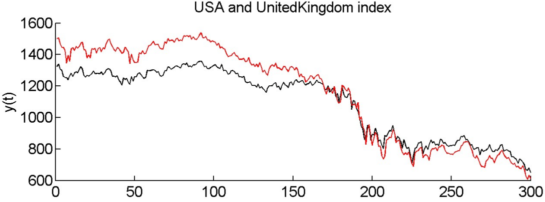
Analysis of multi-variate time series by means of networks

Dimitris Kugiumtzis

January 28, 2015

Spurious cross correlations see [1]: Sec 7.3



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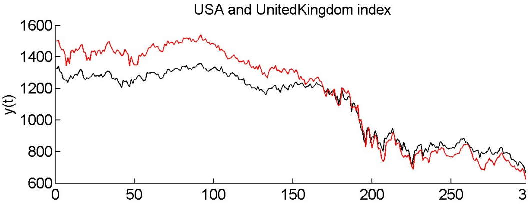

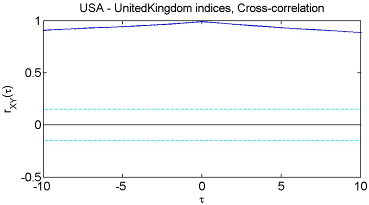

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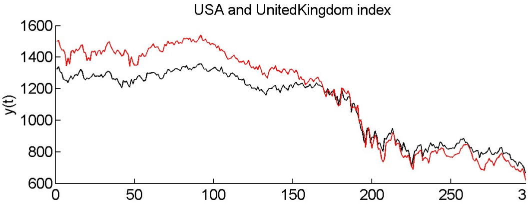

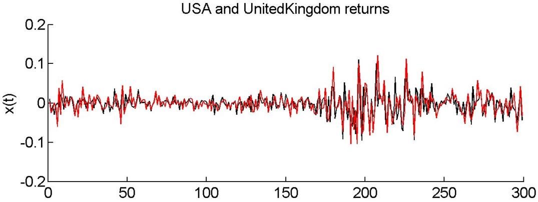

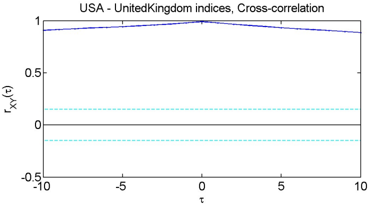

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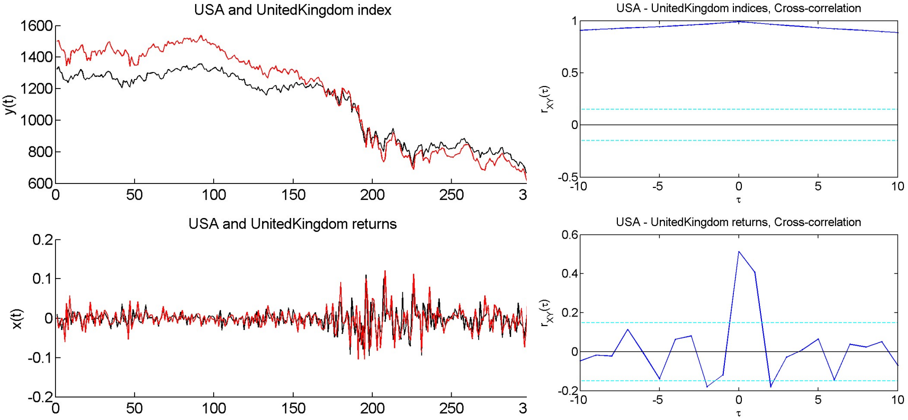

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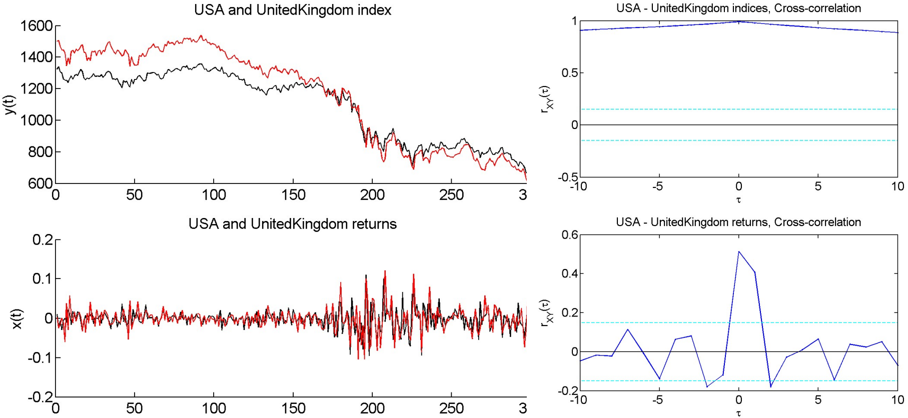

Time series of indices (strongly autocorrelated): large cross-correlation

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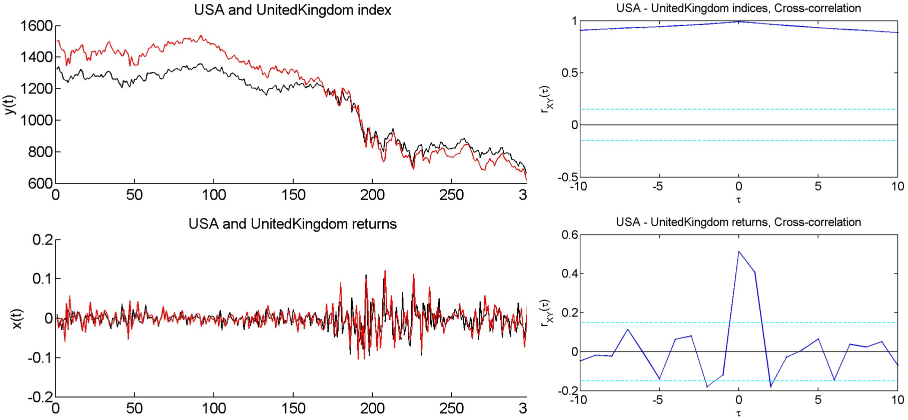

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Time series of returns (weakly or no autocorrelated): small cross-correlation

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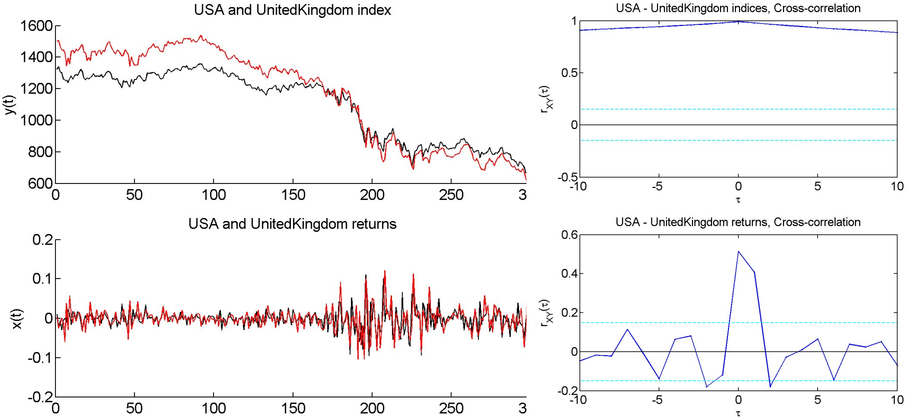

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Autocorrelation may cause spurious cross-correlations

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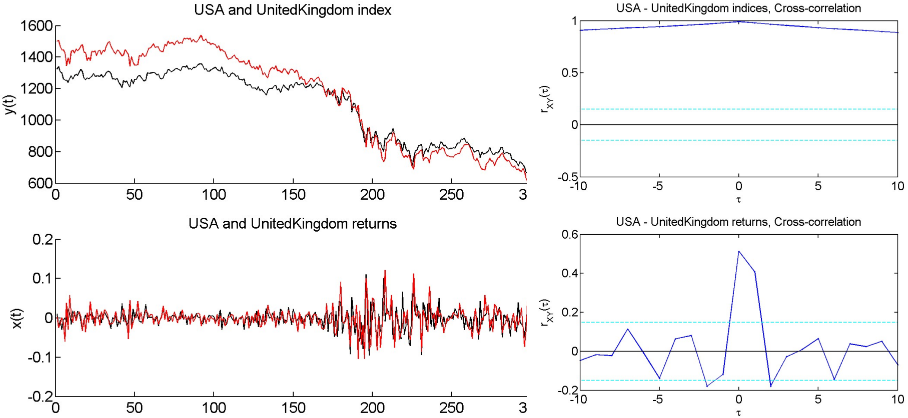

Time series of indices (strongly autocorrelated): large cross-correlation

Time series of returns (weakly or no autocorrelated): small cross-correlation

Autocorrelation may cause spurious cross-correlations

$\Longrightarrow$ prewhiten the time series to have zero autocorrelation.

Example: Two independent AR(1) processes

Time series $\{x_t\}_{t=1}^n$, $\{y_t\}_{t=1}^n$ from two independent AR(1) processes:
$X_t = 0.95X_{t-1} + \epsilon_t^X \quad Y_t = 0.85Y_{t-1} + \epsilon_t^Y$

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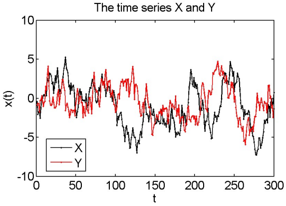

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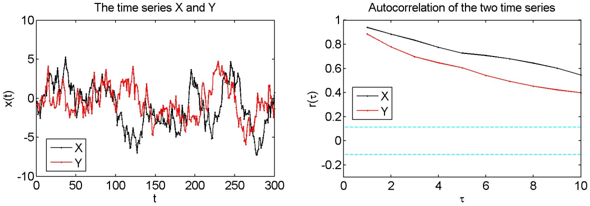

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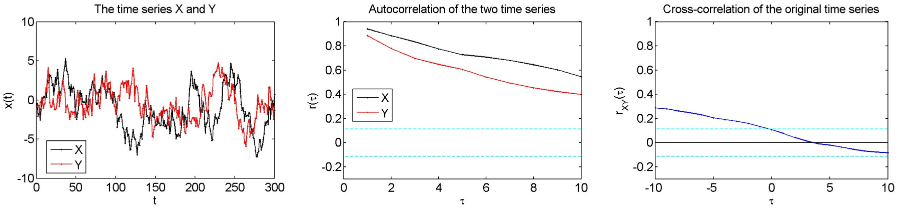

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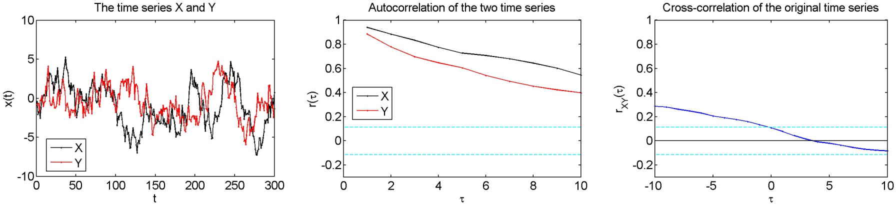

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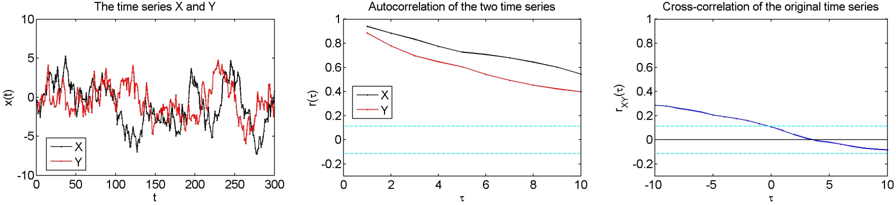

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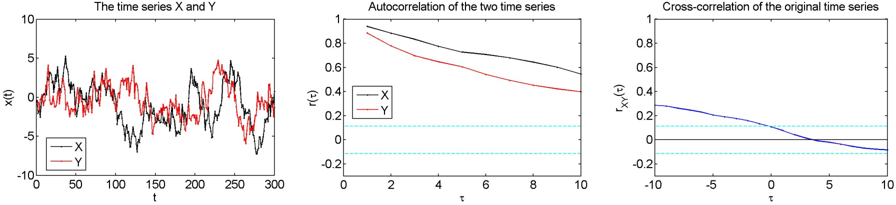

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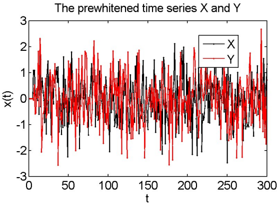

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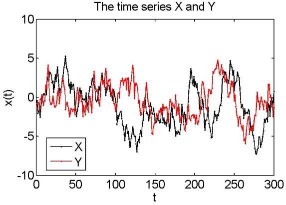

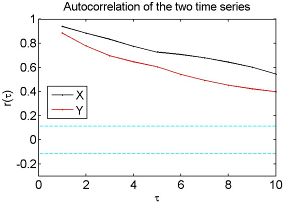

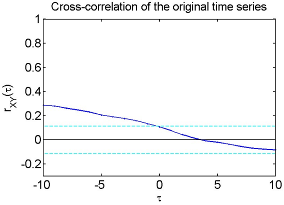

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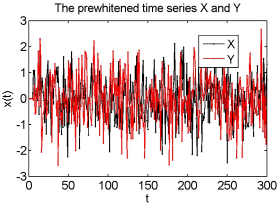

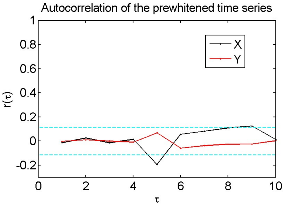

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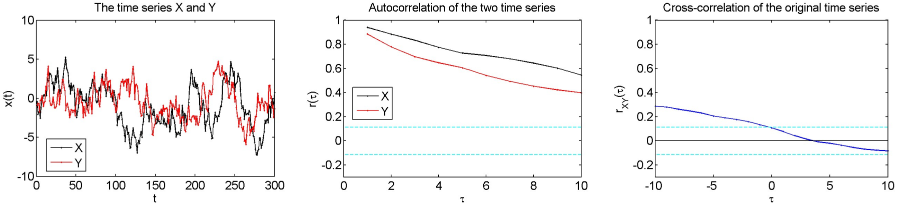

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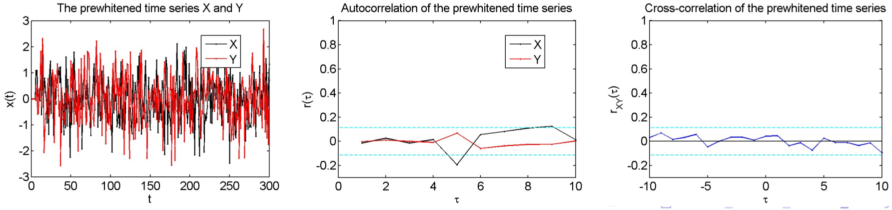

Example: Two dependent AR(1) processes - 1

The first AR(1) process drives the second AR(1) process:

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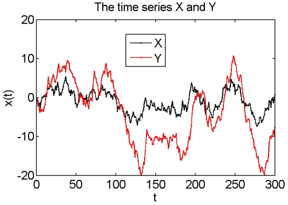

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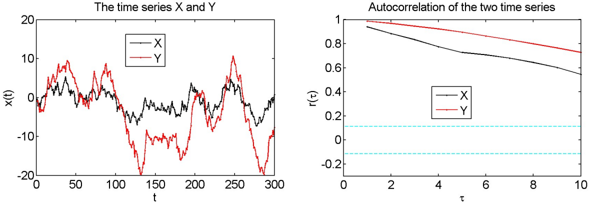

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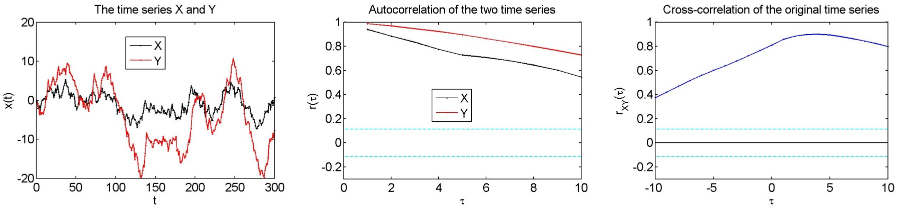

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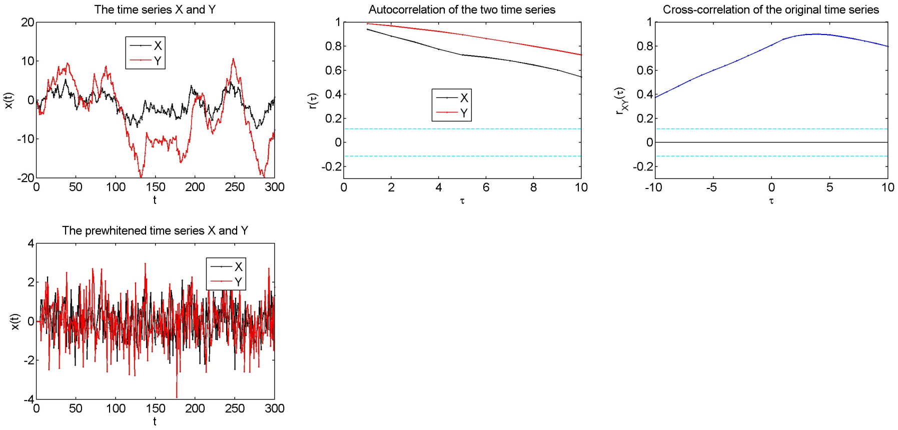

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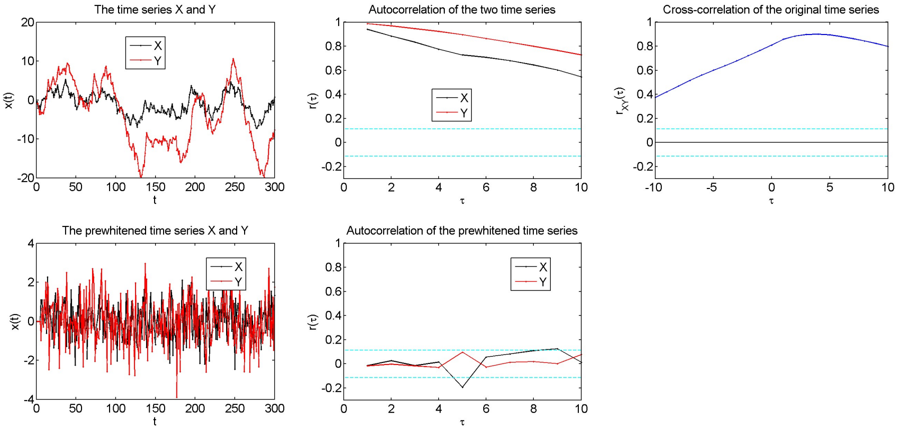

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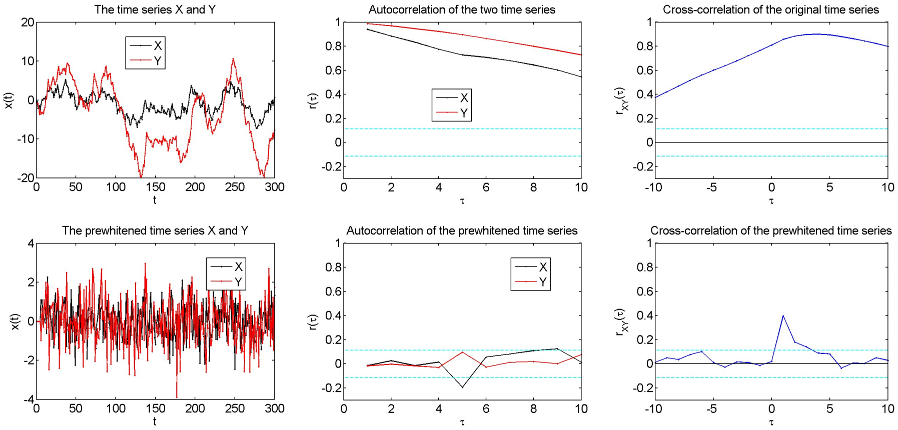

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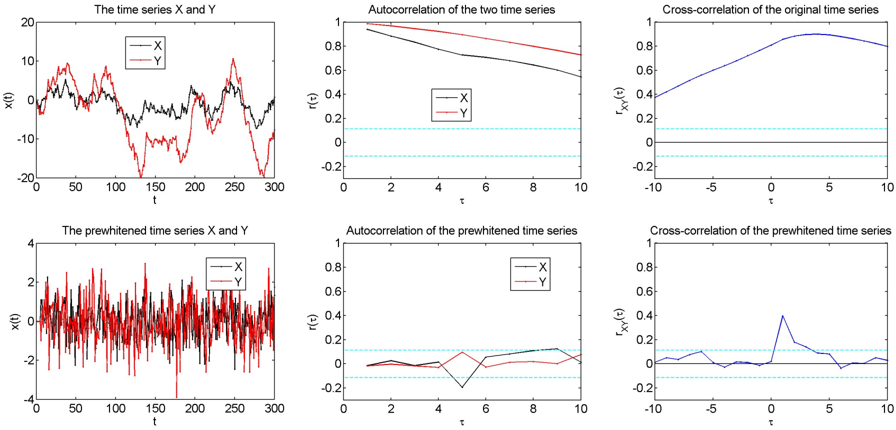

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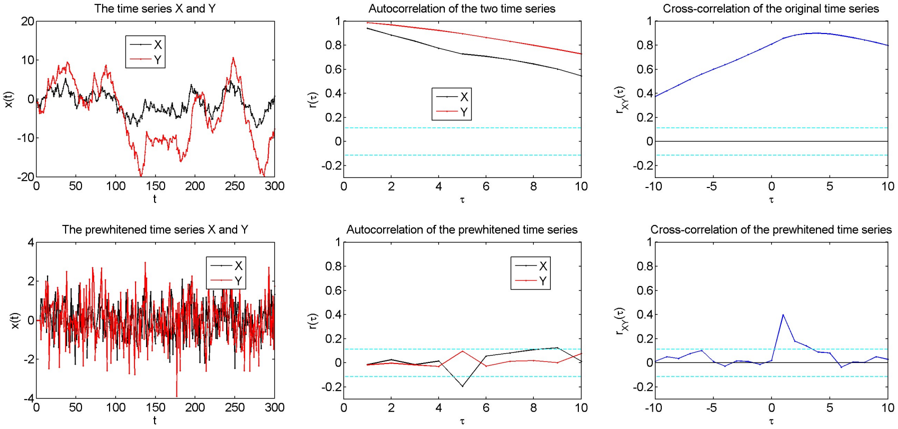

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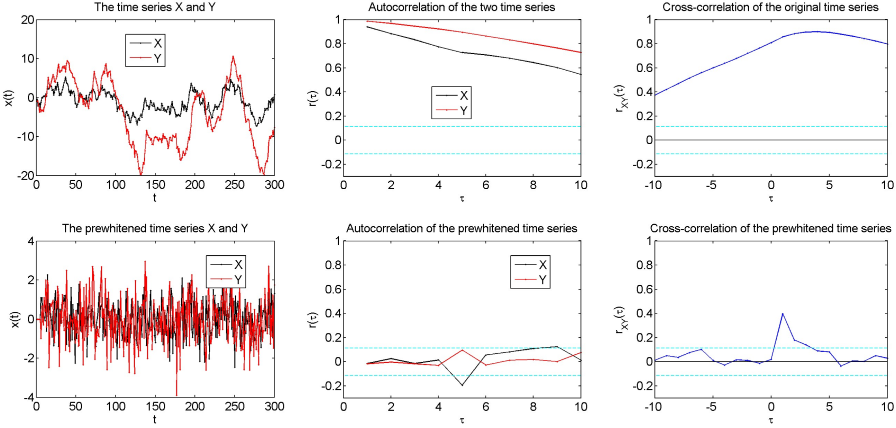

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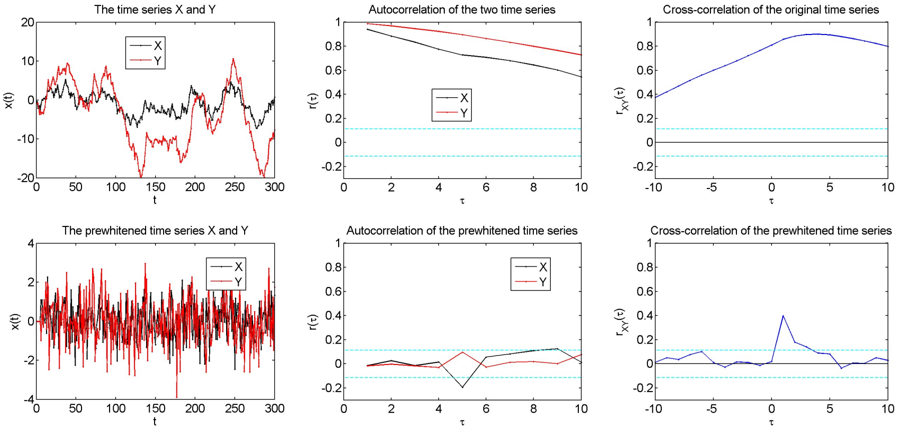

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$\Longrightarrow$ X_t is correlated to $Y_{t+\tau}$, but not the opposite

$\Longrightarrow$ direction of correlation $\Longrightarrow$ (Granger) causality

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The two AR(1) processes are inter-dependent:

$X_t = 1.2X_{t-1} - 0.5Y_{t-1} + \epsilon_t^X \qquad Y_t = 0.6X_{t-1} + 0.3Y_{t-1} + \epsilon_t^Y$

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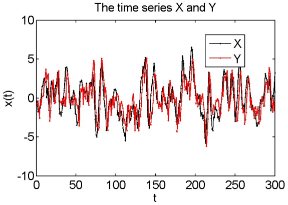

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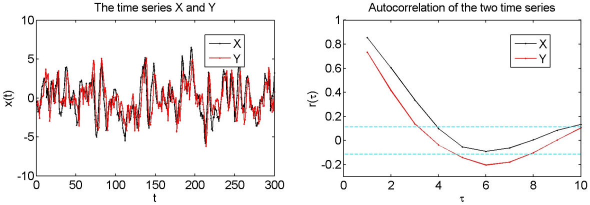

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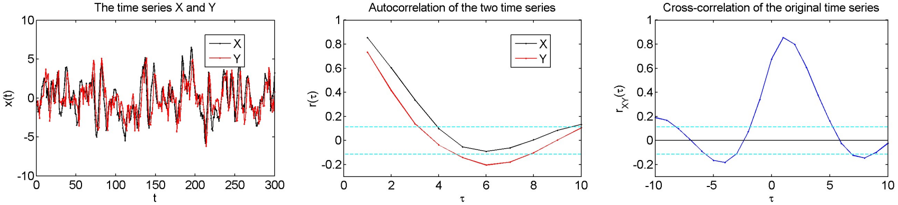

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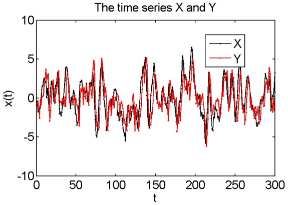

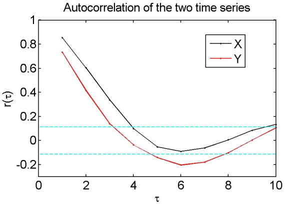

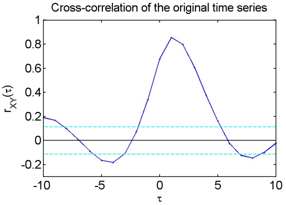

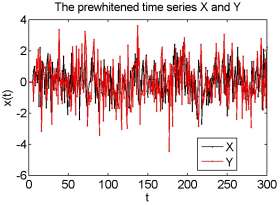

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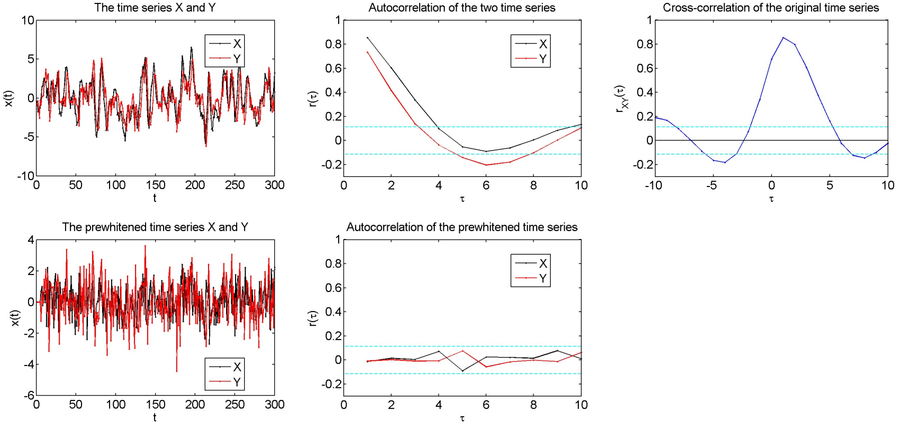

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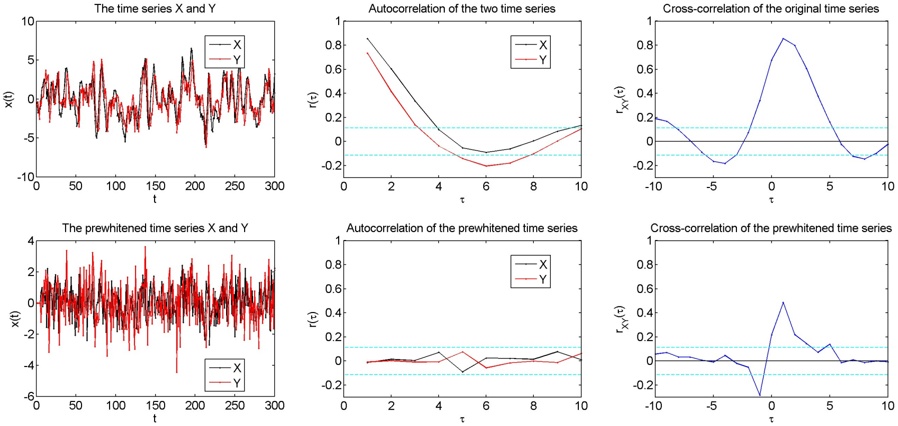

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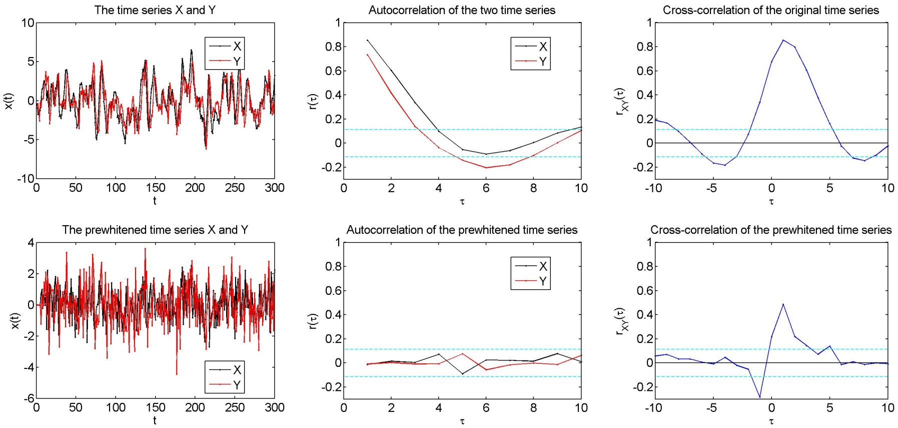

After prewhitening, the statistically significant cross-correlations are for both positive and negative delays

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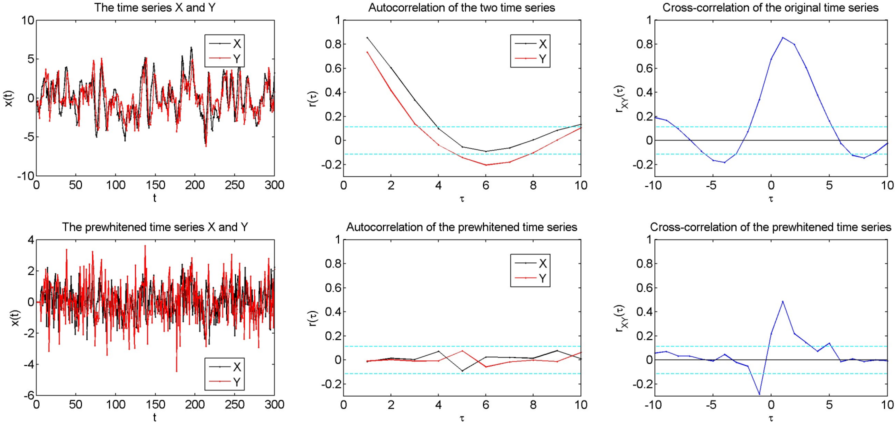

After prewhitening, the statistically significant cross-correlations are for both positive and negative delays

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Example: Two dependent AR(1) processes - 2

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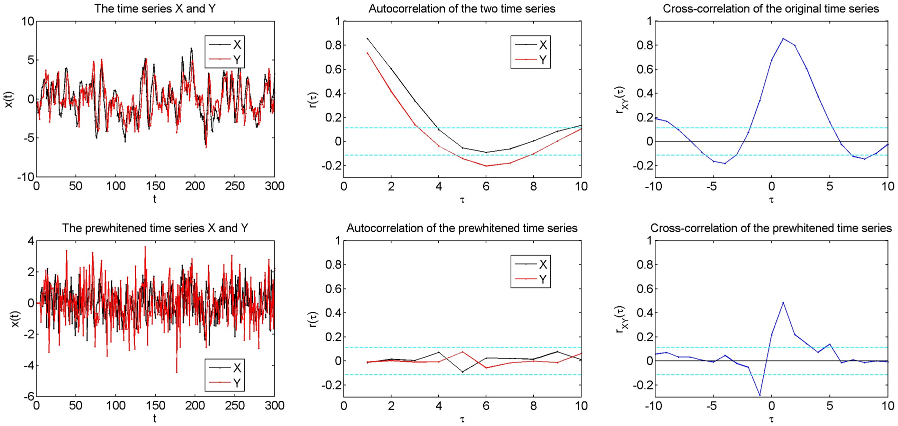

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Given time series $\{x_t\}_{t=1}^n$, $\{y_t\}_{t=1}^n$: see [1]: Chp 12, [2]: Chp 7

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and in matrix form

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Nonlinear dynamical systems

see [1]: Chp 11, [2]: Sec 10.3

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Henon map

$$X_t = 1.4 - X_{t-1}^2 + Y_{t-1} \qquad Y_t = 0.3X_{t-1}$$

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... a nonlinear AR(2) model.

Example: Two independent Henon maps, linear measures

Time series $\{x_t\}_{t=1}^n$, $\{y_t\}_{t=1}^n$, $n = 300$ from two independent Henon maps:
$X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} \quad Y_t = 1.4 - Y_{t-1}^2 + 0.3Y_{t-2}$

Example: Two independent Henon maps, linear measures

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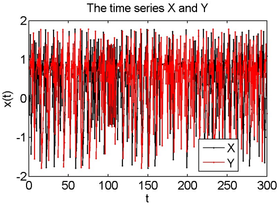

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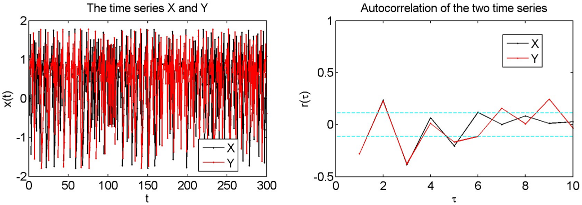

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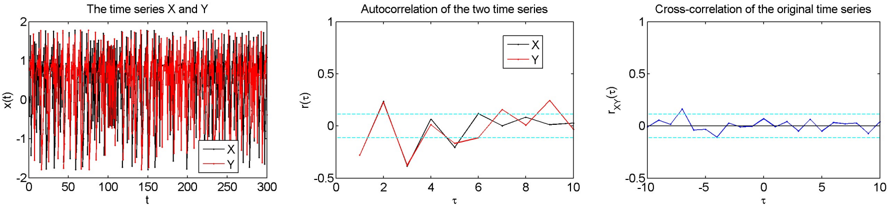

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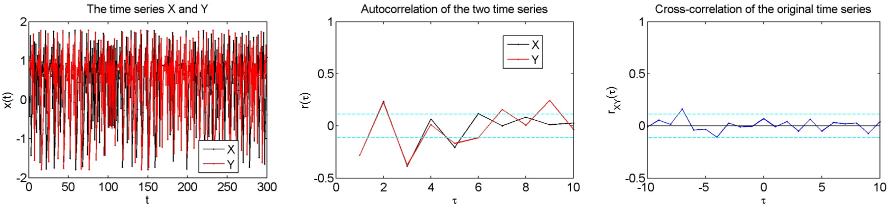

Alternating autocorrelation, zero cross-correlation (correctly!)

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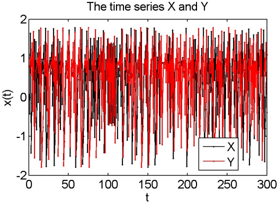

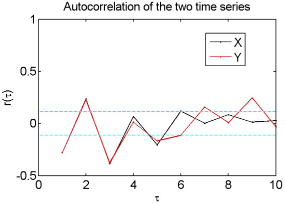

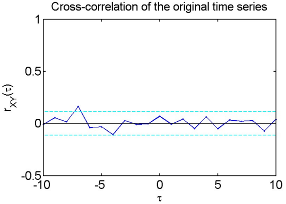

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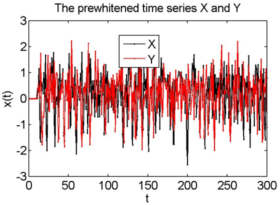

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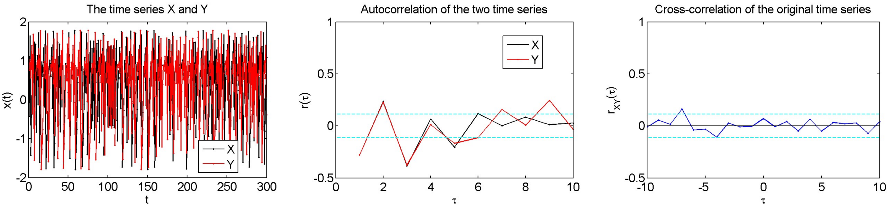

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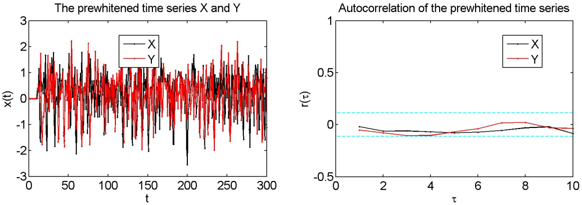

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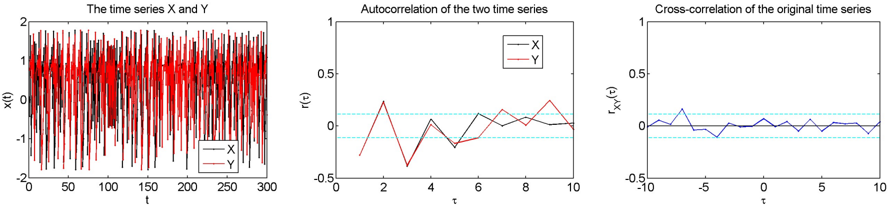

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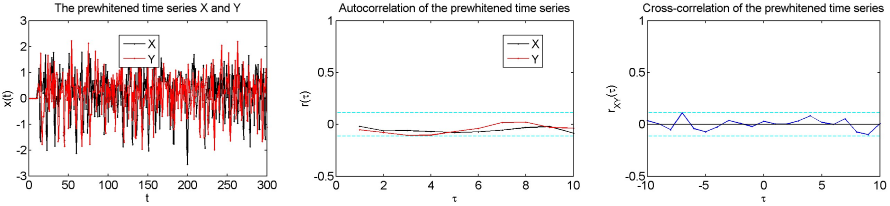

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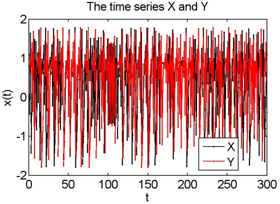

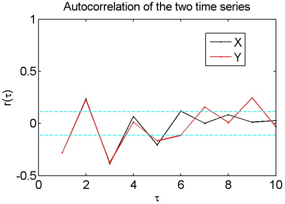

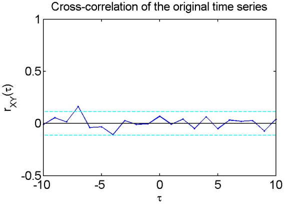

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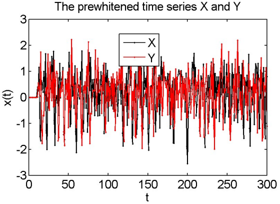

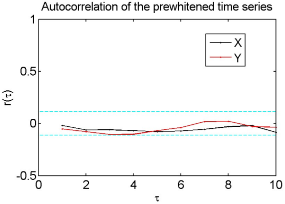

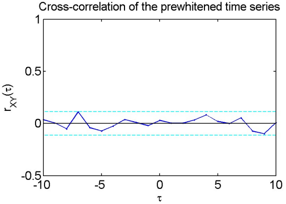

After prewhitening, zero autocorrelation, zero cross-correlation

Example: Independent Henon maps, nonlinear measures

Delayed mutual information $I_X(\tau)$ and $I_Y(\tau)$ and cross mutual information $I_{XY}(\tau)$

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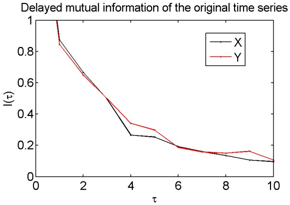

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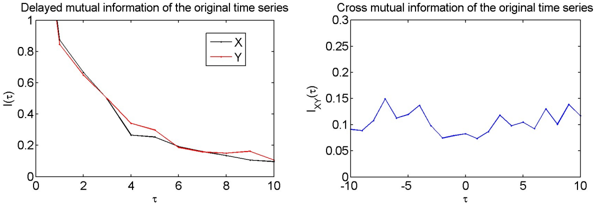

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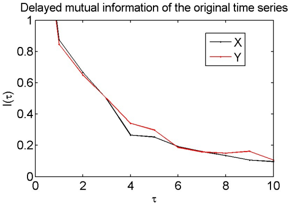

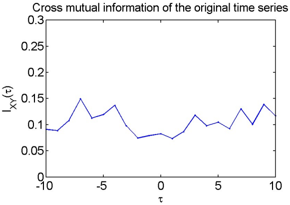

Significant delayed mutual information (for small lags),

Example: Independent Henon maps, nonlinear measures

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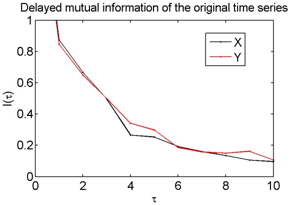

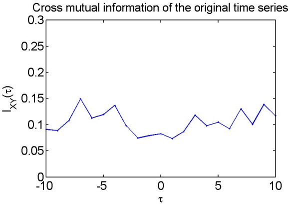

Significant delayed mutual information (for small lags),
Insignificant cross mutual information **?**

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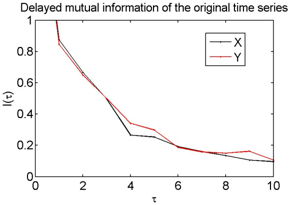

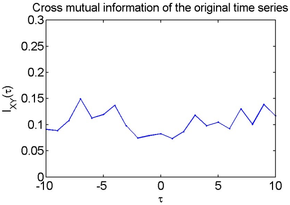

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$I_X(\tau)$, $I_Y(\tau)$ and $I_{XY}(\tau)$ after prewhitening

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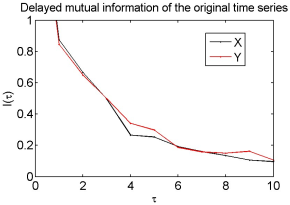

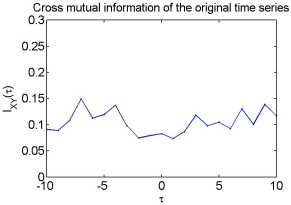

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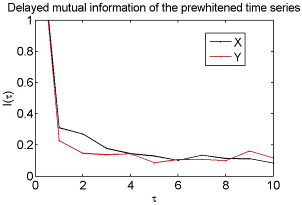

Example: Independent Henon maps, nonlinear measures

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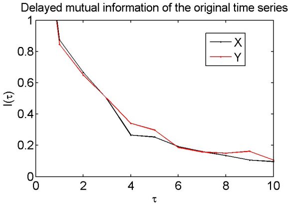

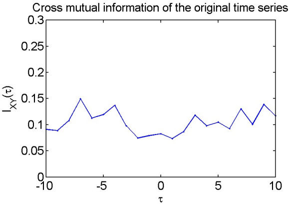

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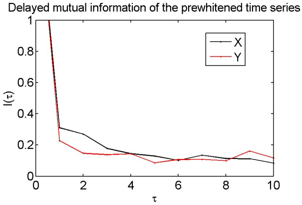

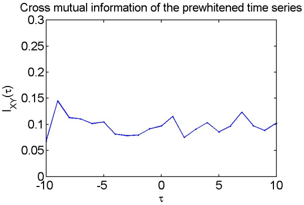

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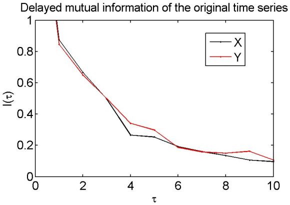

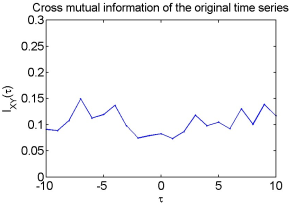

Significant delayed mutual information (for small lags),
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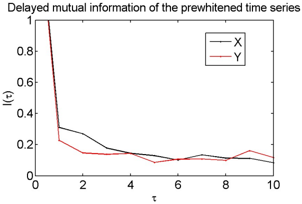
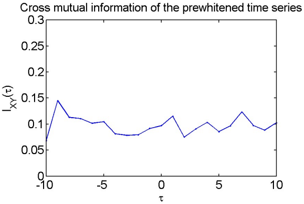
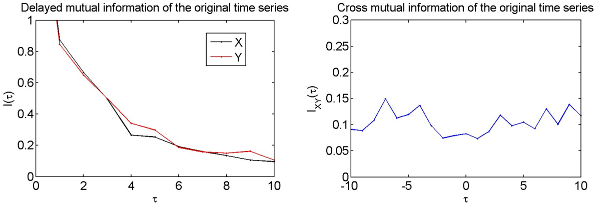




Smaller but still significant delayed mutual information (for small lags),

Example: Independent Henon maps, nonlinear measures

Delayed mutual information $I_X(\tau)$ and $I_Y(\tau)$ and cross mutual information $I_{XY}(\tau)$



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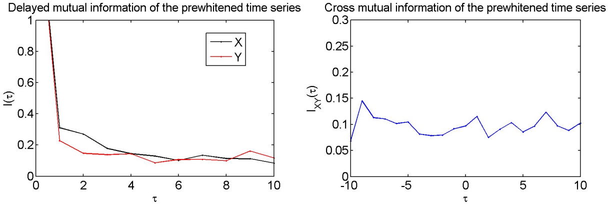

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Example: Two dependent Henon maps - 1

$$X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2}$$
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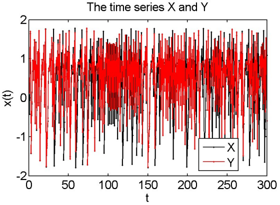

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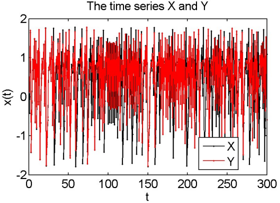

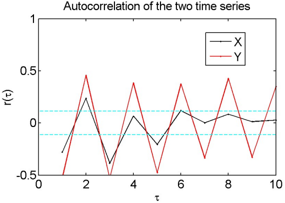

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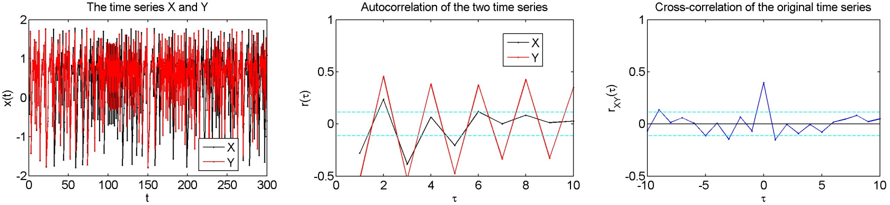

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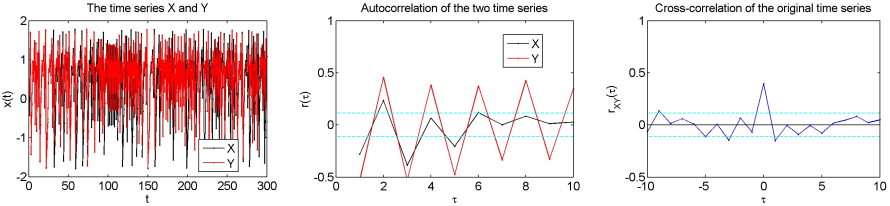

Alternating autocorrelation, significant cross-correlation at $\tau = 0$

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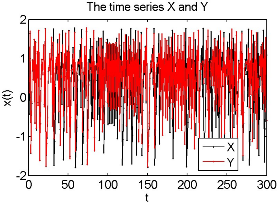

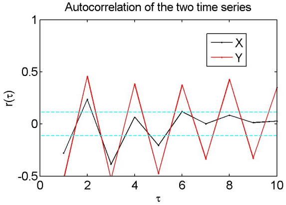

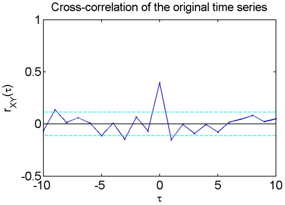

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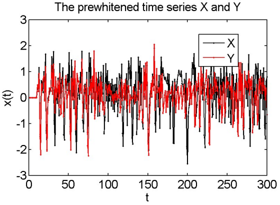

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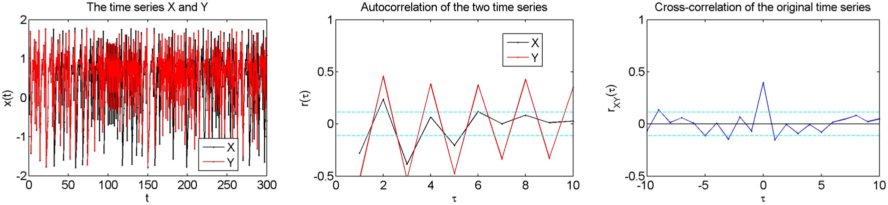

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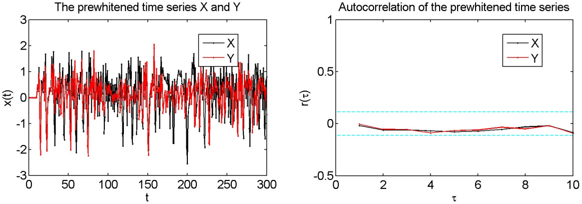

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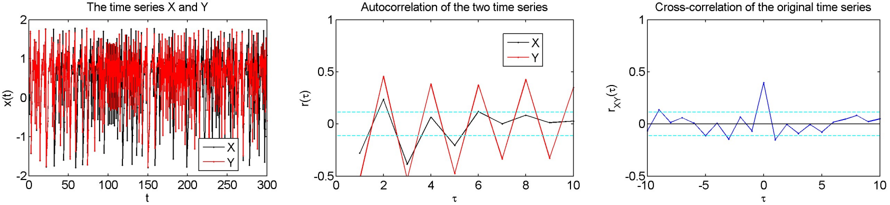

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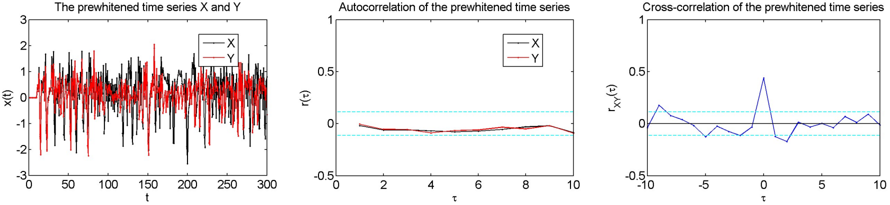

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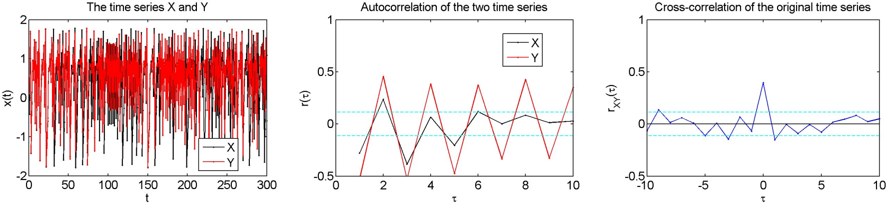
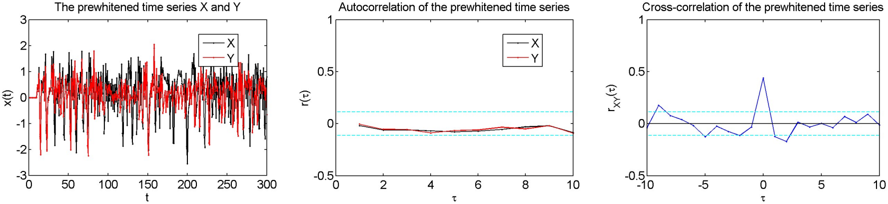
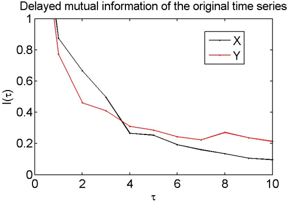


Alternating autocorrelation, significant cross-correlation at $\tau = 0$

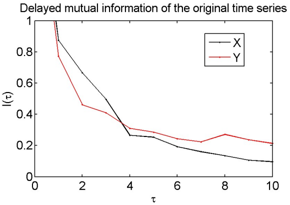


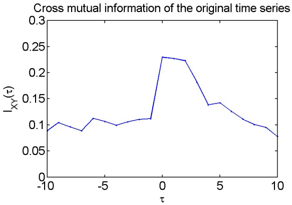
After prewhitening, zero autocorrelation, significant cross-correlation at $\tau = 0$

Example: Two dependent Henon maps - 1

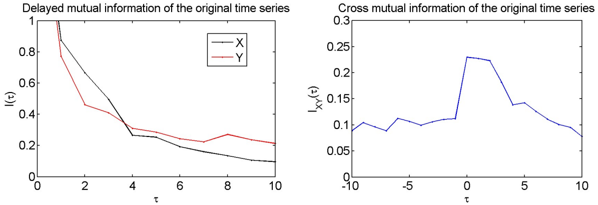


Example: Two dependent Henon maps - 1



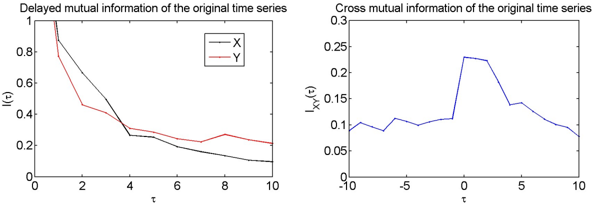


Example: Two dependent Henon maps - 1



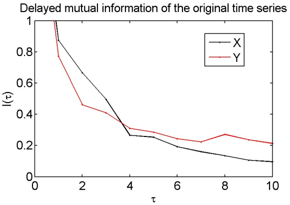
Significant $I_X(\tau)$, $I_Y(\tau)$,

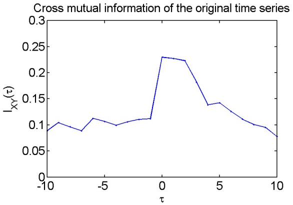
Example: Two dependent Henon maps - 1



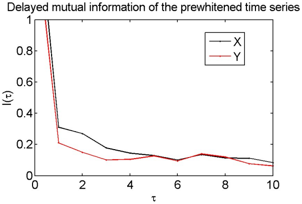
Significant $I_X(\tau)$, $I_Y(\tau)$,
Significant $I_{XY}(\tau)$ for $\tau \geq 0$, X_t is “correlated” to $Y_{t+\tau}$

Example: Two dependent Henon maps - 1

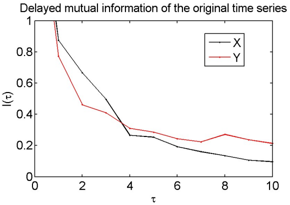


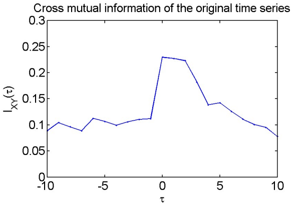


Significant $I_X(\tau)$, $I_Y(\tau)$,
Significant $I_{XY}(\tau)$ for $\tau \geq 0$, X_t is "correlated" to $Y_{t+\tau}$

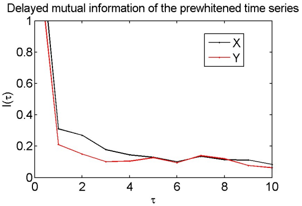


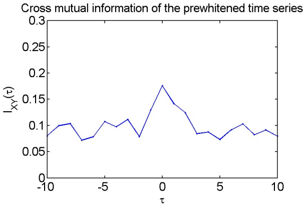
Example: Two dependent Henon maps - 1

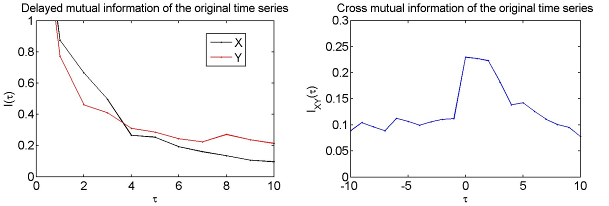




Significant $I_X(\tau)$, $I_Y(\tau)$,
Significant $I_{XY}(\tau)$ for $\tau \geq 0$, X_t is "correlated" to $Y_{t+\tau}$

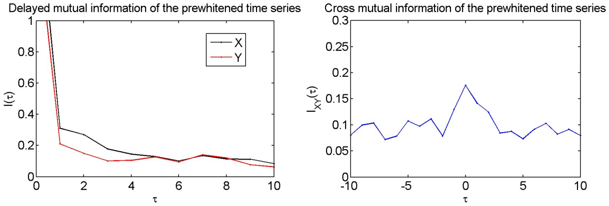






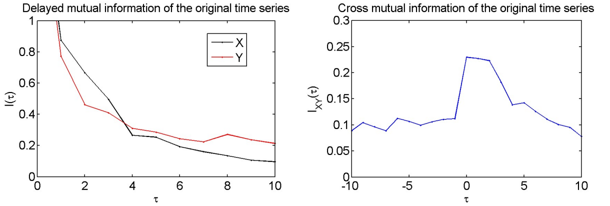
Significant $I_X(\tau)$, $I_Y(\tau)$,

Significant $I_{XY}(\tau)$ for $\tau \geq 0$, X_t is "correlated" to $Y_{t+\tau}$

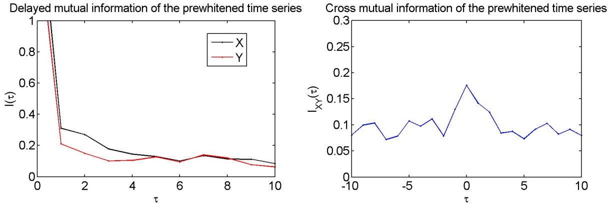


Significant $I_X(\tau)$, $I_Y(\tau)$,

Example: Two dependent Henon maps - 1



Significant $I_X(\tau)$, $I_Y(\tau)$,
Significant $I_{XY}(\tau)$ for $\tau \geq 0$, X_t is "correlated" to $Y_{t+\tau}$



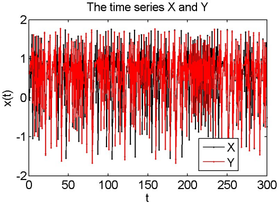
Significant $I_X(\tau)$, $I_Y(\tau)$,
Small $I_{XY}(\tau)$ for $\tau \geq 0$, is it significant?

Example: Two dependent Henon maps - 2

$$X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} + 0.14(X_{t-1}^2 - Y_{t-1}^2)$$
$$Y_t = 1.4 - Y_{t-1}^2 + 0.3Y_{t-2} + 0.08(Y_{t-1}^2 - X_{t-1}^2)$$

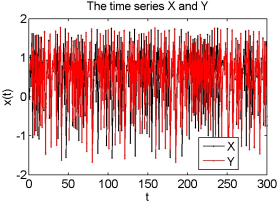
Example: Two dependent Henon maps - 2

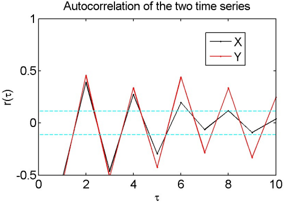
$$X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} + 0.14(X_{t-1}^2 - Y_{t-1}^2)$$
$$Y_t = 1.4 - Y_{t-1}^2 + 0.3Y_{t-2} + 0.08(Y_{t-1}^2 - X_{t-1}^2)$$



Example: Two dependent Henon maps - 2

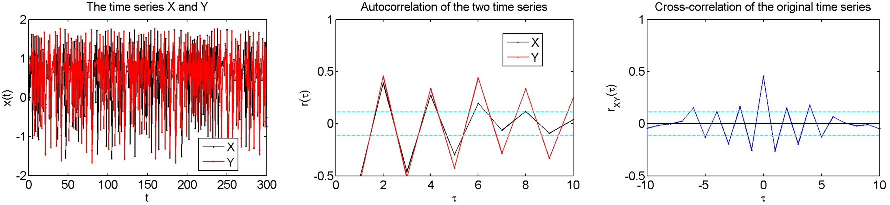
$$X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} + 0.14(X_{t-1}^2 - Y_{t-1}^2)$$
$$Y_t = 1.4 - Y_{t-1}^2 + 0.3Y_{t-2} + 0.08(Y_{t-1}^2 - X_{t-1}^2)$$





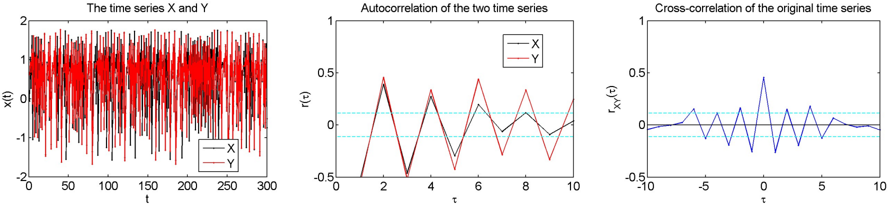
Example: Two dependent Henon maps - 2

$$X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} + 0.14(X_{t-1}^2 - Y_{t-1}^2)$$
$$Y_t = 1.4 - Y_{t-1}^2 + 0.3Y_{t-2} + 0.08(Y_{t-1}^2 - X_{t-1}^2)$$



Example: Two dependent Henon maps - 2

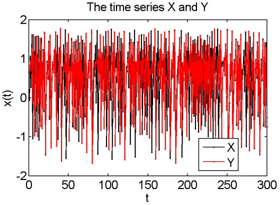
$$X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} + 0.14(X_{t-1}^2 - Y_{t-1}^2)$$
$$Y_t = 1.4 - Y_{t-1}^2 + 0.3Y_{t-2} + 0.08(Y_{t-1}^2 - X_{t-1}^2)$$

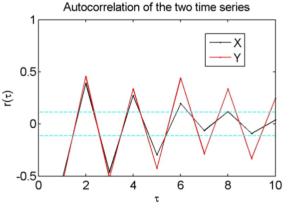


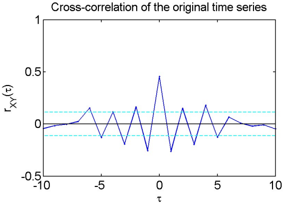
Alternating autocorrelation, alternating cross-correlation

Example: Two dependent Henon maps - 2

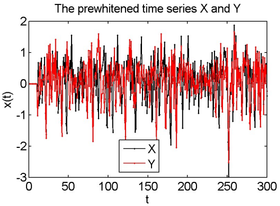
$$X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} + 0.14(X_{t-1}^2 - Y_{t-1}^2)$$
$$Y_t = 1.4 - Y_{t-1}^2 + 0.3Y_{t-2} + 0.08(Y_{t-1}^2 - X_{t-1}^2)$$



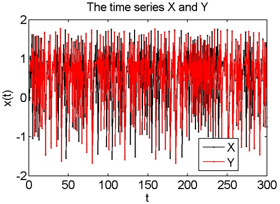


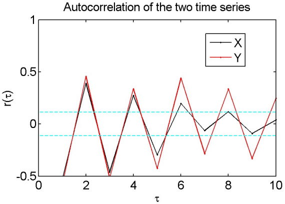


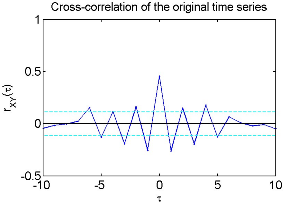
Alternating autocorrelation, alternating cross-correlation



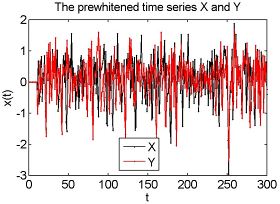
$$X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} + 0.14(X_{t-1}^2 - Y_{t-1}^2)$$
$$Y_t = 1.4 - Y_{t-1}^2 + 0.3Y_{t-2} + 0.08(Y_{t-1}^2 - X_{t-1}^2)$$

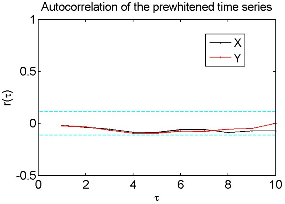






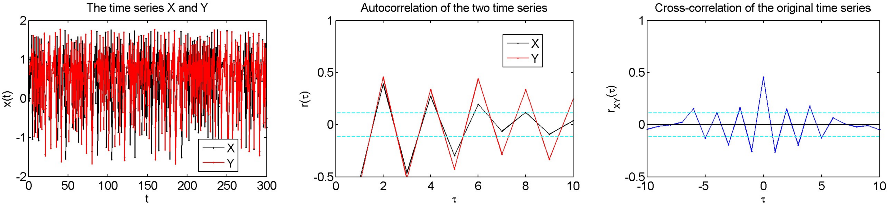
Alternating autocorrelation, alternating cross-correlation



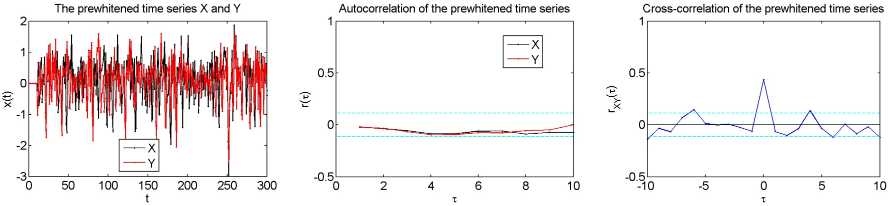


Example: Two dependent Henon maps - 2

$$X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} + 0.14(X_{t-1}^2 - Y_{t-1}^2)$$
$$Y_t = 1.4 - Y_{t-1}^2 + 0.3Y_{t-2} + 0.08(Y_{t-1}^2 - X_{t-1}^2)$$

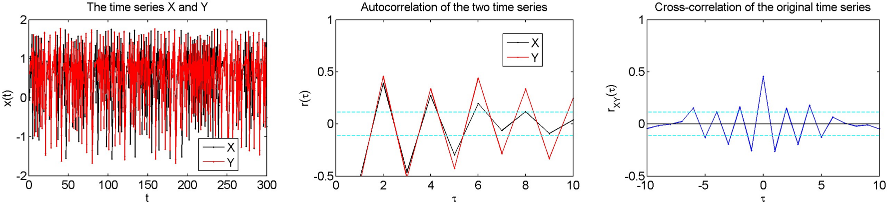


Alternating autocorrelation, alternating cross-correlation

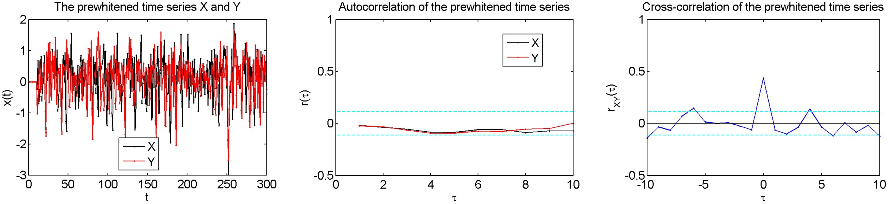


Example: Two dependent Henon maps - 2

$$X_t = 1.4 - X_{t-1}^2 + 0.3X_{t-2} + 0.14(X_{t-1}^2 - Y_{t-1}^2)$$
$$Y_t = 1.4 - Y_{t-1}^2 + 0.3Y_{t-2} + 0.08(Y_{t-1}^2 - X_{t-1}^2)$$

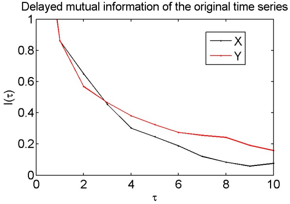


Alternating autocorrelation, alternating cross-correlation

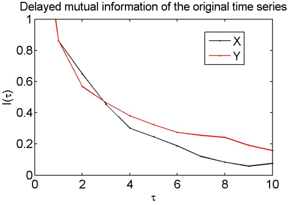


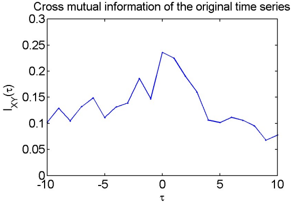
After prewhitening, zero autocorrelation, significant cross-correlation at $\tau = 0$

Example: Two dependent Henon maps - 2

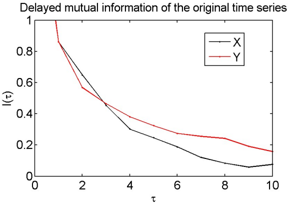


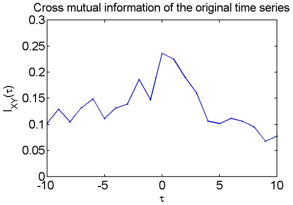
Example: Two dependent Henon maps - 2





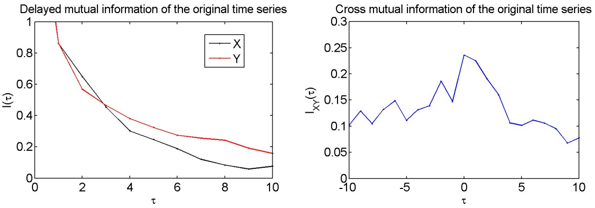
Example: Two dependent Henon maps - 2





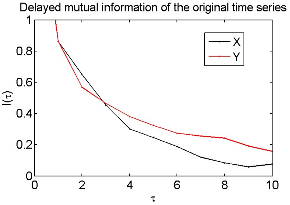
Significant $I_X(\tau)$, $I_Y(\tau)$,

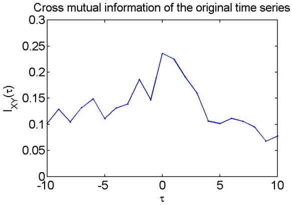
Example: Two dependent Henon maps - 2



Significant $I_X(\tau)$, $I_Y(\tau)$,
Significant $I_{XY}(\tau)$ for $\tau < 0$, $\tau \geq 0$, X_t is "correlated" to $Y_{t+|\tau|}$ and $Y_{t-|\tau|}$

Example: Two dependent Henon maps - 2





Significant $I_X(\tau)$, $I_Y(\tau)$,
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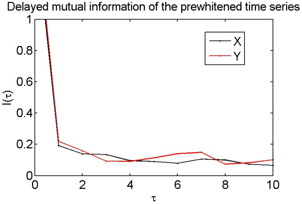
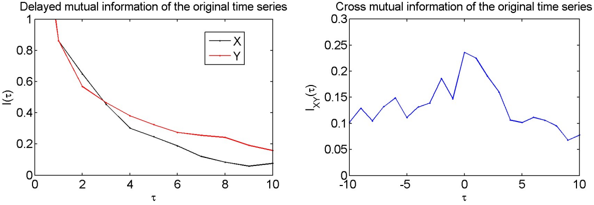


Example: Two dependent Henon maps - 2



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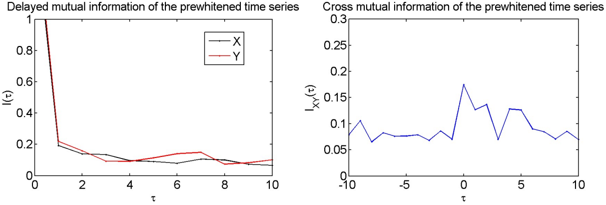
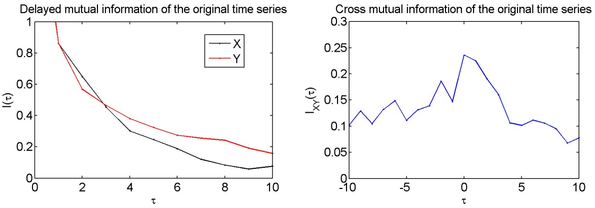
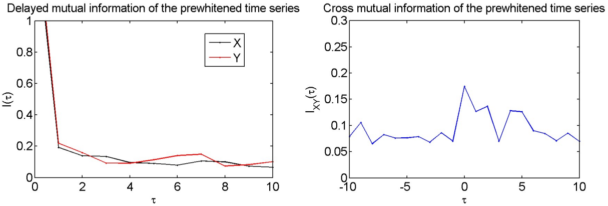


Example: Two dependent Henon maps - 2



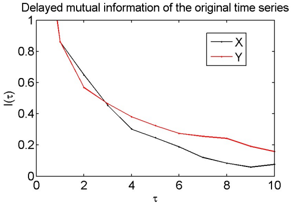
Significant $I_X(\tau)$, $I_Y(\tau)$,

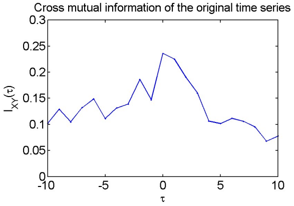
Significant $I_{XY}(\tau)$ for $\tau < 0$, $\tau \geq 0$, X_t is "correlated" to $Y_{t+|\tau|}$ and $Y_{t-|\tau|}$



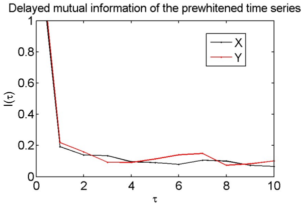
Significant $I_X(\tau)$, $I_Y(\tau)$,

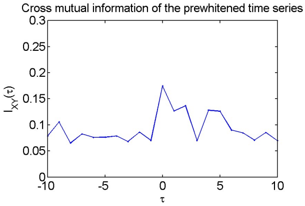
Example: Two dependent Henon maps - 2





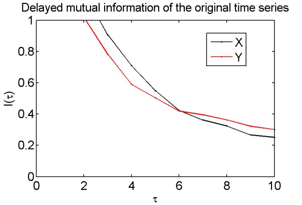
Significant $I_X(\tau)$, $I_Y(\tau)$,
Significant $I_{XY}(\tau)$ for $\tau < 0$, $\tau \geq 0$, X_t is “correlated” to $Y_{t+|\tau|}$ and $Y_{t-|\tau|}$

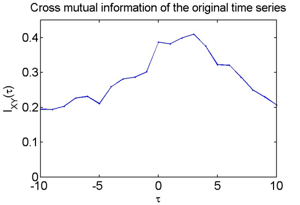




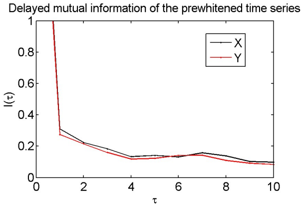
Significant $I_X(\tau)$, $I_Y(\tau)$,
Small $I_{XY}(\tau)$ for $\tau \geq 0$, is it significant?

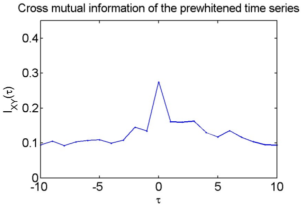
The same but for $n = 4000$





Significant $I_X(\tau)$, $I_Y(\tau)$,
Significant $I_{XY}(\tau)$ for $\tau < 0$, $\tau \geq 0$, X_t is "correlated" to $Y_{t+|\tau|}$ and $Y_{t-|\tau|}$

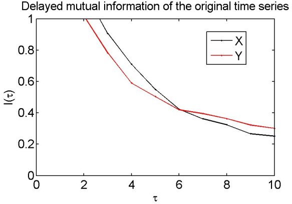


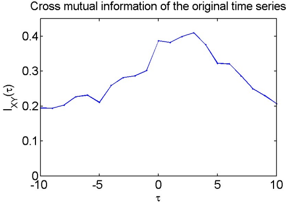


Significant $I_X(\tau)$, $I_Y(\tau)$,

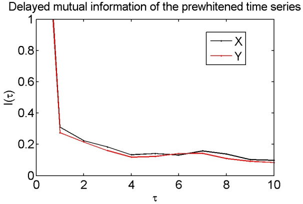
Example: Two dependent Henon maps - 2, large n

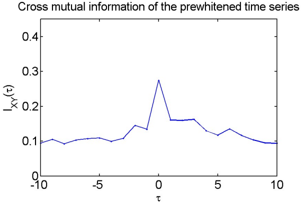
The same but for $n = 4000$





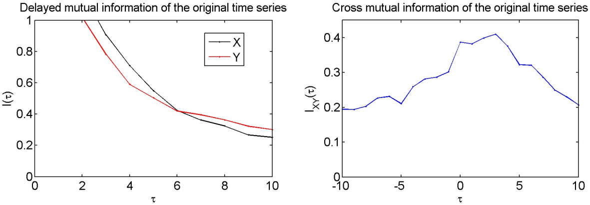
Significant $I_X(\tau)$, $I_Y(\tau)$,
Significant $I_{XY}(\tau)$ for $\tau < 0$, $\tau \geq 0$, X_t is "correlated" to $Y_{t+|\tau|}$ and $Y_{t-|\tau|}$



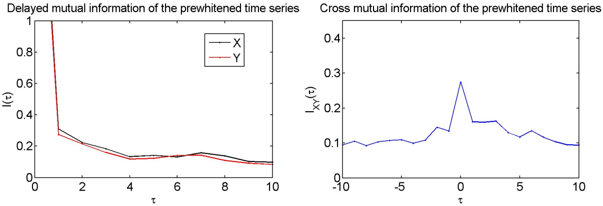


Significant $I_X(\tau)$, $I_Y(\tau)$,
Small $I_{XY}(\tau)$ for $\tau \geq 0$

The same but for $n = 4000$



Significant $I_X(\tau)$, $I_Y(\tau)$,
Significant $I_{XY}(\tau)$ for $\tau < 0$, $\tau \geq 0$, X_t is "correlated" to $Y_{t+|\tau|}$ and $Y_{t-|\tau|}$



Significant $I_X(\tau)$, $I_Y(\tau)$,
Small $I_{XY}(\tau)$ for $\tau \geq 0$ ... but also for $\tau < 0$

Example: VAR model, $K = 3$

$$\begin{bmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{bmatrix} = \begin{bmatrix} 0.95 & -0.5 & -0.3 \\ 0 & 0.85 & 0.3 \\ 0 & 0 & 0.9 \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \\ X_{3,t-1} \end{bmatrix} + \cdots + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{bmatrix}$$

$$\mathbf{X}_t = [X_{1,t}, X_{2,t}, X_{3,t}]' \qquad \mathbf{X}_t = A_1 \mathbf{X}_{t-1} + \epsilon_t$$

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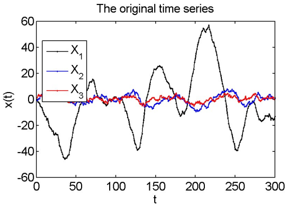

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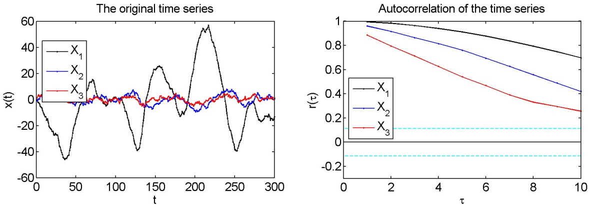

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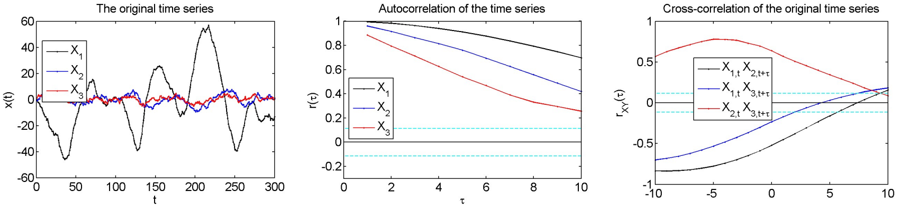

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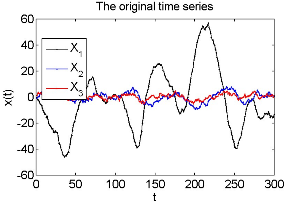

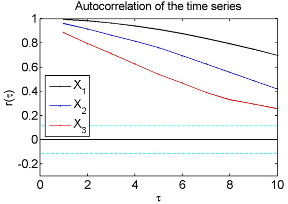

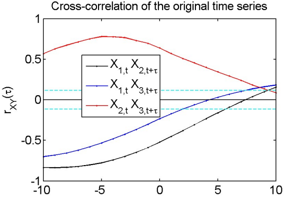

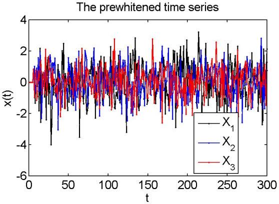

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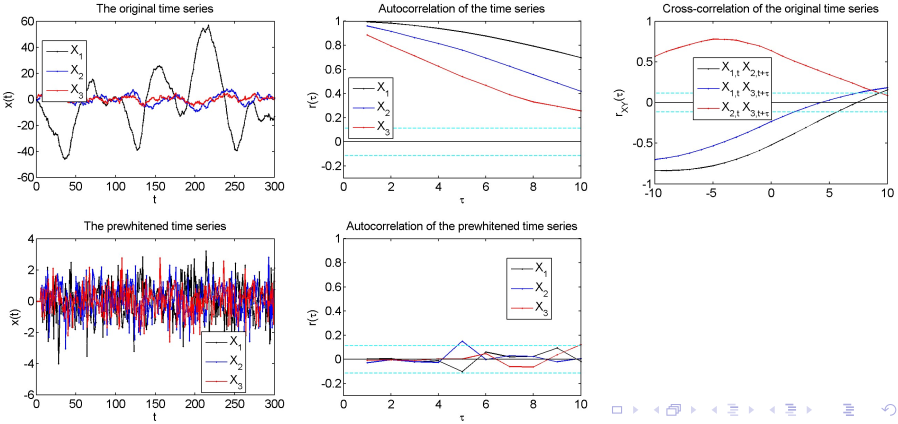

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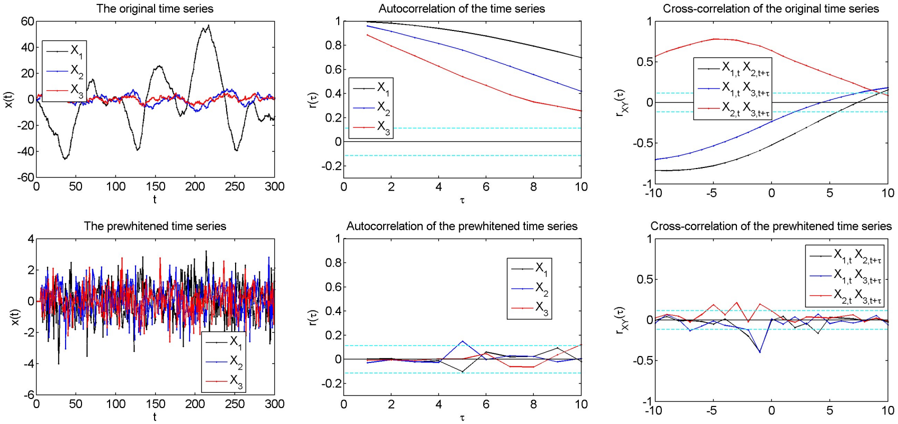

Similarity measure for time series network

N variables (nodes) $X_1, X_2, \ldots, X_N$

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What τ to choose?

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1. $\tau = 0$ correlation of $X_{i,t}$ and $X_{j,t}$

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1. $\tau = 0$ correlation of $X_{i,t}$ and $X_{j,t}$
2. $\tau > 0$ correlation of $X_{i,t}$ and $X_{j,t+\tau}$, X_i influences the evolution of X_j

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3. $\tau < 0$ correlation of $X_{i,t}$ and $X_{j,t-|\tau|}$, X_j influences the evolution of X_i

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3. $\tau < 0$ correlation of $X_{i,t}$ and $X_{j,t-|\tau|}$, X_j influences the evolution of X_i

X_i influences the evolution of $X_j \implies X_i$ (Granger) causes X_j

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There are other measures more appropriate to measure Granger causality.

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Cross correlation matrix $R(\tau)$

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Cross correlation matrix $R(\tau)$

$$R(0) = \begin{bmatrix} & -0.00 & 0.01 \\ -0.00 & & 0.11 \\ 0.01 & 0.11 & \end{bmatrix}$$

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$$R(0) = \begin{bmatrix} & -0.00 & 0.01 \\ -0.00 & & 0.11 \\ 0.01 & 0.11 & \end{bmatrix}$$

$$R(1) = \begin{bmatrix} & 0.05 & -0.05 \\ -0.39 & & 0.01 \\ -0.40 & 0.20 & \end{bmatrix}$$

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$$R(1) = \begin{bmatrix} & 0.05 & -0.05 \\ \color{red}{-0.39} & & 0.01 \\ \color{red}{-0.40} & \color{red}{0.20} & \end{bmatrix}$$

$$R(2) = \begin{bmatrix} & -0.09 & 0.04 \\ \color{red}{-0.20} & & -0.03 \\ \color{red}{-0.12} & -0.02 & \end{bmatrix}$$

Example: VAR model, $K = 3$

$$\begin{bmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{bmatrix} = \begin{bmatrix} 0.95 & -0.5 & -0.3 \\ 0 & 0.85 & 0.3 \\ 0 & 0 & 0.9 \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \\ X_{3,t-1} \end{bmatrix} + \cdots + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{bmatrix}$$

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Adjacency matrix

threshold $\pm 2/\sqrt{n} = \pm 0.11$

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Adjacency matrix

threshold $\pm 2/\sqrt{n} = \pm 0.11$

$$A(0) = \begin{bmatrix} & 0 & 0 \\ 0 & & 0 \\ 0 & 0 & \end{bmatrix}$$

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Adjacency matrix

threshold $\pm 2/\sqrt{n} = \pm 0.11$

$$A(0) = \begin{bmatrix} & 0 & 0 \\ 0 & & 0 \\ 0 & 0 & \end{bmatrix}$$

$$A(1) = \begin{bmatrix} & 0 & 0 \\ 1 & & 0 \\ 1 & 1 & \end{bmatrix}$$

$$\begin{bmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{bmatrix} = \begin{bmatrix} 0.95 & -0.5 & -0.3 \\ 0 & 0.85 & 0.3 \\ 0 & 0 & 0.9 \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \\ X_{3,t-1} \end{bmatrix} + \cdots + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{bmatrix}$$

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$$R(0) = \begin{bmatrix} & -0.00 & 0.01 \\ -0.00 & & 0.11 \\ 0.01 & 0.11 & \end{bmatrix}$$

$$R(1) = \begin{bmatrix} & 0.05 & -0.05 \\ {\color{red}-0.39} & & 0.01 \\ {\color{red}-0.40} & {\color{red}0.20} & \end{bmatrix}$$

$$R(2) = \begin{bmatrix} & -0.09 & 0.04 \\ {\color{red}-0.20} & & -0.03 \\ {\color{red}-0.12} & -0.02 & \end{bmatrix}$$

Adjacency matrix
threshold $\pm 2/\sqrt{n} = \pm 0.11$

$$A(0) = \begin{bmatrix} & 0 & 0 \\ 0 & & 0 \\ 0 & 0 & \end{bmatrix}$$

$$A(1) = \begin{bmatrix} & 0 & 0 \\ 1 & & 0 \\ 1 & 1 & \end{bmatrix}$$

$$A(2) = \begin{bmatrix} & 0 & 0 \\ 1 & & 0 \\ 1 & 0 & \end{bmatrix}$$

Example: VAR model, $K = 5$

$$\mathbf{X}_t = [X_{1,t}, X_{2,t}, X_{3,t}, X_{4,t}, X_{5,t}]' \qquad \mathbf{X}_t = A_1 \mathbf{X}_{t-1} + \epsilon_t$$

$$A_1 = \begin{bmatrix} -0.95 & 0.2 & -0.3 & 0.4 & -0.8 \\ 0 & -0.2 & -0.3 & -0.4 & 0.9 \\ 0 & 0 & -0.1 & -0.1 & 0.8 \\ 0 & 0 & 0 & -0.8 & -0.9 \\ 0 & 0 & 0 & 0 & 0.8 \end{bmatrix}$$

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$$R(0) = \begin{bmatrix} & -0.62 & -0.47 & 0.40 & 0.07 \\ -0.62 & & 0.58 & -0.36 & 0.05 \\ -0.47 & 0.58 & & -0.42 & 0.04 \\ 0.40 & -0.36 & -0.42 & & -0.04 \\ 0.07 & 0.05 & 0.04 & -0.04 & \end{bmatrix}$$

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$$R(1) = \begin{bmatrix} & -0.01 & 0.03 & -0.04 & 0.02 \\ 0.04 & & -0.12 & 0.09 & 0.02 \\ 0.29 & -0.12 & & 0.20 & 0.02 \\ -0.53 & 0.48 & 0.26 & & 0.06 \\ 0.49 & -0.53 & -0.62 & 0.65 & \end{bmatrix}$$

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Example: World market indices see [3]: Chp14, [4]

Detect information flow between stock indices

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Indices | correlation coefficient | transfer entropy

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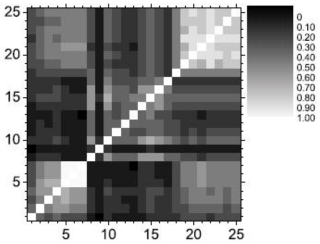
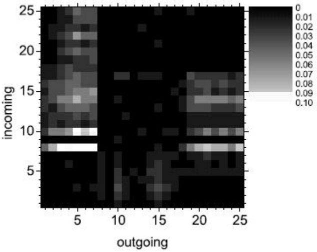
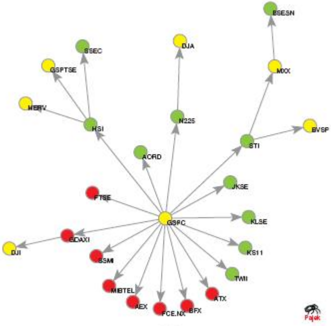
Americas	1	MERV	Argentina
	2	BVSP	Brazil
	3	GSPTSE	Canada
	4	MXX	Mexico
	5	GSPC	US
	6	DJA	US
	7	DJI	US
Asia/Pacific	8	AORD	Australia
	9	SSEC	China
	10	HSI	China
	11	BSESN	India
	12	JKSE	Indonesia
	13	KLSE	Malaysia
	14	N225	Japan
	15	STI	Singapore
	16	KS11	Korea
	17	TWII	Taiwan
Europe	18	ATX	Austria
	19	BFX	Belgium
	20	FCE.NX	France
	21	GDAXI	Germany
	22	AEX	Holland
	23	MIBTEL	Italy
	24	SSMI	Switzerland
	25	FTSE	UK





Example: World market indices

Draw the network of "outgoing" transfer entropy and "incoming" transfer entropy.



(a)

Example: World market indices

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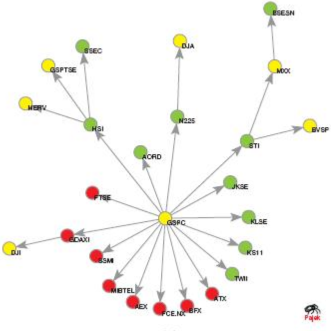
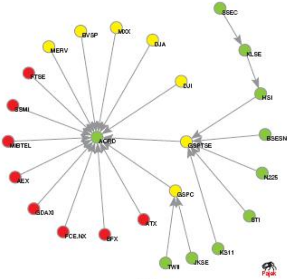


(a)



(b)

Fig. 4: (Color online) Minimum spanning tree for (a) the outgoing transfer entropy and (b) the incoming transfer entropy. The minimum spanning tree is drawn by Pajek

Draw the network of “outgoing” transfer entropy and “incoming” transfer entropy.

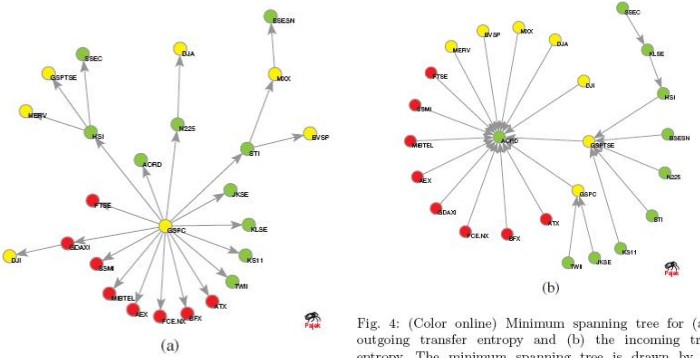


Fig. 4: (Color online) Minimum spanning tree for (a) the outgoing transfer entropy and (b) the incoming transfer entropy. The minimum spanning tree is drawn by Pajek

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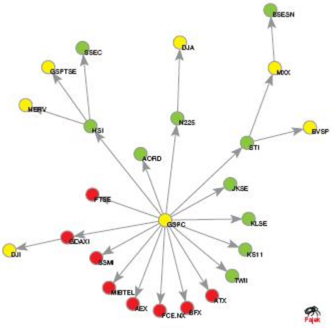

(a)

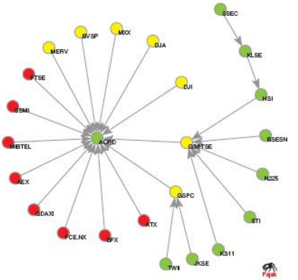

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Fig. 4: (Color online) Minimum spanning tree for (a) the outgoing transfer entropy and (b) the incoming transfer entropy. The minimum spanning tree is drawn by Pajek

- GSPC (Standard and Poor 500) is the information source of the system
- AORD (Australian index) is the information receiver

Literature

[1] Chatfield C (2004) *The Analysis of Time Series, An Introduction*, Sixth Edition, Chapman & Hall.

[2] Brockwell PJ and Davis RA (2002) *Introduction to Time Series and Forecasting*, Second Edition, Springer.

[3] Kantz H and Schreiber T (2003) *Nonlinear Time Series Analysis*, Second Edition, Cambridge.

[4] Kwon O and Yang J.-S. (2008) Information flow between stock indices, Europhysics Letters, 82: 68003, doi: 10.1209/0295-5075/82/68003