, t\right)$$

where at $(*)$ we have used the fact that the set $\{v^{-1}\left(r'_t\right) \mid r'_t \in R_{\mathcal{M}'}(t,q), \forall w \in dom_a(t) : r'_t(w) = q_a\}$ is a partitioning of $R_{\mathcal{M}}(t,q)$. At $(**)$ we have used the following fact: for every $r'_t \in R_{\mathcal{M}'}(t,q), r_t \in v^{-1}\left(r'_t\right)$, and $w \in dom(t) \setminus sur_a(t)$, we have $wt(r_t, w) = wt'(r'_t, w)$ since $r_t(w1), \ldots, r_t(wk), r_t(w) \in Q$. ■

In the next subsection we investigate properties of Φ-recognizable tree series.

3.3 Properties of Φ-recognizable tree series

Proposition 11 *(i) The class $K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$ is closed under sum, scalar product, and Hadamard product.*

(ii) Let Σ, Γ be two ranked alphabets and $h : \Sigma \to \Gamma$ be a relabeling. Furthermore, for the Φ-discounting over Σ and K we assume that $\Phi_\sigma = \Phi_{\sigma'}$ whenever $h(\sigma) = h(\sigma')$ for every $\sigma, \sigma' \in \Sigma_k, k \geq 0$. If the tree series $S \in K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$ is Φ-recognizable, then the tree series $h(S)$ is Φ'-recognizable where $\Phi' = (\Phi'_k)_{k\geq 0}$ is a discounting over Γ and K determined for every $\gamma \in \Gamma_k$ $(k \geq 0)$ by $\Phi'_\gamma = \Phi_\sigma$ for every $\sigma \in \Sigma_k$ $(k \geq 0)$ with $h(\sigma) = \gamma$. Furthermore, if $T \in K^{\Phi'-rec}\langle\langle T_\Gamma\rangle\rangle$, then the tree series $h^{-1}(T) \in K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$ is Φ-recognizable.

(iii) Let $L \subseteq T_\Sigma$ be a recognizable tree language. Then its characteristic series $1_L \in K\langle\langle T_\Sigma\rangle\rangle$ is Φ-recognizable.

Proof. The closure property under sum and scalar product is proved by using well-known constructions from classical tree automata: for the sum we consider the disjoint union of two automata, whereas for the scalar product we multiply the terminal distribution of a wbuta with the given scalar.

We deal now with the Hadamard product case. Let $\mathcal{M} = (Q, wt, ter)$ and $\mathcal{M}' = (Q', wt', ter')$ be two wbuta over Σ and K. We consider the wbuta $\widetilde{\mathcal{M}} = (\widetilde{Q}, \widetilde{wt}, \widetilde{ter})$ with state set $\widetilde{Q} = Q \times Q'$, and weight transition mapping given by

$$\widetilde{wt}(((q_1, q'_1), \ldots, (q_k, q'_k)), \sigma, (q, q')) = wt((q_1, \ldots, q_k), \sigma, q)wt'((q'_1, \ldots, q'_k), \sigma, q')$$

for every $k \geq 0, \sigma \in \Sigma_k, (q, q'), (q_1, q_1'), \ldots, (q_k, q_k') \in \widetilde{Q}$.

The final distribution $\widetilde{ter}$ of $\widetilde{\mathcal{M}}$ is determined for every $(q, q') \in \widetilde{Q}$ by

$$\widetilde{ter}(q, q') = ter(q)ter'(q').$$

Consider now a tree $t \in T_\Sigma$ and let $\widetilde{r}_t$ be a run of $\widetilde{\mathcal{M}}$ over t. Then, there are runs r_t and r_t' of $\mathcal{M}$ and $\mathcal{M}'$ over t respectively, obtained by projections of $\widetilde{r}_t$ on Q and Q' in the obvious way. Conversely, for every run r_t of $\mathcal{M}$ over t and every run r_t' of $\mathcal{M}'$ over t there is a uniquely composed run $\widetilde{r}_t$ of $\widetilde{\mathcal{M}}$ over t. Moreover, the definitions of $\widetilde{wt}$ and $\widetilde{ter}$ and the commutativity of K imply that $weight_{\widetilde{\mathcal{M}}}(\widetilde{r}_t) = weight_{\mathcal{M}}(r_t)weight_{\mathcal{M}'}(r_t')$. Thus, for every $t \in T_\Sigma$ we get

$$\begin{aligned}
\left(\left\|\widetilde{\mathcal{M}}\right\|, t\right) &= \sum_{\widetilde{r}_t \in R_{\widetilde{\mathcal{M}}}(t)} weight_{\widetilde{\mathcal{M}}}(\widetilde{r}_t) \\
&= \sum_{r_t \in R_{\mathcal{M}}(t), r_t' \in R_{\mathcal{M}'}(t)} weight_{\mathcal{M}}(r_t)weight_{\mathcal{M}'}(r_t') \\
&\overset{(*)}{=} \left(\sum_{r_t \in R_{\mathcal{M}}(t)} weight_{\mathcal{M}}(r_t)\right)\left(\sum_{r_t' \in R_{\mathcal{M}'}(t)} weight_{\mathcal{M}'}(r_t')\right) \\
&= (\|\mathcal{M}\|, t)(\|\mathcal{M}'\|, t)
\end{aligned}$$

where the equality $(*)$ is obtained by the distributivity laws of K. Thus, we have shown

$$\left\|\widetilde{\mathcal{M}}\right\| = \|\mathcal{M}\| \odot \|\mathcal{M}'\|$$

as required.

Next, we proceed with the second claim of our proposition. Let $S \in K^{\Phi-rec}\langle\langle T_\Sigma \rangle\rangle$ and $\mathcal{M} = (Q, wt, ter)$ be a wbuta over Σ and K with behavior S. We consider the wbuta $\widehat{\mathcal{M}} = (Q, \widehat{wt}, ter)$ over Γ and K, where for every $n \geq 0, \gamma \in \Gamma_n, q_1, \ldots, q_n, q \in Q$ we set $\widehat{wt}((q_1, \ldots, q_n), \gamma, q) = \sum_{\sigma \in h^{-1}(\gamma)} wt((q_1, \ldots, q_n), \sigma, q)$.

Let $s \in T_\Gamma$ and $t \in h^{-1}(s)$. Then $dom(t) = dom(s)$, and by our assumptions for every $w \in dom(t)$ we have $\Phi_{t(w)} = \Phi'_{s(w)}$ which implies $\Phi_w^t = \Phi_w'^s$. The set of runs of $\widehat{\mathcal{M}}$ over s coincides with the set of runs of $\mathcal{M}$ over t, i.e., for every run $\widehat{r_s}$ of $\widehat{\mathcal{M}}$ over s there is a run r_t of $\mathcal{M}$ over t with $r_t = \widehat{r_s}$ and vice versa. Furthermore, for every $w \in dom(s)$ we have

$$\begin{aligned}
\widehat{wt}(\widehat{r_s}, w) &= \widehat{wt}((\widehat{r_s}(w1), \ldots, \widehat{r_s}(w.rk_\Gamma(s(w)))), s(w), \widehat{r_s}(w)) \\
&= \sum_{\sigma \in h^{-1}(s(w))} wt((\widehat{r_s}(w1), \ldots, \widehat{r_s}(w \cdot rk_\Gamma(s(w)))), \sigma, \widehat{r_s}(w)).
\end{aligned}$$

Therefore

$$
\begin{aligned}
weight_{\widehat{\mathcal{M}}}(\widehat{r_s}) &= \prod_{w\in dom(s)} \Phi_w'^s\left(\widehat{wt}(\widehat{r_s}, w)\right)\cdot ter\left(\widehat{r_s}(\varepsilon)\right)\\
&= \prod_{w\in dom(s)} \Phi_w'^s\left(\sum_{\sigma\in h^{-1}(s(w))} wt((\widehat{r_s}(w1),\ldots,\widehat{r_s}(w\cdot rk_\Gamma(s(w)))),\sigma,\widehat{r_s}(w))\right)\cdot ter(\widehat{r_s}(\varepsilon))\\
&= \prod_{w\in dom(s)} \left(\sum_{\sigma\in h^{-1}(s(w))} \Phi_w'^s\left(wt((\widehat{r_s}(w1),\ldots,\widehat{r_s}(w.rk_\Gamma(s(w)))),\sigma,\widehat{r_s}(w))\right)\right)\cdot ter(\widehat{r_s}(\varepsilon))\\
&\overset{(*)}{=} \sum_{t\in h^{-1}(s)} \left(\prod_{w\in dom(s)} \Phi_w^t\left(wt((r_t(w1),\ldots,r_t(w.rk_\Sigma(t(w)))),t(w),r_t(w))\right)\right)\cdot ter(r_t(\varepsilon))\\
&= \sum_{t\in h^{-1}(s)} \left(\prod_{w\in dom(t)} \Phi_w^t(wt(r_t,w))\cdot ter(r_t(\varepsilon))\right)\\
&= \sum_{t\in h^{-1}(s)} weight_{\mathcal{M}}(r_t)
\end{aligned}
$$

where the equality $(*)$ holds by the distribution laws of K taking into account the above remark. Then, we get

$$
\begin{aligned}
\left(\left\|\widehat{\mathcal{M}}\right\|, s\right) &= \sum_{\widehat{r_s}\in R_{\widehat{\mathcal{M}}}(t)} weight_{\widehat{\mathcal{M}}}(\widehat{r_s}) = \sum_{\widehat{r_s}\in R_{\widehat{\mathcal{M}}}(t)} \left(\sum_{t\in h^{-1}(s)} weight_{\mathcal{M}}(r_t)\right)\\
&= \sum_{r_t\in R_{\mathcal{M}}(t)} \left(\sum_{t\in h^{-1}(s)} weight_{\mathcal{M}}(r_t)\right) = \sum_{t\in h^{-1}(s)} \left(\sum_{r_t\in R_{\mathcal{M}}(t)} weight_{\mathcal{M}}(r_t)\right)\\
&= \sum_{t\in h^{-1}(s)} (\|\mathcal{M}\|, t) = \sum_{t\in h^{-1}(s)} (S,t) = (h(S), s)
\end{aligned}
$$

for every $s\in T_\Gamma$.

Next, let $T\in K^{\Phi'-rec}\langle\langle T_\Gamma\rangle\rangle$. We will show that the tree series $h^{-1}(T)\in K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$. Assume that $\mathcal{N}=(Q,wt,ter)$ is a wbuta over Γ and K accepting T. We construct the wbuta $\widetilde{\mathcal{N}}=(Q,\widetilde{wt},ter)$ over Σ and K by setting $\widetilde{wt}((q_1,\ldots,q_k),\sigma,q)=wt\left((q_1,\ldots,q_k),h(\sigma),q\right)$ for every $k\geq 0, \sigma\in\Sigma_k, q_1,\ldots,q_k,q\in Q$.

Let $t\in T_\Sigma$. Clearly for every run $\widetilde{r_t}$ of $\widetilde{\mathcal{N}}$ over t there exists a run $r_{h(t)}$ of $\mathcal{N}$ over $h(t)$ with $r_{h(t)}=\widetilde{r_t}$ and vice versa. Furthermore, $weight_{\widetilde{\mathcal{N}}}(\widetilde{r_t}) = weight_{\mathcal{N}}\left(r_{h(t)}\right)$. We conclude

$$
\begin{aligned}
\left(\left\|\widetilde{\mathcal{N}}\right\|, t\right) &= \sum_{\widetilde{r_t}\in R_{\widetilde{\mathcal{N}}}(t)} weight_{\widetilde{\mathcal{N}}}\left(\widetilde{r_t}\right) = \sum_{r_{h(t)}\in R_{\mathcal{N}}(t)} weight_{\mathcal{N}}\left(r_{h(t)}\right)\\
&= (\|\mathcal{N}\|, h(t)) = (T, h(t)) = \left(h^{-1}(T), t\right)
\end{aligned}
$$

for every $t\in T_\Sigma$.

Now, it remains to prove that the characteristic series of a recognizable tree language $L \subseteq T_\Sigma$ is Φ-recognizable. To this end we consider a deterministic buta $\mathcal{M} = (Q, \Sigma, \delta, F)$ accepting L. Then, we construct the wbuta $\widetilde{\mathcal{M}} = (Q, wt, ter)$ over Σ and K with

$$wt\left(\left(q_1, \ldots, q_k\right), \sigma, q\right) = \begin{cases} 1 & \text{if } \delta_\sigma\left(q_1, \ldots, q_k\right) = q \\ 0 & \text{otherwise} \end{cases}$$

for every $k \geq 0, \sigma \in \Sigma_k, q_1, \ldots, q_k, q \in Q$, and

$$ter(q) = \begin{cases} 1 & \text{if } q \in F \\ 0 & \text{otherwise} \end{cases}$$

for every $q \in Q$.
Obviously, $\left(\left\|\widetilde{\mathcal{M}}\right\|, t\right) = 1$ if $t \in L$ and $\left(\left\|\widetilde{\mathcal{M}}\right\|, t\right) = 0$ if $t \notin L$, and thus $\left\|\widetilde{\mathcal{M}}\right\| = 1_L$ which concludes our proof. ■

A tree series $S : T_\Sigma \to K$ is called a *recognizable step function* if $S = \sum_{1 \leq j \leq n} k_j 1_{L_j}$ where $k_j \in K$ and $L_j \subseteq T_\Sigma$ ($1 \leq j \leq n$ and $n \in \mathbb{N}$) are recognizable tree languages. By Proposition 11 such a tree series is Φ-recognizable. The class of recognizable tree languages is closed under the Boolean operations, therefore in the sequel for every recognizable step function $S = \sum_{1 \leq j \leq n} k_j 1_{L_j}$, we may assume the family $\left(L_j\right)_{j \in J}$ to be a partition of T_Σ.

Proposition 12 *(i) The class of all recognizable step functions over Σ and K is closed under sum, scalar product, and Hadamard product.*

(ii) Let Σ, Γ be two ranked alphabets and $h : \Sigma \to \Gamma$ be a relabeling. If the tree series $T \in K\left\langle\left\langle T_\Gamma \right\rangle\right\rangle$ is a recognizable step function, then $h^{-1}(T) \in K\left\langle\left\langle T_\Sigma \right\rangle\right\rangle$ is also a recognizable step function.

Proof. (i) Closure under sum and scalar product is trivially proved. The closure under Hadamard product is easily obtained by the closure property of recognizable tree languages under intersection.

(ii) Let $T = \sum_{1 \leq j \leq n} k_j 1_{L_j}$ where $k_j \in K$ and $L_j \subseteq T_\Gamma$ ($1 \leq j \leq n$) are recognizable tree languages. Then

$$h^{-1}(T) = \sum_{1 \leq j \leq n} k_j h^{-1}\left(1_{L_j}\right) = \sum_{1 \leq j \leq n} k_j 1_{h^{-1}\left(L_j\right)}.$$

The class of recognizable tree languages is closed inverse relabelings and this concludes our proof. ■

Next we wish to show that also relabelings preserve recognizable step functions provided that the underlying semiring is additively locally finite. We shall need the following two lemmas. Especially, Lemma 14 has its own interest.

Lemma 13 *(see [3, 24]) Let Σ be a ranked alphabet.*

(i) [42] Let $S \in \mathbb{Z}^{rec}\left\langle\left\langle T_\Sigma \right\rangle\right\rangle$ and $a, b \in \mathbb{Z}$ with $b \neq 0$. Then $S^{-1}(a + b\mathbb{Z})$ is a recognizable tree language.

(ii) *Let $S \in \mathbb{N}^{rec}\langle\langle T_\Sigma\rangle\rangle$ and $a \in \mathbb{N}$. Then the tree languages $S^{-1}(\{n \in \mathbb{N} \mid n \geq a\})$, $S^{-1}(\{n \in \mathbb{N} \mid n \leq a\})$, and $S^{-1}(a)$ are recognizable.*

Lemma 14 *Let K be additively locally finite. Let also Σ, Γ be two ranked alphabets, $1_L \in K\langle\langle T_\Sigma\rangle\rangle$ be the characteristic series of a recognizable tree language $L \subseteq T_\Sigma$, and $h : \Sigma \to \Gamma$ be a relabeling. Then the tree series $h(1_L) \in K\langle\langle T_\Gamma\rangle\rangle$ is a recognizable step function.*

Proof. For every $k \in K$ and $n \geq 0$ we define the value $n \otimes k \in K$ inductively by $0 \otimes k = 0$ (of K) and $(n+1)\otimes k = k + n\otimes k$. Thus $n \otimes k = k + \ldots + k$ with n times k. For every $s \in T_\Gamma$, let $m(s) = \left|h^{-1}(s) \cap L\right|$. Then $(h(1_L), s) = m(s) \otimes 1$. The additive monoid $\langle 1\rangle$ generated by $\{1\}$ (of K) is finite. We choose a minimal element $a \in \mathbb{N}$ such that $a \otimes 1 = (a + x) \otimes 1$ for some $x > 0$ and we let b to be the smallest such x. Then $\langle 1\rangle = \{0, 1, 2\otimes 1, \ldots, (a+b-1)\otimes 1\}$. Now for every $s \in T_\Gamma$ we have $m(s) \otimes 1 = d(s) \otimes 1$ for a uniquely determined $d(s) \in \mathbb{N}$ with $0 \leq d(s) \leq a + b - 1$. Note that if $0 \leq d < a$, then $m(s) \otimes 1 = d \otimes 1$ iff $m(s) = d$, and if $a \leq d < a + b$, then $m(s) \otimes 1 = d \otimes 1$ iff $m(s) \in d + b\mathbb{N}$. For every $0 \leq d < a + b$ let $M_d = \{s \in T_\Gamma \mid d(s) = d\}$. Then

$$(h(1_L), s) = d(s) \otimes 1 = (d(s) \otimes 1) \cdot \left(1_{M_{d(s)}}, s\right) = \sum_{0 \leq d < a+b} (d \otimes 1) \cdot (1_{M_d}, s).$$

Now let $1'_L : T_\Sigma \to \mathbb{N}$ be the characteristic series of L with values $0, 1 \in \mathbb{N}$. Then $1'_L \in \mathbb{N}^{rec}\langle\langle T_\Sigma\rangle\rangle$, and by Proposition 11 the tree series $S = h(1'_L) \in \mathbb{N}^{rec}\langle\langle T_\Gamma\rangle\rangle$ is also recognizable, and $(S, s) = \sum\limits_{t \in h^{-1}(s)} (1'_L, t) = m(s)$ $(s \in T_\Gamma)$. Hence $M_d = \{s \in T_\Gamma \mid m(s) = d\} = S^{-1}(d)$ if $0 \leq d < a$, and $M_d = \{s \in T_\Gamma \mid m(s) \in d + b\mathbb{N}\} = S^{-1}(d + b\mathbb{N})$ if $a \leq d < a + b$. In any case, M_d is recognizable by Lemma 13. Thus $h(1_L)$ is a recognizable step function. ■

Proposition 15 *Let K be additively locally finite. Let Σ, Γ be two ranked alphabets, $h : \Sigma \to \Gamma$ be a relabeling, and $S \in K\langle\langle T_\Sigma\rangle\rangle$ be a recognizable step function. Then the tree series $h(S) \in K\langle\langle T_\Gamma\rangle\rangle$ is also a recognizable step function.*

Proof. Let $S = \sum\limits_{1 \leq j \leq n} k_j 1_{L_j}$ where $k_j \in K$ and $L_j \subseteq T_\Sigma$ $(1 \leq j \leq n)$ are recognizable tree languages. Then for every $s \in T_\Gamma$

$$\begin{aligned}(h(S), s) &= \sum_{t \in h^{-1}(s)} (S, t) = \sum_{t \in h^{-1}(s)} \sum_{1 \leq j \leq n} k_j \left(1_{L_j}, t\right) \\ &= \sum_{1 \leq j \leq n} k_j \left(\sum_{t \in h^{-1}(s)} \left(1_{L_j}, t\right)\right) = \sum_{1 \leq j \leq n} k_j \left(h\left(1_{L_j}\right), s\right)\end{aligned}$$

i.e.,

$$h(S) = \sum_{1 \leq j \leq n} k_j h\left(1_{L_j}\right)$$

which by Lemma 14 implies that $h(S)$ is a recognizable step function. ■

4 Φ-rational tree series operations and a Kleene theorem

In this section we introduce the Φ-rational operations on formal tree series, and we show a Kleene theorem for Φ-recognizable tree series. For the reader's convenience, we first recall well-known operations on trees and tree languages that we will need in the sequel.

Let $t \in T_\Sigma, a \in \Sigma_0$ and assume that a occurs n times ($n \geq 0$) in t. For every $s_1, \ldots, s_n \in T_\Sigma$, we denote by $t \cdot_a (s_1, \ldots, s_n)$ the result of substituting in t, every occurrence of a from left to right, by $s_1, \ldots, s_n$, respectively. Furthermore, for every $L \subseteq T_\Sigma$ we set

$$t \cdot_a L = \{t \cdot_a (s_1, \ldots, s_n) \mid s_i \in L \text{ for } 1 \leq i \leq n\}.$$

Then for $L_1, L_2 \subseteq T_\Sigma$ and $a \in \Sigma_0$, the *a-concatenation of* L_1 and L_2 is the tree language

$$L_1 \cdot_a L_2 = \bigcup_{t \in L_1} t \cdot_a L_2.$$

It should be clear that the a-concatenation coincides with the OI-substitution of the language L_2 in L_1 at a (see [27]). This implies that the a-concatenation operation of tree languages is associative.

Let $k > 0, \sigma \in \Sigma_k$. The *Φ-top-concatenation with* σ is an operation $\sigma_\Phi : K\langle\langle T_\Sigma\rangle\rangle^k \to K\langle\langle T_\Sigma\rangle\rangle$ on trees series defined for every $S_1, \ldots, S_k \in K\langle\langle T_\Sigma\rangle\rangle$ and $t \in T_\Sigma$ by

$$(\sigma_\Phi(S_1, \ldots, S_k), t) = \begin{cases} \Phi_\sigma^1((S_1, t_1)) \cdot \ldots \cdot \Phi_\sigma^k((S_k, t_k)) & \text{if } t = \sigma(t_1, \ldots, t_k) \\ 0 & \text{otherwise.} \end{cases}$$

Let also $S, T \in K\langle\langle T_\Sigma\rangle\rangle$ and $a \in \Sigma_0$. The *a, Φ-concatenation of S and T* is the tree series $S \cdot_{a,\Phi} T \in K\langle\langle T_\Sigma\rangle\rangle$ defined for every $t \in T_\Sigma$ by

$$(S \cdot_{a,\Phi} T, t) = \sum_{\substack{s, t_1, \ldots, t_r \in T_\Sigma, t = s \cdot_a (t_1, \ldots, t_r) \\ s(w_1) = \ldots = s(w_r) = a}} (S, s) \cdot \Phi_{w_1}^t((T, t_1)) \cdot \ldots \cdot \Phi_{w_r}^t((T, t_r)).$$

Proposition 16 *The a, Φ-concatenation of tree series is associative, i.e., for every $S, T, R \in K\langle\langle T_\Sigma\rangle\rangle$ it holds $S \cdot_{a,\Phi} (T \cdot_{a,\Phi} R) = (S \cdot_{a,\Phi} T) \cdot_{a,\Phi} R$.*

Proof. For every $t \in T_\Sigma$ we have

$$\begin{aligned} &(S \cdot_{a,\Phi} (T \cdot_{a,\Phi} R), t) \\ &= \sum_{\substack{s, t_1, \ldots, t_r \in T_\Sigma, t = s \cdot_a (t_1, \ldots, t_r) \\ s(w_1) = \ldots = s(w_r) = a}} (S, s) \cdot \prod_{i=1}^{r} \Phi_{w_i}^t((T \cdot_{a,\Phi} R, t_i)) \end{aligned}$$

$$= \sum_{\substack{s,t_1,\ldots,t_r\in T_\Sigma, t=s\cdot_a(t_1,\ldots,t_r)\\ s(w_1)=\ldots=s(w_r)=a}} (S,s)$$

$$\cdot \prod_{i=1}^{r} \Phi^t_{w_i}\left(\sum_{\substack{v_i,u_{i1},\ldots,u_{in_i}\in T_\Sigma, t_i=v_i\cdot_a\left(u_{i1},\ldots,u_{in_i}\right)\\ v_i\left(w_1^i\right)=\ldots=v_i\left(w_{n_i}^i\right)=a}} (T,v_i)\cdot\prod_{j_i=1}^{n_i}\Phi^{t_i}_{w^i_{j_i}}\left((R,u_{ij_i})\right)\right)$$

$$= \sum_{\substack{s,t_1,\ldots,t_r\in T_\Sigma, t=s\cdot_a(t_1,\ldots,t_r)\\ s(w_1)=\ldots=s(w_r)=a}} (S,s)$$

$$\cdot \prod_{i=1}^{r}\left(\sum_{\substack{v_i,u_{i1},\ldots,u_{in_i}\in T_\Sigma, t_i=v_i\cdot_a\left(u_{i1},\ldots,u_{in_i}\right)\\ v_i\left(w_1^i\right)=\ldots=v_i\left(w_{n_i}^i\right)=a}} \Phi^t_{w_i}\left((T,v_i)\cdot\prod_{j_i=1}^{n_i}\Phi^{t_i}_{w^i_{j_i}}\left((R,u_{ij_i})\right)\right)\right)$$

$$= \sum_{\substack{s,t_1,\ldots,t_r\in T_\Sigma, t=s\cdot_a(t_1,\ldots,t_r)\\ s(w_1)=\ldots=s(w_r)=a}} (S,s)$$

$$\cdot \prod_{i=1}^{r}\left(\sum_{\substack{v_i,u_{i1},\ldots,u_{in_i}\in T_\Sigma, t_i=v_i\cdot_a\left(u_{i1},\ldots,u_{in_i}\right)\\ v_i\left(w_1^i\right)=\ldots=v_i\left(w_{n_i}^i\right)=a}} \Phi^t_{w_i}\left((T,v_i)\right)\cdot\prod_{j_i=1}^{n_i}\Phi^t_{w_i}\circ\Phi^{t_i}_{w^i_{j_i}}\left((R,u_{ij_i})\right)\right)$$

$$= \sum_{\substack{s,t_1,\ldots,t_r,v_i,u_{i1},\ldots,u_{in_i}\in T_\Sigma\\ t=s\cdot_a(t_1,\ldots,t_r), t_i=v_i\cdot_a\left(u_{i1},\ldots,u_{in_i}\right)\\ s(w_1)=\ldots=s(w_r)=v_i\left(w_1^i\right)=\ldots=v_i\left(w_{n_i}^i\right)=a}} (S,s)$$

$$\cdot \prod_{i=1}^{r}\left(\Phi^t_{w_i}\left((T,v_i)\right)\cdot\prod_{j_i=1}^{n_i}\Phi^t_{w_iw^i_{j_i}}\left((R,u_{ij_i})\right)\right).$$

On the other side

$$\left((S\cdot_{a,\Phi}T)\cdot_{a,\Phi}R,t\right)$$

$$= \sum_{\substack{v,u_1,\ldots,u_q\in T_\Sigma, t=v\cdot_a(u_1,\ldots,u_q)\\ v(w_1)=\ldots=v(w_q)=a}}\left((S\cdot_{a,\Phi}T,v)\cdot\prod_{j=1}^{q}\Phi^t_{w_j}\left((R,u_j)\right)\right)$$

$$
= \sum_{\substack{v,u_1,\ldots,u_q \in T_\Sigma, t = v \cdot_a (u_1,\ldots,u_q) \\ v(w_1)=\ldots=v(w_q)=a}} \left(\left(\sum_{\substack{s,s_1,\ldots,s_r \in T_\Sigma, v = s \cdot_a (s_1,\ldots,s_r) \\ s(w'_1)=\ldots=s(w'_r)=a}} (S,s) \cdot \prod_{i=1}^{r} \Phi^{v}_{w'_i}((T,s_i)) \right) \cdot \prod_{j=1}^{q} \Phi^{t}_{w_j}((R,u_j)) \right)
$$

$$
= \sum_{\substack{v,u_1,\ldots,u_q \in T_\Sigma, t = v \cdot_a (u_1,\ldots,u_q) \\ v(w_1)=\ldots=v(w_q)=a}} \left(\sum_{\substack{s,s_1,\ldots,s_r \in T_\Sigma, v = s \cdot_a (s_1,\ldots,s_r) \\ s(w'_1)=\ldots=s(w'_r)=a}} (S,s) \cdot \prod_{i=1}^{r} \Phi^{v}_{w'_i}((T,s_i)) \cdot \prod_{j=1}^{q} \Phi^{t}_{w_j}((R,u_j)) \right)
$$

$$
= \sum_{\substack{v,u_1,\ldots,u_q,s,s_1,\ldots,s_r \in T_\Sigma \\ t = v \cdot_a (u_1,\ldots,u_q), v = s \cdot_a (s_1,\ldots,s_r) \\ v(w_1)=\ldots=v(w_q)=s(w'_1)=\ldots=s(w'_r)=a}} (S,s) \cdot \prod_{i=1}^{r} \Phi^{t}_{w'_i}((T,s_i)) \cdot \prod_{j=1}^{q} \Phi^{t}_{w_j}((R,u_j)).
$$

For the last obtained equality we used the observation that every node of v is also a node of t. Clearly, there is an one to one correspondence between the two ways of decomposing t. This also implies that the occurred endomorphisms at each node of the runs of the two corresponding decompositions, coincide. We conclude $S \cdot_{a,\Phi} (T \cdot_{a,\Phi} R) = (S \cdot_{a,\Phi} T) \cdot_{a,\Phi} R$. ■

Lemma 17 *Let $a \in \Sigma_0$.*

(i) The a, Φ-concatenation is right-distributive, i.e., for every $S, T, R \in K\langle\langle T_\Sigma \rangle\rangle$, it holds $(S + T) \cdot_{a,\Phi} R = S \cdot_{a,\Phi} R + T \cdot_{a,\Phi} R$.

(ii) Assume that K is idempotent. Then for every $S, T, R \in K\langle\langle T_\Sigma \rangle\rangle$ the approximation $S \cdot_{a,\Phi} (T + R) \geq S \cdot_{a,\Phi} T + S \cdot_{a,\Phi} R$ holds.

Proof. Let $t \in T_\Sigma$. Then

$$
\begin{aligned}
&((S+T) \cdot_{a,\Phi} R, t) \\
&= \sum_{\substack{s,t_1,\ldots,t_r \in T_\Sigma, t = s \cdot_a (t_1,\ldots,t_r) \\ s(w_1)=\ldots=s(w_r)=a}} ((S+T), s) \cdot \prod_{i=1}^{r} \Phi^{t}_{w_i}((R,t_i)) \\
&= \sum_{\substack{s,t_1,\ldots,t_r \in T_\Sigma, t = s \cdot_a (t_1,\ldots,t_r) \\ s(w_1)=\ldots=s(w_r)=a}} ((S,s) + (T,s)) \cdot \prod_{i=1}^{r} \Phi^{t}_{w_i}((R,t_i))
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\substack{s,t_1,\ldots,t_r\in T_\Sigma, t=s\cdot_a(t_1,\ldots,t_r)\\ s(w_1)=\ldots=s(w_r)=a}} \left((S,s)\cdot\prod_{i=1}^{r}\Phi^t_{w_i}\left((R,t_i)\right) + (T,s)\cdot\prod_{i=1}^{r}\Phi^t_{w_i}\left((R,t_i)\right)\right)\\
&= \left(\sum_{\substack{s,t_1,\ldots,t_r\in T_\Sigma, t=s\cdot_a(t_1,\ldots,t_r)\\ s(w_1)=\ldots=s(w_r)=a}} (S,s)\cdot\prod_{i=1}^{r}\Phi^t_{w_i}\left((R,t_i)\right)\right)\\
&\quad + \left(\sum_{\substack{s,t_1,\ldots,t_r\in T_\Sigma, t=s\cdot_a(t_1,\ldots,t_r)\\ s(w_1)=\ldots=s(w_r)=a}} (T,s)\cdot\prod_{i=1}^{r}\Phi^t_{w_i}\left((R,t_i)\right)\right)\\
&= \left(S\cdot_{a,\Phi} R,t\right) + \left(T\cdot_{a,\Phi} R,t\right).
\end{aligned}
$$

For our second claim, we compute for every $t\in T_\Sigma$

$$
\begin{aligned}
\left(S\cdot_{a,\Phi}\left(T+R\right),t\right) &= \sum_{\substack{s,t_1,\ldots,t_r\in T_\Sigma, t=s\cdot_a(t_1,\ldots,t_r)\\ s(w_1)=\ldots=s(w_r)=a}} (S,s)\cdot\prod_{i=1}^{r}\Phi^t_{w_i}\left(\left(\left(T+R\right),t_i\right)\right)\\
&= \sum_{\substack{s,t_1,\ldots,t_r\in T_\Sigma, t=s\cdot_a(t_1,\ldots,t_r)\\ s(w_1)=\ldots=s(w_r)=a}} (S,s)\cdot\prod_{i=1}^{r}\Phi^t_{w_i}\left((T,t_i)+(R,t_i)\right)\\
&= \sum_{\substack{s,t_1,\ldots,t_r\in T_\Sigma, t=s\cdot_a(t_1,\ldots,t_r)\\ s(w_1)=\ldots=s(w_r)=a}} (S,s)\cdot\prod_{i=1}^{r}\left(\Phi^t_{w_i}\left((T,t_i)\right)+\Phi^t_{w_i}\left((R,t_i)\right)\right)\\
&\geq \sum_{\substack{s,t_1,\ldots,t_r\in T_\Sigma, t=s\cdot_a(t_1,\ldots,t_r)\\ s(w_1)=\ldots=s(w_r)=a}} (S,s)\cdot\left(\prod_{i=1}^{r}\Phi^t_{w_i}\left((T,t_i)\right)+\prod_{i=1}^{r}\Phi^t_{w_i}\left((R,t_i)\right)\right)\\
&= \left(\sum_{\substack{s,t_1,\ldots,t_r\in T_\Sigma, t=s\cdot_a(t_1,\ldots,t_r)\\ s(w_1)=\ldots=s(w_r)=a}} (S,s)\cdot\prod_{i=1}^{r}\Phi^t_{w_i}\left((T,t_i)\right)\right)\\
&\quad + \left(\sum_{\substack{s,t_1,\ldots,t_r\in T_\Sigma, t=s\cdot_a(t_1,\ldots,t_r)\\ s(w_1)=\ldots=s(w_r)=a}} (S,s)\cdot\prod_{i=1}^{r}\Phi^t_{w_i}\left((R,t_i)\right)\right)\\
&= \left(S\cdot_{a,\Phi} T,t\right) + \left(S\cdot_{a,\Phi} R,t\right).
\end{aligned}
$$

The above inequality is derived by the obvious argument $\prod_{i=1}^{r}(a_i+b_i)\geq\prod_{i=1}^{r}a_i+\prod_{i=1}^{r}b_i$ for every $r>0$ and every $a_i,b_i\in K$ $(1\leq i\leq r)$. ■

Lemma 18 *Let Σ be an a-monadic alphabet and $S, T, R \in K\langle\langle T_\Sigma\rangle\rangle$. Then $S \cdot_{a,\Phi} (T+R) = S \cdot_{a,\Phi} T + S \cdot_{a,\Phi} R$.*

Proof. For every $t \in T_\Sigma$

$$
\begin{aligned}
(S \cdot_{a,\Phi} (T+R), t) &= \sum_{\substack{s,u \in T_\Sigma, t = s \cdot_a u \\ s(w)=a}} (S,s) \cdot \Phi_w^t (((T+R), u)) \\
&= \sum_{\substack{s,u \in T_\Sigma, t = s \cdot_a u \\ s(w)=a}} (S,s) \cdot \left(\Phi_w^t ((T,u)) + \Phi_w^t ((R,u))\right) \\
&= \left(\sum_{\substack{s,u \in T_\Sigma, t = s \cdot_a u \\ s(w)=a}} (S,s) \cdot \Phi_w^t ((T,u)) \right) + \left(\sum_{\substack{s,u \in T_\Sigma, t = s \cdot_a u \\ s(w)=a}} (S,s) \cdot \Phi_w^t ((R,u)) \right) \\
&= (S \cdot_{a,\Phi} T, t) + (S \cdot_{a,\Phi} R, t).
\end{aligned}
$$

■

Lemma 19 *Let K be idempotent. Then the a, Φ-concatenation respects the partial order $\leq$, i.e., for every $S, T, R \in K\langle\langle T_\Sigma\rangle\rangle$ if $S \leq T$, then $S \cdot_{a,\Phi} R \leq T \cdot_{a,\Phi} R$ and $R \cdot_{a,\Phi} S \leq R \cdot_{a,\Phi} T$.*

Proof. Consider $S, T, R \in K\langle\langle T_\Sigma\rangle\rangle$ such that $S \leq T$. Observe that $S \leq T$ iff $S + T = T$. Then by Lemma 17 we have $S \cdot_{a,\Phi} R \leq S \cdot_{a,\Phi} R + T \cdot_{a,\Phi} R = (S+T) \cdot_{a,\Phi} R = T \cdot_{a,\Phi} R$ and $R \cdot_{a,\Phi} S \leq R \cdot_{a,\Phi} S + R \cdot_{a,\Phi} T \leq R \cdot_{a,\Phi} (S+T) = R \cdot_{a,\Phi} T$. ■

Next, we wish to introduce the Φ-star operation of tree series. This can be done by extending either the Engelfriet's nth a-E-iteration [26] or the nth a-TW-iteration of a tree language due to Thatcher and Wright [50] (see also [34]).

Let $S \in K\langle\langle T_\Sigma\rangle\rangle$ and $a \in \Sigma_0$. The *nth a, Φ-E-iteration of* S is the tree series $S_{a,\Phi}^{n,E} \in K\langle\langle T_\Sigma\rangle\rangle$ defined inductively as follows:

(i) $S_{a,\Phi}^{0,E} = 1a$ and

(ii) $S_{a,\Phi}^{n+1,E} = S_{a,\Phi}^{n,E} \cdot_{a,\Phi} (S + 1a)$ for every $n \geq 0$.

Let $S \in K\langle\langle T_\Sigma\rangle\rangle$ and $a \in \Sigma_0$. The *nth a, Φ-TW-iteration* of S is the tree series $S_{a,\Phi}^{n,TW} \in K\langle\langle T_\Sigma\rangle\rangle$ defined inductively as follows:

(i) $S_{a,\Phi}^{0,TW} = 1a$

(ii) $S_{a,\Phi}^{n+1,TW} = (S + 1a) \cdot_{a,\Phi} S_{a,\Phi}^{n,TW}$ for every $n \geq 0$.

Next we show that for every $n \geq 0$, the nth iterations of the above two types coincide.

Lemma 20 *For every $S \in K\langle\langle T_\Sigma\rangle\rangle$ and $n \geq 0$, we have $S_{a,\Phi}^{n,E} = S_{a,\Phi}^{n,TW}$.*

Proof. We have already proved that the a,Φ-concatenation is associative. Thus, for every $n,m\geq 0$, we have

$$\begin{aligned}S_{a,\Phi}^{n+1,E}\cdot_{a,\Phi}S_{a,\Phi}^{m,TW} &= \left(S_{a,\Phi}^{n,E}\cdot_{a,\Phi}(S+1a)\right)\cdot_{a,\Phi}S_{a,\Phi}^{m,TW}=S_{a,\Phi}^{n,E}\cdot_{a,\Phi}\left((S+1a)\cdot_{a,\Phi}S_{a,\Phi}^{m,TW}\right)\\ &= S_{a,\Phi}^{n,E}\cdot_{a,\Phi}S_{a,\Phi}^{m+1,TW}. \qquad (1)\end{aligned}$$

Now

$$S_{a,\Phi}^{n,E}=S_{a,\Phi}^{n,E}\cdot_{a,\Phi}1a=S_{a,\Phi}^{n,E}\cdot_{a,\Phi}S_{a,\Phi}^{0,TW}=S_{a,\Phi}^{0,E}\cdot_{a,\Phi}S_{a,\Phi}^{n,TW}=1a\cdot_{a,\Phi}S_{a,\Phi}^{n,TW}=S_{a,\Phi}^{n,TW}$$

where the third equality is obtained by applying (1) n times. ■

In [21] it is shown that considering either the a-E-iterations or the a-TW-iterations to define an a-star operation on tree series, we should require our semiring K to be idempotent. Thus, following [21] we face this problem by introducing a new tree series iteration. In fact, it is the same iteration used in [21] but here we employ the Φ-discounting. This iteration is considered below to define the Φ-star operation.

Definition 21 *Let $S\in K\langle\langle T_\Sigma\rangle\rangle$ and $a\in\Sigma_0$. The* nth a,Φ-iteration *of S is the tree series $S_{a,\Phi}^{n}\in K\langle\langle T_\Sigma\rangle\rangle$ defined inductively as follows:*

(i) $S_{a,\Phi}^{0}=\mathbf{0}$ *and*

(ii) $S_{a,\Phi}^{n+1}=S\cdot_{a,\Phi}S_{a,\Phi}^{n}+1a$ *for every* $n\geq 0$.

Lemma 22 *Let $S\in K^a\langle\langle T_\Sigma\rangle\rangle$ and $t\in T_\Sigma$. If $n\geq ht(t)+1$, then* $\left(S_{a,\Phi}^{n+1},t\right)=\left(S_{a,\Phi}^{n},t\right)$.

Proof. The proof is by induction on $ht(t)$. First we show that $\left(S_{a,\Phi}^{n},a\right)=1$ for every $n\geq 1$. Indeed for $n\geq 0$

$$\left(S_{a,\Phi}^{n+1},a\right)=\left(S\cdot_{a,\Phi}S_{a,\Phi}^{n},a\right)+(1a,a)=(S,a)\cdot\Phi_{\varepsilon}^{a}\left(\left(S_{a,\Phi}^{n},a\right)\right)+(1a,a)=0\cdot\left(S_{a,\Phi}^{n},a\right)+1=1.$$

Next we show that for every other $b\in\Sigma_0$ with $b\neq a$, $\left(S_{a,\Phi}^{n},b\right)=(S,b)$ for every $n\geq 1$. For $n\geq 0$

$$\left(S_{a,\Phi}^{n+1},b\right)=\left(S\cdot_{a,\Phi}S_{a,\Phi}^{n},b\right)+(1a,b)=\left(S\cdot_{a,\Phi}S_{a,\Phi}^{n},b\right)=(S,b).$$

Now, assume that $ht(t)>0$ and let $n\geq ht(t)+1$. Then

$$\begin{aligned}\left(S_{a,\Phi}^{n+1},t\right) &= \left(S\cdot_{a,\Phi}S_{a,\Phi}^{n},t\right)+(1a,t)=\left(S\cdot_{a,\Phi}S_{a,\Phi}^{n},t\right)\\ &= \sum_{\substack{s,t_1,\dots,t_r\in T_\Sigma\\ t=s\cdot_a(t_1,\dots,t_r),s\neq a\\ s(w_1)=\dots=s(w_r)=a}}(S,s)\cdot\prod_{i=1}^{r}\Phi_{w_i}^{t}\left(\left(S_{a,\Phi}^{n},t_i\right)\right)\\ &= \sum_{\substack{s,t_1,\dots,t_r\in T_\Sigma\\ t=s\cdot_a(t_1,\dots,t_r),s\neq a\\ s(w_1)=\dots=s(w_r)=a}}(S,s)\cdot\prod_{i=1}^{r}\Phi_{w_i}^{t}\left(\left(S_{a,\Phi}^{n-1},t_i\right)\right)\\ &= \left(S\cdot_{a,\Phi}S_{a,\Phi}^{n-1},t\right)+(1a,t)=\left(S_{a,\Phi}^{n},t\right).\end{aligned}$$

Note that the restriction of the above summation to all $s \neq a$ is possible since S is a-proper. Therefore, $ht(t_i) < ht(t)$ $(1 \leq i \leq n)$ and the induction hypothesis can be applied. ■

Now we are ready to define the a, Φ-Kleene-star of a-proper tree series.

Definition 23 *Let $S \in K^a \langle\langle T_\Sigma \rangle\rangle$. The a, Φ-*Kleene-star *(or simply a, Φ-*star*) of S is a tree series $S^*_{a,\Phi} \in K \langle\langle T_\Sigma \rangle\rangle$ which is defined in the following way. For every $t \in T_\Sigma$ we set $\left(S^*_{a,\Phi}, t\right) = \left(S^{ht(t)+1}_{a,\Phi}, t\right)$.*

Lemma 24 *Let $S \in K^a \langle\langle T_\Sigma \rangle\rangle$. Then $S^*_{a,\Phi} = S \cdot_{a,\Phi} S^*_{a,\Phi} + 1a$.*

Proof. Let $t \in T_\Sigma$ and $n = ht(t) + 1$. Then

$$\left(S^*_{a,\Phi}, t\right) = \left(S^{n+1}_{a,\Phi}, t\right) = \left(S \cdot_{a,\Phi} S^n_{a,\Phi}, t\right) + (1a, t)$$

and

$$\begin{aligned}\left(S \cdot_{a,\Phi} S^n_{a,\Phi}, t\right) &= \sum_{\substack{s,u_1,\ldots,u_r \in T_\Sigma \\ t = s\cdot_a(u_1,\ldots,u_r), s\neq a \\ s(w_1)=\ldots=s(w_r)=a}} (S, s) \cdot \prod_{i=1}^{r} \Phi^t_{w_i}\left(\left(S^n_{a,\Phi}, u_i\right)\right) \\ &= \sum_{\substack{s,u_1,\ldots,u_r \in T_\Sigma \\ t = s\cdot_a(u_1,\ldots,u_r), s\neq a \\ s(w_1)=\ldots=s(w_r)=a}} (S, s) \cdot \prod_{i=1}^{r} \Phi^t_{w_i}\left(\left(S^*_{a,\Phi}, u_i\right)\right) \\ &= \left(S \cdot_{a,\Phi} S^*_{a,\Phi}, t\right)\end{aligned}$$

where the second equality is true by Lemma 22 since $n = ht(t) + 1 > ht(u_i) + 1$. ■

The subsequent proposition shows the relation of the iterations $S^n_{a,\Phi}$ and $S^{n,TW}_{a,\Phi}$ whenever the underlying semiring is idempotent.

Proposition 25 *Let K be idempotent and $S \in K^a \langle\langle T_\Sigma \rangle\rangle$. For every $n \geq 0$, $S^n_{a,\Phi} \leq S^{n,TW}_{a,\Phi} \leq S^{n+1}_{a,\Phi}$.*

Proof. For every $n \geq 0$, by definition of the nth a, Φ-TW-iteration, we have

$$S^{n+1,TW}_{a,\Phi} = (S + 1a) \cdot_{a,\Phi} S^{n,TW}_{a,\Phi} = S \cdot_{a,\Phi} S^{n,TW}_{a,\Phi} + 1a \cdot_{a,\Phi} S^{n,TW}_{a,\Phi} = S \cdot_{a,\Phi} S^{n,TW}_{a,\Phi} + S^{n,TW}_{a,\Phi}$$

which implies $S^{n,TW}_{a,\Phi} \leq S^{n+1,TW}_{a,\Phi}$. Next, we show by induction on n that $S^n_{a,\Phi} \leq S^{n+1}_{a,\Phi}$. Indeed, for $n = 0$

$$S^0_{a,\Phi} = \mathbf{0} \leq S^1_{a,\Phi}$$

whereas for $n \geq 0$

$$S^{n+1}_{a,\Phi} = S \cdot_{a,\Phi} S^n_{a,\Phi} + 1a \leq S \cdot_{a,\Phi} S^{n+1}_{a,\Phi} + 1a = S^{n+2}_{a,\Phi}.$$

Once more by induction on n, we conclude our claim $S^n_{a,\Phi} \leq S^{n,TW}_{a,\Phi} \leq S^{n+1}_{a,\Phi}$. Indeed, for $n = 0$

$$\mathbf{0} \leq 1a \leq S \cdot_{a,\Phi} \mathbf{0} + 1a.$$

Since $1a = S_a^{0,TW} \leq S_a^{n-1,TW}$ for $n \geq 1$, by the induction step we get

$$S_{a,\Phi}^n = S \cdot_{a,\Phi} S_{a,\Phi}^{n-1} + 1a \leq S \cdot_{a,\Phi} S_{a,\Phi}^{n-1,TW} + S_{a,\Phi}^{n-1,TW} = S_{a,\Phi}^{n,TW}.$$

Furthermore,

$$\begin{aligned} S_{a,\Phi}^{n,TW} &= S \cdot_{a,\Phi} S_{a,\Phi}^{n-1,TW} + S_{a,\Phi}^{n-1,TW} \leq S \cdot_{a,\Phi} S_{a,\Phi}^n + S_{a,\Phi}^n \\ &\leq S_{a,\Phi}^{n+1} + S_{a,\Phi}^n \leq S_{a,\Phi}^{n+1} + S_{a,\Phi}^{n+1} = S_{a,\Phi}^{n+1} \end{aligned}$$

and our proof is completed. ■

Definition 26 *The set of* Φ-rational tree series expressions over Σ and K*, denoted by* Φ-*Rat*(Σ, K)*, and the tree series* $\|\zeta\| \in K\langle\langle T_\Sigma \rangle\rangle$ *derived by such an expression* ζ *are defined inductively as follows:*

- $\Sigma_0 \subseteq \Phi\text{-}Rat(\Sigma, K)$*, and for every* $a \in \Sigma_0$*,* $\|a\| = 1a$*,*

- *for every* $k \geq 1, \sigma \in \Sigma_k$*, and* $\zeta_1, \ldots, \zeta_k \in \Phi\text{-}Rat(\Sigma, K)$*, the expression* $\sigma_\Phi(\zeta_1, \ldots, \zeta_k) \in \Phi\text{-}Rat(\Sigma, K)$ *and* $\|\sigma_\Phi(\zeta_1, \ldots, \zeta_k)\| = \sigma_\Phi(\|\zeta_1\|, \ldots, \|\zeta_k\|)$*,*

- *for every* $\zeta \in \Phi\text{-}Rat(\Sigma, K)$ *and* $k \in K$*, the expression* $k\zeta \in \Phi\text{-}Rat(\Sigma, K)$ *and* $\|k\zeta\| = k \cdot \|\zeta\|$*,*

- *for every* $\zeta_1, \zeta_2 \in \Phi\text{-}Rat(\Sigma, K)$*, the expression* $\zeta_1 + \zeta_2 \in \Phi\text{-}Rat(\Sigma, K)$ *and* $\|\zeta_1 + \zeta_2\| = \|\zeta_1\| + \|\zeta_2\|$*,*

- *for every* $\zeta_1, \zeta_2 \in \Phi\text{-}Rat(\Sigma, K)$ *and* $a \in \Sigma_0$*, the expression* $\zeta_1 \cdot_{a,\Phi} \zeta_2 \in \Phi\text{-}Rat(\Sigma, K)$ *and* $\|\zeta_1 \cdot_{a,\Phi} \zeta_2\| = \|\zeta_1\| \cdot_{a,\Phi} \|\zeta_2\|$*, and*

- *for every* $\zeta \in \Phi\text{-}Rat(\Sigma, K)$ *and* $a \in \Sigma_0$ *such that* $\|\zeta\|$ *is a-proper, the expression* $\zeta_{a,\Phi}^* \in \Phi\text{-}Rat(\Sigma, K)$ *and* $\left\|\zeta_{a,\Phi}^*\right\| = \|\zeta\|_{a,\Phi}^*$*.*

A tree series $S \in K\langle\langle T_\Sigma \rangle\rangle$ is called Φ-*rational over* Σ *and* K if there is a $\zeta \in \Phi\text{-}Rat(\Sigma, K)$ such that $S = \|\zeta\|$. The class of all Φ-rational tree series over Σ and K is denoted by $K^{\Phi-rat}\langle\langle T_\Sigma \rangle\rangle$.

Clearly, $K^{\Phi-rat}\langle\langle T_\Sigma \rangle\rangle$ is the smallest subclass of $K\langle\langle T_\Sigma \rangle\rangle$ which contains the polynomials and is closed under the Φ-rational tree series operations.

Next, we wish to show a Kleene theorem showing the coincidence of Φ-recognizable and Φ-rational tree series. For this, we shall need to extend the definition of a wbuta, so that the input trees may contain the states of the automaton as constants. Let $\mathcal{M} = (Q, wt, ter)$ be a wbuta over Σ and K and consider a tree $t \in T_\Sigma(Q)$ (without any loss we assume that $\Sigma \cap Q = \emptyset$). We set $dom_Q(t) = \{w \in dom(t) \mid t(w) \in Q\}$. Let $P \subseteq Q$. A *run of* $\mathcal{M}$ *over* t *using* P is a mapping $r_t : dom(t) \to Q$ such that

- $r_t(w) \in P$ for every $w \in dom(t) \setminus (dom_Q(t) \cup \{\varepsilon\})$

- $r_t(w) = t(w)$ for every $w \in dom_Q(t)$.

The set of all runs $r_t \in R_{\mathcal{M}}(t,q)$ using P will be denoted by $R^P_{\mathcal{M}}(t,q)$. Observe that whenever $P=Q$ we have $R^P_{\mathcal{M}}(t,q) = R_{\mathcal{M}}(t,q)$.

Let $w \in dom(t)$. The definition of the weight of r_t at w, is now extended as follows

$$wt(r_t, w) = \begin{cases} wt\left((r_t(w1), \dots, r_t(w.rk_\Sigma(t(w)))), t(w), r_t(w)\right) & \text{if } t(w) \in \Sigma_k, k \geq 0 \\ 1 & \text{if } t(w) \in Q. \end{cases}$$

A mapping $U : R_{\mathcal{M}}(t) \to \mathcal{P}(dom(t))$ is called a *node property of* t. Then, for every $r_t \in R_{\mathcal{M}}(t)$ the *U-decomposition* $dec_U(t, r_t) = (t', r_{t'}, (w_1, t_1, r_{t_1}), \dots, (w_m, t_m, r_{t_m}))$ *of t and* r_t is determined by the following statements:

- $\{w_1, \dots, w_m\} = \{w \in U(r_t) \setminus \{\varepsilon\} \mid$ for every $u \in dom(t) \setminus \{w, \varepsilon\}$ such that $u \leq w$ then $u \notin U(r_t)\}$, i.e., the w_i's are the topmost nodes of t, excluding the root, for which the node property $U(r_t)$ holds
- $t' = t[w_1 \leftarrow r_t(w_1), \dots, w_m \leftarrow r_t(w_m)]$ and for every $u \in dom(t')$, we put $r_{t'}(u) = r_t(u)$
- for every $1 \leq i \leq m$, $t_i = t|_{w_i}$ and for every $u \in dom(t_i)$ we let $r_{t_i}(u) = r_t(w_i u)$.

If we permit that a decomposition U can take place also at the root, i.e., $\{w_1, \dots, w_m\} = \{w \in U(r_t) \mid$ for every $u \in dom(t) \setminus \{w\}$ such that $u \leq w$ then $u \notin U(r_t)\}$ then we call U the *U, ε-decomposition of t and r_t*.

In order to simplify our notations, for every $t \in T_\Sigma(Q)$ and every run r_t of $\mathcal{M}$ over t, we define the *running weight $rweight(r_t)$ of r_t* by

$$rweight(r_t) = \prod_{w \in dom(t)} \Phi^t_w(wt(r_t, w)).$$

Then

$$weight(r_t) = rweight(r_t) \cdot ter(r_t(\varepsilon)).$$

Now assume that our tree automaton $\mathcal{M} = (Q, wt, q_f)$ is final weight normalized. Clearly for every $t \in T_\Sigma$ and every run r_t of $\mathcal{M}$ over t we have $rweight(r_t) = weight(r_t)$ whenever $r_t(\varepsilon) = q_f$. Let $t \in T_\Sigma(Q), P \subseteq Q, q \in Q$, and U be a node property of t. Moreover, let $r_t \in R^P_{\mathcal{M}}(t,q)$ and $dec_U(t, r_t) = (t', r_{t'}, (w_1, t_1, r_{t_1}), \dots, (w_m, t_m, r_{t_m}))$. Then

- $r_{t'} \in R^P_{\mathcal{M}}(t', q)$
- for every $1 \leq i \leq m$, $r_{t_i} \in R^P_{\mathcal{M}}(t_i, r_{t'}(w_i))$
- $t = t'[w_1 \leftarrow t_1, \dots, w_m \leftarrow t_m]$
- $rweight(r_t) = rweight(r_{t'}) \cdot \Phi^t_{w_1}(rweight(r_{t_1})) \cdot \ldots \cdot \Phi^t_{w_m}(rweight(r_{t_m}))$.

Clearly, if $t = \sigma(t_1, \dots, t_k)$ and $U(r_t) = \{1, \dots, k\}$, then $t' = \sigma(r_t(1), \dots, r_t(k))$ and $rweight(r_t) = wt((r_t(1), \dots, r_t(k)), \sigma, r_t(\varepsilon)) \cdot \prod_{i=1}^{k} \Phi^t_i(rweight(r_{t_i}))$. Now, for every $Q', P \subseteq Q$ and $q \in Q$ we define the tree series $\|\mathcal{M}\|(Q', P, q) \in K\langle\langle T_\Sigma(Q')\rangle\rangle$ by

$$(\|\mathcal{M}\|(Q', P, q), t) = \begin{cases} \sum\limits_{r_t \in R^P_{\mathcal{M}}(t,q)} rweight(r_t) & \text{if } t \in T_\Sigma(Q') \setminus Q' \\ 0 & \text{otherwise} \end{cases}$$

for every $t \in T_\Sigma(Q')$. Obviously, $\|\mathcal{M}\|(Q', P, q)$ is p-proper for every $p \in Q'$.

Lemma 27 *Let $\mathcal{M} = (Q, wt, ter)$ be a wbuta. Let $Q', P \subseteq Q, q \in Q$, and $p \in Q' \setminus P$. Then*

$$\|\mathcal{M}\|\left(Q', P \cup \{p\}, q\right) = \|\mathcal{M}\|\left(Q', P, q\right) \cdot_{p,\Phi} \|\mathcal{M}\|\left(Q', P, p\right)^*_{p,\Phi}$$

Proof. Let $t \in T_\Sigma(Q') \setminus Q'$. We shall prove our claim by induction on the structure of t. Therefore, we calculate

$$\begin{aligned}
&\left(\|\mathcal{M}\|\left(Q', P \cup \{p\}, q\right), t\right) \\
&= \sum_{r_t \in R^{P\cup\{p\}}_{\mathcal{M}}(t,q)} rweight\left(r_t\right) \\
&\overset{(*)}{=} \sum_{\substack{t = s\cdot_p(t_1,\ldots,t_k), s\neq p \\ t(w_1)=\ldots=t(w_k)=p \\ r_s \in R^{P}_{\mathcal{M}}(s,q) \\ r_{t_i}\in R^{P\cup\{p\}}_{\mathcal{M}}(t_i,p)}} rweight\left(r_s\right) \cdot \prod_{i=1}^{k} \Phi^t_{w_i}\left(rweight\left(r_{t_i}\right)\right) \\
&= \sum_{\substack{t = s\cdot_p(t_1,\ldots,t_k), s\neq p \\ t(w_1)=\ldots=t(w_k)=p}} \left(\sum_{r_s \in R^{P}_{\mathcal{M}}(s,q)} rweight\left(r_s\right)\right) \cdot \left(\prod_{i=1}^{k} \sum_{r_{t_i}\in R^{P\cup\{p\}}_{\mathcal{M}}(t_i,p)} \Phi^t_{w_i}\left(rweight\left(r_{t_i}\right)\right)\right) \\
&= \sum_{\substack{t = s\cdot_p(t_1,\ldots,t_k), s\neq p \\ t(w_1)=\ldots=t(w_k)=p}} \left(\|\mathcal{M}\|\left(Q', P, q\right), s\right) \cdot \left(\prod_{i=1}^{k} \sum_{r_{t_i}\in R^{P\cup\{p\}}_{\mathcal{M}}(t_i,p)} \Phi^t_{w_i}\left(rweight\left(r_{t_i}\right)\right)\right) \\
&= \sum_{\substack{t = s\cdot_p(t_1,\ldots,t_k), s\neq p \\ t(w_1)=\ldots=t(w_k)=p}} \left(\|\mathcal{M}\|\left(Q', P, q\right), s\right) \cdot \left(\prod_{i=1}^{k} \Phi^t_{w_i}\left(\sum_{r_{t_i}\in R^{P\cup\{p\}}_{\mathcal{M}}(t_i,p)} rweight\left(r_{t_i}\right)\right)\right) \\
&= \sum_{\substack{t = s\cdot_p(t_1,\ldots,t_k), s\neq p \\ t(w_1)=\ldots=t(w_k)=p}} \left(\|\mathcal{M}\|\left(Q', P, q\right), s\right) \cdot \prod_{i=1}^{k} \Phi^t_{w_i}\left(\left(\|\mathcal{M}\|\left(Q', P \cup \{p\}, p\right) + 1p, t_i\right)\right)
\end{aligned}$$

which by the induction hypothesis equals

$$\begin{aligned}
&\sum_{\substack{t = s\cdot_p(t_1,\ldots,t_k), s\neq p \\ t(w_1)=\ldots=t(w_k)=p}} \left(\|\mathcal{M}\|\left(Q', P, q\right), s\right) \cdot \prod_{i=1}^{k} \Phi^t_{w_i}\left(\left(\|\mathcal{M}\|\left(Q', P, p\right) \cdot_{p,\Phi} \|\mathcal{M}\|\left(Q', P, p\right)^*_{p,\Phi} + 1p, t_i\right)\right) \\
&= \sum_{\substack{t = s\cdot_p(t_1,\ldots,t_k), s\neq p \\ t(w_1)=\ldots=t(w_k)=p}} \left(\|\mathcal{M}\|\left(Q', P, q\right), s\right) \cdot \prod_{i=1}^{k} \Phi^t_{w_i}\left(\left(\|\mathcal{M}\|\left(Q', P, p\right)^*_{p,\Phi}, t_i\right)\right) \\
&= \left(\|\mathcal{M}\|\left(Q', P, q\right) \cdot_{p,\Phi} \|\mathcal{M}\|\left(Q', P, p\right)^*_{p,\Phi}, t\right).
\end{aligned}$$

We have considered the U-decomposition $dec_U(t, r_t)$ with $U(r_t) = \{w \in dom(t) \mid r_t(w) = p\}$. At $(*)$ (from left to right) we have used the fact that for every $r_s \in R^P_{\mathcal{M}}(s,q), r_{t_1} \in R^{P\cup\{p\}}_{\mathcal{M}}(t_1,p), \ldots, r_{t_k} \in R^{P\cup\{p\}}_{\mathcal{M}}(t_k,p)$ with $t = s \cdot_p (t_1,\ldots,t_k)$ and $s \neq p$, there is exactly one $r_t \in R^{P\cup\{p\}}_{\mathcal{M}}(t,q)$ such that $dec_U(t,r) = (s, r_s, (w_1, t_1, r_{t_1}), \ldots, (w_k, t_k, r_{t_k}))$ where $\{w_1,\ldots,w_k\} = dom_{\{p\}}(s)$. ■

We consider a universal state set Q_∞, such that the state set Q of every wbuta $\mathcal{M}$ is a subset of Q_∞. We set

$$K^{\Phi-rec}\langle\langle T_\Sigma(Q_\infty)\rangle\rangle = \bigcup_{Q \text{ finite}} K^{\Phi-rec}\langle\langle T_\Sigma(Q)\rangle\rangle$$

and

$$K^{\Phi-rat}\langle\langle T_\Sigma(Q_\infty)\rangle\rangle = \bigcup_{Q \text{ finite}} K^{\Phi-rat}\langle\langle T_\Sigma(Q)\rangle\rangle.$$

Now, we are ready to prove one half of our Kleene theorem.

Proposition 28 $K^{\Phi-rec}\langle\langle T_\Sigma(Q_\infty)\rangle\rangle \subseteq K^{\Phi-rat}\langle\langle T_\Sigma(Q_\infty)\rangle\rangle$.

Proof. First, we show that $K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle \subseteq K^{\Phi-rat}\langle\langle T_\Sigma(Q_\infty)\rangle\rangle$. Let $\mathcal{M} = (Q, wt, q_f)$ be a final weight normalized wbuta with $Q = \{q_1,\ldots,q_n\}$, and let $q_i = q_f$ for some $1 \le i \le n$. We shall prove that $\|\mathcal{M}\| \in K^{\Phi-rat}\langle\langle T_\Sigma(Q)\rangle\rangle$. For every $P \subseteq Q$ and $q \in Q$ we define the tree series $\|\mathcal{M}\|(P,q) = \|\mathcal{M}\|(Q,P,q) \in K\langle\langle T_\Sigma(Q)\rangle\rangle$. As it is already noticed, the tree series $\|\mathcal{M}\|(P,q)$ is p-proper for every $p \in Q$. Observe that

$$\|\mathcal{M}\| = (\ldots((\|\mathcal{M}\|(Q,q_f) \cdot_{\Phi,q_1} \mathbf{0}) \cdot_{\Phi,q_2} \mathbf{0})\ldots) \cdot_{\Phi,q_n} \mathbf{0}.$$

Thus it remains to prove that for every $P \subseteq Q$ and $q \in Q$, the tree series $\|\mathcal{M}\|(P,q) \in K^{\Phi-rat}\langle\langle T_\Sigma(Q)\rangle\rangle$. To this end, we apply induction on the number of elements of P. Let $P = \emptyset$. For every $k \ge 0, \sigma \in \Sigma_k$, and $q_1,\ldots,q_k \in Q$, we define the run $r^\sigma_{q_1,\ldots,q_k,q} : dom(\sigma(q_1,\ldots,q_k)) \to Q$ of $\mathcal{M}$ over $\sigma(q_1,\ldots,q_k)$ using $\emptyset$, such that $r^\sigma_{q_1,\ldots,q_k,q}(\varepsilon) = q$, and $r^\sigma_{q_1,\ldots,q_k,q}(i) = q_i$ for every $1 \le i \le k$. Then by definition we have

$$R^\emptyset_{\mathcal{M}}(t,q) = \begin{cases} \{r^\sigma_{q_1,\ldots,q_k,q}\} & \text{if } t = \sigma(q_1,\ldots,q_k) \text{ for some } k \ge 0, \sigma \in \Sigma_k, q_1,\ldots,q_k \in Q \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus $supp(\|\mathcal{M}\|(\emptyset,q)) \subseteq \Sigma(Q)$ where $\Sigma(Q) = \{\sigma(q_1,\ldots,q_k) \mid k \ge 0, \sigma \in \Sigma_k, q_1,\ldots,q_k \in Q\}$, i,e., $\|\mathcal{M}\|(\emptyset,q)$ is a polynomial, and hence a Φ-rational tree series.

For the induction step, we assume that, for every $q \in Q$ the tree series $\|\mathcal{M}\|(P,q)$ is Φ-rational over $\Sigma \cup Q$ and K. Let $p \in Q \setminus P$. Then, by Lemma 27 we conclude that $\|\mathcal{M}\|(P \cup \{p\}, q)$ is also Φ-rational over $\Sigma \cup Q$ and K which in turn implies that $\|\mathcal{M}\| \in K^{\Phi-rat}\langle\langle T_\Sigma(Q)\rangle\rangle$. Now, the definitions of the classes $K^{\Phi-rec}\langle\langle T_\Sigma(Q_\infty)\rangle\rangle$ and $K^{\Phi-rec}\langle\langle T_\Sigma(Q)\rangle\rangle$ for Q finite, conclude our proof. ■

In the sequel, we establish the inclusion $K^{\Phi-rat}\langle\langle T_\Sigma\rangle\rangle \subseteq K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$. For this, it suffices to show that the class $K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$ contains the tree series $1a$ (for every $a \in \Sigma_0$) and is closed under the Φ-rational operations.

Lemma 29 *For every $a \in \Sigma_0$, the tree series $1a \in K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$.*

Proof. We consider the wbuta $\mathcal{M} = (Q, wt, ter)$ over Σ and K with $Q = \{q\}$ and $ter(q) = 1$. For every $k \geq 0, \sigma \in \Sigma_k$, we let

$$wt\left(\left(\underbrace{q, \ldots, q}_{k \text{ times}}\right), \sigma, q\right) = \begin{cases} 1 & \text{if } \sigma = a \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\|\mathcal{M}\| = 1a$. ■

Lemma 30 *The class $K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$ is closed under Φ-top-concatenation.*

Proof. Let $\sigma \in \Sigma_k$ with $k \geq 1$ and $\mathcal{M}_i = (Q_i, wt_i, q_{f_i})$ $(1 \leq i \leq k)$ be final weight normalized wbuta such that $Q_i \cap Q_j = \varnothing$ whenever $i \neq j$. We construct the final weight normalized wbuta $\mathcal{M} = (Q, wt, q_f)$ with $Q = \{q_f\} \cup \left(\bigcup_{1\leq i\leq k} Q_i\right)$ where q_f is a new state. For every $m \geq 0, \gamma \in \Sigma_m, q_1, \ldots, q_m, q \in Q$, we set

$$wt((q_1, \ldots, q_m), \gamma, q) = \begin{cases} wt_i((q_1, \ldots, q_m), \gamma, q) & \text{if } q_1, \ldots, q_m, q \in Q_i \\ 1 & \text{if } m = k, \gamma = \sigma, q = q_f \text{ and } q_i = q_{f_i} \ (1 \leq i \leq k) \\ 0 & \text{otherwise.} \end{cases}$$

Then, for every $t = \sigma(t_1, \ldots, t_k) \in T_\Sigma$, the coefficient $(\|\mathcal{M}\|, t)$ equals

$$\begin{aligned}
&\sum_{r_t \in R_{\mathcal{M}}(t)} weight(r_t) \\
&= \sum_{r_t \in R_{\mathcal{M}}(t, q_f)} weight(r_t) = \sum_{r_t \in R_{\mathcal{M}}(t, q_f)} rweight(r_t) \\
&= \sum_{r_{t_1} \in R_{\mathcal{M}}(t_1, q_{f_1})} \cdots \sum_{r_{t_k} \in R_{\mathcal{M}}(t_k, q_{f_k})} \Phi_\sigma^1(rweight_{\mathcal{M}_1}(r_{t_1})) \cdot \ldots \cdot \Phi_\sigma^k(rweight_{\mathcal{M}_k}(r_{t_k})) \\
&= \prod_{1\leq i\leq k}\left(\sum_{r_{t_i} \in R_{\mathcal{M}}(t_i, q_{f_i})} \Phi_\sigma^i(rweight_{\mathcal{M}_i}(r_{t_i}))\right) = \prod_{1\leq i\leq k}\left(\Phi_\sigma^i\left(\sum_{r_{t_i} \in R_{\mathcal{M}}(t_i, q_{f_i})} rweight_{\mathcal{M}_i}(r_{t_i})\right)\right) \\
&= \Phi_\sigma^1((\|\mathcal{M}_1\|, t_1)) \cdot \ldots \cdot \Phi_\sigma^k((\|\mathcal{M}_k\|, t_k)) \\
&= (\sigma_\Phi(\|\mathcal{M}_1\|, \ldots, \|\mathcal{M}_k\|), t).
\end{aligned}$$

Clearly for every t with $t(\varepsilon) \neq \sigma$, we have $(\|\mathcal{M}\|, t) = 0$. We conclude $\|\mathcal{M}\| = \sigma_\Phi(\|\mathcal{M}_1\|, \ldots, \|\mathcal{M}_k\|)$, as required. ■

Lemma 31 *Let $S_1, S_2 \in K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$ and $a \in \Sigma_0$. Then the a, Φ-concatenation of S_2 and S_1 is a Φ-recognizable tree series, i.e., $S_2 \cdot_{a,\Phi} S_1 \in K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$.*

Proof. Let $\mathcal{M}_1 = (Q_1, wt_1, q_{f_1})$ and $\mathcal{M}_2 = (Q_2, wt_2, q_{f_2})$ be final weight normalized wbuta with $\|\mathcal{M}_1\| = S_1$ and $\|\mathcal{M}_2\| = S_2$, and let us assume that $Q_1 \cap Q_2 = \emptyset$. We consider the final

weight normalized wbuta $\mathcal{M} = (Q, wt, q_{f_2})$ with $Q = (Q_1 \cup Q_2) \setminus \{q_{f_1}\}$.
For every $k \geq 0, \sigma \in \Sigma_k, q_1, \ldots, q_k, q \in Q$ we set

$$wt\left(\left(q_1, \ldots, q_k\right), \sigma, q\right) = \begin{cases} wt_1\left(\left(q_1, \ldots, q_k\right), \sigma, q\right) & \text{if } q_1, \ldots, q_k, q \in Q_1 \\ wt_1\left(\left(q_1, \ldots, q_k\right), \sigma, q_{f_1}\right) \cdot wt_2\left(a, q\right) & \text{if } k \neq 0,\ q_1, \ldots, q_k \in Q_1, \text{and } q \in Q_2 \\ wt_2\left(\left(q_1, \ldots, q_k\right), \sigma, q\right) & \text{if } k \neq 0,\ q_1, \ldots, q_k, q \in Q_2 \\ wt_2\left(\sigma, q\right) + wt_1(\sigma, q_{f_1}) \cdot wt_2(a, q) & \text{if } k = 0, \sigma \neq \alpha, \text{ and } q \in Q_2 \\ wt_1(a, q_{f_1}) \cdot wt_2(a, q) & \text{if } k = 0, \sigma = a, \text{ and } q \in Q_2 \\ 0 & \text{otherwise.} \end{cases}$$

Let $t \in T_\Sigma, q \in Q$, and $r_t \in R_{\mathcal{M}}(t, q)$. For every $k \geq 0$ we set

$$V(r_t, k) = \{w \in dom(t) \mid t(w) \in \Sigma_k, r_t(w) \in Q_2, r_t(w1), \ldots, r_t(wk) \in Q_1\}$$

and

$$H(r_t) = \left\{ dec_{U,\varepsilon}(t, r_t) \mid U(r_t) = \bigcup_{k \geq 1} V(r_t, k) \cup V \text{ and } V \subseteq V(r_t, 0) \right\}.$$

The U, ε-decomposition here is needed for the case where $a \in supp(S_2)$. Let $t \in T_\Sigma$ and $r_t \in R_{\mathcal{M}}(t, q_{f_2})$ with $rweight_{\mathcal{M}}(r_t) \neq 0$. Then, the set $H(r_t)$ coincides with the set of all possible decompositions $(s, r_s, (w_1, t_1, r_{t_1}), \ldots, (w_m, t_m, r_{t_m}))$ of t and r_t where $r_s \in R_{\mathcal{M}_2}(s)$ and for every $1 \leq i \leq m$, $r'_{t_i} \in R_{\mathcal{M}_1}(t_i)$ with r'_{t_i} obtained from r_{t_i} by setting $r'_{t_i}(\varepsilon) = q_{f_1}$. It should be clear that the non-unique way for decomposing t and r_t is showing up because of Case 4 in the above definition of wt. Now, we have

$$\begin{aligned} rweight_{\mathcal{M}}(r_t) &= \prod_{w \in dom(t)} \Phi^t_w\left(wt\left(r_t, w\right)\right) \\ &= \prod_{\substack{w \in dom(t), r_t(w) \in Q_2 \\ rk_\Sigma(t(w)) = k > 0 \\ 1 \leq j \leq k, r_t(wj) \in Q_2}} \Phi^t_w\left(wt\left(r_t, w\right)\right) \cdot \prod_{\substack{w \in dom(t), r_t(w) \in Q_2 \\ rk_\Sigma(t(w)) = k > 0 \\ 1 \leq j \leq k, r_t(wj) \in Q_1}} \Phi^t_w\left(wt(r_t, w)\right) \\ &\quad \cdot \prod_{\substack{w \in dom(t) \\ r_t(w) \in Q_1}} \Phi^t_w\left(wt\left(r_t, w\right)\right) \cdot \prod_{\substack{w \in lv(t) \\ r_t(w) \in Q_2}} \Phi^t_w\left(wt\left(r_t, w\right)\right) \\ &= \prod_{\substack{w \in dom(t), r_t(w) \in Q_2 \\ rk_\Sigma(t(w)) = k > 0 \\ 1 \leq j \leq k, r_t(wj) \in Q_2}} \Phi^t_w\left(wt\left(r_t, w\right)\right) \cdot \prod_{\substack{w \in dom(t), r_t(w) \in Q_2 \\ rk_\Sigma(t(w)) = k > 0 \\ 1 \leq j \leq k, r_t(wj) \in Q_1}} \Phi^t_w\left(wt(r_t, w)\right) \\ &\quad \cdot \prod_{\substack{w \in dom(t) \\ r_t(w) \in Q_1}} \Phi^t_w\left(wt\left(r_t, w\right)\right) \cdot \prod_{\substack{w \in lv(t), t(w) = a \\ r_t(w) \in Q_2}} \Phi^t_w\left(wt\left(r_t, w\right)\right) \\ &\quad \cdot \prod_{\substack{w \in lv(t), t(w) \in \Sigma_0 \setminus \{a\} \\ r_t(w) \in Q_2}} \Phi^t_w\left(wt_2(t(w), r_t(w)) + wt_1\left(t\left(w\right), q_{f_1}\right) \cdot wt_2(a, r_t(w))\right) \end{aligned}$$

The definitions of the weight assigning mapping wt of $\mathcal{M}$ and the set $H(r_t)$, the distributivity of K, and the latter product of the last line above imply

$$\begin{aligned}
rweight_{\mathcal{M}}(r_t) = & \sum_{(s,r_s,(w_1,t_1,r_{t_1}),\ldots,(w_m,t_m,r_{t_m}))\in H(r_t)} \prod_{w\in dom(s)} \Phi^t_w(wt(r_s,w)) \\
& \cdot \prod_{1\leq i\leq m} \prod_{u\in dom(t_i)} \Phi^t_{w_i u}(wt(r'_{t_i},u)) \\
= & \sum_{(s,r_s,(w_1,t_1,r_{t_1}),\ldots,(w_m,t_m,r_{t_m}))\in H(r_t)} \prod_{w\in dom(s)} \Phi^s_w(wt_2(r_s,w)) \\
& \cdot \prod_{1\leq i\leq m} \Phi^t_{w_i} \left(\prod_{u\in dom(t_i)} \Phi^{t_i}_u(wt_1(r'_{t_i},u)) \right) \\
= & \sum_{(s,r_s,(w_1,t_1,r_{t_1}),\ldots,(w_m,t_m,r_{t_m}))\in H(r_t)} rweight_{\mathcal{M}_2}(r_s) \cdot \prod_{1\leq i\leq m} \Phi^t_{w_i}\left(rweight_{\mathcal{M}_1}(r'_{t_i})\right).
\end{aligned}$$

Let now $t \in T_\Sigma$ and $r_t \in R_{\mathcal{M}}(t)$. Our latter proved statement implies that whenever $rweight_{\mathcal{M}}(r_t) \neq 0$ there exists at least one decomposition $(s, r_s, (w_1, t_1, r_{t_1}), \ldots, (w_m, t_m, r_{t_m})) \in H(r_t)$ such that $rweight_{\mathcal{M}_2}(r_s) \neq 0$ and $rweight_{\mathcal{M}_1}(r'_{t_i}) \neq 0$ for every $1 \leq i \leq m$. Therefore, we compute

$$\begin{aligned}
(\|\mathcal{M}\|, t) = & \sum_{r_t\in R_{\mathcal{M}}(t)} weight_{\mathcal{M}}(r_t) = \sum_{r_t\in R_{\mathcal{M}}(t,q_{f_2})} rweight_{\mathcal{M}}(r_t) \\
= & \sum_{r_t\in R_{\mathcal{M}}(t,q_{f_2})} \sum_{(s,r_s,(w_1,t_1,r_{t_1}),\ldots,(w_m,t_m,r_{t_m}))\in H(r_t)} rweight_{\mathcal{M}_2}(r_s) \cdot \prod_{1\leq i\leq m} \Phi^t_{w_i}\left(rweight_{\mathcal{M}_1}(r'_{t_i})\right) \\
= & \sum_{\substack{s,t_1,\ldots,t_m\in T_\Sigma \\ t=s\cdot_a(t_1,\ldots,t_m) \\ s(w_1)=\ldots=s(w_m)=a}} \sum_{r_s\in R_{\mathcal{M}_2}(s,q_{f_2})} \sum_{r'_{t_1}\in R_{\mathcal{M}_1}(t_1,q_{f_1})} \cdots \sum_{r'_{t_m}\in R_{\mathcal{M}_1}(t_m,q_{f_1})} rweight_{\mathcal{M}_2}(r_s) \\
& \cdot \prod_{1\leq i\leq m} \Phi^t_{w_i}\left(rweight_{\mathcal{M}_1}(r'_{t_i})\right)
\end{aligned}$$

which by distributivity equals

$$\begin{aligned}
& \sum_{\substack{s,t_1,\ldots,t_m\in T_\Sigma \\ t=s\cdot_a(t_1,\ldots,t_m) \\ s(w_1)=\ldots=s(w_m)=a}} \left(\sum_{r_s\in R_{\mathcal{M}_2}(s,q_{f_2})} rweight_{\mathcal{M}_2}(r_s) \right) \cdot \prod_{1\leq i\leq m} \Phi^t_{w_i} \left(\sum_{r'_{t_i}\in R_{\mathcal{M}_1}(t_i,q_{f_1})} rweight_{\mathcal{M}_1}(r'_{t_i}) \right) \\
= & \sum_{\substack{s,t_1,\ldots,t_m\in T_\Sigma \\ t=s\cdot_a(t_1,\ldots,t_m) \\ s(w_1)=\ldots=s(w_m)=a}} \left(\sum_{r_s\in R_{\mathcal{M}_2}(s)} weight_{\mathcal{M}_2}(r_s) \right) \cdot \prod_{1\leq i\leq m} \Phi^t_{w_i} \left(\sum_{r'_{t_i}\in R_{\mathcal{M}_1}(t_i)} weight_{\mathcal{M}_1}(r'_{t_i}) \right) \\
= & \sum_{\substack{s,t_1,\ldots,t_m\in T_\Sigma \\ t=s\cdot_a(t_1,\ldots,t_m) \\ s(w_1)=\ldots=s(w_m)=a}} (\|\mathcal{M}_2\|, s) \cdot \prod_{1\leq i\leq m} \Phi^t_{w_i}\left((\|\mathcal{M}_1\|, t_i)\right).
\end{aligned}$$

We conclude

$$\|\mathcal{M}\| = \|\mathcal{M}_2\| \cdot_{a,\Phi} \|\mathcal{M}_1\|$$

as required. ■

Lemma 32 *Let $a \in \Sigma_0$ and $\mathcal{M} = (Q, wt, q_f)$ be a final weight normalized wbuta such that $\|\mathcal{M}\| \in K^a \langle\langle T_\Sigma \rangle\rangle$. Then $\|\mathcal{M}\|^*_{a,\Phi} \in K^{\Phi-rec} \langle\langle T_\Sigma \rangle\rangle$.*

Proof. By Lemma 10 we can assume that $\mathcal{M} = (Q, wt, q_f)$ is initial a-state-normalized. Let $I_\alpha = \{q_a\}$ with $wt(a, q_a) = 1$. We construct the wbuta $\mathcal{M}^* = (Q^*, wt^*, ter^*)$ with $Q^* = Q \setminus \{q_f\}$ and $ter^*(q) = 1$ if $q = q_a$ and $ter^*(q) = 0$ otherwise. The weight assigning mapping is given for every $k \geq 0, \sigma \in \Sigma_k, q_1, \ldots, q_k, q \in Q$ by

$$wt^*((q_1, \ldots q_k), \sigma, q) = \begin{cases} wt((q_1, \ldots q_k), \sigma, q) & \text{if } \sigma \neq a \text{ and } q \neq q_a \\ wt((q_1, \ldots q_k), \sigma, q_f) & \text{if } \sigma \neq a \text{ and } q = q_a \\ wt(a, q_a) & \text{if } k = 0, \sigma = a \text{ and } q = q_a \\ 0 & \text{otherwise.} \end{cases}$$

We consider also the wbuta $\overline{\mathcal{M}^*}$ obtained from $\mathcal{M}^*$ by first removing the constant a from the input alphabet Σ and then identifying the state q_a with a. Thus, $\|\mathcal{M}^*\| = \|\overline{\mathcal{M}^*}\| (\{q_a\}, Q^*, q_a) + 1q_a$. Now it is easily seen that

$$\|\mathcal{M}\| = \|\overline{\mathcal{M}^*}\| (\{q_a\}, Q^* \setminus \{q_a\}, q_a)$$

and by Lemma 27 we get

$$\begin{aligned} \|\overline{\mathcal{M}^*}\| (\{q_a\}, Q^*, q_a) &= \|\overline{\mathcal{M}^*}\| (\{q_a\}, Q^* \setminus \{q_a\}, q_a) \cdot_{q_a,\Phi} \|\overline{\mathcal{M}^*}\| (\{q_a\}, Q^* \setminus \{q_a\}, q_a)^*_{q_a,\Phi} \\ &= \|\mathcal{M}\| \cdot_{q_a,\Phi} \|\mathcal{M}\|^*_{q_a,\Phi}. \end{aligned}$$

Therefore, by Lemma 24 we conclude

$$\|\mathcal{M}^*\| = \|\mathcal{M}\| \cdot_{q_a,\Phi} \|\mathcal{M}\|^*_{q_a,\Phi} + 1q_a = \|\mathcal{M}\|^*_{q_a,\Phi} = \|\mathcal{M}\|^*_{a,\Phi}.$$

■

Theorem 33 *Let Σ be a ranked alphabet, K be a commutative semiring, and Φ be a discounting over Σ and K. Then*

(i) $K^{\Phi-rec} \langle\langle T_\Sigma \rangle\rangle$ is closed under the Φ-rational tree series operations.

(ii) $K^{\Phi-rat} \langle\langle T_\Sigma \rangle\rangle \subseteq K^{\Phi-rec} \langle\langle T_\Sigma \rangle\rangle$.

Proof. We induce statement (i) by Proposition 11 and Lemmas 30, 31, and 32. For statement (ii) we use statement (i), Lemma 29, and the fact that $K^{\Phi-rat} \langle\langle T_\Sigma \rangle\rangle$ is the smallest set of tree series over Σ and K which contains the series $1a$ for every $a \in \Sigma_0$, and is closed under the Φ-rational operations. ■

Now we are ready to state our first main result, namely the Kleene theorem for Φ-recognizable tree series.

Theorem 34 (Kleene theorem) *Let Σ be a ranked alphabet, K be a commutative semiring, and Φ be a discounting over Σ and K. Then*

$$K^{\Phi-rec}\langle\langle T_\Sigma(Q_\infty)\rangle\rangle = K^{\Phi-rat}\langle\langle T_\Sigma(Q_\infty)\rangle\rangle\,.$$

Proof. One half of our claim is obtained by Proposition 28. For the converse inclusion observe that for a finite set Q, we have $T_\Sigma(Q) = T_{\Sigma\cup Q}$ and by the previous theorem $K^{\Phi-rat}\langle\langle T_\Sigma(Q)\rangle\rangle \subseteq K^{\Phi-rec}\langle\langle T_\Sigma(Q)\rangle\rangle$. ■

5 Weighted MSO logic over finite trees

In this section, we introduce a weighted monadic second order logic (abbreviated to weighted MSO logic) with Φ-discounting over finite trees, and characterize the class $K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$ of Φ-recognizable tree series over Σ and K in terms of this logic. The syntax of our MSO-formulas is the one used in [24] but here we exclude second order universal quantifiers since we do not need them for the logical description of our automata. For the semantics of our MSO-formulas, we employ the Φ-discounting. Let us first recall some basic terminology and definitions from [24].

Let $\mathcal{V}$ be a finite set of first and second order variables. A tree $t \in T_\Sigma$ is represented by the relational structure $(dom(t), edge_1, \ldots, edge_{\deg(\Sigma)}, (R^t_\sigma)_{\sigma\in\Sigma})$ where $R^t_\sigma = \{w \in dom(t) \mid t(w) = \sigma\}$ for $\sigma \in \Sigma$, and for every $w, u \in dom(t)$, $j \in \{1, \ldots, \deg(\Sigma)\}$, $edge_j(w, u)$ holds true iff $u = wj$. A $(t, \mathcal{V})$-*assignment* ρ is a mapping assigning elements of $dom(t)$ to first order variables from $\mathcal{V}$, and subsets of $dom(t)$ to second order variables from $\mathcal{V}$. Let x be a first order variable and $w \in dom(t)$. Then $\rho[x \to w]$ denotes the $(t, \mathcal{V} \cup \{x\})$-assignment which associates w to x and acts as ρ on $\mathcal{V} \setminus \{x\}$. The notation $\rho[X \to I]$ for a second order variable X and a set $I \subseteq dom(t)$ has a similar meaning.

Now, we consider the ranked alphabet $\Sigma_\mathcal{V} = \Sigma \times \{0,1\}^\mathcal{V}$ with $rk_{\Sigma_\mathcal{V}}(\sigma, f) = rk_\Sigma(\sigma)$ for every $\sigma \in \Sigma$ and $f \in \{0,1\}^\mathcal{V}$. For every $(\sigma, f) \in \Sigma_\mathcal{V}$ we denote by $(\sigma, f)_1$ and $(\sigma, f)_2$ the symbols σ and f, respectively. A tree $s \in T_{\Sigma_\mathcal{V}}$ is called *valid* if for every first order variable $x \in \mathcal{V}$, there is exactly one node w of s such that $(s(w)_2)(x) = 1$. The set of all valid finite trees over $\Sigma_\mathcal{V}$ is denoted by $T^v_{\Sigma_\mathcal{V}}$. Every valid tree $s \in T_{\Sigma_\mathcal{V}}$ corresponds to a pair (t, ρ) where $t \in T_\Sigma$ and ρ is a $(t, \mathcal{V})$-assignment, in the following way. It holds $dom(t) = dom(s)$ and $t(w) = s(w)_1$ for every $w \in dom(s)$, and for every first order variable x, second order variable X, and every node $w \in dom(s)$, we have that $\rho(x) = w$ iff $(s(w)_2)(x) = 1$, and $w \in \rho(X)$ iff $(s(w)_2)(X) = 1$. Then, we say that s and (t, ρ) *correspond to each other*. In the following, we identify every valid tree s with its corresponding pair (t, ρ).

Lemma 35 *[24] The tree language $T^v_{\Sigma_\mathcal{V}}$ is recognizable.*

Corollary 36 *The characteristic series*

$$1_{T^v_{\Sigma_\mathcal{V}}} : T^v_{\Sigma_\mathcal{V}} \to K$$

is Φ-recognizable.

Now let φ be an MSO-formula over trees [51, 52] with $Free(\varphi) \subseteq \mathcal{V}$. As usual we shall write Σ_φ for $\Sigma_{Free(\varphi)}$. The well-known result of Thatcher and Wright [50], and Doner [17] states that the tree language

$$\mathcal{L}_\mathcal{V}(\varphi) = \{(t, \rho) \in T^v_{\Sigma_\mathcal{V}} \mid (t, \rho) \models \varphi\}$$

is recognizable; conversely, for every recognizable tree language $L \subseteq T_\Sigma$ there exists an MSO-sentence φ, such that $L = \mathcal{L}(\varphi)$ where $\mathcal{L}(\varphi) = \mathcal{L}_{Free(\varphi)}(\varphi)$.

Next we introduce our weighted MSO logic with Φ-discounting over trees. For this we need to extend our Φ-discounting over Σ and K to a discounting over $\Sigma_\mathcal{V}$ and K, for every finite set of first and second order variables $\mathcal{V}$. For simplicity we shall use the same symbol Φ. More precisely, for every $(\sigma, f) \in \Sigma_\mathcal{V}$ we set $\Phi_{(\sigma,f)} = \Phi_\sigma$.

Definition 37 *The set $MSO(K, \Sigma)$ of all formulas of the weighted MSO logic with Φ-discounting over Σ and K on finite trees is defined to be the smallest set F such that*

- *F contains all atomic formulas $k, label_\sigma(x), edge_i(x, y), x \in X$ and the negations $\neg label_\sigma(x)$, $\neg edge_i(x, y), \neg(x \in X)$, and*
- *if $\varphi, \psi \in F$, then also $\varphi \vee \psi, \varphi \wedge \psi, \exists x \,.\, \varphi, \exists X \,.\, \varphi, \forall x \,.\, \varphi \in F$*

where $k \in K$, $\sigma \in \Sigma$, $1 \leq i \leq \deg(\Sigma)$, x, y are first order variables, and X is a second order variable.

Next we represent the semantics of the formulas in $MSO(K, \Sigma)$ as tree series over the extended alphabet $\Sigma_\mathcal{V}$ and the semiring K. As in the word case [22], we employ the Φ-discounting only in the semantics of first order universal quantifications.

Definition 38 *Let $\varphi \in MSO(K, \Sigma)$ and $\mathcal{V}$ be a finite set of variables with $Free(\varphi) \subseteq \mathcal{V}$. The* Φ-semantics *of φ is a tree series $\|\varphi\|_\mathcal{V} \in K\langle\langle T_{\Sigma_\mathcal{V}}\rangle\rangle$. Let $(t, \rho) \in T_{\Sigma_\mathcal{V}}$. If (t, ρ) is not a valid tree, then we set $(\|\varphi\|_\mathcal{V}, (t, \rho)) = 0$. Otherwise, we inductively define $(\|\varphi\|_\mathcal{V}, (t, \rho)) \in K$ as follows:*

- $(\|k\|_\mathcal{V}, (t, \rho)) = k$
- $(\|label_\sigma(x)\|_\mathcal{V}, (t, \rho)) = \begin{cases} 1 & if \;\; t(\rho(x)) = \sigma \\ 0 & otherwise \end{cases}$
- $(\|edge_i(x, y)\|_\mathcal{V}, (t, \rho)) = \begin{cases} 1 & if \;\; \rho(y) = \rho(x)i \\ 0 & otherwise \end{cases}$
- $(\|x \in X\|_\mathcal{V}, (t, \rho)) = \begin{cases} 1 & if \;\; \rho(x) \in \rho(X) \\ 0 & otherwise \end{cases}$
- $(\|\neg\varphi\|_\mathcal{V}, (t, \rho)) = \begin{cases} 1 & if \;\; (\|\varphi\|_\mathcal{V}, (t, \rho)) = 0 \\ 0 & if \;\; (\|\varphi\|_\mathcal{V}, (t, \rho)) = 1 \end{cases}$, *provided that φ is of the form $label_\sigma(x)$, $edge_i(x, y)$, or $x \in X$*
- $(\|\varphi \vee \psi\|_\mathcal{V}, (t, \rho)) = (\|\varphi\|_\mathcal{V}, (t, \rho)) + (\|\psi\|_\mathcal{V}, (t, \rho))$
- $(\|\varphi \wedge \psi\|_\mathcal{V}, (t, \rho)) = (\|\varphi\|_\mathcal{V}, (t, \rho)) \cdot (\|\psi\|_\mathcal{V}, (t, \rho))$

- $(\|\exists x \centerdot \varphi\|_{\mathcal{V}}, (t,\rho)) = \sum_{w \in dom(t)} \left(\|\varphi\|_{\mathcal{V}\cup\{x\}}, (t, \rho[x \to w]) \right)$

- $(\|\exists X \centerdot \varphi\|_{\mathcal{V}}, (t,\rho)) = \sum_{I \subseteq dom(t)} \left(\|\varphi\|_{\mathcal{V}\cup\{X\}}, (t, \rho[X \to I]) \right)$

- $(\|\forall x \centerdot \varphi\|_{\mathcal{V}}, (t,\rho)) = \prod_{w \in dom(t)} \Phi_w^t \left(\left(\|\varphi\|_{\mathcal{V}\cup\{x\}}, (t, \rho[x \to w]) \right) \right)$.

We shall simply write $\|\varphi\|$ for $\|\varphi\|_{Free(\varphi)}$. If φ has no free variables, i.e., if it is a sentence, then $\|\varphi\| \in K\langle\langle T_\Sigma \rangle\rangle$. One should observe that the Φ-semantics $\|\varphi\|_{\mathcal{V}}$ of every formula $\varphi \in MSO(K,\Sigma)$ is defined according to a finite set of variables $\mathcal{V}$ containing $Free(\varphi)$. Actually, this is not an essential restriction. In the subsequent proposition, we show that $\|\varphi\|_{\mathcal{V}}$ depends only on $Free(\varphi)$.

Proposition 39 *Let $\varphi \in MSO(K,\Sigma)$ and $\mathcal{V}$ be a finite set of variables such that $Free(\varphi) \subseteq \mathcal{V}$. Then*

$$(\|\varphi\|_{\mathcal{V}}, (t,\rho)) = \left(\|\varphi\|, (t, \rho|_{Free(\varphi)})\right)$$

for every $(t,\rho) \in T^v_{\Sigma_{\mathcal{V}}}$. Moreover, the tree series $\|\varphi\|$ is Φ-recognizable (resp. a recognizable step function) iff $\|\varphi\|_{\mathcal{V}}$ is a Φ-recognizable tree series (resp. a recognizable step function).

Proof. We state the proof of the equality by induction on the structure of φ. For atomic formulas our claim holds by definition. For disjunctions and conjunctions it follows directly by induction. Let now $\varphi = \exists x.\psi$. If ρ is a valid $(t,\mathcal{V})$-assignment, then $\rho[x \to w]$ is a valid $(t, \mathcal{V}\cup\{x\})$-assignment for every $w \in dom(t)$. Since $Free(\psi) \subseteq \mathcal{V}\cup\{x\}$, we get by induction

$$\left(\|\psi\|_{\mathcal{V}\cup\{x\}}, (t, \rho[x \to w]) \right) = \left(\|\psi\|, \left(t, \rho[x \to w]|_{Free(\psi)}\right)\right).$$

Now, $\rho|_{Free(\varphi)}[x \to w]$ is a valid $(t, Free(\varphi) \cup \{x\})$-assignment for every $w \in dom(t)$. Since $Free(\psi) \subseteq Free(\varphi) \cup \{x\}$, we get by induction

$$\left(\|\psi\|_{Free(\varphi)\cup\{x\}}, \left(t, \rho|_{Free(\varphi)}[x \to w]\right) \right) = \left(\|\psi\|, \left(t, \rho[x \to w]|_{Free(\psi)}\right)\right)$$

and thus

$$\left(\|\psi\|_{\mathcal{V}\cup\{x\}}, (t, \rho[x \to w]) \right) = \left(\|\psi\|_{Free(\varphi)\cup\{x\}}, \left(t, \rho|_{Free(\varphi)}[x \to w]\right)\right).$$

Therefore

$$\begin{aligned}
(\|\varphi\|_{\mathcal{V}}, (t,\rho)) &= \sum_{w \in dom(t)} \left(\|\psi\|_{\mathcal{V}\cup\{x\}}, (t, \rho[x \to w]) \right) \\
&= \sum_{w \in dom(t)} \left(\|\psi\|_{Free(\varphi)\cup\{x\}}, \left(t, \rho|_{Free(\varphi)}[x \to w]\right) \right) \\
&= \left(\|\varphi\|, (t, \rho|_{Free(\varphi)}) \right).
\end{aligned}$$

The other cases of quantifications are proved in a similar way. Next, we verify our second claim. Let us first assume that $\|\varphi\|$ is Φ-recognizable (resp. a recognizable step function), and consider the projection $h : \Sigma_{\mathcal{V}} \to \Sigma_\varphi$ given by $h((t,\rho)) = (t, \rho|_{Free(\varphi)})$. Obviously, h is a relabeling and it holds that

$$\|\varphi\|_{\mathcal{V}} = h^{-1}(\|\varphi\|) \odot 1_{T^v_{\Sigma_{\mathcal{V}}}}.$$

Then, by Proposition 11 (resp. Proposition 12) and Corollary 36, the series $\|\varphi\|_{\mathcal{V}}$ is Φ-recognizable (resp. a recognizable step function).

Conversely, let $R \subseteq T^v_{\Sigma_{\mathcal{V}}}$ be the tree language

$$R = \left\{(t,\rho) \in T^v_{\Sigma_{\mathcal{V}}} \mid \forall x \in \mathcal{V} \setminus Free(\varphi), \forall X \in \mathcal{V} \setminus Free(\varphi),\ \rho(x) = \varepsilon \text{ and } \rho(X) = \{\varepsilon\}\right\}.$$

By standard constructions on tree automata, we can prove that R is recognizable which in turn implies that its characteristic series 1_R is Φ-recognizable. Then

$$\|\varphi\| = h\left(\|\varphi\|_{\mathcal{V}} \odot 1_R\right).$$

and if $\|\varphi\|_{\mathcal{V}}$ is Φ-recognizable, then by Proposition 11 the tree series $\|\varphi\| \in K\langle\langle T_{\Sigma_\varphi}\rangle\rangle$ is also Φ-recognizable.

Finally, observe that $\|\varphi\|$ takes on the same non-zero values as $\|\varphi\|_{\mathcal{V}}$. Furthermore, for every $k \in K \backslash \{0\}$

$$\|\varphi\|^{-1}(k) = h(\|\varphi\|_{\mathcal{V}}^{-1}(k))$$

which by Proposition 12 implies that if $\|\varphi\|_{\mathcal{V}}$ is a recognizable step function, then also $\|\varphi\|$ is a recognizable step function. ■

Definition 40 *A formula $\varphi \in MSO(K,\Sigma)$ is called* restricted *if whenever φ contains a universal first order quantification $\forall x \,.\, \psi$, then $\|\psi\|$ is a recognizable step function.*

Definition 41 *A formula $\varphi \in MSO(K,\Sigma)$ is called* almost existential *if φ contains a universal first order quantification $\forall x \,.\, \psi$ and ψ contains a universal first order quantification $\forall y \,.\, \psi'$, then ψ' is composed from conjunctions of negations of atomic formulas of the form $edge_i(z,z')$, where $1 \leq i \leq \deg(\Sigma)$.*

We let $RMSO\,(K,\Sigma)$ comprise all restricted formulas of $MSO\,(K,\Sigma)$. Furthermore, let $REMSO\,(K,\Sigma)$ contain all restricted existential $MSO\,(K,\Sigma)$-formulas, i.e., formulas of the form $\exists X_1,\ldots,X_n \,.\, \psi$ with $\psi \in RMSO\,(K,\Sigma)$ containing no set quantification. We shall denote by $AEMSO\,(K,\Sigma)$ the collection of all almost existential formulas of $MSO\,(K,\Sigma)$. A tree series $S \in K\,\langle\langle T_\Sigma\rangle\rangle$ is called *RMSO-Φ-definable* (resp. *REMSO-Φ-definable*, *AEMSO-Φ-definable*) if there is a sentence $\varphi \in RMSO\,(K,\Sigma)$ (resp. $\varphi \in REMSO\,(K,\Sigma)$, $\varphi \in AEMSO\,(K,\Sigma)$) such that $S = \|\varphi\|$. We let $K^{\Phi-rmso}\,\langle\langle T_\Sigma\rangle\rangle$ (resp. $K^{\Phi-remso}\,\langle\langle T_\Sigma\rangle\rangle$, $K^{\Phi-aemso}\,\langle\langle T_\Sigma\rangle\rangle$) comprise all *RMSO*-$\Phi$-definable (resp. *REMSO*-Φ-definable, *AEMSO*-Φ-definable) tree series over Σ and K.

In the sequel, we investigate the relation between the class $K^{\Phi-rec}\,\langle\langle T_\Sigma\rangle\rangle$ of Φ-recognizable tree series and the classes $K^{\Phi-rmso}\,\langle\langle T_\Sigma\rangle\rangle$, $K^{\Phi-remso}\,\langle\langle T_\Sigma\rangle\rangle$, and $K^{\Phi-aemso}\,\langle\langle T_\Sigma\rangle\rangle$. More precisely, we shall show the subsequent theorem which is our second main result.

Theorem 42 *Let Σ be a ranked alphabet and K be a commutative semiring. Then*

(i) $K^{\Phi-rec}\,\langle\langle T_\Sigma\rangle\rangle = K^{\Phi-rmso}\,\langle\langle T_\Sigma\rangle\rangle = K^{\Phi-remso}\,\langle\langle T_\Sigma\rangle\rangle$

(ii) if K is additively locally finite, then $K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle = K^{\Phi-aemso}\langle\langle T_\Sigma\rangle\rangle$.

First, we prove by induction on the structure of formulas φ that $K^{\Phi-rmso}\langle\langle T_\Sigma\rangle\rangle \subseteq K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$, and whenever K is additively locally finite, then $K^{\Phi-aemso}\langle\langle T_\Sigma\rangle\rangle \subseteq K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$. We denote by $\mathcal{V}$ a finite set of first and second order variables.

Lemma 43 *Let $\varphi \in MSO(K,\Sigma)$ be atomic or the negation of an atomic $MSO(K,\Sigma)$-formula. Then $\|\varphi\|$ is a recognizable step function.*

Proof. Assume that $Free(\varphi) \subseteq \mathcal{V}$. Let $\varphi = k$ for some $k \in K$. Then $\|\varphi\| = k1_{T_\Sigma^v}$ which is a recognizable step function. Now, let φ be one of the other atomic formulas or their negations. Clearly, we may consider φ as a classical MSO-formula. Then its tree language $\mathcal{L}(\varphi)$ is recognizable, and so the tree series $\|\varphi\| = 1_{\mathcal{L}(\varphi)}$ is a recognizable step function. ■

Lemma 44 *Let $\varphi, \psi \in MSO(K,\Sigma)$ such that $\|\varphi\|$ and $\|\psi\|$ are Φ-recognizable (resp. recognizable step functions). Then $\|\varphi \vee \psi\|$ and $\|\varphi \wedge \psi\|$ are also Φ-recognizable (resp. recognizable step functions).*

Proof. We let $\mathcal{V} = Free(\varphi) \cup Free(\psi)$. By definition, we have $\|\varphi \vee \psi\| = \|\varphi\|_{\mathcal{V}} + \|\psi\|_{\mathcal{V}}$ and $\|\varphi \wedge \psi\| = \|\varphi\|_{\mathcal{V}} \odot \|\psi\|_{\mathcal{V}}$. Then our claim for Φ-recognizable series (resp. recognizable step functions) follows from Propositions 11 (resp. 12) and 39. ■

Lemma 45 *Let $\varphi \in MSO(K,\Sigma)$.*

(i) If $\|\varphi\|$ is a Φ-recognizable tree series, then $\|\exists x \centerdot \varphi\|$ and $\|\exists X \centerdot \varphi\|$ are also Φ-recognizable.

(ii) Let K be additively locally finite and $\|\varphi\|$ be a recognizable step function. Then $\|\exists x \centerdot \varphi\|$ and $\|\exists X \centerdot \varphi\|$ are recognizable step functions.

Proof. We verify the case $\exists x \centerdot \varphi$. The other one is established in a similar way. Let $\mathcal{V} = Free(\exists x \centerdot \varphi)$; then $x \notin \mathcal{V}$. We consider the relabeling

$$h : T_{\Sigma_{\mathcal{V}\cup\{x\}}} \to T_{\Sigma_{\mathcal{V}}}$$

erasing only the x-component. Then, for every $(t,\rho) \in T_{\Sigma_{\mathcal{V}}}$, we have that

$$(\|\exists x \centerdot \varphi\|_{\mathcal{V}}, (t,\rho)) = \sum_{w \in dom(t)} \left(\|\varphi\|_{\mathcal{V}\cup\{x\}}, (t, \rho[x \to w])\right) = \left(h\left(\|\varphi\|_{\mathcal{V}\cup\{x\}}\right), (t,\rho)\right)$$

i.e.,

$$\|\exists x \centerdot \varphi\|_{\mathcal{V}} = h\left(\|\varphi\|_{\mathcal{V}\cup\{x\}}\right).$$

Since $Free(\varphi) \subseteq \mathcal{V} \cup \{x\}$, if $\|\varphi\|$ is Φ-recognizable (resp. a recognizable step function), then by Proposition 39 $\|\varphi\|_{\mathcal{V}\cup\{x\}}$ is also Φ-recognizable (resp. a recognizable step function). Now, by Proposition 11 (resp. Proposition 15) we get that $\|\exists x \centerdot \varphi\|_{\mathcal{V}}$ is Φ-recognizable (resp. a recognizable step function). Finally, by Proposition 39 we conclude that the tree series $\|\exists x \centerdot \varphi\|$ is also Φ-recognizable (resp. a recognizable step function). ■

Lemma 46 *Let $\varphi \in MSO(K,\Sigma)$ such that $\|\varphi\|$ is a recognizable step function. Then the tree series $\|\forall x \,.\, \varphi\|$ is Φ-recognizable.*

Proof. Let $\mathcal{W} = Free\,(\varphi)$ and $\mathcal{V} = Free\,(\forall x \,.\, \varphi) = \mathcal{W} \setminus \{x\}$. Let also $\|\varphi\| = \sum_{j=1}^{n} k_j 1_{L_j}$, where $k_j \in K$ and $L_j \subseteq T^v_{\Sigma_{\mathcal{W}}}$ is a recognizable tree language for every $1 \leq j \leq n$. Furthermore, we assume that the family $(L_j)_{1\leq j\leq n}$ is a partition of $T^v_{\Sigma_{\mathcal{W}}}$.

First assume that $x \in \mathcal{W}$. Let $\widetilde{\Sigma} = \Sigma \times \{1,\ldots,n\}$ be a ranked alphabet with $rk_{\widetilde{\Sigma}}\left((\sigma,j)\right) = rk_{\Sigma}(\sigma)$ for every $(\sigma,j) \in \widetilde{\Sigma}$. Every tree $s \in T^v_{\widetilde{\Sigma}_{\mathcal{V}}}$ can be written as a triple (t,v,ρ) where $(t,\rho) \in T^v_{\Sigma_{\mathcal{V}}}$ and v is a mapping $v : dom\,(s) \to \{1,\ldots,n\}$ determined by $v\,(w) = j$ whenever $s\,(w) = (\sigma,j,f)$ for some $\sigma \in \Sigma$ and $f \in \{0,1\}^{\mathcal{V}}$. Clearly $dom(t) = dom(s)$. Conversely, every such triple (t,v,ρ) corresponds to a tree $s \in T^v_{\widetilde{\Sigma}_{\mathcal{V}}}$. Hence in the sequel, we write the elements of $T^v_{\widetilde{\Sigma}_{\mathcal{V}}}$ in the form (t,v,ρ). Let

$$\widetilde{L} = \left\{(t,v,\rho) \in T^v_{\widetilde{\Sigma}_{\mathcal{V}}} \mid \forall w \in dom\,(t)\,, \forall 1 \leq j \leq n \text{ if } v\,(w) = j, \text{ then } (t,\rho\,[x \to w]) \in L_j\right\}.$$

Since $(L_j)_{1\leq j\leq n}$ is a partition of $T^v_{\Sigma_{\mathcal{W}}}$, for every $(t,\rho) \in T^v_{\Sigma_{\mathcal{V}}}$ there is a unique $v : dom\,(t) \to \{1,\ldots,n\}$ such that $(t,v,\rho) \in \widetilde{L}$. By Lemma 5.5 in [24], we obtain that $\widetilde{L}$ is recognized by a deterministic buta $\widetilde{\mathcal{M}} = \left(Q,\widetilde{\Sigma}_{\mathcal{V}},\delta,F\right)$. We consider now the wbuta $\mathcal{M} = (Q, wt, ter)$ over $\widetilde{\Sigma}_{\mathcal{V}}$ and K, with weight assigning mapping wt defined for every $m \geq 0, (\sigma,j,f) \in \left(\widetilde{\Sigma}_{\mathcal{V}}\right)_m, q_1,\ldots,q_m, q \in Q$ by

$$wt\left((q_1,\ldots,q_m)\,,(\sigma,j,f)\,,q\right) = \begin{cases} k_j & \text{if } \delta_{(\sigma,j,f)}\,(q_1,\ldots,q_m) = q \\ 0 & \text{otherwise.} \end{cases}$$

The final distribution ter is determined by

$$ter\,(q) = \begin{cases} 1 & \text{if } q \in F \\ 0 & \text{otherwise} \end{cases}$$

for every $q \in Q$.

Since $\widetilde{\mathcal{M}}$ is deterministic, for every $(t,v,\rho) \in T^v_{\widetilde{\Sigma}_{\mathcal{V}}}$ there is at most one run $r_{(t,v,\rho)}$ of $\widetilde{\mathcal{M}}$ over (t,v,ρ). Moreover, since $\left|\widetilde{\mathcal{M}}\right| = \widetilde{L}$ we get

$$\left(\|\mathcal{M}\|\,,(t,v,\rho)\right) = \begin{cases} \prod\limits_{w \in dom((t,v,\rho))} \Phi^{(t,v,\rho)}_{w}\left(wt\left(r_{(t,v,\rho)},w\right)\right) \cdot ter\left(r_{(t,v,\rho)}\,(\varepsilon)\right) & \text{if } (t,v,\rho) \in \widetilde{L} \\ 0 & \text{otherwise.} \end{cases}$$

Let $(t,v,\rho) \in \widetilde{L}$. For every $w \in dom\,(t)$ with $v\,(w) = j$, we have $wt\left(r_{(t,v,\rho)},w\right) = k_j$, and $(t,\rho\,[x \to w]) \in L_j$ which in turn implies that $\left(\|\varphi\|_{\mathcal{V}\cup\{x\}}\,,(t,\rho\,[x \to w])\right) = k_j$. We consider the relabeling $h : \widetilde{\Sigma}_{\mathcal{V}} \to \Sigma_{\mathcal{V}}$ by $h\left((\sigma,j,f)\right) = (\sigma,f)$ for every $(\sigma,j,f) \in \widetilde{\Sigma}_{\mathcal{V}}$. Then for every

$(t,\rho) \in T^v_{\Sigma_{\mathcal{V}}}$,

$$\begin{aligned}
(h(\|\mathcal{M}\|),(t,\rho)) &= \sum_{(t,v,\rho)\in h^{-1}((t,\rho))} (\|\mathcal{M}\|,(t,v,\rho)) = (\|\mathcal{M}\|,(t,v,\rho)) \qquad (\text{where } (t,v,\rho)\in \widetilde{L}) \\
&= \prod_{w\in dom((t,v,\rho))} \Phi_w^{(t,v,\rho)}\left(wt\left(r_{(t,v,\rho)},w\right)\right)\cdot ter\left(r_{(t,v,\rho)}(\varepsilon)\right) \\
&= \prod_{w\in dom(t)} \Phi^t_w\left(\left(\|\varphi\|_{\mathcal{V}\cup\{x\}},(t,\rho[x\to w])\right)\right) \\
&= (\|\forall x\,.\,\varphi\|,(t,\rho)).
\end{aligned}$$

We conclude that $\|\forall x\,.\,\varphi\| = h(\|\mathcal{M}\|)$ which by Proposition 11 implies that $\|\forall x\,.\,\varphi\|$ is Φ-recognizable.

Now, assume that $x \notin \mathcal{W}$, i.e., $\mathcal{V} = \mathcal{W}$. We consider the formula $\varphi' = \varphi \wedge (\neg edge_i(x,x))$. By Proposition 11 and Lemmas 43, 44 we have that $\|\varphi'\|$ is Φ-recognizable, in particular $\|\varphi'\|$ is a recognizable step function. Thus $\|\forall x\,.\,\varphi'\|$ is Φ-recognizable by the previous arguments. Clearly $\|\varphi'\|_{\mathcal{V}\cup\{x\}} = \|\varphi\|_{\mathcal{V}\cup\{x\}}$ and so $\|\forall x\,.\,\varphi'\| = \|\forall x\,.\,\varphi\|$ which concludes our proof. ■

Lemma 47 *Let $\varphi \in MSO(K,\Sigma)$ with $\|\varphi\| = 1_L$, where $L \subseteq T^v_{\Sigma_\varphi}$ is a recognizable tree language. Then, the tree series $\|\forall x\,.\,\varphi\|$ is a recognizable step function.*

Proof. Assume first that $x \in Free(\varphi)$. Since L is recognizable there exists an MSO-formula ψ with $Free(\varphi) = Free(\psi)$ such that $\mathcal{L}(\psi) = L$. Then, the tree language $\mathcal{L}(\forall x\,.\,\psi)$ is recognizable. Without any loss, we assume that all the occurring negations of ψ are applied only to atomic subformulas of ψ. Clearly, ψ can be considered as a weighted formula in $MSO(K,\Sigma)$, and thus $\|\psi\|_{Free(\varphi)} = 1_L$. Now, for every $(t,\rho) \in T^v_{\Sigma_\varphi}$ we have

$$\begin{aligned}
(\|\forall x\,.\,\varphi\|,(t,\rho)) &= \prod_{w\in dom(t)} \Phi^t_w\left(\left(\|\varphi\|_{Free(\varphi)},(t,\rho[x\to w])\right)\right) \\
&= \begin{cases} 1 & \text{if } \left(\|\varphi\|_{Free(\varphi)},(t,\rho[x\to w])\right) = 1 \ \ \forall w \in dom(t) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } (t,\rho[x\to w]) \in L \ \ \forall w \in dom(t) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } (t,\rho[x\to w]) \in \mathcal{L}(\psi) \ \ \forall w \in dom(t) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} 1 & \text{if } (t,\rho) \in \mathcal{L}(\forall x\,.\,\psi) \\ 0 & \text{otherwise} \end{cases} \\
&= \left(1_{\mathcal{L}(\forall x.\psi)},(t,\rho)\right),
\end{aligned}$$

i.e.,

$$\|\forall x\,.\,\varphi\| = 1_{\mathcal{L}(\forall x.\psi)}$$

which implies that $\|\forall x\,.\,\varphi\|$ is a recognizable step function, as required.

The case $x \notin Free(\varphi)$ is treated in a similar way as the corresponding case in Lemma 46.

■

By combining Lemmas 43, 44, 45, 46, and 47 we get the subsequent result.

Proposition 48

- $K^{\Phi-rmso}\langle\langle T_\Sigma\rangle\rangle \subseteq K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$
- *if K is additively locally finite, then* $K^{\Phi-aemso}\langle\langle T_\Sigma\rangle\rangle \subseteq K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$.

Next, we wish to show the converse inclusions, namely

Proposition 49 $K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle \subseteq K^{\Phi-remso}\langle\langle T_\Sigma\rangle\rangle \cap K^{\Phi-aemso}\langle\langle T_\Sigma\rangle\rangle$.

First, we need to disambiguate disjunctions of $MSO(K,\Sigma)$-formulas. More precisely, let φ,ψ be two $MSO(K,\Sigma)$-formulas whose semantics $\|\varphi\|$ and $\|\psi\|$ take on only 0 and 1. Then clearly, it is not always the case that the series $\|\varphi\vee\psi\|$ takes on only 0 and 1. Thus, following [24], for every $MSO(K,\Sigma)$-formula φ which does not contain any constant $k\in K$ and any quantifier, we define the $MSO(K,\Sigma)$-formulas φ^+ and φ^- inductively as follows:

- if φ is an atomic or the negation of an atomic formula, then we set $\varphi^+=\varphi$ and $\varphi^-=\neg\varphi$ with the convention that $\neg\neg\varphi=\varphi$,
- $(\varphi\vee\psi)^+=\varphi^+\vee(\varphi^-\wedge\psi^+)$ and $(\varphi\vee\psi)^-=\varphi^-\wedge\psi^-$, and
- $(\varphi\wedge\psi)^-=\varphi^-\vee(\varphi^+\wedge\psi^-)$ and $(\varphi\wedge\psi)^+=\varphi^+\wedge\psi^+$.

It should be clear that $\mathcal{L}_\mathcal{V}(\varphi^+)=\mathcal{L}_\mathcal{V}(\varphi)$ and $\mathcal{L}_\mathcal{V}(\varphi^-)=T^v_{\Sigma_\mathcal{V}}\setminus\mathcal{L}_\mathcal{V}(\varphi)$ for every set of variables $\mathcal{V}$ containing $Free(\varphi)$.

Lemma 50 *[24] Let φ be a classical MSO-formula which does not contain any quantifier. Then, $\|\varphi^+\|=1_{\mathcal{L}(\varphi)}$ and $\|\varphi^-\|=1_{\mathcal{L}(\neg\varphi)}$.*

Next, we introduce abbreviations of several $MSO(K,\Sigma)$-formulas. More precisely, for every atomic $MSO(K,\Sigma)$-formula of the form $label_\sigma(x), edge_i(x,y)$, or $x\in X$, and every $MSO(K,\Sigma)$-formula ψ, we set

- $\varphi\to\psi=\varphi^-\vee(\varphi^+\wedge\psi)$.

In particular, if $\varphi=x\in X$ and $\psi=k$ with $k\in K$, then for every $(t,\rho)\in T^v_{\Sigma_\mathcal{V}}$ we have

$$(\|(x\in X)\to k\|_\mathcal{V},(t,\rho))=\begin{cases} k & \text{if } \rho(x)\in\rho(X)\\ 1 & \text{otherwise.}\end{cases}$$

Therefore $\|(x\in X)\to k\|_\mathcal{V}$ is a recognizable step function.
Furthermore, we set

- $edge(x,(y_1,\ldots,y_k))=edge_1(x,y_1)\wedge\ldots\wedge edge_k(x,y_k)$.
- $root(x)=\forall y.\left(\neg(edge_1(y,x))\wedge\ldots\wedge\neg\left(edge_{\deg(\Sigma)}(y,x)\right)\right)$.

Clearly, for every $t\in T_\Sigma$, we have $(\|root(x)\|,(t,[x\to\varepsilon]))=1$ and $(\|root(x)\|,(t,[x\to w]))=0$ for every $w\in dom(t)\setminus\{\varepsilon\}$.

- $leaf(x) = \forall y. \left(\neg \left(edge_1 (x,y)\right) \wedge \ldots \wedge \neg \left(edge_{\deg(\Sigma)} (x,y)\right)\right).$

Finally, for set variables $X_1, \ldots, X_n$ we put

- $partition(X_1, \ldots, X_n) = \forall x \,\boldsymbol{.}\, \left(\bigvee_{1 \le i \le n} \left((x \in X_i) \wedge \bigwedge_{\substack{1 \le j \le n \\ j \ne i}} \neg (x \in X_j) \right) \right).$

Proof of Proposition 49. Let $\mathcal{M} = (Q, wt, ter)$ be a wbuta over Σ and K, and let $T = \bigcup_{k \ge 0} Q^k \times \Sigma_k \times Q$ be the collection of all the transitions of $\mathcal{M}$. We abbreviate every transition $((q_1, \ldots, q_k), \sigma, q)$ of T by $(\overrightarrow{q}, \sigma, q)$ and we set $\mathcal{V} = \left\{X_{\overrightarrow{q},\sigma,q} \mid (\overrightarrow{q}, \sigma, q) \in T\right\}$. Let $X_1, \ldots, X_n$ be an enumeration of $\mathcal{V}$ where $n = \sum_{k \ge 0} |Q|^{k+1} \cdot |\Sigma_k|$. We consider the formula

$$\psi_1(x) = \bigwedge_{\substack{(\overrightarrow{q},\sigma,q) \in T \\ \overrightarrow{q} = (q_1, \ldots, q_k), k > o}} \left(\begin{array}{c} \left(x \in X_{\overrightarrow{q},\sigma,q}\right) \to \exists y_1 \ldots \exists y_k \,\boldsymbol{.} \\ edge(x, (y_1, \ldots, y_k)) \wedge \\ \left(\left(\bigvee_{\substack{(\overrightarrow{p}_1, \sigma_1, q_1) \in T, \\ \vdots \\ (\overrightarrow{p}_k, \sigma_k, q_k) \in T}} \left(y_1 \in X_{\overrightarrow{p}_1,\sigma_1,q_1}\right) \wedge \ldots \wedge \left(y_k \in X_{\overrightarrow{p}_k,\sigma_k,q_k}\right) \right)^+ \right) \end{array} \right)$$

and we let

$$\psi(X_1, \ldots, X_n) = partition(X_1, \ldots, X_n) \wedge \bigwedge_{(\overrightarrow{q},\sigma,q) \in T} \forall x \,\boldsymbol{.}\, \left(\left(x \in X_{\overrightarrow{q},\sigma,q}\right) \to label_\sigma (x) \right)$$
$$\wedge \forall x \,\boldsymbol{.}\, (leaf(x) \vee \psi_1(x)).$$

We show that for every tree $t \in T_\Sigma$, there exists a bijection between the set $R_{\mathcal{M}}(t)$ of runs of $\mathcal{M}$ over t, and the set of $(t, \mathcal{V})$-assignments ρ satisfying $\psi(X_1, \ldots, X_n)$, i.e., such that $(\|\psi(X_1, \ldots, X_n)\|, (t, \rho)) = 1$. Let $r_t \in R_{\mathcal{M}}(t)$. We consider the $(t, \mathcal{V})$-assignment ρ_{r_t} defined in the following way. For every $X_{\overrightarrow{q},\sigma,q} \in \mathcal{V}$ with $\overrightarrow{q} = (q_1, \ldots, q_k)$, we set $\rho_{r_t}\left(X_{\overrightarrow{q},\sigma,q}\right) = \{w \in dom(t) \mid t(w) = \sigma, r_t(w) = q, r_t(w1) = q_1, \ldots, r_t(wk) = q_k\}$. Clearly $(t, \rho_{r_t}) \models \psi(X_1, \ldots, X_n)$. Conversely, let ρ be a $(t, \mathcal{V})$-assignment such that $(t, \rho) \models \psi(X_1, \ldots, X_n)$. We define the run r_{t_ρ} of $\mathcal{M}$ over t as follows. Due to $partition(X_1, \ldots, X_n)$, for every $w \in dom(t)$ there are uniquely determined $k \ge 0, \sigma \in \Sigma_k, \overrightarrow{q} \in Q^k, q \in Q$ such that $w \in \rho\left(X_{\overrightarrow{q},\sigma,q}\right)$. Then we set $r_{t_\rho}(w) = q$.

Consider now the formula

$$\varphi(X_1, \ldots, X_n) = \psi(X_1, \ldots, X_n) \wedge \bigwedge_{(\overrightarrow{q},\sigma,q) \in T} \forall x \,\boldsymbol{.}\, \left(\left(x \in X_{\overrightarrow{q},\sigma,q}\right) \to wt\left(\overrightarrow{q}, \sigma, q\right)\right)$$
$$\wedge \exists y \,\boldsymbol{.}\, \left(root\,(y) \wedge \bigvee_{(\overrightarrow{q},\sigma,q) \in T} \left(y \in X_{\overrightarrow{q},\sigma,q}\right) \wedge ter\,(q) \right).$$

Let $t \in T_\Sigma$ and $r_t \in R_{\mathcal{M}}(t)$. Then, for the uniquely determined assignment ρ_{r_t} associated to r_t (as described above) we have:

$$\begin{aligned}
(\|\varphi(X_1,\ldots,X_n)\|,(t,\rho_{r_t})) &= \prod_{\substack{(\overrightarrow{q},\sigma,q)\in T\\ w\in\rho_{r_t}\left(X_{\overrightarrow{q},\sigma,q}\right)}} \Phi_w^t\left(wt(\overrightarrow{q},\sigma,q)\right)\cdot ter\left(r_t\left(\varepsilon\right)\right)\\
&= \prod_{w\in dom(t)} \Phi_w^t\left(wt\left(\left(r_t\left(w1\right),\ldots,r_t\left(w\cdot rk_\Sigma\left(t\left(w\right)\right)\right)\right),t\left(w\right),r_t\left(w\right)\right)\right)\\
&\qquad\cdot ter\left(r_t\left(\varepsilon\right)\right)\\
&= \prod_{w\in dom(t)} \Phi_w^t\left(wt\left(r_t,w\right)\right)\cdot ter\left(r_t\left(\varepsilon\right)\right)\\
&= weight\left(r_t\right).
\end{aligned}$$

Now, let $\xi = \exists X_1\ldots\exists X_n \centerdot \varphi(X_1,\ldots,X_n)$. Then $\xi \in REMSO\left(K,\Sigma\right)\cap AEMSO\left(K,\Sigma\right)$. We show that $\|\xi\| = \|\mathcal{M}\|$. Indeed, for every $t \in T_\Sigma$ we get

$$\begin{aligned}
(\|\xi\|,t) &= \sum_{\rho\ (t,\mathcal{V})\text{-assignment}} (\|\varphi(X_1,\ldots,X_n)\|,(t,\rho))\\
&= \sum_{\substack{\rho\ (t,\mathcal{V})\text{-assignment}\\ (t,\rho)\models\psi(X_1,\ldots,X_n)}} (\|\varphi(X_1,\ldots,X_n)\|,(t,\rho))\\
&= \sum_{r_t\in R_{\mathcal{M}}(t)} weight\left(r_t\right)\\
&= (\|\mathcal{M}\|,t)
\end{aligned}$$

as required. ■

Proof of Theorem 42. It is immediate by Propositions 48 and 49. ■

6 Weighted Muller tree automata with discounting

In this section, we investigate weighted Muller tree automata acting on infinite trees. The underlying semiring is the max-plus semiring $\mathbb{R}_{\max} = (\mathbb{R}_+ \cup \{-\infty\}, \sup, +, -\infty, 0)$, where we consider sup instead of max since we need to compute over infinite trees. Our results can be applied to the min-plus semiring $(\mathbb{R}_+ \cup \{\infty\}, \inf, +, \infty, 0)$ as well. A weighted Muller tree automaton computes the weight of a run (of an input infinite tree) by applying a Φ-discounting over $\mathbb{R}_{\max}$. By considering suitable endomorphisms for Φ, we do not require any completeness axioms (for the sum operation) in $\mathbb{R}_{\max}$ (see [47]). For a study on weighted Muller automata with discounting over infinite words see [22]. In the sequel, we firstly recall classical Muller tree automata.

A *(nondeterministic) Muller tree automaton* (*mta* for short) is a tuple $\mathcal{M} = (Q, \Sigma, q_0, \Delta, \mathcal{F})$ with *finite state set* Q, *input ranked alphabet* Σ, *initial state* q_0, *set of transitions* $\Delta \subseteq \bigcup_{k\geq 0} Q \times \Sigma_k \times Q^k$, and *family of final state sets* $\mathcal{F} \subseteq \mathcal{P}(Q)$.

Let $t \in T_\Sigma^\omega$. A *run of* $\mathcal{M}$ *over* t is a mapping $r_t : dom(t) \to Q$ such that $r_t(\varepsilon) = q_0$, and for every $w \in dom(t)$, it holds that $(r_t(w), t(w), (r_t(w1), \ldots, r_t(w.rk_\Sigma(t(w))))) \in \Delta$. Every infinite prefix-closed chain $\pi \subseteq dom(r_t)$ is called an *infinite path of* r_t. We denote by $In^Q(r_t|_\pi)$ the set of states that appear infinitely often in $r_t|_\pi$. Then, the run r_t is called *successful* if for every infinite path π of r_t, the set of states that appear infinitely often along π, constitutes a final state set, i.e., $In^Q(r_t|_\pi) \in \mathcal{F}$. The tree $t \in T_\Sigma^\omega$ is *recognized* (or *accepted*) by $\mathcal{M}$ if it has a successful run. The *infinitary tree language of* $\mathcal{M}$ consists of all trees accepted by $\mathcal{M}$ and is denoted by $|\mathcal{M}|$.

An infinitary tree language $L \subseteq T_\Sigma^\omega$ is called *Muller recognizable* (or *ω-recognizable*) if there is an mta $\mathcal{M} = (Q, \Sigma, q_0, \Delta, \mathcal{F})$, such that $|\mathcal{M}| = L$. It is well-known that the class of ω-recognizable tree languages is closed under the Boolean operations [51, 52].

Let now Σ, Γ be two ranked alphabets and $h : \Sigma \to \Gamma$ be a relabeling. Then h is extended to a mapping $h : T_\Sigma^\omega \to T_\Gamma^\omega$ as follows:

$$h(t) = \{s \in T_\Gamma^\omega \mid dom(s) = dom(t) \text{ and } s(w) = h(t(w)),\ \forall w \in dom(s)\}$$

for every $t \in T_\Sigma^\omega$.

By standard automata constructions, it can be easily shown that the class of ω-recognizable tree languages is closed under relabelings and inverse relabelings.

Let now Σ be a ranked alphabet and $\Phi = (\Phi_k)_{k \geq 1}$ be a discounting over Σ and $\mathbb{R}_{\max}$. Here, we do not assume endomorphisms for elements of Σ with rank 0, since in our weighted Muller tree automata we do not assign a final weight to the leaves of the input trees. In fact, this is not an essential restriction. We actually simplify our notations and computations since in the case of infinite trees we are mainly interested in their infinite paths. As already noticed (see [20], and Example 2 above) every endomorphism of $\mathbb{R}_{\max}$ is of the form $\overline{p} : \mathbb{R}_{\max} \to \mathbb{R}_{\max}$ where $p \in \mathbb{R}_+$ and $x \longmapsto p \cdot x$ for every $x \in \mathbb{R}_+ \cup \{-\infty\}$, with the convention that $p \cdot (-\infty) = -\infty$. For our Φ-discounting here, we require for every $k \geq 1, \sigma \in \Sigma_k$ that $\Phi_\sigma^i = \overline{p_\sigma^i}$ with $0 \leq p_\sigma^i < 1$. Then, we simply write $\Phi_\sigma = \overline{p_\sigma} = \left(\overline{p_\sigma^1}, \ldots, \overline{p_\sigma^k}\right)$ for every $k \geq 1, \sigma \in \Sigma_k$. Furthermore, for every infinite tree $t \in T_\Sigma^\omega$ and every $w \in dom(t)$ we write $\overline{p_w^t}$ for Φ_w^t. Then we set

$$p_w^t = \begin{cases} 1 & \text{if } w = \varepsilon \\ p_{t(\varepsilon)}^{i_1} \cdot p_{t(i_1)}^{i_2} \cdot \ldots \cdot p_{t(i_1 \ldots i_{n-1})}^{i_n} & \text{if } w = i_1 \ldots i_n \text{ with } i_1, \ldots, i_n \in \mathbb{N}_+, n > 0. \end{cases}$$

We let $m_\Phi = \max\left\{p_\sigma^i \mid k \geq 1, \sigma \in \Sigma_k, 1 \leq i \leq k\right\}$. In the sequel, we shall use also the concatenation notation for the multiplication in $\mathbb{R}_+ \cup \{-\infty\}$.

Definition 51 *A* weighted Muller tree automaton *(*wmta *for short)* over Σ and $\mathbb{R}_{\max}$ *is a quadruple* $\mathcal{M} = (Q, in, wt, \mathcal{F})$*, where* Q *is the* finite state set*,* $in : Q \to \mathbb{R}_{\max}$ *is the* initial distribution*,* $wt : \bigcup\limits_{k \geq 0} Q \times \Sigma_k \times Q^k \to \mathbb{R}_{\max}$ *is the* mapping assigning weights to the transitions *of the automaton, and* $\mathcal{F} \subseteq \mathcal{P}(Q)$ *is the* family of final states sets*.*

Let $t \in T_\Sigma^\omega$. A *run of* $\mathcal{M}$ *over* t is a mapping $r_t : dom(t) \to Q$. The *weight of* r_t *at* $w \in dom\,(t)$ is the value

$$wt\,(r_t, w) = wt\,(r_t\,(w)\,, t\,(w)\,, (r_t\,(w1)\,, \ldots, r_t\,(w.rk_\Sigma\,(t\,(w))))).$$

Then the Φ-*weight* (or simply *weight*) *of* r_t, which is denoted by $weight\,(r_t)$ (or by $weight_{\mathcal{M}}\,(r_t)$ whenever we want to notify the wmta $\mathcal{M}$), is defined by

$$weight\,(r_t) = in\,(r_t\,(\varepsilon)) + \sum_{w \in dom(t)} \frac{p_w^t}{\deg\,(\Sigma)^{|w|}} \cdot wt\,(r_t, w)\,.$$

Observe that $weight\,(r_t)$ is well-defined. Indeed, let $M = \max \left\{ wt\,(\tau) \mid \tau \in \bigcup_{n \geq 0} Q \times \Sigma_n \times Q^n \right\}$. Then, we have

$$\begin{aligned}
\sum_{w \in dom(t)} \frac{p_w^t}{\deg\,(\Sigma)^{|w|}} \cdot wt\,(r_t, w) \leq M \cdot \sum_{w \in dom(t)} \frac{p_w^t}{\deg\,(\Sigma)^{|w|}} &\leq M \cdot \sum_{w \in dom(t)} \frac{m_\Phi^{|w|}}{\deg\,(\Sigma)^{|w|}} \\
&= M \cdot \sum_{n \geq 0} \sum_{\substack{w \in dom(t) \\ |w| = n}} \frac{m_\Phi^{|w|}}{\deg\,(\Sigma)^{|w|}} \leq M \cdot \sum_{n \geq 0} \deg(\Sigma)^n \frac{m_\Phi^n}{\deg\,(\Sigma)^n} \\
&= M \cdot \sum_{n \geq 0} m_\Phi^n = M \cdot \frac{1}{1 - m_\Phi}.
\end{aligned}$$

The run r_t is called *successful* if for every infinite path π of r_t, the set of states that appear infinitely often along π, constitutes a final state set, i.e., $In^Q(r_t|_\pi) \in \mathcal{F}$. We shall denote by $R_{\mathcal{M}}(t)$ the set of all runs of $\mathcal{M}$ over t, and by $R_{\mathcal{M}}^{suc}(t)$ the set of all successful runs in $R_{\mathcal{M}}(t)$.

The Φ-*behavior* (or simply *behavior*) *of* $\mathcal{M}$ is the infinitary formal tree series

$$\|\mathcal{M}\| : T_\Sigma^\omega \to \mathbb{R}_{\max}$$

whose coefficients are determined for every $t \in T_\Sigma^\omega$ by

$$(\|\mathcal{M}\|\,, t) = \sup_{r_t \in R_{\mathcal{M}}^{suc}(t)} (weight\,(r_t))\,.$$

Clearly, this supremum exists in $\mathbb{R}_{\max}$ since the values $weight(r_t)$ are bounded by $N + M \cdot \frac{1}{1 - m_\Phi}$, where $N = \max\{in(q) \mid q \in Q\}$.

A tree series $S : T_\Sigma^\omega \to \mathbb{R}_{\max}$ is said to be Φ-*Muller recognizable* (or Φ-ω-*recognizable*) if there exists a wmta $\mathcal{M}$ over Σ and $\mathbb{R}_{\max}$ such that $S = \|\mathcal{M}\|$. The family of all Φ-Muller recognizable tree series over Σ and $\mathbb{R}_{\max}$ is denoted by $\mathbb{R}_{\max}^{\Phi-\omega-rec}\langle\langle T_\Sigma^\omega \rangle\rangle$. In the sequel, we shall investigate closure properties of the class $\mathbb{R}_{\max}^{\Phi-\omega-rec}\langle\langle T_\Sigma^\omega \rangle\rangle$.

Let $L \subseteq T_\Sigma^\omega$. The *characteristic series* $1_L : T_\Sigma^\omega \to \mathbb{R}_{\max}$ *of* L is defined in a similar way as for finitary tree languages. Furthermore, let $S, T \in \mathbb{R}_{\max}\langle\langle T_\Sigma^\omega \rangle\rangle$ and $k \in \mathbb{R}_{\max}$. Then, the $\max(S, T)$, the scalar sum $k + S$ and the sum $S + T$ are defined for every $t \in T_\Sigma^\omega$ by

$$\begin{aligned}
(\max(S, T), t) &= \max((S, t), (T, t)), \\
(k + S, t) &= k + (S, t), \\
(S + T, t) &= (S, t) + (T, t).
\end{aligned}$$

Let $S \in \mathbb{R}_{\max}\langle\langle T_\Sigma^\omega \rangle\rangle$. The *image* $\mathrm{Im}(S)$ *of* S is the set $\mathrm{Im}(S) = \{k \in \mathbb{R}_+ \cup \{-\infty\} \mid \exists t \in T_\Sigma^\omega$ with $(S, t) = k\}$. We say that S has *bounded image* if there is an $m \in \mathbb{R}_+$ such that $k \leq m$

for every $k \in \mathrm{Im}(S)$. Clearly, every Φ-ω-recognizable tree series $S \in \mathbb{R}_{\max}\langle\langle T_\Sigma^\omega \rangle\rangle$ has bounded image.

Let now Σ, Γ be two ranked alphabets and $h : \Sigma \to \Gamma$ be a relabeling. Then h can be extended to a partial mapping $h : \mathbb{R}_{\max}\langle\langle T_\Sigma^\omega \rangle\rangle \to \mathbb{R}_{\max}\langle\langle T_\Gamma^\omega \rangle\rangle$ in the following way. For every infinitary tree series $S \in \mathbb{R}_{\max}\langle\langle T_\Sigma^\omega \rangle\rangle$ with bounded image, we define the infinitary tree series $h(S) \in \mathbb{R}_{\max}\langle\langle T_\Gamma^\omega \rangle\rangle$ by $(h(S), s) = \sup_{t \in h^{-1}(s)} ((S,t))$ for every $s \in T_\Gamma^\omega$. Furthermore, for every $T \in \mathbb{R}_{\max}\langle\langle T_\Gamma^\omega \rangle\rangle$, the series $h^{-1}(T) \in \mathbb{R}_{\max}\langle\langle T_\Sigma^\omega \rangle\rangle$ is determined by $\big(h^{-1}(T), t\big) = (T, h(t))$ for every $t \in T_\Sigma^\omega$.

Proposition 52 *(i) The class $\mathbb{R}_{\max}^{\Phi-\omega-rec}\langle\langle T_\Sigma^\omega \rangle\rangle$ is closed under max, scalar sum, and sum.*

(ii) Let Σ, Γ be two ranked alphabets and $h : \Sigma \to \Gamma$ be a relabeling. Furthermore, for the Φ-discounting over Σ and $\mathbb{R}_{\max}$ we assume that $\overline{p_\sigma} = \overline{p_{\sigma'}}$ whenever $h(\sigma) = h(\sigma')$ for every $\sigma, \sigma' \in \Sigma_k, k \geq 1$. If the tree series $S \in \mathbb{R}_{\max}^{\Phi-\omega-rec}\langle\langle T_\Sigma^\omega \rangle\rangle$ is Φ-ω-recognizable, then the tree series $h(S)$ is Φ'-ω-recognizable where $\Phi' = (\Phi'_k)_{k \geq 1}$ is a discounting over Γ and $\mathbb{R}_{\max}$ determined for every $\gamma \in \Gamma_k$ ($k \geq 1$) by $\overline{p'_\gamma} = \overline{p_\sigma}$ for every $\sigma \in \Sigma_k$ ($k \geq 1$) with $h(\sigma) = \gamma$. Furthermore, if $T \in \mathbb{R}_{\max}^{\Phi'-\omega-rec}\langle\langle T_\Gamma^\omega \rangle\rangle$, then the tree series $h^{-1}(T) \in \mathbb{R}_{\max}^{\Phi-\omega-rec}\langle\langle T_\Sigma^\omega \rangle\rangle$ is Φ-ω-recognizable.

(iii) Let $L \subseteq T_\Sigma^\omega$ be an ω-recognizable tree language. Then its characteristic series $1_L \in \mathbb{R}_{\max}\langle\langle T_\Sigma^\omega \rangle\rangle$ is Φ-ω-recognizable.

Proof. (i) We can state the proof for max and scalar sum by using well-known arguments from classical tree automata: for max we consider the disjoint union of two automata, whereas for the scalar sum we add the given scalar to the initial distribution. Next, we show closure under sum. In fact, we recall a construction from [47] (used for the closure under Hadamard product).

Let $\mathcal{M} = (Q, in, wt, \mathcal{F})$ and $\mathcal{M}' = (Q', in', wt', \mathcal{F}')$ be two wmta over Σ and $\mathbb{R}_{\max}$. We construct the wmta $\widetilde{\mathcal{M}} = (\widetilde{Q}, \widetilde{in}, \widetilde{wt}, \widetilde{\mathcal{F}})$ with

- $\widetilde{Q} = Q \times Q'$
- $\widetilde{in}(q, q') = in(q) + in'(q')$ for every $(q, q') \in \widetilde{Q}$
- $\widetilde{wt}((q, q'), \sigma, ((q_1, q_1'), \ldots, (q_k, q_k'))) = wt(q, \sigma, (q_1, \ldots, q_k)) + wt'(q', \sigma, (q_1', \ldots, q_k'))$ for every $k \geq 0, \sigma \in \Sigma_k, (q, q'), (q_1, q_1'), \ldots, (q_k, q_k') \in \widetilde{Q}$, and
- $\widetilde{\mathcal{F}} = \{\widetilde{F} \mid pr(\widetilde{F}) \in \mathcal{F},\ pr'(\widetilde{F}) \in \mathcal{F}'\}$ where $pr : \widetilde{Q} \to Q$, $pr' : \widetilde{Q} \to Q'$ are the projections of $\widetilde{Q}$ on Q and Q', respectively.

Consider an infinite tree $t \in T_\Sigma^\omega$ and let $\widetilde{r}_t$ be a run of $\widetilde{\mathcal{M}}$ over t. Then, there are runs r_t and r_t' of $\mathcal{M}$ and $\mathcal{M}'$ respectively, obtained by projections of $\widetilde{r}_t$ in the obvious way. Let us assume that $\widetilde{r}_t$ is successful, i.e., for every infinite path π of $\widetilde{r}_t$ there exists $\widetilde{F} \in \widetilde{\mathcal{F}}$ so that $In^{\widetilde{Q}}(\widetilde{r}_t|_\pi) = \widetilde{F}$. Therefore, there are sets $F \in \mathcal{F}$ and $F' \in \mathcal{F}'$ such that $In^Q(r_t|_\pi) = F$ and $In^{Q'}(r_t'|_\pi) = F'$ (observe that $dom(r_t) = dom(r_t') = dom(\widetilde{r}_t)$). This implies that the runs r_t and r_t' are successful. Conversely, keeping the same notations, let us assume that r_t and r_t' are successful runs of $\mathcal{M}$ and $\mathcal{M}'$ over t, respectively. Then, for every infinite path π of r_t

and r'_t, we have $In^Q(r_t|_\pi) = F$ and $In^{Q'}(r'_t|_\pi) = F'$, for some $F \in \mathcal{F}$ and $F' \in \mathcal{F}'$. Now let $\widetilde{r}_t$ be the run of $\widetilde{\mathcal{M}}$ over t composed by r_t and r'_t. By construction, for every infinite path π of $\widetilde{r}_t$ it holds that $In^{\widetilde{Q}}(\widetilde{r}_t|_\pi) \in \widetilde{\mathcal{F}}$, and thus $\widetilde{r}_t$ is successful.

We thus have shown that for every successful run $\widetilde{r}_t$ of $\widetilde{\mathcal{M}}$ over $t \in T_\Sigma^\omega$, there are two uniquely determined successful runs r_t of $\mathcal{M}$ and r'_t of $\mathcal{M}'$ over t, and vice versa. Now we compute

$$weight_{\widetilde{\mathcal{M}}}(\widetilde{r}_t) = \widetilde{in}(\widetilde{r}_t(\varepsilon)) + \sum_{w \in dom(t)} \frac{p_w^t}{\deg(\Sigma)^{|w|}} \widetilde{wt}(\widetilde{r}_t, w)$$

$$\begin{aligned}
&= in(r_t(\varepsilon)) + in'\left(r'_t(\varepsilon)\right) + \sum_{w \in dom(t)} \frac{p_w^t}{\deg(\Sigma)^{|w|}} \left(wt(r_t, w) + wt'\left(r'_t, w\right)\right) \\
&= in(r_t(\varepsilon)) + \sum_{w \in dom(t)} \frac{p_w^t}{\deg(\Sigma)^{|w|}} wt(r_t, w) \\
&\qquad + in'\left(r'_t(\varepsilon)\right) + \sum_{w \in dom(t)} \frac{p_w^t}{\deg(\Sigma)^{|w|}} wt'\left(r'_t, w\right) \\
&= weight_{\mathcal{M}}(r_t) + weight_{\mathcal{M}'}\left(r'_t\right).
\end{aligned}$$

Then, for every $t \in T_\Sigma^\omega$ we have

$$\begin{aligned}
\left(\left\|\widetilde{\mathcal{M}}\right\|, t\right) &= \sup_{\widetilde{r}_t \in R_{\widetilde{\mathcal{M}}}^{suc}(t)} \left(weight_{\widetilde{\mathcal{M}}}(\widetilde{r}_t)\right) \\
&= \sup_{r_t \in R_{\mathcal{M}}^{suc}(t), r'_t \in R_{\mathcal{M}'}^{suc}(t)} \left(weight_{\mathcal{M}}(r_t) + weight_{\mathcal{M}'}\left(r'_t\right)\right) \\
&= \sup_{r_t \in R_{\mathcal{M}}^{suc}(t)} \left(weight_{\mathcal{M}}(r_t)\right) + \sup_{r'_t \in R_{\mathcal{M}'}^{suc}(t)} \left(weight_{\mathcal{M}'}\left(r'_t\right)\right) \\
&= (\|\mathcal{M}\|, t) + \left(\left\|\mathcal{M}'\right\|, t\right)
\end{aligned}$$

as required.

(ii) Let $S \in \mathbb{R}_{\max}^{\Phi-\omega-rec}\langle\langle T_\Sigma^\omega\rangle\rangle$ and $\mathcal{M} = (Q, in, wt, \mathcal{F})$ be a wmta over Σ and $\mathbb{R}_{\max}$ with behavior S. We construct the wmta $\widehat{\mathcal{M}} = \left(Q, in, \widehat{wt}, \mathcal{F}\right)$ over Γ and $\mathbb{R}_{\max}$, where for every $k \geq 0, \gamma \in \Gamma_k, q, q_1, \ldots, q_k \in Q$ we set $\widehat{wt}(q, \gamma, (q_1, \ldots, q_k)) = \max_{\sigma \in h^{-1}(\gamma)} (wt(q, \sigma, (q_1, \ldots, q_k)))$.

Let now $s \in T_\Gamma^\omega$ and $t \in h^{-1}(s)$. Since $dom(t) = dom(s)$, by our assumptions we have $\overline{p_{t(w)}} = \overline{p'_{s(w)}}$ for every $w \in dom(t)$, which in turn implies that $p_w^t = p_w'^s$. Furthermore, the set of (successful) runs of $\widehat{\mathcal{M}}$ over s coincides with the set of (successful) runs of $\mathcal{M}$ over t, i.e., for every (successful) run $\widehat{r}_s$ of $\widehat{\mathcal{M}}$ over s there is a unique (successful) run r_t of $\mathcal{M}$ over t with $r_t = \widehat{r}_s$ and vice versa. Then, for every $w \in dom(s)$ we have

$$\begin{aligned}
\widehat{wt}(\widehat{r}_s, w) &= \widehat{wt}(\widehat{r}_s(w), s(w), (\widehat{r}_s(w1), \ldots, \widehat{r}_s(w.rk_\Gamma(s(w))))) \\
&= \max_{\sigma \in h^{-1}(s(w))} (wt(\widehat{r}_s(w), \sigma, (\widehat{r}_s(w1), \ldots, \widehat{r}_s(w.rk_\Gamma(s(w)))))).
\end{aligned}$$

Therefore,

$$\begin{aligned}
weight_{\widehat{\mathcal{M}}}\left(\widehat{r_s}\right) &= in\left(\widehat{r_s}\left(\varepsilon\right)\right) + \sum_{w\in dom(s)} \frac{p_w'^s}{\deg\left(\Gamma\right)^{|w|}} \widehat{wt}\left(\widehat{r_s}, w\right) \\
&= in\left(\widehat{r_s}\left(\varepsilon\right)\right) \\
&\quad + \sum_{w\in dom(s)} \frac{p_w'^s}{\deg\left(\Gamma\right)^{|w|}} \left(\max_{\sigma\in h^{-1}(s(w))} \left(wt\left(\widehat{r_s}\left(w\right), \sigma, \left(\widehat{r_s}\left(w1\right), \ldots, \widehat{r_s}\left(w.rk_{\Gamma}\left(s\left(w\right)\right)\right)\right)\right)\right)\right)
\end{aligned}$$

$$\begin{aligned}
&= in\left(r_t\left(\varepsilon\right)\right) \\
&\quad + \sup_{t\in h^{-1}(s)} \left(\sum_{w\in dom(t)} \frac{p_w^t}{\deg\left(\Sigma\right)^{|w|}} wt\left(r_t\left(w\right), \sigma, \left(r_t\left(w1\right), \ldots, r_t\left(w.rk_{\Sigma}\left(t\left(w\right)\right)\right)\right)\right)\right) \\
&= \sup_{t\in h^{-1}(s)} \left(weight_{\mathcal{M}}\left(r_t\right)\right).
\end{aligned}$$

Then, we get

$$\begin{aligned}
\left(\left\|\widehat{\mathcal{M}}\right\|, s\right) &= \sup_{\widehat{r_s}\in R^{suc}_{\widehat{\mathcal{M}}}(s)} \left(weight_{\widehat{\mathcal{M}}}\left(\widehat{r_s}\right)\right) \\
&= \sup_{\widehat{r_s}\in R^{suc}_{\widehat{\mathcal{M}}}(s)} \left(\sup_{t\in h^{-1}(s)} \left(weight_{\mathcal{M}}\left(r_t\right)\right)\right) \\
&= \sup_{t\in h^{-1}(s)} \left(\sup_{r_t\in R^{suc}_{\mathcal{M}}(t)} \left(weight_{\mathcal{M}}\left(r_t\right)\right)\right) \\
&= \sup_{t\in h^{-1}(s)} \left(\left(\left\|\mathcal{M}\right\|, t\right)\right) = \left(h\left(\left\|\mathcal{M}\right\|\right), s\right).
\end{aligned}$$

Let now $T \in \mathbb{R}_{\max}^{\Phi'-\omega-rec}\langle\langle T_\Gamma^\omega\rangle\rangle$ and $\mathcal{N} = (Q, in, wt, \mathcal{F})$ be a wmta over Γ and $\mathbb{R}_{\max}$ with $\|\mathcal{N}\| = T$. We consider the wmta $\widetilde{\mathcal{N}} = \left(Q, in, \widetilde{wt}, \mathcal{F}\right)$ over Σ and $\mathbb{R}_{\max}$ by setting $\widetilde{wt}\left(q, \sigma, (q_1, \ldots, q_k)\right) = wt\left(q, h\left(\sigma\right), (q_1, \ldots, q_k)\right)$ for every $k \geq 0, \sigma \in \Sigma_k, q, q_1, \ldots, q_k \in Q$.

Consider an infinite tree $t \in T_\Sigma^\omega$ and let $\widetilde{r_t}$ be a (successful) run of $\widetilde{\mathcal{N}}$ over t. Then, there exists a (successful) run $r_{h(t)}$ of $\mathcal{N}$ over $h\left(t\right)$ with $r_{h(t)} = \widetilde{r_t}$ and vice versa. Furthermore, we have

$$\begin{aligned}
weight_{\widetilde{\mathcal{N}}}\left(\widetilde{r_t}\right) &= in\left(\widetilde{r_t}\left(\varepsilon\right)\right) + \sum_{w\in dom(t)} \frac{p_w^t}{\deg\left(\Sigma\right)^{|w|}} \cdot \widetilde{wt}\left(\widetilde{r_t}, w\right) \\
&= in\left(\widetilde{r_t}\left(\varepsilon\right)\right) + \sum_{w\in dom(t)} \frac{p_w^t}{\deg\left(\Sigma\right)^{|w|}} \cdot \widetilde{wt}\left(\widetilde{r_t}\left(w\right), t\left(w\right), \left(\widetilde{r_t}\left(w1\right), \ldots, \widetilde{r_t}\left(w.rk_{\Sigma}\left(t\left(w\right)\right)\right)\right)\right) \\
&= in\left(r_{h(t)}\left(\varepsilon\right)\right) + \sum_{w\in dom(h(t))} \frac{p_w'^{h(t)}}{\deg\left(\Gamma\right)^{|w|}} \\
&\qquad \cdot wt\left(r_{h(t)}\left(w\right), h\left(t\left(w\right)\right), \left(r_{h(t)}\left(w1\right), \ldots, r_{h(t)}\left(w.rk_{\Gamma}\left(h\left(t\right)\left(w\right)\right)\right)\right)\right) \\
&= weight_{\mathcal{N}}\left(r_{h(t)}\right).
\end{aligned}$$

Therefore,

$$\left(\left\|\widetilde{\mathcal{N}}\right\|, t\right) = \sup_{\widetilde{r}_t \in R^{suc}_{\widetilde{\mathcal{N}}}(t)} \left(weight_{\widetilde{\mathcal{N}}}\left(\widetilde{r}_t\right)\right) = \sup_{r_{h(t)} \in R^{suc}_{\mathcal{N}}(h(t))} \left(weight_{\mathcal{N}}\left(r_{h(t)}\right)\right) = \left(\|\mathcal{N}\|, h(t)\right)$$

for every $t \in T^{\omega}_{\Sigma}$.

(iii) Finally, let $L \subseteq T^{\omega}_{\Sigma}$ be an ω-recognizable tree language, and let $\mathcal{M} = (Q, \Sigma, q_0, \Delta, \mathcal{F})$ be an mta with $|\mathcal{M}| = L$. Consider the wmta $\widetilde{\mathcal{M}} = (Q, in, wt, \mathcal{F})$ over Σ and $\mathbb{R}_{\max}$, with

$$wt\left(q, \sigma, (q_1, \ldots, q_k)\right) = \begin{cases} 0 & \text{if } (q, \sigma, (q_1, \ldots, q_k)) \in \Delta \\ -\infty & \text{otherwise} \end{cases}$$

for every $k \geq 0, \sigma \in \Sigma_k, q, q_1, \ldots, q_k \in Q$, and

$$in(q) = \begin{cases} 0 & \text{if } q = q_0 \\ -\infty & \text{otherwise} \end{cases}$$

for every $q \in Q$.

Clearly, for every $t \in T^{\omega}_{\Sigma}$ we have $\left(\left\|\widetilde{\mathcal{M}}\right\|, t\right) = 0$ if $t \in L$, and $\left(\left\|\widetilde{\mathcal{M}}\right\|, t\right) = -\infty$ if $t \notin L$, i.e., $\left\|\widetilde{\mathcal{M}}\right\| = 1_L$ and this concludes our proof. ■

An infinitary tree series $S \in \mathbb{R}_{\max}\langle\langle T^{\omega}_{\Sigma}\rangle\rangle$ is called a *Muller recognizable step function* (or *ω-recognizable step function*) if $S = \max\limits_{1 \leq j \leq n}\left(k_j + 1_{L_j}\right)$ where $k_j \in \mathbb{R}_+ \cup \{-\infty\}$ and L_j is an ω-recognizable tree language for every $1 \leq j \leq n$. Since the class of ω-recognizable tree languages is closed under the Boolean operations, we may consider the family $\left(L_j\right)_{1 \leq j \leq n}$ to be a partition of T^{ω}_{Σ}. Clearly, every ω-recognizable step function is a Φ-ω-recognizable tree series.

Proposition 53 *(i) The class of ω-recognizable step functions over Σ and $\mathbb{R}_{\max}$ is closed under max, scalar sum, and sum.*

(ii) Let Σ, Γ be two ranked alphabets and $h : T^{\omega}_{\Sigma} \rightarrow T^{\omega}_{\Gamma}$ be a relabeling. Then $h : \mathbb{R}_{\max}\langle\langle T^{\omega}_{\Sigma}\rangle\rangle \rightarrow \mathbb{R}_{\max}\langle\langle T^{\omega}_{\Gamma}\rangle\rangle$ and $h^{-1} : \mathbb{R}_{\max}\langle\langle T^{\omega}_{\Gamma}\rangle\rangle \rightarrow \mathbb{R}_{\max}\langle\langle T^{\omega}_{\Sigma}\rangle\rangle$ preserve ω-recognizable step functions.

Proof. (ii) Let $S = \max\limits_{1 \leq j \leq n}\left(k_j + 1_{L_j}\right)$ be an ω-recognizable step function over Σ and $\mathbb{R}_{\max}$, where $k_j \in \mathbb{R}_+ \cup \{-\infty\}$ and L_j is a non-empty ω-recognizable tree language for every $1 \leq j \leq n$. Then, for every $s \in T^{\omega}_{\Gamma}$ we have

$$\begin{aligned}
\left(h(S), s\right) &= \sup_{t \in h^{-1}(s)} \max_{1 \leq j \leq n}\left(k_j + \left(1_{L_j}, t\right)\right) \\
&= \max_{1 \leq j \leq n} \sup_{t \in h^{-1}(s)}\left(k_j + \left(1_{L_j}, t\right)\right) \\
&= \max_{1 \leq j \leq n}\left(k_j + \sup_{t \in h^{-1}(s)}\left(1_{L_j}, t\right)\right) \\
&= \max_{1 \leq j \leq n}\left(k_j + \left(1_{h(L_j)}, s\right)\right).
\end{aligned}$$

The class of ω-recognizable tree languages is closed under relabelings and thus the tree series $h(S)$ is an ω-recognizable step function.

Next, assume that $T = \max\limits_{1 \le i \le m} \left(k_i' + 1_{R_i'}\right)$ is an ω-recognizable step function over Γ and $\mathbb{R}_{\max}$, where for every $1 \le i \le m, k_i' \in \mathbb{R}_+ \cup \{-\infty\}$ and $R_i' \subseteq T_\Gamma^\omega$ is an ω-recognizable tree language. Then

$$h^{-1}(T) = \max_{1 \le i \le m} \left(k_i' + 1_{h^{-1}(R_i')}\right).$$

The class of ω-recognizable tree languages is closed under inverse relabelings, hence $h^{-1}(T)$ is an ω-recognizable step function. ■

7 Weighted MSO logic over infinite trees

In this section we deal with weighted MSO logic with discounting over the semiring $\mathbb{R}_{\max}$, and we interpret the semantics of weighted MSO-formulas as formal series over infinite trees. In our logic here, we add the atomic formula $x = y$ and its negation as well as second order universal quantifiers. In fact, we use the MSO logic of [47]. The atomic formula $x = y$ has been also considered by other authors as an atomic formula in MSO logic over finite trees (see for instance [4, 39]). First, we briefly recall from [51, 52] several notions from classical MSO logic over infinite trees.

Let Σ be a ranked alphabet and $\mathcal{V}$ be a finite set of first and second order variables. As for finite trees every infinite tree $t \in T_\Sigma^\omega$ is represented by the relational structure $(dom(t), edge_1, \ldots, edge_{\deg(\Sigma)}, (R_\sigma^t)_{\sigma \in \Sigma})$. The notions of a $(t, \mathcal{V})$-assignment and the set $T_{\Sigma_\mathcal{V}}^{\omega,v}$ of all valid infinite trees over $\Sigma_\mathcal{V}$ are defined as in the case of finite trees. Then, the infinitary tree language $T_{\Sigma_\mathcal{V}}^{\omega,v}$ is ω-recognizable (see [47]), and thus the characteristic series $1_{T_{\Sigma_\mathcal{V}}^{\omega,v}} : T_{\Sigma_\mathcal{V}}^\omega \to \mathbb{R}_{\max}$ is Φ-ω-recognizable.

Let now φ be an MSO-formula [51, 52] over trees. As usual we shall write Σ_φ instead of $\Sigma_{Free(\varphi)}$. Then for $Free(\varphi) \subseteq \mathcal{V}$, the well-known result of Rabin [44] states that the tree language

$$\mathcal{L}_\mathcal{V}^\omega(\varphi) = \{(t, \rho) \in T_{\Sigma_\mathcal{V}}^{\omega,v} \mid (t, \rho) \models \varphi\}$$

is ω-recognizable; conversely, for every ω-recognizable tree language $L \subseteq T_\Sigma^\omega$ there exists an MSO-sentence φ, such that $L = \mathcal{L}^\omega(\varphi)$, where we simply write $\mathcal{L}^\omega(\varphi)$ for $\mathcal{L}_{Free(\varphi)}^\omega(\varphi)$.

Definition 54 *The set $MSO(\mathbb{R}_{\max}, \Sigma)$ of all formulas of the weighted MSO logic with Φ-discounting over Σ and $\mathbb{R}_{\max}$ on infinite trees is defined to be the smallest set F such that*

- *F contains all atomic formulas $k, label_\sigma(x), edge_i(x, y), x = y, x \in X$ and the negations $\neg label_\sigma(x), \neg edge_i(x, y), \neg(x = y), \neg(x \in X)$, and*
- *if $\varphi, \psi \in F$, then also $\varphi \vee \psi, \varphi \wedge \psi, \exists x . \varphi, \exists X . \varphi, \forall x . \varphi, \forall X . \varphi \in F$*

where $k \in K$, $\sigma \in \Sigma$, $1 \le i \le \deg(\Sigma)$, x, y are first order variables, and X is a second order variable.

Next we represent the semantics of the formulas in $MSO(\mathbb{R}_{\max}, \Sigma)$ as infinitary formal tree series over the extended alphabet $\Sigma_\mathcal{V}$ and the semiring $\mathbb{R}_{\max}$.

Definition 55 *Let $\varphi \in MSO(\mathbb{R}_{\max}, \Sigma)$ and $\mathcal{V}$ be a finite set of variables with $Free(\varphi) \subseteq \mathcal{V}$. The* Φ-semantics *of φ is a formal tree series $\|\varphi\|_{\mathcal{V}} \in \mathbb{R}_{\max}\left\langle\left\langle T^{\omega}_{\Sigma_{\mathcal{V}}}\right\rangle\right\rangle$. Let $(t,\rho) \in T^{\omega}_{\Sigma_{\mathcal{V}}}$. If (t,ρ) is not a valid tree, then we set $(\|\varphi\|_{\mathcal{V}}, (t,\rho)) = -\infty$. Otherwise, we inductively define $(\|\varphi\|_{\mathcal{V}}, (t,\rho))$ as follows:*

- $(\|k\|_{\mathcal{V}}, (t,\rho)) = k$

- $(\|label_\sigma(x)\|_{\mathcal{V}}, (t,\rho)) = \begin{cases} 0 & \text{if } \ t(\rho(x)) = \sigma \\ -\infty & \text{otherwise} \end{cases}$

- $(\|edge_i(x,y)\|_{\mathcal{V}}, (t,\rho)) = \begin{cases} 0 & \text{if } \ \rho(y) = \rho(x)i \\ -\infty & \text{otherwise} \end{cases}$

- $(\|x = y\|_{\mathcal{V}}, (t,\rho)) = \begin{cases} 0 & \text{if } \ \rho(x) = \rho(y) \\ -\infty & \text{otherwise} \end{cases}$

- $(\|x \in X\|_{\mathcal{V}}, (t,\rho)) = \begin{cases} 0 & \text{if } \ \rho(x) \in \rho(X) \\ -\infty & \text{otherwise} \end{cases}$

- $(\|\neg\varphi\|_{\mathcal{V}}, (t,\rho)) = \begin{cases} 0 & \text{if } \ (\|\varphi\|_{\mathcal{V}}, (t,\rho)) = -\infty \\ -\infty & \text{if } \ (\|\varphi\|_{\mathcal{V}}, (t,\rho)) = 0 \end{cases}$, *provided that φ is of the form $label_\sigma(x)$, $edge_i(x,y)$, $x = y$, or $x \in X$*

- $(\|\varphi \vee \psi\|_{\mathcal{V}}, (t,\rho)) = \max((\|\varphi\|_{\mathcal{V}}, (t,\rho)), (\|\psi\|_{\mathcal{V}}, (t,\rho)))$

- $(\|\varphi \wedge \psi\|_{\mathcal{V}}, (t,\rho)) = (\|\varphi\|_{\mathcal{V}}, (t,\rho)) + (\|\psi\|_{\mathcal{V}}, (t,\rho))$

- $(\|\exists x \,.\, \varphi\|_{\mathcal{V}}, (t,\rho)) = \sup\limits_{w \in dom(t)} \left(\left(\|\varphi\|_{\mathcal{V}\cup\{x\}}, (t, \rho[x \to w])\right)\right)$

- $(\|\exists X \,.\, \varphi\|_{\mathcal{V}}, (t,\rho)) = \sup\limits_{I \subseteq dom(t)} \left(\left(\|\varphi\|_{\mathcal{V}\cup\{X\}}, (t, \rho[X \to I])\right)\right)$

- $(\|\forall x \,.\, \varphi\|_{\mathcal{V}}, (t,\rho)) = \sum\limits_{w \in dom(t)} \frac{p^t_w}{\deg(\Sigma)^{|w|}} \left(\|\varphi\|_{\mathcal{V}\cup\{x\}}, (t, \rho[x \to w])\right)$.

Now we assume that $\|\varphi\|_{\mathcal{V}\cup\{X\}} = 1_L$, where L is an ω-recognizable tree language. Then, we define the semantics of the formula $\forall X \,.\, \varphi$ as follows:

- $(\|\forall X \,.\, \varphi\|_{\mathcal{V}}, (t,\rho)) = \sum\limits_{I \subseteq dom(t)} \left(\|\varphi\|_{\mathcal{V}\cup\{X\}}, (t, \rho[X \to I])\right)$.

We shall simply write $\|\varphi\|$ for $\|\varphi\|_{Free(\varphi)}$. As in the case of finitary tree series, the above definition of $\|\varphi\|_{\mathcal{V}}$ depends on the set $\mathcal{V}$. The subsequent proposition shows that actually this is not an essential restriction.

Proposition 56 *Let $\varphi \in MSO(\mathbb{R}_{\max}, \Sigma)$ and $\mathcal{V}$ be a finite set of variables such that $Free(\varphi) \subseteq \mathcal{V}$. Then*

$$(\|\varphi\|_{\mathcal{V}}, (t,\rho)) = \left(\|\varphi\|, (t, \rho|_{Free(\varphi)})\right)$$

for every $(t,\rho) \in T^{\omega,v}_{\Sigma_{\mathcal{V}}}$. Moreover, the tree series $\|\varphi\|$ is Φ-ω-recognizable (resp. an ω-recognizable step function) iff $\|\varphi\|_{\mathcal{V}}$ is Φ-ω-recognizable (resp. an ω-recognizable step function).

Proof. For the first claim, we use induction on the structure of φ. For our second claim, we use the same arguments as in Proposition 39 taking into account Propositions 52, 53 and the fact that the tree language $T^{\omega,v}_{\Sigma_{\mathcal{V}}}$ is ω-recognizable. ■

Definition 57 *A formula $\varphi \in MSO(\mathbb{R}_{\max}, \Sigma)$ is called* restricted *if whenever φ contains a universal first order quantification $\forall x \,.\, \psi$, then $\|\psi\|$ is an ω-recognizable step function, and whenever φ contains a subformula $\forall X \,.\, \psi$, then $\|\psi\|$ is the characteristic series of an ω-recognizable tree language.*

Definition 58 *A formula $\varphi \in MSO(\mathbb{R}_{\max}, \Sigma)$ is called* incomplete universal *if*

(i) whenever φ contains a universal first order quantification $\forall x \,.\, \psi$, then ψ may contain universal quantifiers but then it does not contain any constant $k \in \mathbb{R}_+ \cup \{-\infty\}$, and

(ii) not any constant $k \in \mathbb{R}_+ \cup \{-\infty\}$ is in the scope of any universal second order quantifier.

We denote by $RMSO(\mathbb{R}_{\max}, \Sigma)$ (resp. $IUMSO(\mathbb{R}_{\max}, \Sigma)$) the collection of all restricted (resp. incomplete universal) formulas of $MSO(\mathbb{R}_{\max}, \Sigma)$. A tree series $S \in \mathbb{R}_{\max} \langle\langle T^{\omega}_{\Sigma} \rangle\rangle$ is called *$RMSO$-Φ-definable* (resp. *$IUMSO$-Φ-definable*) if there is a sentence $\varphi \in RMSO(\mathbb{R}_{\max}, \Sigma)$ (resp. $\varphi \in IUMSO(\mathbb{R}_{\max}, \Sigma)$) such that $S = \|\varphi\|$. We let $\mathbb{R}^{\Phi-rmso}_{\max} \langle\langle T^{\omega}_{\Sigma} \rangle\rangle$ (resp. $\mathbb{R}^{\Phi-iumso}_{\max} \langle\langle T^{\omega}_{\Sigma} \rangle\rangle$) comprise all $RMSO$-Φ-definable (resp. $IUMSO$-Φ-definable) infinitary tree series.

The main result of this section is the subsequent Rabin-type theorem, namely

Theorem 59 *Let Σ be a ranked alphabet. Then*

$$\mathbb{R}^{\Phi-\omega-rec}_{\max} \langle\langle T^{\omega}_{\Sigma} \rangle\rangle = \mathbb{R}^{\Phi-rmso}_{\max} \langle\langle T^{\omega}_{\Sigma} \rangle\rangle = \mathbb{R}^{\Phi-iumso}_{\max} \langle\langle T^{\omega}_{\Sigma} \rangle\rangle .$$

First, by using induction on the structure of formulas, we shall show the inclusions $\mathbb{R}^{\Phi-rmso}_{\max} \langle\langle T^{\omega}_{\Sigma} \rangle\rangle \subseteq \mathbb{R}^{\Phi-\omega-rec}_{\max} \langle\langle T^{\omega}_{\Sigma} \rangle\rangle$ and $\mathbb{R}^{\Phi-iumso}_{\max} \langle\langle T^{\omega}_{\Sigma} \rangle\rangle \subseteq \mathbb{R}^{\Phi-\omega-rec}_{\max} \langle\langle T^{\omega}_{\Sigma} \rangle\rangle$. The subsequent Lemmas 60-63, are established using similar arguments with the proofs of Lemmas 43-46 in Section 5. Thus, we only indicate the constructions whenever it is required. We denote by $\mathcal{V}$ a finite set of first and second order variables.

Lemma 60 *Let $\varphi \in MSO(\mathbb{R}_{\max}, \Sigma)$ be atomic or the negation of an atomic $MSO(\mathbb{R}_{\max}, \Sigma)$-formula. Then $\|\varphi\|$ is an ω-recognizable step function.*

Proof. Assume that $Free(\varphi) \subseteq \mathcal{V}$. Let $\varphi = k$ for some $k \in \mathbb{R}_+ \cup \{-\infty\}$. Then, $\|\varphi\| = k + 1_{T^{\omega,v}_{\Sigma}}$ which is an ω-recognizable step function. Now, let φ be one of the other atomic formulas or their negations. Obviously, we may consider φ as a classical MSO-formula. Therefore, its infinitary tree language $\mathcal{L}^{\omega}(\varphi)$ is ω-recognizable, hence the tree series $\|\varphi\| = 1_{\mathcal{L}^{\omega}(\varphi)}$ is an ω-recognizable step function. ■

Lemma 61 *Let $\varphi, \psi \in MSO(\mathbb{R}_{\max}, \Sigma)$ such that $\|\varphi\|$ and $\|\psi\|$ are Φ-ω-recognizable tree series (resp. ω-recognizable step functions). Then $\|\varphi \vee \psi\|$ and $\|\varphi \wedge \psi\|$ are also Φ-ω-recognizable tree series (resp. ω-recognizable step functions).*

Proof. We let $\mathcal{V} = Free(\varphi) \cup Free(\psi)$. By definition $\|\varphi \vee \psi\| = \max(\|\varphi\|_{\mathcal{V}}, \|\psi\|_{\mathcal{V}})$ and $\|\varphi \wedge \psi\| = \|\varphi\|_{\mathcal{V}} + \|\psi\|_{\mathcal{V}}$. Then our claim for Φ-ω-recognizable tree series (resp. ω-recognizable step functions) follows from Propositions 52 (resp. 53) and 56. ■

Lemma 62 *Let $\varphi \in MSO(\mathbb{R}_{\max}, \Sigma)$ with $\|\varphi\|$ being Φ-ω-recognizable (resp. an ω-recognizable step function). Then the tree series $\|\exists x \centerdot \varphi\|$ and $\|\exists X \centerdot \varphi\|$ are also Φ-ω-recognizable (resp. ω-recognizable step functions).*

Proof. We use the same arguments as in the proof of Proposition 45 taking into account Propositions 52, 53, and 56. ■

Lemma 63 *Let $\varphi \in MSO(\mathbb{R}_{\max}, \Sigma)$ such that $\|\varphi\|$ is an ω-recognizable step function. Then the tree series $\|\forall x \centerdot \varphi\|$ is Φ-ω-recognizable.*

Proof. Let $\mathcal{W} = Free(\varphi)$ and $\mathcal{V} = Free(\forall x \centerdot \varphi) = \mathcal{W} \setminus \{x\}$. Let also $\|\varphi\| = \max\limits_{1 \le j \le n} \left(k_j + 1_{L_j}\right)$ where $k_j \in \mathbb{R}_+ \cup \{-\infty\}$ and $L_j \subseteq T^{\omega,v}_{\Sigma_{\mathcal{W}}}$ is an ω-recognizable tree language for every $1 \le j \le n$. Furthermore, we assume that the family $(L_j)_{1 \le j \le n}$ is a partition of $T^{\omega,v}_{\Sigma_{\mathcal{W}}}$.

First assume that $x \in \mathcal{W}$. Let $\widetilde{\Sigma} = \Sigma \times \{1, \ldots, n\}$ be the ranked alphabet with $rk_{\widetilde{\Sigma}}(\sigma, j) = rk_{\Sigma}(\sigma)$ for every $(\sigma, j) \in \widetilde{\Sigma}$. Every tree $s \in T^{\omega,v}_{\widetilde{\Sigma}_{\mathcal{V}}}$ can be written as a triple (t, v, ρ) where $(t, \rho) \in T^{\omega,v}_{\Sigma_{\mathcal{V}}}$, and v is a mapping $v : dom(t) \to \{1, \ldots, n\}$ determined by $v(w) = j$ whenever $s(w) = (\sigma, j, f)$ for some $\sigma \in \Sigma$ and $f \in \{0,1\}^{\mathcal{V}}$. Clearly $dom(t) = dom(s)$. We consider the infinitary tree language

$$\widetilde{L} = \left\{(t, v, \rho) \in T^{\omega,v}_{\widetilde{\Sigma}_{\mathcal{V}}} \mid \forall w \in dom(t), \forall j \in \{1, \ldots, n\} \text{ if } v(w) = j, \text{ then } (t, \rho[x \to w]) \in L_j\right\}.$$

Observe that for every $(t, \rho) \in T^{\omega,v}_{\Sigma_{\mathcal{V}}}$ there is a unique v such that $(t, v, \rho) \in \widetilde{L}$, due to the partitioning of $T^{\omega,v}_{\Sigma_{\mathcal{W}}}$ by $(L_j)_{1 \le j \le n}$.

Next we prove that the tree language $\widetilde{L}$ is ω-recognizable. In fact it holds that $\widetilde{L} = \bigcap\limits_{1 \le j \le n} \widetilde{L}_j$, where for every $j \in \{1, \ldots, n\}$ we let

$$\widetilde{L}_j = \left\{(t, v, \rho) \in T^{\omega,v}_{\widetilde{\Sigma}_{\mathcal{V}}} \mid \forall w \in dom(t) \text{ if } v(w) = j, \text{ then } (t, \rho[x \to w]) \in L_j\right\}.$$

The class of ω-recognizable tree languages is closed under intersection, thus it suffices to show that every tree language $\widetilde{L}_j$ is ω-recognizable. To this end we fix a $j \in \{1, \ldots, n\}$. Since L_j is ω-recognizable, there is an MSO-sentence φ_j over $\Sigma_{\mathcal{W}}$ such that $L_j = \mathcal{L}(\varphi_j)$. Now we construct from φ_j a sentence φ'_j over $\widetilde{\Sigma}_{\mathcal{W}}$ by replacing every occurrence of $label_{(\sigma,s)}(y)$ in φ_j where $\sigma \in \Sigma$ and $s \in \{0,1\}^{\mathcal{W}}$ by $\bigvee\limits_{1 \le l \le n} label_{(\sigma,l,s)}(y)$. For every $(t, v, \tau) \in T^{\omega,v}_{\widetilde{\Sigma}_{\mathcal{W}}}$ we conclude that $(t, v, \tau) \models \varphi'_j$ iff $(t, \tau) \models \varphi_j$. Next, we modify φ'_j to obtain a formula φ''_j over $\widetilde{\Sigma}_{\mathcal{V}}$, as follows. Every occurrence of $label_{(\sigma,l,s)}(y)$ in φ'_j is replaced by $label_{(\sigma,l,s')}(y) \wedge (x = y)$ if $s(x) = 1$, and by $label_{(\sigma,l,s')}(y) \wedge \neg(x = y)$ if $s(x) = 0$, where s' is the restriction of s to $\mathcal{V} = \mathcal{W} \setminus \{x\}$. Then φ''_j has x as its only free variable. Moreover, for every $(t, v, \rho)[x \to w] \in \left(T^{\omega,v}_{(\widetilde{\Sigma}_{\mathcal{V}})_{\varphi''_j}}\right)$ and $w \in dom(t)$ we have

$$(t, v, \rho)[x \to w] \models \varphi''_j \quad \text{iff} \quad (t, v, \rho[x \to w]) \models \varphi'_j.$$

We set

$$\widetilde{\varphi_j} = \forall x \,.\, \left(\left(\bigvee_{(\sigma,s')\in\Sigma_{\mathcal{V}}} label_{(\sigma,j,s')}(x)\right) \to \varphi''_j\right)$$

where $\to$ is the usual implication. Then $\widetilde{\varphi_j}$ is an MSO-sentence over $\widetilde{\Sigma}_{\mathcal{V}}$. We claim that $\mathcal{L}(\widetilde{\varphi_j}) = \widetilde{L}_j$.

Indeed, let $(t,v,\rho) \in T^{\omega,v}_{\widetilde{\Sigma}_{\mathcal{V}}}$. Then $(t,v,\rho) \models \widetilde{\varphi_j}$ iff for every $w \in dom(t)$, whenever $v(w) = j$, then

$$(t,v,\rho)[x \to w] \models \varphi''_j.$$

As already noted, the latter holds

$$\begin{aligned}
&\text{iff} \quad (t,v,\rho[x \to w]) \models \varphi'_j \\
&\text{iff} \quad (t,\rho[x \to w]) \models \varphi_j \\
&\text{iff} \quad (t,\rho[x \to w]) \in L_j.
\end{aligned}$$

This implies our claim, and hence $\widetilde{L}_j$ is ω-recognizable.

Let $\widetilde{\mathcal{M}} = (Q, \widetilde{\Sigma}_{\mathcal{V}}, q_0, \Delta, \mathcal{F})$ be an mta accepting $\widetilde{L}$. We consider the wmta $\mathcal{M} = (Q, in, wt, \mathcal{F})$ over $\widetilde{\Sigma}_{\mathcal{V}}$ and $\mathbb{R}_{\max}$, with its initial distribution in defined by

$$in(q) = \begin{cases} 0 & \text{if } q = q_0 \\ -\infty & \text{otherwise} \end{cases}$$

for every $q \in Q$, and its weight assigning mapping wt determined for every $m \geq 0, (\sigma, j, f) \in \left(\widetilde{\Sigma}_{\mathcal{V}}\right)_m, q, q_1, \ldots, q_m \in Q$ by

$$wt(q, (\sigma,j,f), (q_1,\ldots,q_m)) = \begin{cases} k_j & \text{if } (q, (\sigma,j,f), (q_1,\ldots,q_m)) \in \Delta \\ -\infty & \text{otherwise.} \end{cases}$$

Let $(t,v,\rho) \in T^{\omega,v}_{\widetilde{\Sigma}_{\mathcal{V}}}$. If $(t,v,\rho) \in \widetilde{L}$, then for every run $r_{(t,v,\rho)} \in R_{\mathcal{M}}((t,v,\rho)), 1 \leq j \leq n, w \in dom\,(t)$ with $v\,(w) = j$, we have $wt\left(r_{(t,v,\rho)}, w\right) = k_j$ and $(t, \rho\,[x \to w]) \in L_j$, which in turn implies that $\left(\|\varphi\|_{\mathcal{V}\cup\{x\}}, (t, \rho\,[x \to w])\right) = k_j$. Moreover

$$(\|\mathcal{M}\|, (t,v,\rho)) = \begin{cases} in\left(r_{(t,v,\rho)}(\varepsilon)\right) + \displaystyle\sum_{w \in dom((t,v,\rho))} \frac{p_w^{(t,v,\rho)}}{\deg(\Sigma)^{|w|}} wt\left(r_{(t,v,\rho)}, w\right) & \text{if } (t,v,\rho) \in \widetilde{L} \\ -\infty & \text{otherwise.} \end{cases}$$

If $(t,v,\rho) \notin \widetilde{L}$, then $weight_{\mathcal{M}}(r_{(t,v,\rho)}) = -\infty$. Consider the relabeling $h : \widetilde{\Sigma}_{\mathcal{V}} \to \Sigma_{\mathcal{V}}$ given by $h\,((\sigma,j,f)) = (\sigma, f)$ for every $(\sigma,j,f) \in \widetilde{\Sigma}_{\mathcal{V}}$. Then, for every $(t,\rho) \in T^{\omega,v}_{\Sigma_{\mathcal{V}}}$ we have

$$\begin{aligned}
(h\,(\|\mathcal{M}\|), (t,\rho)) &= \sup_{(t,v,\rho)\in h^{-1}((t,\rho))} ((\|\mathcal{M}\|, (t,v,\rho))) = (\|\mathcal{M}\|, (t,v,\rho)) \quad (\text{where } (t,v,\rho) \in \widetilde{L}) \\
&= in\left(r_{(t,v,\rho)}(\varepsilon)\right) + \sum_{w \in dom((t,v,\rho))} \frac{p_w^{(t,v,\rho)}}{\deg(\Sigma)^{|w|}} wt\left(r_{(t,v,\rho)}, w\right) \\
&= \sum_{w \in dom(t)} \frac{p_w^t}{\deg(\Sigma)^{|w|}} \left(\|\varphi\|_{\mathcal{V}\cup\{x\}}, (t, \rho\,[x \to w])\right) \\
&= (\|\forall x \,.\, \varphi\|, (t,\rho))\,.
\end{aligned}$$

We conclude that $\|\forall x \,.\, \varphi\| = h(\|\mathcal{M}\|)$ which by Proposition 52 implies that $\|\forall x \,.\, \varphi\|$ is Φ-ω-recognizable.

Now, assume that $x \notin \mathcal{W}$, hence $\mathcal{V} = \mathcal{W}$. We consider the formula $\varphi' = \varphi \wedge (x = x)$. By Lemmas 60, 61 we have that $\|\varphi'\|$ is Φ-ω-recognizable, in particular $\|\varphi'\|$ is an ω-recognizable step function. Thus $\|\forall x \,.\, \varphi'\|$ is Φ-ω-recognizable by what we have shown above. Clearly $\|\varphi'\|_{\mathcal{V}\cup\{x\}} = \|\varphi\|_{\mathcal{V}\cup\{x\}}$ and so $\|\forall x \,.\, \varphi'\| = \|\forall x \,.\, \varphi\|$ which concludes our proof. ■

Lemma 64 *[47] Let $\varphi \in MSO(\mathbb{R}_{\max}, \Sigma)$ with $\|\varphi\| = 1_L$, where $L \subseteq T^{\omega,v}_{\Sigma_\varphi}$ is an ω-recognizable tree language. Then, the tree series $\|\forall X \,.\, \varphi\|$ is a Muller recognizable step function.*

Proof. First, let us assume that $X \in Free(\varphi)$. Since L is ω-recognizable, by Rabin's theorem there exists an MSO-formula ψ with $Free(\varphi) = Free(\psi)$, such that $\mathcal{L}(\psi) = L$. Without any loss, we assume that if any negation occurs in ψ then it is applied to an atomic formula. Then, the infinitary tree language $\mathcal{L}(\forall X \,.\, \psi)$ is ω-recognizable. Moreover, ψ can be considered as a weighted formula in $MSO(\mathbb{R}_{\max}, \Sigma)$, and thus $\|\psi\|_{Free(\varphi)} = 1_L$. Then, for every $(t, \rho) \in T^{\omega,v}_{\Sigma_\varphi}$ we have

$$\begin{aligned}
(\|\forall X \,.\, \varphi\|, (t,\rho)) &= \sum_{I \subseteq dom(t)} \Big(\|\varphi\|_{Free(\varphi)}, (t, \rho[X \to I]) \Big) \\
&= \begin{cases} 0 & \text{if } \Big(\|\varphi\|_{Free(\varphi)}, (t, \rho[X \to I]) \Big) = 0 \;\; \forall I \subseteq dom(t) \\ -\infty & \text{otherwise} \end{cases} \\
&= \begin{cases} 0 & \text{if } (t, \rho[X \to I]) \in L \;\; \forall I \subseteq dom(t) \\ -\infty & \text{otherwise} \end{cases} \\
&= \begin{cases} 0 & \text{if } (t, \rho[X \to I]) \in \mathcal{L}(\psi) \;\; \forall I \subseteq dom(t) \\ -\infty & \text{otherwise} \end{cases} \\
&= \begin{cases} 0 & \text{if } (t, \rho) \in \mathcal{L}(\forall X \,.\, \psi) \\ -\infty & \text{otherwise} \end{cases} \\
&= \left(1_{\mathcal{L}(\forall X . \psi)}, (t,\rho)\right),
\end{aligned}$$

i.e.,

$$\|\forall X \,.\, \varphi\| = 1_{\mathcal{L}(\forall X . \psi)}$$

and so $\|\forall X \,.\, \varphi\|$ is an ω-recognizable step function, as required.

The case $X \notin Free(\varphi)$ is treated in the same way as in the proof of Lemma 63. ■

By combining now Lemmas 60, 61, 62, 63, and 64 we get one half of our Rabin-type theorem.

Proposition 65 $\mathbb{R}_{\max}^{\Phi-rmso} \langle\langle T_\Sigma^\omega \rangle\rangle \subseteq \mathbb{R}_{\max}^{\Phi-\omega-rec} \langle\langle T_\Sigma^\omega \rangle\rangle$ *and* $\mathbb{R}_{\max}^{\Phi-iumso} \langle\langle T_\Sigma^\omega \rangle\rangle \subseteq \mathbb{R}_{\max}^{\Phi-\omega-rec} \langle\langle T_\Sigma^\omega \rangle\rangle$.

Next, we wish to show also the converse inclusions. More precisely, we shall prove the subsequent proposition.

Proposition 66 $\mathbb{R}_{\max}^{\Phi-\omega-rec} \langle\langle T_\Sigma^\omega \rangle\rangle \subseteq \mathbb{R}_{\max}^{\Phi-rmso} \langle\langle T_\Sigma^\omega \rangle\rangle \cap \mathbb{R}_{\max}^{\Phi-iumso} \langle\langle T_\Sigma^\omega \rangle\rangle$.

For our proof we shall need the following auxiliary result.

Lemma 67 *[47] For every classical MSO-formula φ we can effectively construct two $MSO(\mathbb{R}_{\max}, \Sigma)$-formulas φ^+ and φ^- such that $\|\varphi^+\| = 1_{\mathcal{L}^\omega(\varphi)}$ and $\|\varphi^-\| = 1_{\mathcal{L}^\omega(\neg\varphi)}$.*

Then, for every classical MSO-formulas φ, ψ we have

- $\varphi \to \psi = \varphi^- \vee (\varphi^+ \wedge \psi)$.

We shall also need some more macros. First, for a set variable X we set $downward-closed(X) = \forall z \,\mathbf{.}\, (z \in X \to (\forall z' \,\mathbf{.}\, ((edge_1(z, z') \vee \ldots \vee edge_{\deg(\Sigma)}(z, z')) \to z' \in X)))$.

Then

- $x < y = \neg(x = y) \wedge \forall X \,\mathbf{.}\, ((x \in X \wedge downward-closed(X)) \to y \in X)$.
- $x \leq y = (x < y) \vee (x = y)$.
- $X \subseteq Y = \forall x \,\mathbf{.}\, (x \in X \to x \in Y)$.
- $(X = Y) = (X \subseteq Y) \wedge (Y \subseteq X)$.
- $Chain(X) = \forall x \forall y \,\mathbf{.}\, ((x \in X \wedge y \in X) \to (x < y \vee x = y \vee y < x))$.
- $Path(X) = Chain(X) \wedge \forall Y \,\mathbf{.}\, ((Chain(Y) \wedge (X \subseteq Y)) \to (X = Y))$.

Note that by Lemma 67 all the above defined formulas are in $RMSO(\mathbb{R}_{\max}, \Sigma)$. Furthermore, they are contained in $IUMSO(\mathbb{R}_{\max}, \Sigma)$.

Proof of Proposition 66. Let $\mathcal{M} = (Q, in, wt, \mathcal{F})$ be a wmta over Σ and $\mathbb{R}_{\max}$ and let $T = \bigcup_{k \geq 0} Q \times \Sigma_k \times Q^k$ be the collection of all transitions of $\mathcal{M}$. We abbreviate every transition $(q, \sigma, (q_1, \ldots, q_k))$ of T by $(q, \sigma, \overrightarrow{q})$ and we set $\mathcal{V} = \left\{X_{q,\sigma,\overrightarrow{q}} \mid (q, \sigma, \overrightarrow{q}) \in T\right\}$. Let $X_1, \ldots, X_n$ be an enumeration of $\mathcal{V}$ where $n = \sum_{k \geq 0} |Q|^{k+1} \cdot |\Sigma_k|$. We consider the formula

$$\psi_1(x) = \bigwedge_{\substack{(q,\sigma,\overrightarrow{q}) \in T \\ \overrightarrow{q} = (q_1, \ldots, q_k), k > 0}} \left(\begin{array}{c} \left(x \in X_{q,\sigma,\overrightarrow{q}}\right) \to \exists y_1 \ldots \exists y_k \,\mathbf{.} \\ \left(\begin{array}{c} edge(x, (y_1, \ldots, y_k)) \wedge \\ \left(\bigvee_{\substack{(q_1,\sigma_1,\overrightarrow{p}_1) \in T, \\ \vdots \\ (q_k,\sigma_k,\overrightarrow{p}_k) \in T}} \left(y_1 \in X_{q_1,\sigma_1,\overrightarrow{p}_1}\right) \wedge \ldots \wedge \left(y_k \in X_{q_k,\sigma_k,\overrightarrow{p}_k}\right) \right) \end{array} \right) \end{array} \right)$$

and we let

$$\psi(X_1, \ldots, X_n) = partition(X_1, \ldots, X_n) \wedge \bigwedge_{(q,\sigma,\overrightarrow{q}) \in T} \forall x \,\mathbf{.}\, \left(\left(x \in X_{q,\sigma,\overrightarrow{q}}\right) \to label_\sigma(x)\right) \wedge \forall x \,\mathbf{.}\, (\psi_1(x) \vee leaf(x)).$$

Note that the $MSO(\mathbb{R}_{\max}, \Sigma)$-formula $\psi(X_1, \ldots, X_n)$ does not contain any set quantification, and whenever it contains a subformula $\forall x \centerdot \varphi$, then by Lemma 67 $\|\varphi\|$ is an ω-recognizable step function. Let now $t \in T_\Sigma^\omega$. Using similar arguments as in the proof of Proposition 49 we can show that there exists a bijection between the set of runs $R_{\mathcal{M}}(t)$ of $\mathcal{M}$ over t, and the set of $(t, \mathcal{V})$-assignments ρ satisfying ψ, i.e., such that $(t, \rho) \models \psi(X_1, \ldots, X_n)$. Let $r_t \in R_{\mathcal{M}}(t)$. We shall denote by ρ_{r_t} the uniquely determined $(t, \mathcal{V})$-assignment associated to r_t. Now we consider the formula

$$\begin{aligned}
\varphi(X_1, \ldots, X_n) = \psi(X_1, \ldots, X_n) \wedge \left(\bigwedge_{(q,\sigma,\overrightarrow{q}) \in T} \forall x \centerdot \left(\left(x \in X_{q,\sigma,\overrightarrow{q}} \right) \to wt(q, \sigma, \overrightarrow{q}) \right) \right) \wedge \\
\left(\exists y \centerdot \left(root(y) \wedge \bigvee_{(q,\sigma,\overrightarrow{q}) \in T} \left(\left(y \in X_{q,\sigma,\overrightarrow{q}} \right) \wedge in(q) \right) \right) \right) \wedge \forall X \centerdot Path(X) \to \\
\left(\bigvee_{F \in \mathcal{F}} \exists x \centerdot (x \in X) \wedge \left(\forall y \centerdot (y \in X \wedge x \leq y) \to \left(\left(\bigvee_{\substack{(q,\sigma,\overrightarrow{q}) \in T \\ q \in F}} y \in X_{q,\sigma,\overrightarrow{q}} \right) \wedge \bigwedge_{q \in F} \exists z \centerdot \left((z \in X) \wedge (y \leq z) \wedge \bigvee_{\substack{\overrightarrow{q} \in Q \\ \sigma \in \Sigma}} z \in X_{q,\sigma,\overrightarrow{q}} \right) \right) \right) \right).
\end{aligned}$$

Intuitively, the semantics of the formula following $\forall X$, checks whether a run r_t of $\mathcal{M}$ over an input tree $t \in T_\Sigma^\omega$ is successful or not, by taking the value 0 or $-\infty$, respectively. Clearly, the whole formula starting with $\forall X$ belongs to $IUMSO(\mathbb{R}_{\max}, \Sigma)$.

Consider now an infinite tree $t \in T_\Sigma^\omega$, a run r_t of $\mathcal{M}$ over t, and let ρ_{r_t} be the uniquely determined assignment associated to r_t. If r_t is not successful, then the semantics of the subformula started with $\forall X$, gives $(\|\varphi\| (X_1, \ldots, X_n), (t, \rho_{r_t})) = -\infty$. Otherwise,

$$\begin{aligned}
&\left(\|\varphi\| (X_1, \ldots, X_n)^+, (t, \rho_{r_t}) \right) \\
&= in(r_t(\varepsilon)) \; + \sum_{\substack{(q,\sigma,\overrightarrow{q}) \in T \\ w \in \rho_{r_t}(X_{q,\sigma,\overrightarrow{q}})}} \frac{p_w^t}{\deg(\Sigma)^{|w|}} wt(q, \sigma, \overrightarrow{q}) \\
&= in(r_t(\varepsilon)) + \sum_{w \in dom(t)} \frac{p_w^t}{\deg(\Sigma)^{|w|}} wt(r_t(w), t(w), (r_t(w1), \ldots, r_t(w.rk_\Sigma(t(w))))) \\
&= in(r_t(\varepsilon)) + \sum_{w \in dom(t)} \frac{p_w^t}{\deg(\Sigma)^{|w|}} wt(r_t, w) \\
&= weight(r_t).
\end{aligned}$$

Now, we let $\xi = \exists X_1 \ldots \exists X_n \centerdot \varphi(X_1, \ldots, X_n)^+$. Clearly $\xi \in RMSO(\mathbb{R}_{\max}, \Sigma) \cap IUMSO(\mathbb{R}_{\max}, \Sigma)$.

We show that $\|\xi\| = \|\mathcal{M}\|$. Indeed, for every $t \in T_\Sigma^\omega$ we get

$$\begin{aligned}
(\|\xi\|, t) &= \sup_{\rho\ (t,\mathcal{V})\text{-}assignment} \left(\|\varphi\| (X_1, \dots, X_n)^+, (t, \rho)\right) \\
&= \sup_{\substack{\rho\ (t,\mathcal{V})\text{-}assignment \\ (t,\rho)\models\psi(X_1,\dots,X_n)}} \left(\|\varphi\| (X_1, \dots, X_n)^+, (t, \rho)\right) \\
&= \sup_{r_t\in R_{\mathcal{M}}(t)} \left(\|\varphi\| (X_1, \dots, X_n)^+, (t, \rho_{r_t})\right) \\
&= \sup_{r_t\in R^{suc}_{\mathcal{M}}(t)} weight(r_t) \\
&= (\|\mathcal{M}\|, t)
\end{aligned}$$

as required. ■

Proof of Theorem 59. It is immediate by Propositions 65 and 66. ■

8 An application to word series

In this section, we reconsider our Kleene theorem of Section 4 for tree series over a-monadic ranked alphabets. Then, we reobtain in a natural way the Kleene-Schützenberger theorem for formal power series with discounting over finite words established by Droste and Kuske in [20].

For the rest of this section, Σ will denote an a-monadic ranked alphabet, K an arbitrary semiring, and Φ a discounting over Σ and K. Here, we can relax the commutativity of K since for products running on trees over a-monadic ranked alphabets, we can consider the obvious order for multiplications. It should be clear that Φ can be considered as a discounting over the alphabet Σ_1 and K (see [22]).

Let $t \in T_\Sigma$. Then $t = \sigma_0(\sigma_1(\dots(\sigma_{n-1}(a))\dots))$ with $\sigma_0, \dots, \sigma_{n-1} \in \Sigma_1, n \geq 1$. For simplicity we shall write $t = \sigma_0 \dots \sigma_{n-1} a$. Clearly for every tree $t \in T_\Sigma$ its domain $dom(t) = \{\varepsilon, 1, \dots, 1^{ht(t)}\}$. We recall from [21] the mapping $flat : T_\Sigma \to \Sigma_1^*$ defined by

$$flat(t) = \begin{cases} \varepsilon & \text{if } t = a \\ \sigma_0 \dots \sigma_{n-1} & \text{if } t = \sigma_0 \dots \sigma_{n-1} a \text{ with } \sigma_0, \dots, \sigma_{n-1} \in \Sigma_1, n \geq 1. \end{cases}$$

Clearly $flat$ is a bijection. Now for every tree series $S \in K\langle\langle T_\Sigma\rangle\rangle$ we define the formal power (word) series $flat(S) \in K\langle\langle \Sigma_1^*\rangle\rangle$ by $(flat(S), u) = \left(S, flat^{-1}(u)\right)$ for every $u \in \Sigma_1^*$. Moreover, for every formal power series $R \in K\langle\langle \Sigma_1^*\rangle\rangle$ we define the tree series $flat^{-1}(R) \in K\langle\langle T_\Sigma\rangle\rangle$ by $\left(flat^{-1}(R), t\right) = (R, flat(t))$ for every $t \in T_\Sigma$. Let now $\mathcal{M} = (Q, in, wt, ter)$ be a wtdta-t over Σ and K. For every tree $t \in T_\Sigma$ and every run r_t of $\mathcal{M}$ over t, the Φ-weight of r_t is given by

$$weight_{\mathcal{M}}(r_t) = in(r_t(\varepsilon)) \cdot \prod_{\varepsilon\leq w\leq 1^{ht(t)}} \Phi_w^t(wt(r_t, w)) \cdot \Phi^t_{1^{ht(t)}} \circ \Phi_a\left(ter\left(r_t\left(1^{ht(t)}\right)\right)\right).$$

Next, we show that for every Φ-recognizable tree series S over Σ and K, the series $flat(S)$ is Φ-recognizable (for the definition of Φ-recognizable word series see [20, 22]).

Lemma 68 *Let $S \in K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle$ be a Φ-recognizable tree series over Σ and K. Then the series $flat(S) \in K\langle\langle \Sigma_1^*\rangle\rangle$ is Φ-recognizable.*

Proof. Let $\mathcal{M} = (Q, in, wt, ter)$ be a wtdta-t over Σ and K accepting S. We consider the weighted automaton $\mathcal{M}' = (Q, in, wt', ter')$ over Σ_1 and K, where $wt' = wt|_{Q\times\Sigma_1\times Q}$ and $ter'(q) = wt(q,a)\cdot\Phi_a(ter(q))$ for every $q \in Q$. Let $t = \sigma_0 \ldots \sigma_{n-1}a \in T_\Sigma$ with $n \geq 1$. For every run r_t of $\mathcal{M}$ over t there exists a unique path $P^{r_t}_{flat(t)}$ of $\mathcal{M}'$ over $flat(t)$ obtained from r_t. More precisely,

$$P^{r_t}_{flat(t)} = (r_t(\varepsilon), t(\varepsilon), r_t(1)) \ldots \left(r_t\left(1^{ht(t)-1}\right), t\left(1^{ht(t)-1}\right), r_t\left(1^{ht(t)}\right)\right).$$

Conversely, for every $u = \sigma_0 \ldots \sigma_{n-1} \in \Sigma_1^+$ $(n \geq 1)$ and every path P_u of $\mathcal{M}'$ over u there is a unique run $r^{P_u}_{flat^{-1}(u)}$ of $\mathcal{M}$ over $flat^{-1}(u)$ defined in the obvious way. Moreover, we have

$$\begin{aligned}
&weight_{\mathcal{M}'}\left(P^{r_t}_{flat(t)}\right)\\
&= in(r_t(\varepsilon)) \cdot \prod_{0\leq i\leq |flat(t)|-1} \Phi_{flat(t)_{\leq i}}\left(wt'\left(r_t\left(1^i\right), t\left(1^i\right), r_t\left(1^{i+1}\right)\right)\right) \cdot \Phi_{flat(t)}\left(ter'\left(r_t\left(1^{flat(t)}\right)\right)\right)\\
&= in(r_t(\varepsilon)) \cdot \prod_{\varepsilon\leq w\leq 1^{ht(t)-1}} \Phi^t_w\left(wt\left(r_t\left(1^i\right), t\left(1^i\right), r_t\left(1^{i+1}\right)\right)\right)\\
&\qquad \cdot \Phi^t_{1^{ht(t)}}\left(wt\left(r_t\left(1^{ht(t)}\right), a\right) \cdot \Phi_a\left(ter\left(r_t\left(1^{ht(t)}\right)\right)\right)\right)\\
&= in\left(r_t\left(\varepsilon\right)\right) \cdot \prod_{\varepsilon\leq w\leq 1^{ht(t)}} \Phi^t_w\left(wt\left(r_t, w\right)\right) \cdot \Phi^t_{1^{ht(t)}} \circ \Phi_a\left(ter\left(r_t\left(1^{ht(t)}\right)\right)\right)\\
&= weight_{\mathcal{M}}\left(r_t\right).
\end{aligned}$$

Thus, for every $t = \sigma_0 \ldots \sigma_{n-1}a \in T_\Sigma$ with $n \geq 1$ we have $(\|\mathcal{M}'\|, flat(t)) = (\|\mathcal{M}\|, t)$. On the other hand, if $t = a$, then $flat(t) = \varepsilon$ and

$$\begin{aligned}
(\|\mathcal{M}'\|, \varepsilon) &= \sum_{q\in Q} in(q) \cdot ter'(q)\\
&= \sum_{q\in Q} in(q) \cdot wt(q,a) \cdot \Phi_a\left(ter(q)\right)\\
&= (\|\mathcal{M}\|, a).
\end{aligned}$$

We conclude that $\|\mathcal{M}'\| = flat(S)$, i.e., $flat(S)$ is Φ-recognizable, as required. ■

Next, we show that also the converse result holds. More precisely,

Lemma 69 *Let Φ be a discounting over Σ_1 and K, and $R \in K^{\Phi-rec}\langle\langle \Sigma_1^* \rangle\rangle$. Then, the tree series $flat^{-1}(R) \in K\langle\langle T_\Sigma \rangle\rangle$ is Φ'-recognizable where Φ' is a discounting over Σ and K given by $\Phi'_\sigma = \Phi_\sigma$ for every $\sigma \in \Sigma_1$ and $\Phi'_a = id$.*

Proof. Let $\mathcal{N} = (Q, in, wt, ter)$ be a weighted automaton over Σ_1 and K accepting R. We consider the wtdta-t $\overline{\mathcal{N}} = (Q, in, \overline{wt}, ter)$ over Σ and K, with weight assigning mapping $\overline{wt}$ defined by $\overline{wt}(q,\sigma,q') = wt(q,\sigma,q')$ for every $\sigma \in \Sigma_1, q, q' \in Q$, and $\overline{wt}(q,a) = 1$ for every $q \in Q$. Using similar arguments as in the proof of Lemma 68, we can show that $\left\|\overline{\mathcal{N}}\right\| = flat^{-1}(R)$ which implies that $flat^{-1}(R)$ is a Φ'-recognizable tree series. ■

By combining now Lemmas 68 and 69 we get

Proposition 70 *Let Σ be an a-monadic ranked alphabet, K be an arbitrary semiring, and Φ be a discounting over Σ and K. Then*

$$flat\left(K^{\Phi-rec}\langle\langle T_\Sigma\rangle\rangle\right) = K^{\Phi-rec}\langle\langle \Sigma_1^*\rangle\rangle .$$

In the sequel, we show that the mapping $flat$ "preserves" the Φ-rational series operations. Firstly, let us recall from [20] the Φ-rational operations on word series.

Let $P, R \in K\langle\langle \Sigma_1^*\rangle\rangle$. The Φ-*skew product of* P *and* R is the series $P \cdot_\Phi R \in K\langle\langle \Sigma_1^*\rangle\rangle$ defined by

$$(P \cdot_\Phi R, u) = \sum_{\substack{u_1,u_2\in\Sigma_1^* \\ u_1u_2=u}} (P,u_1)\cdot \Phi_{u_1}((R,u_2))$$

for every $u \in \Sigma_1^*$.

Then, the *nth power* R_Φ^n *of* R is defined inductively by

(i) $R_\Phi^0 = 1\varepsilon$, and

(ii) $R_\Phi^{n+1} = R \cdot_\Phi R_\Phi^n$ for every $n \geq 0$.

Furthermore, if the series R is proper, i.e., $(R,\varepsilon) = 0$, then we define the Φ-*Kleene star* $R_\Phi^* \in K\langle\langle \Sigma_1^*\rangle\rangle$ *of* R by

$$R_\Phi^* = R_\Phi^+ + 1\varepsilon$$

where R_Φ^+ is the formal series over Σ_1 and K determined by $\left(R_\Phi^+, u\right) = \sum_{1\leq n\leq |u|} (R_\Phi^n, u)$ for every $u \in \Sigma_1^*$.

The sum, the Φ-skew product, and the Φ-Kleene star constitute the Φ-*rational series operations* (see [20]). We denote by $K^{\Phi-rat}\langle\langle \Sigma_1^*\rangle\rangle$ the class of Φ-*rational series over* Σ_1 *and* K, i.e., the least subclass of $K\langle\langle \Sigma_1^*\rangle\rangle$ which contains the monomials and is closed under the Φ-rational series operations.

Lemma 71 *Let $S, T \in K\langle\langle T_\Sigma\rangle\rangle$. Then*

(i) $flat(S+T) = flat(S) + flat(T)$.

(ii) $flat(S \cdot_{a,\Phi} T) = flat(S) \cdot_\Phi flat(T)$.

(iii) If S is a-proper, then $flat\left(S_{a,\Phi}^*\right) = flat(S)_\Phi^*$.

Proof. Statement (i) is trivially proved. Thus, we deal with statements (ii) and (iii).

For every $u \in \Sigma_1^*$

$$\begin{aligned}
(flat\,(S \cdot_{a,\Phi} T)\,, u) &= \left(S \cdot_{a,\Phi} T, flat^{-1}(u)\right) \\
&= \sum_{\substack{flat^{-1}(u)=t_1 \cdot_a t_2 \\ t_1(w)=a}} (S,t_1)\Phi_w^t\left((T,t_2)\right) \\
&= \sum_{\substack{flat^{-1}(u)=t_1 \cdot_a t_2 \\ flat(t_1)=u_1, flat(t_2)=u_2}} (flat(S),u_1)\Phi_{u_1}\left((flat(T),u_2)\right) \\
&= \sum_{\substack{u \in \Sigma_1^* \\ u_1u_2=u}} (flat(S),u_1)\Phi_{u_1}\left((flat(T),u_2)\right) \\
&= (flat(S) \cdot_\Phi flat(T), u)\,.
\end{aligned}$$

Therefore, we conclude statement (ii). Finally, in order to prove statement (iii), we firstly show by induction that for every $n \geq 1$ it holds $flat\left(S^n_{a,\Phi}\right) = \sum_{0 \leq i \leq n-1} flat(S)^i_\Phi$. Indeed, for $n=1$ we have $flat\left(S^1_{a,\Phi}\right) = flat\,(S \cdot_{a,\Phi} \mathbf{0} + 1a) = flat(1a) = 1\varepsilon$. Then, for every $n \geq 1$

$$\begin{aligned}
flat\left(S^{n+1}_{a,\Phi}\right) &= flat\left(S \cdot_{a,\Phi} S^n_{a,\Phi} + 1a\right) \\
&= flat\left(S \cdot_{a,\Phi} S^n_{a,\Phi}\right) + flat(1a) \\
&= flat\,(S) \cdot_\Phi flat\left(S^n_{a,\Phi}\right) + flat(1a) \\
&= flat(S) \cdot_\Phi \sum_{0 \leq i \leq n-1} flat(S)^i_\Phi + 1\varepsilon \\
&\overset{(*)}{=} \sum_{1 \leq i \leq n} flat(S)^i_\Phi + 1\varepsilon \\
&= \sum_{0 \leq i \leq n} flat(S)^i_\Phi
\end{aligned}$$

where the equality $(*)$ holds true since the Φ-skew product distributes over sum (see [20]).

Now, we require that $S \in K\left\langle\left\langle T_\Sigma \right\rangle\right\rangle$ is a-proper. Then, clearly $flat(S)$ is proper. Consider a word $u \in \Sigma_1^+$ with $|u| = n$, $n \geq 1$. We have

$$\begin{aligned}
\left(flat\left(S^*_{a,\Phi}\right), u\right) &= \left(S^*_{a,\Phi}, ua\right) = \left(S^{n+1}_{a,\Phi}, ua\right) \\
&= \left(flat\left(S^{n+1}_{a,\Phi}\right), u\right) = \sum_{0 \leq i \leq n} \left(flat(S)^i_\Phi, u\right) \\
&= \left(flat(S)^*_\Phi, u\right).
\end{aligned}$$

Thus, $flat\left(S^*_{a,\Phi}\right) = flat(S)^*_\Phi$ and our proof is completed. ■

Proposition 72 *Let Σ be an a-monadic ranked alphabet, K be an arbitrary semiring, and Φ be a discounting over Σ and K. Then*

$$flat\left(K^{\Phi-rat}\left\langle\left\langle T_\Sigma\right\rangle\right\rangle\right) = K^{\Phi-rat}\left\langle\left\langle \Sigma_1^*\right\rangle\right\rangle.$$

Proof. The inclusion $flat\left(K^{\Phi-rat}\langle\langle T_\Sigma\rangle\rangle\right) \subseteq K^{\Phi-rat}\langle\langle \Sigma_1^*\rangle\rangle$ is obtained by Lemma 71 and the observation that every monomial in $K\langle T_\Sigma\rangle$ is projected by $flat$ to a monomial in $K\langle\Sigma_1^*\rangle$. For the other half of our proposition, we consider the Φ-rational expression of every series $R \in K^{\Phi-rat}\langle\langle \Sigma_1^*\rangle\rangle$. This is composed of finite monomials in $K\langle\Sigma_1^*\rangle$ with finitely many applications of the Φ-rational series operations. Then, taking into account that every monomial in $K\langle\Sigma_1^*\rangle$ is the projection (via the mapping $flat$) of a unique monomial in $K\langle T_\Sigma\rangle$, and using once more Lemma 71 we write R as the projection (via the mapping $flat$) of a Φ-rational tree series in $K^{\Phi-rat}\langle\langle T_\Sigma\rangle\rangle$. ■

Now we are ready to state the Kleene-Schützenberger theorem for formal power series with discounting.

Theorem 73 *[20] Let Σ_1 be an alphabet, K be an arbitrary semiring, and Φ be a discounting over Σ_1 and K. Then*

$$K^{\Phi-rec}\langle\langle \Sigma_1^*\rangle\rangle = K^{\Phi-rat}\langle\langle \Sigma_1^*\rangle\rangle .$$

Proof. We combine Theorem 34 and Propositions 70, 72. ■

References

[1] A. Alexandrakis, S. Bozapalidis, Weighted grammars and Kleene's theorem, *Inform. Process. Lett.* 24(1987) 1-4.

[2] R. Balbes, P. Dwinger, *Distributive Lattices*, University of Missouri Press, 1974.

[3] J. Berstel, C. Reutenauer, Recognizable formal power series on trees, *Theoret. Comput. Sci.* 18(2)(1982) 115–148.

[4] R. Bloem, J. Engelfriet, A comparison of tree transductions defined by monadic second order logic and by attribute grammars, *J. Comput. Systems Sci.* 61(2000) 1–50.

[5] B. Borchardt, *The Theory of Recognizable Tree Series*, PhD Thesis, Dresden University of Technology, 2004.

[6] B. Borchardt, Code selection for tree series transducers, in: *Proceedings of CIAA'04, LNCS* 3317(2005) 57–67.

[7] S. Bozapalidis, Effective construction of the syntactic algebra of a recognizable series on trees, *Acta Inform.* 28(1991) 351–363.

[8] S. Bozapalidis, Equational Elements in Additive Algebras, *Theory of Computing Systems* 32(1999) 1–33.

[9] S. Bozapalidis, G. Rahonis, On two families of forests, *Acta Inform.* 31(1994) 235-260.

[10] S. Bozapalidis, G. Rahonis, On the closure of recognizable tree series under tree homomorphisms, *J. Autom. Lang. Comb.* 10(2005) 2/3 185–202.

[11] G. Bruns, P. Godefroid, Model-checking with multi-valued logics, in: *Proceedings of ICALP 2004, LNCS* 3142(2004) 281–293.

[12] K. Chatterjee, L. Doyen, T.A. Henzinger, Quantitative languages, in: *Proceedings of CSL 2008, LNCS* 5213(2008) 385-400.

[13] K. Chatterjee, L. Doyen, T.A. Henzinger, Composition and alternation for weighted automata, *EPLF Technical Report MTC-REPORT-2008-004.*

[14] M. Chechik, B. Devereux, A. Gurfinkel, Model-checking infinite state-space systems with fine-grained abstractions using SPIN, *Proceedings of SPIN 2001, LNCS* 2057(2001) 16–36.

[15] B. Courcelle, Fundamental properties of infinite trees, *Theoret. Comput. Sci.* 25 (1983) 95–169.

[16] L. de Alfaro, T.A. Henzinger, R. Majumda, Discounting the future in systems theory, in: *Proceeedings of ICALP 2003, LNCS* 2719(2003) 1022-1037.

[17] J. Doner, Tree acceptors and some of their applications, *J. Comput. Systems Sci.* 4(1970) 406–451.

[18] M. Droste, P. Gastin, Weighted automata and weighted logics, *Theoret. Comput. Sci.* 380(2007) 69-86; extended abstract in: *Proceedings of ICALP 2005*, *LNCS* 3580(2005) 513–525.

[19] M. Droste, P. Gastin, Weighted automata and weighted logics, in: *Handbook of Weighted Automata* (M. Droste, W.Kuich, H.Vogler, eds.) Springer-Verlag 2009.

[20] M. Droste, D. Kuske, Skew and infinitary formal power series, *Theoret. Comput. Sci.* 366(2006) 189-227; extended abstract in: *Proceedings of ICALP 2003*, *LNCS* 2719(2003) 426–438.

[21] M. Droste, C. Pech, H. Vogler, A Kleene theorem for weighted tree automata, *Theory of Computing Systems* 38(2005) 1–38.

[22] M. Droste, G. Rahonis, Weighted automata and weighted logics with discounting, *Theoret. Comput. Sci.,* to appear; extended abstract in: *Proceedings of CIAA 2007*, *LNCS* 4783(2007) 73-84.

[23] M. Droste, J. Sakarovitch, H. Vogler, Weighted automata with discounting, *Inform. Process. Lett.* 108(2008) 23-28.

[24] M. Droste, H. Vogler, Weighted tree automata and weighted logics, *Theoret. Comput. Sci.* 366(2006) 228–247.

[25] M. Droste, H. Vogler, Weighted logics for unranked tree automata, *manuscript* 2008.

[26] J. Engelfriet, Tree automata and tree grammars, *Technical Report DAIMIFN-10*, Aarhus University, 1975.

[27] J. Engelfriet, E. M. Schmidt, IO and OI. I, *J. Comput. System Sci.,* 15(3)(1977) 328-353.

[28] Z. Ésik, W. Kuich, Formal tree series, *J. Autom. Lang. Comb.* 8(2)(2003) 219–285.

[29] C. Ferdinand, H. Seidl, R. Wilhelm, Tree automata for code selection, *Acta Inform.* 31(8)(1994) 741–760.

[30] J. Filar, K. Vrieze, *Competitive Marcov Decision Processes.* Springer-Verlag, 1997.

[31] Z. Fülöp, A. Maletti, H. Vogler, A Kleene theorem for weighted tree automata over distributive multioperator monoids, *Theory of Computing Systems*, to appear.

[32] Z. Fülöp, H. Vogler, Weighted tree automata and tree series transducers, in: *Handbook of Weighted Automata* (M. Droste, W.Kuich, H.Vogler, eds.) Springer-Verlag 2009.

[33] F. Gécseg, M. Steinby, *Tree Automata*, Akadémiai Kiadó, Budapest 1984.

[34] F. Gécseg, M. Steinby, Tree Languages, in: *Handbook of Formal Languages, Vol. III,* G. Rozenberg and A. Salomaa, eds., Springer-Verlag, 1997, pp.1–68.

[35] S. Golan, *Semirings and their Applications,* Kluwer Academic Publishers 1999.

[36] I. Guessarian, Algebraic Semantics, *LNCS* 99(1981).

[37] A. Gurfinkel, M. Chechik, Multi-valued model checking via classical model checking, in: *Proceedings of CONCUR 2003, LNCS* 2761(2003) 266–280.

[38] D. Harel, Effective transformations on infinite trees, with applications to high undecidability, dominoes, and fairness, *J. ACM* 33(1986) 224–248.

[39] B. Khoussainov, A. Nerode, *Automata Theory and its Applications,* Birkhäuser, Boston, 2001.

[40] W. Kuich, Formal power series on trees, in: *Proceedings of DLT'97 (S. Bozapalidis, ed.),* Aristotle University of Thessaloniki, Thessaloniki 1997, pp.61–101.

[41] W. Kuich, On skew formal power series, in: *Proceedings of the 1st International Conference on Algebraic Informatics,* Thessaloniki 2005, pp.7-30.

[42] O. Louscou-Bozapalidou, Some remarks on recognizable tree series, *Internat. J. Computer Math.* 70(1999) 649–655.

[43] C. Pech, *Kleene-Type Results for Weighted Tree Automata,* PhD Thesis, Dresden University of Technology, 2003.

[44] M.O. Rabin, Decidability of second-order theories and automata on infinite trees, *Trans. Amer. Math. Soc.* 141(1969) 1–35.

[45] J.R. Rico-Juan, J. Carela-Rubio, R. Carracso, Stochastic k-testable tree languages, in: *Proceedings of ICGE 2002, LNAI* 2484(2002) 199–212.

[46] J.R. Rico-Juan, J. Carela-Rubio, R. Carracso, Smoothing and compression with stochastic k-testable tree languages, *Pattern Recognition* 38(2002) 1420–1430.

[47] G. Rahonis, Weighted Muller tree automata and weighted logics, *J. Autom. Lang. Comb.* 12(2007) 455-483.

[48] H. Seidl, Finite tree automata with cost functions, *Theoret. Comput. Sci.* 126(1)(1994) 113-142.

[49] L.S. Shapley, Stochastic games, *Roc. National Acad. of Sciences* 39(1953) 1095-1100.

[50] J.W. Thatcher, J.B. Wright, Generalized finite automata theory with application to a decision problem of second-order logic, *Math. Syst. Theory* 2(1)(1968) 57–81.

[51] W. Thomas, Automata on infinite objects, in: *Handbook of Theoretical Computer Science, vol. B* (J. v. Leeuwen, ed.), Elsevier Science Publishers, Amsterdam 1990, pp. 135–191.

[52] W. Thomas, Languages, automata and logic, in: *Handbook of Formal Languages* vol. 3 (G. Rozenberg, A. Salomaa, eds.), Springer, 1997, pp. 389–485.

[53] M.Y. Vardi, An automata-theoretic approach to linear temporal logic, in: *Logics in Concurrency, LNCS* 1043(1996) 238–266.

[54] W. Wechler, *Universal Algebra for Computer Scientists,* EATCS Monographs on Theoretical Computer Science, vol. 25, Springer-Verlag, 1992.