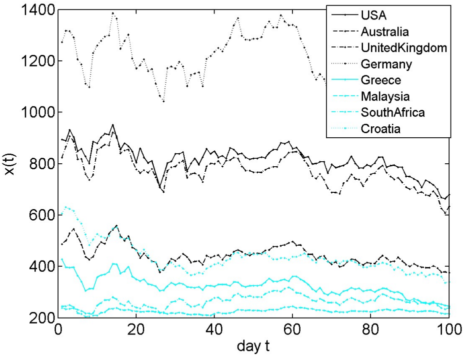
Correlation, complexity, and coupling measures of time series

Dimitris Kugiumtzis

November 4, 2020

Time Series, time dependence



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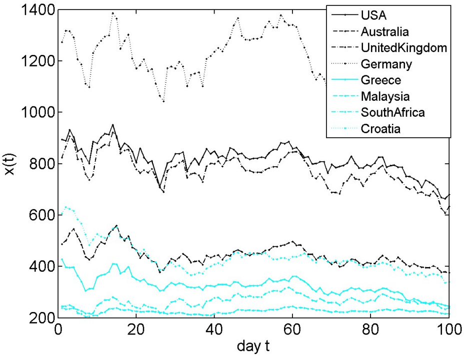

Time dependence? $\Longrightarrow$ Test for independence (see below)

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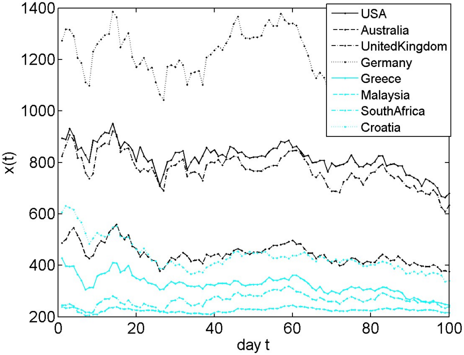

Time dependence? $\Longrightarrow$ Test for independence (see below)

Time dependence

- The time series is a function of time
- The time series is a realization of a stochastic process / dynamical system

Stationarity see [1]: Sec 2.1-2.4, [2] Chp 1 and Sec 2.1

Stationary time series: the statistics do not change by the shift of time

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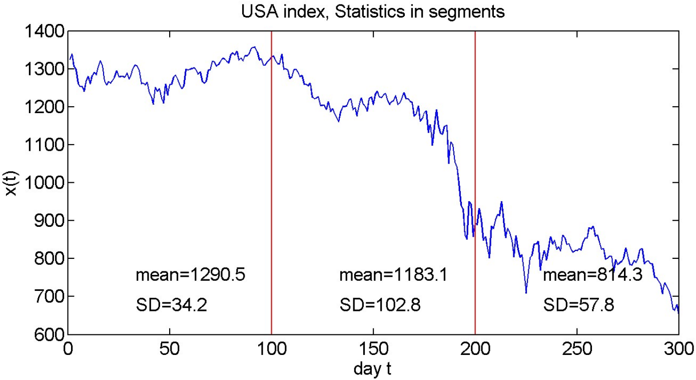

Time series decomposition see [1]: Sec 2.5-2.6 and [2]: Chp 1

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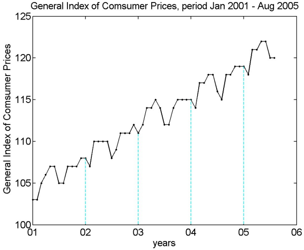
x_t: residual component, stationary time series.

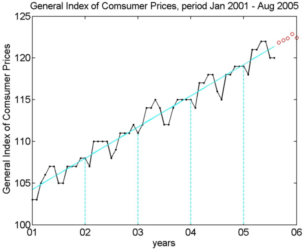
Example: Monthly Greek General Index of Consumer Prices

$\{y_t\}_{t=1}^{56}$



Example: Monthly Greek General Index of Consumer Prices

$\{y_t\}_{t=1}^{56} \quad \mu_t = 103.9 + 0.31t$



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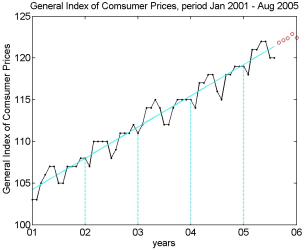

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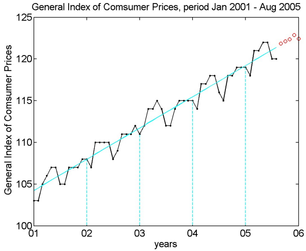

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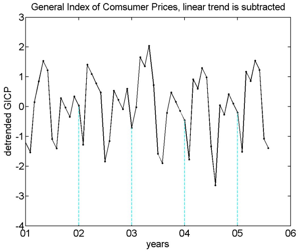

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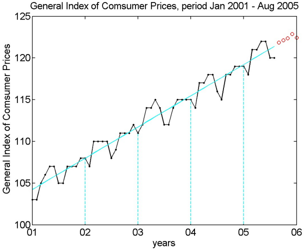

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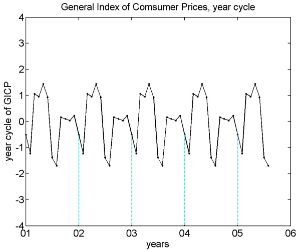

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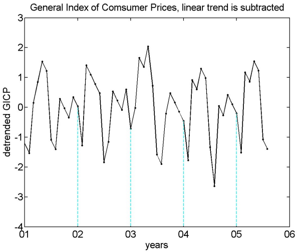

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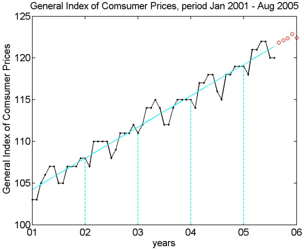

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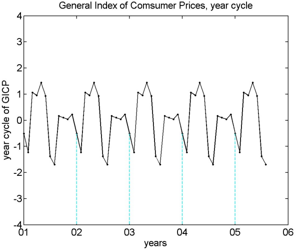

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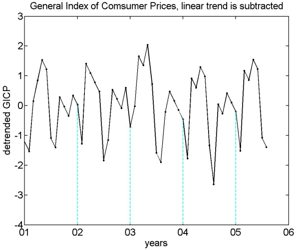

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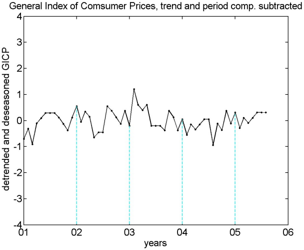

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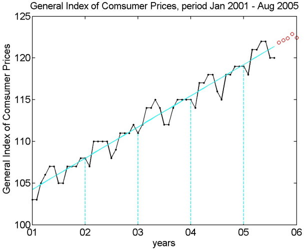

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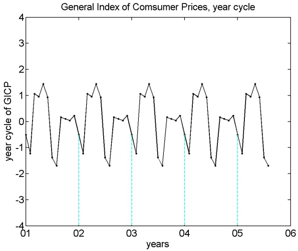

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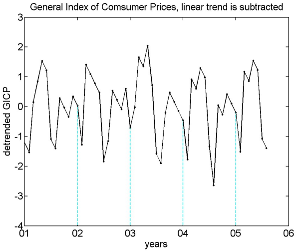

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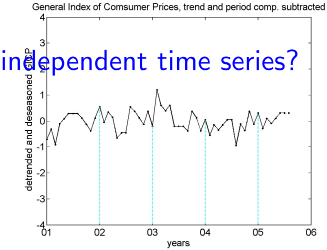

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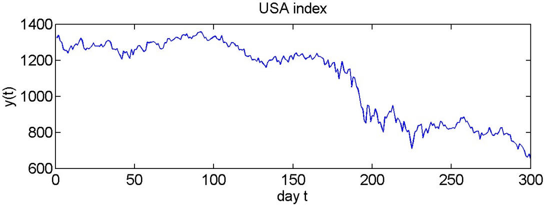

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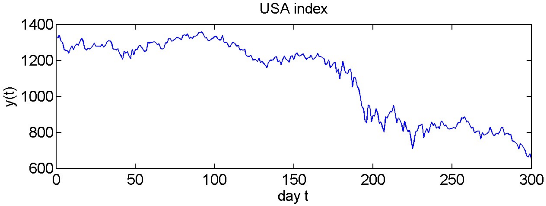

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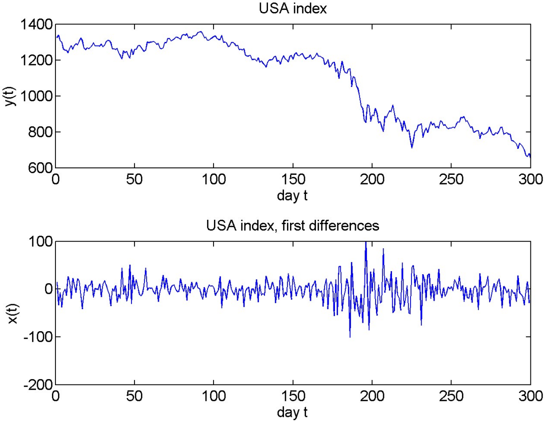

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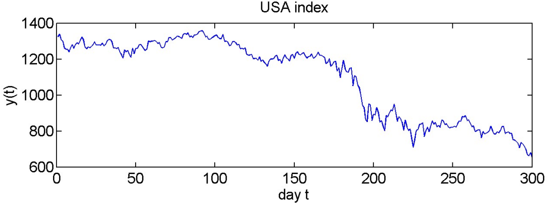

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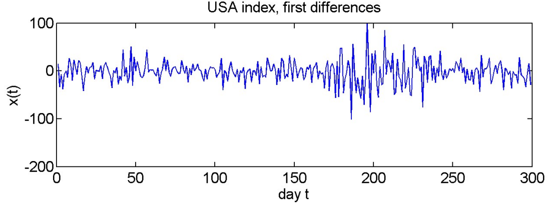

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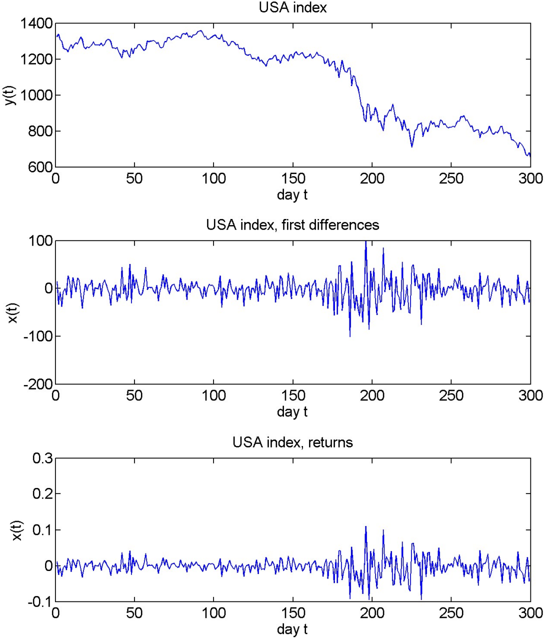

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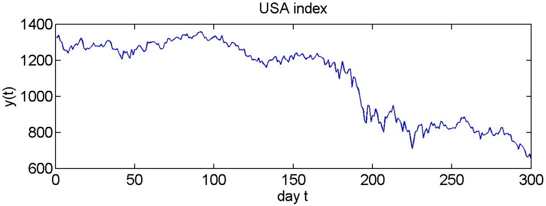

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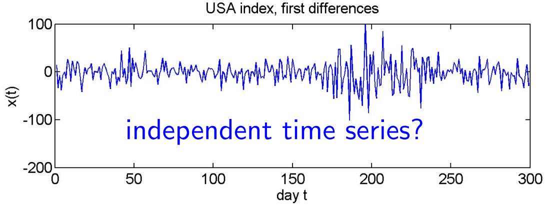

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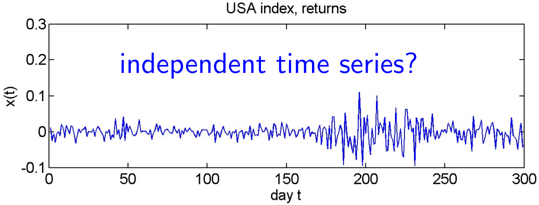

Autocovariance and autocorrelation see [1]: Sec 2.7, [2]: Chp 1

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residuals of GICP

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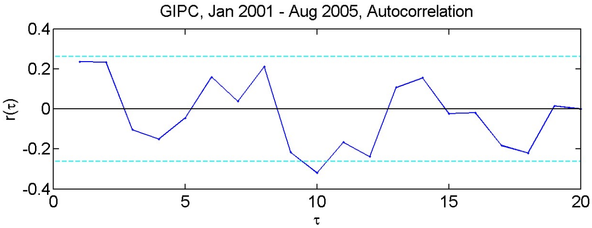

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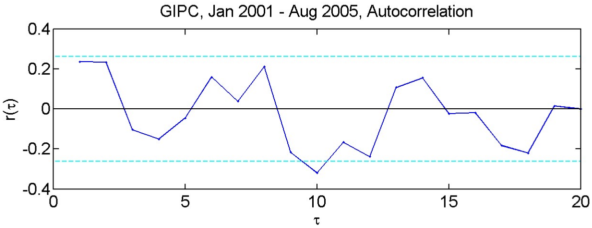

returns of USA stock market index

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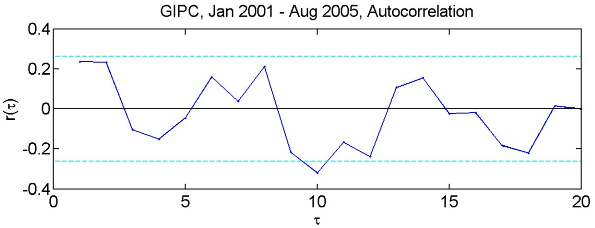

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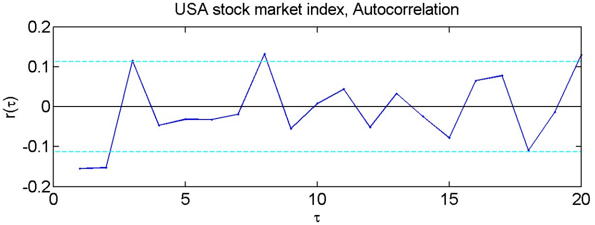

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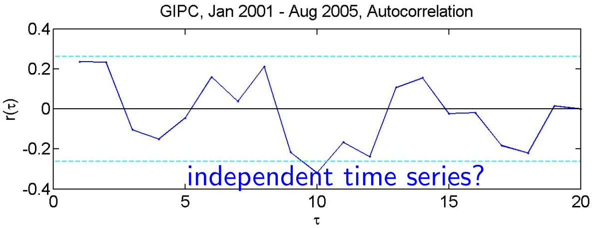

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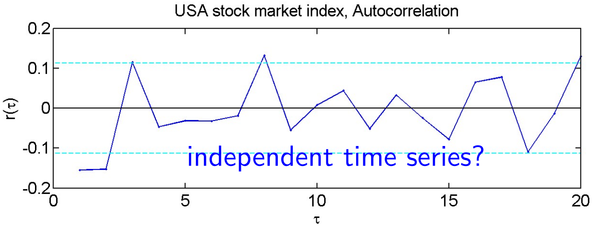

Example: Traces of packet arrivals

A trace containing packet arrivals seen on an Ethernet at the Bellcore Morristown Research and Engineering facility, regarding LAN traffic, period 11:25 on August 29, 1989.

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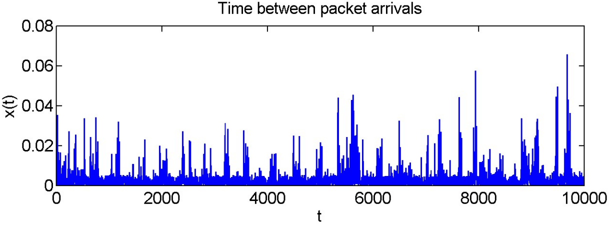

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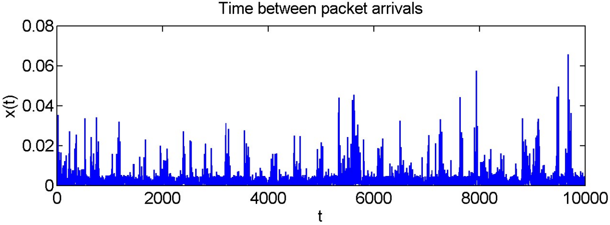

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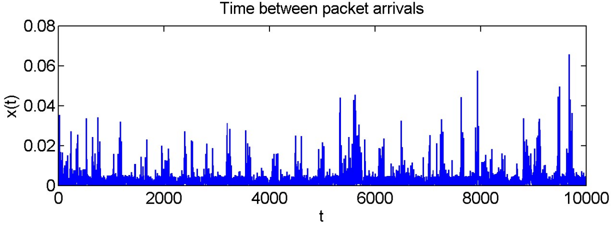

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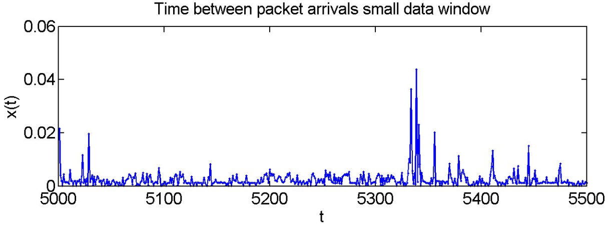

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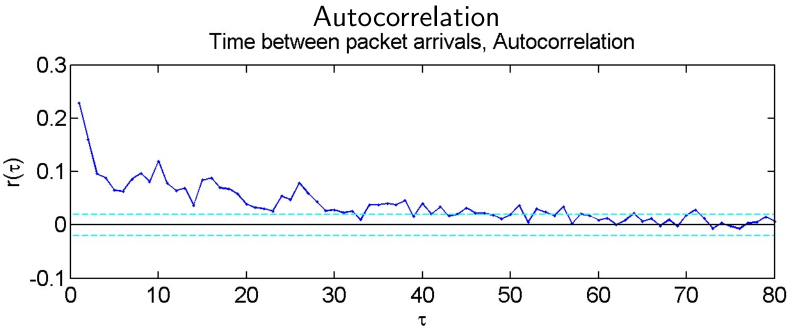

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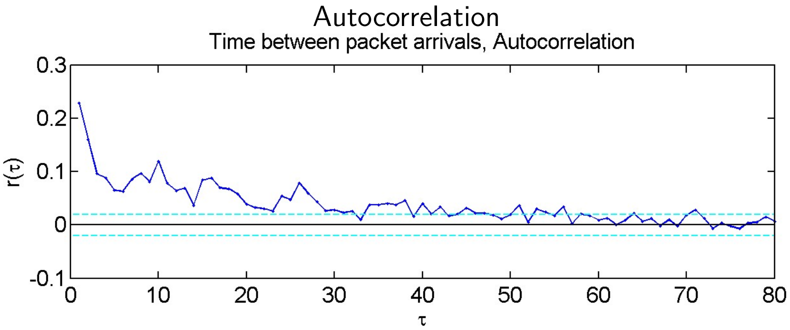

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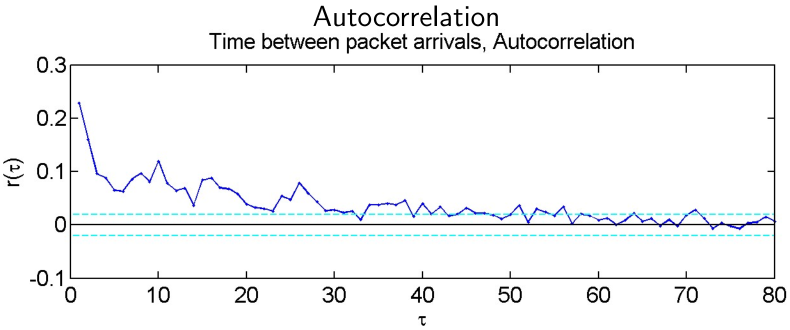

- Are the residuals of GIPC correlated?
- Are the returns of USA stock marker index correlated?

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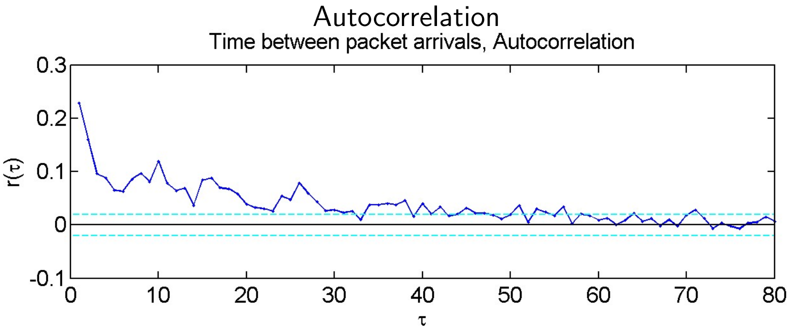

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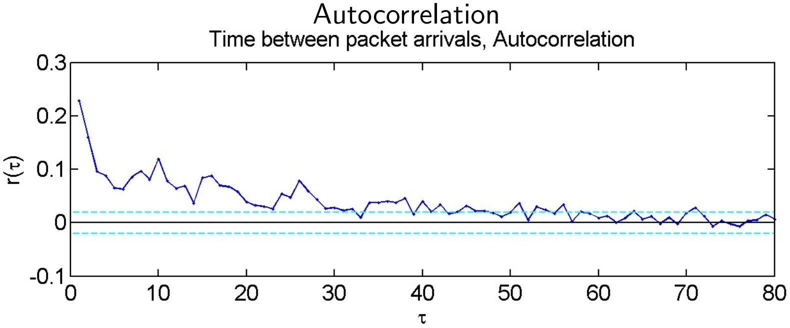

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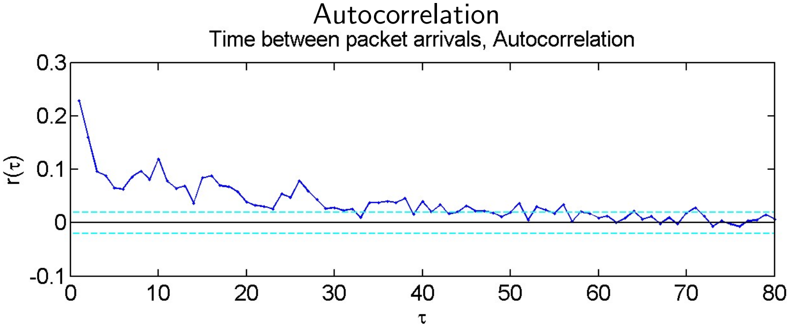

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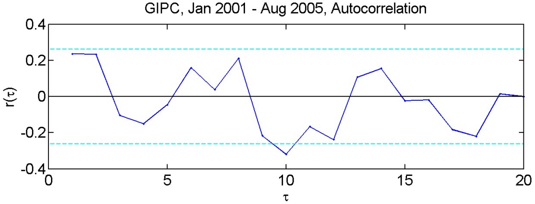
Examples of test for independence



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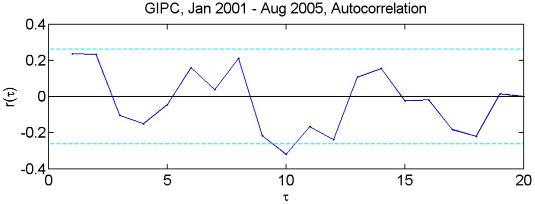

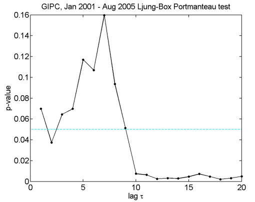

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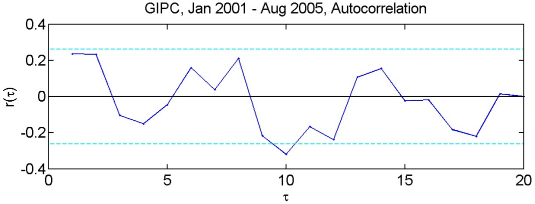

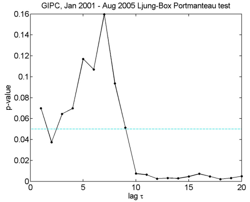

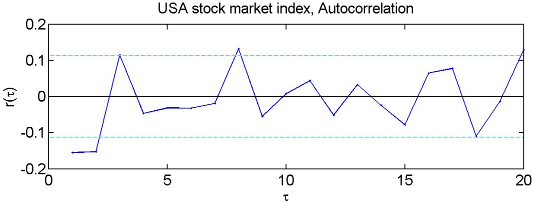

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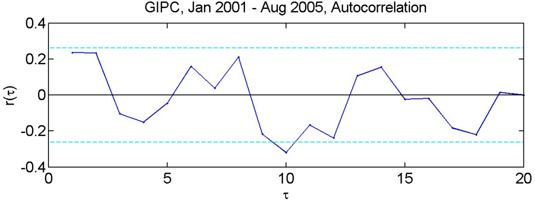

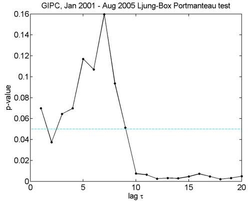

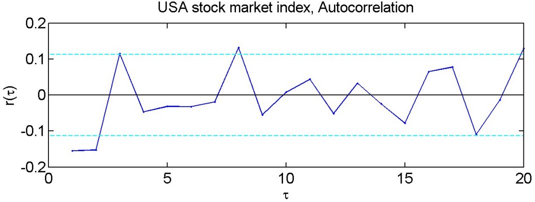

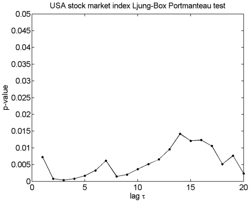

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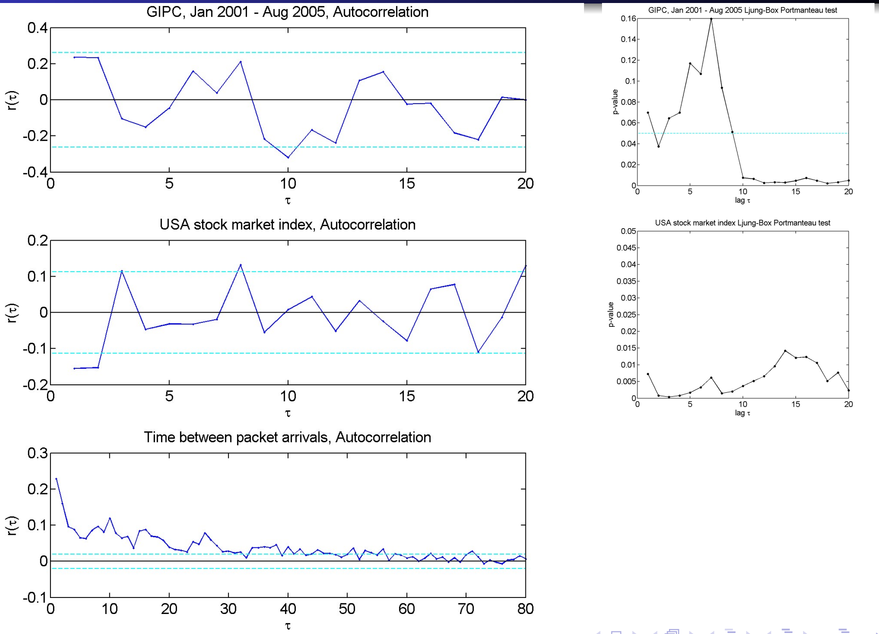

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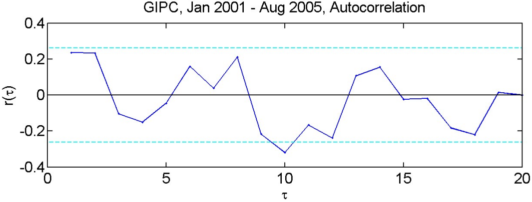

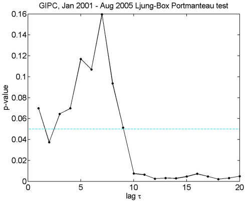

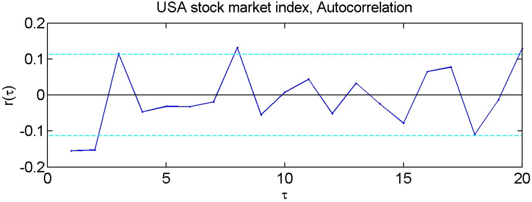

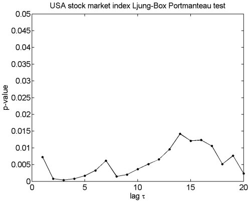

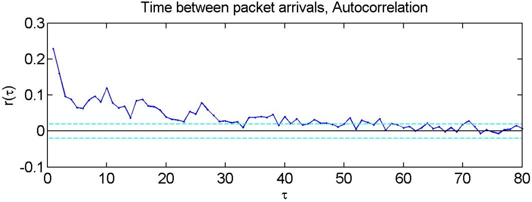

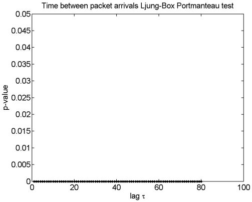

White noise and random walk see [1]: Chp 3, [2]: Chp 1

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- White noise process: Independent stochastic process, a series of iid variables X_t.

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White noise is a stationary process and random walk a non-stationary process.

Linear autoregressive models see [1]: Chp 3, [2]: Chp 2

A linear stochastic process possessing significant autocorrelation is the linear autoregressive process of order p, AR(p)

$$x_t = \phi_0 + \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + \epsilon_t$$

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To estimate the model AR(p) from a time series, we use least squares to compute the coefficients $\phi_0, \phi_1, \ldots \phi_p$ and σ_ϵ^2.

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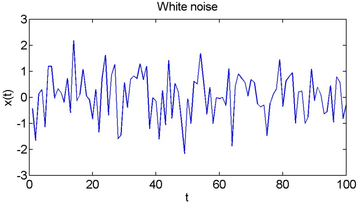
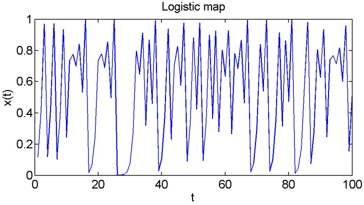
Nonlinear correlations see [3], Chp 1,2,3





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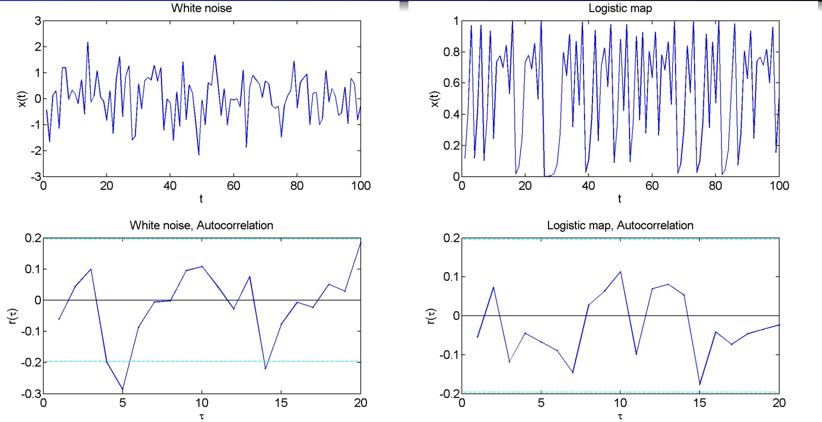

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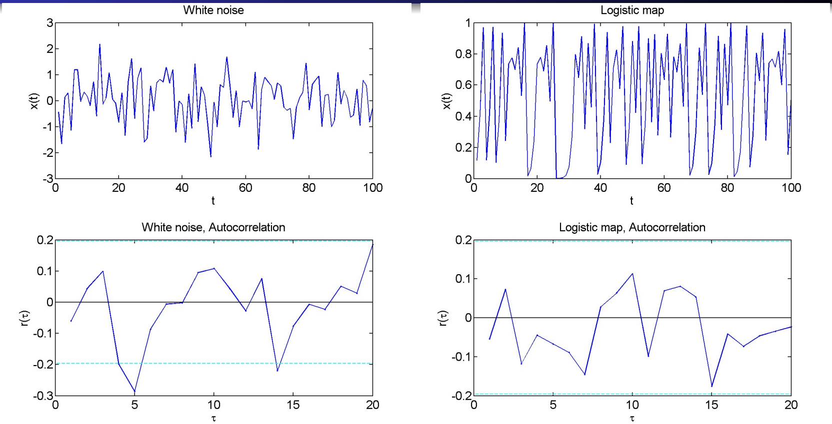

Logistic map:
$X_t = 4X_{t-1}(1 - X_{t-1})$,
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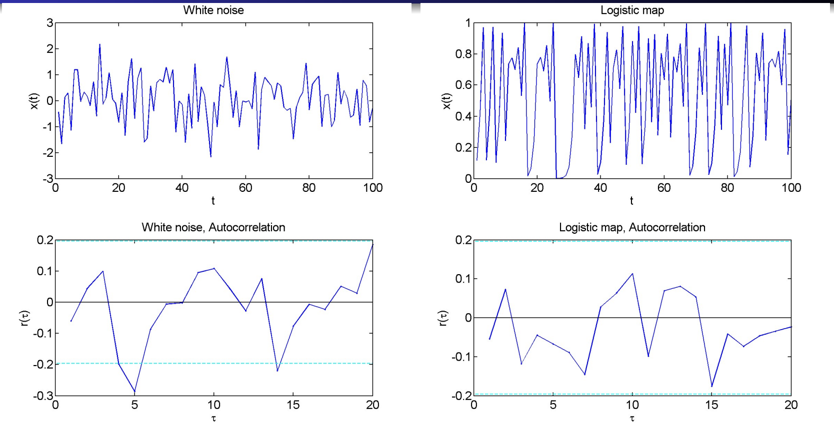

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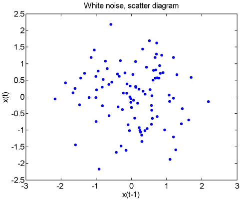

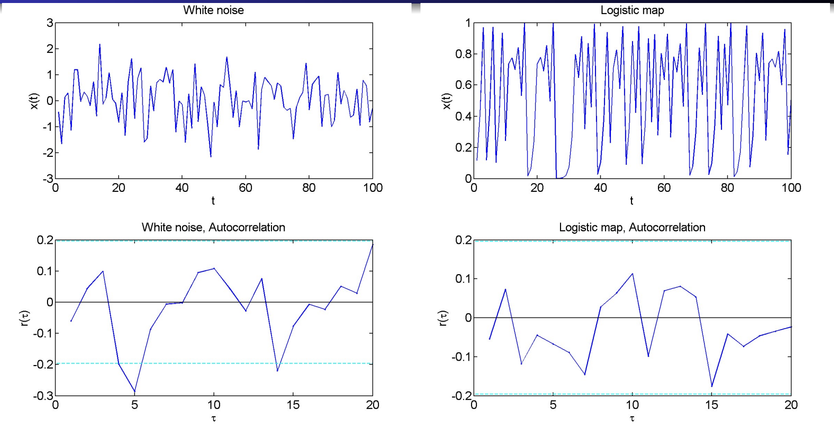

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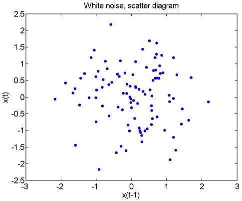

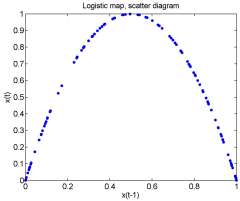

Measure of nonlinear autocorrelation see [3]: Sec 9.2

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Mutual information: white noise and logistic map

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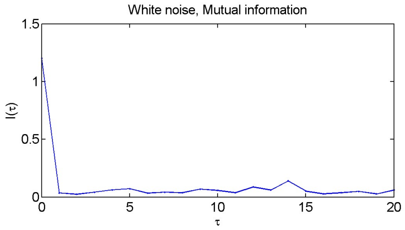
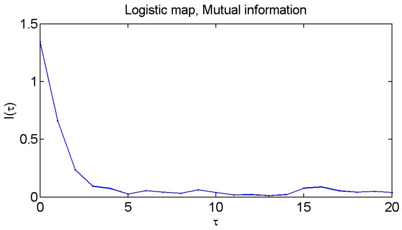
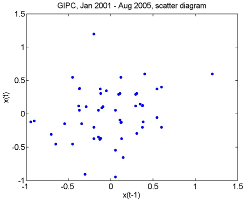
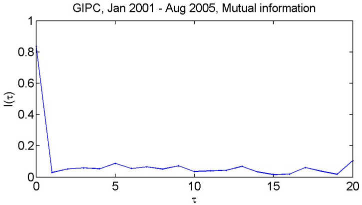




Mutual information: real examples





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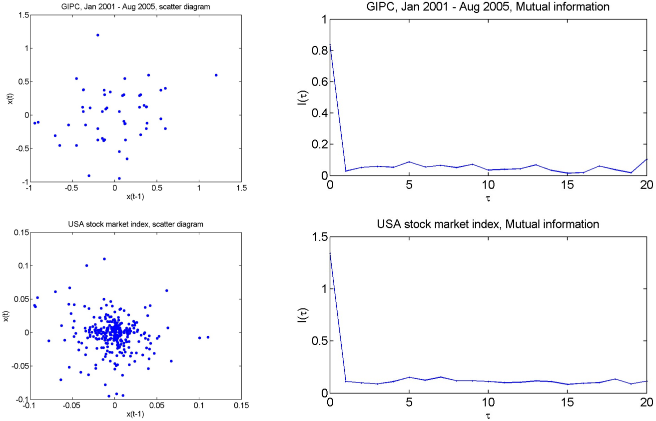

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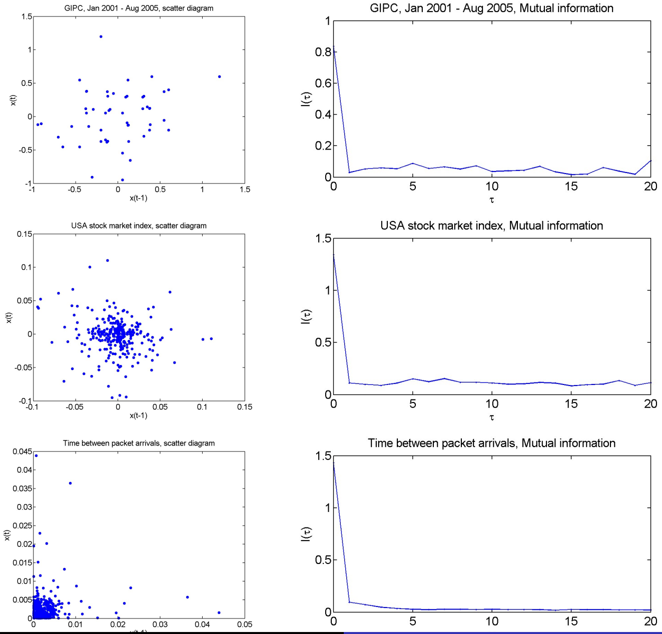

Exercise 5: Nonparametric test for zero mutual information

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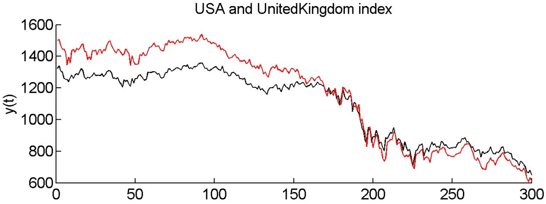
Time evolution of two stock market indices

Bivariate time series

Time evolution of two stock market indices



Bivariate time series

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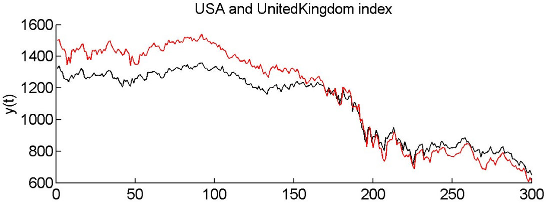

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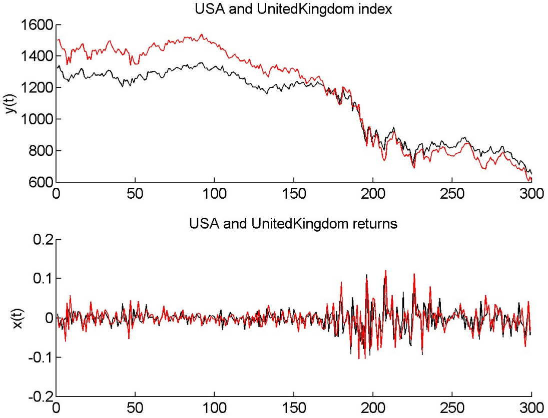

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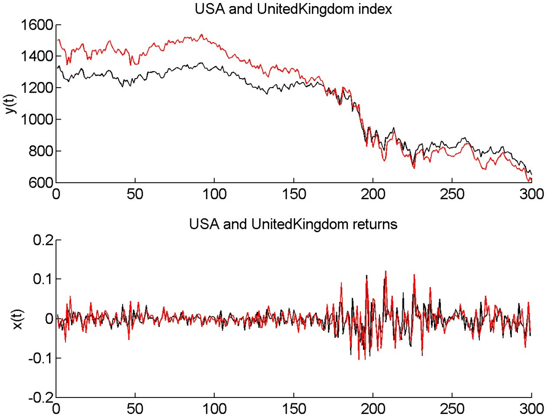

Are there autocorrelations in the two indices?

Bivariate time series

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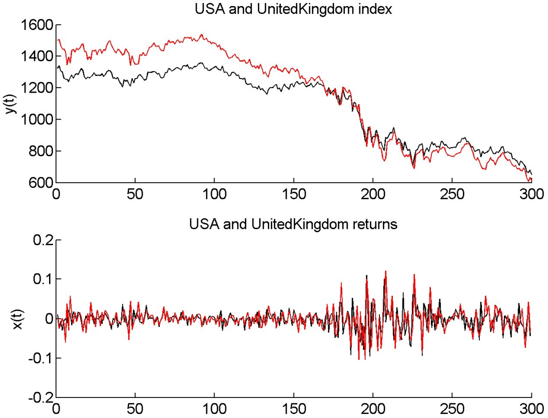

Are there autocorrelations in the two indices?

Are there cross-correlations in the two indices?

Cross-covariance and cross-correlation see [1]: Chp 8, [2]: Chp 7

Stationary processes $\{X_t\}$ and $\{Y_t\}$ and their realizations, time series $\{x_t, y_t\}_{t=1}^{n}$.

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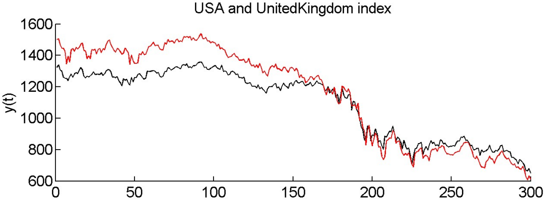
Cross-covariance is not even function: $\gamma_{XY}(\tau) \neq \gamma_{XY}(-\tau)$,
but it holds that $\gamma_{XY}(\tau) = \gamma_{YX}(-\tau)$. Also it holds $|\rho_{XY}(\tau)| \leq 1$.

Example: two world stock indices



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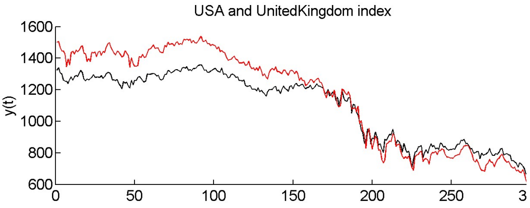

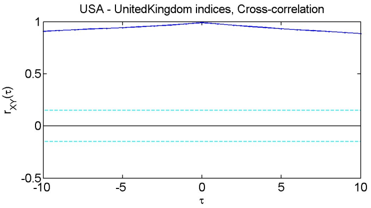

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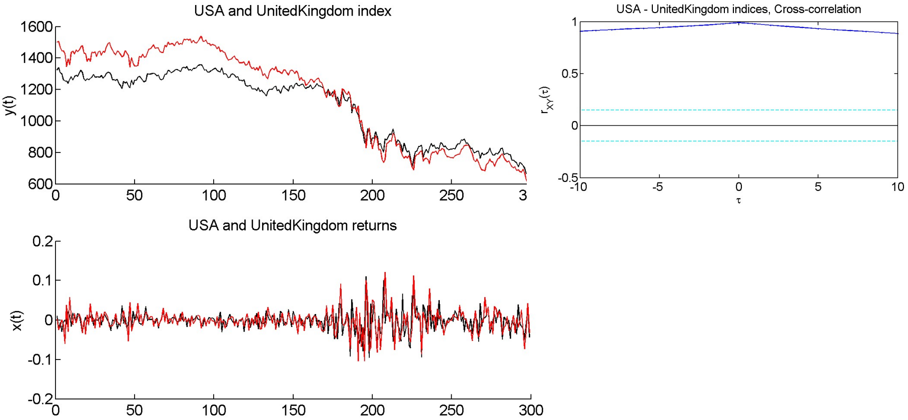

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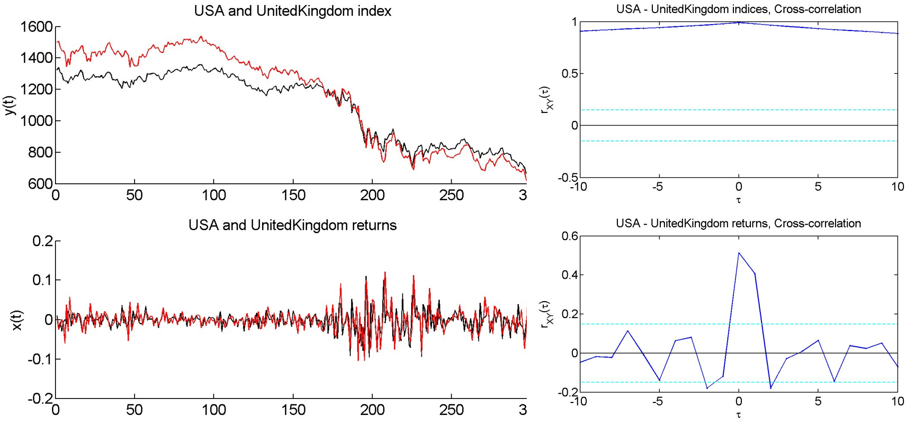

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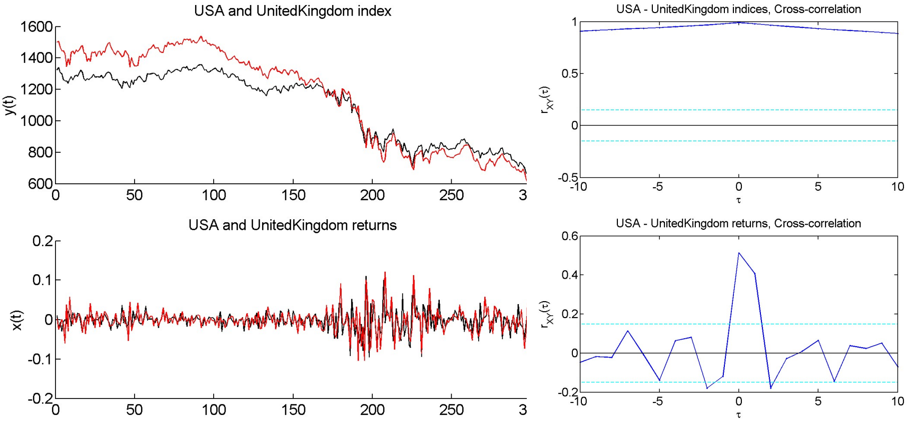

Limits of significance: $\pm z_{\alpha/2}/\sqrt{n}$.

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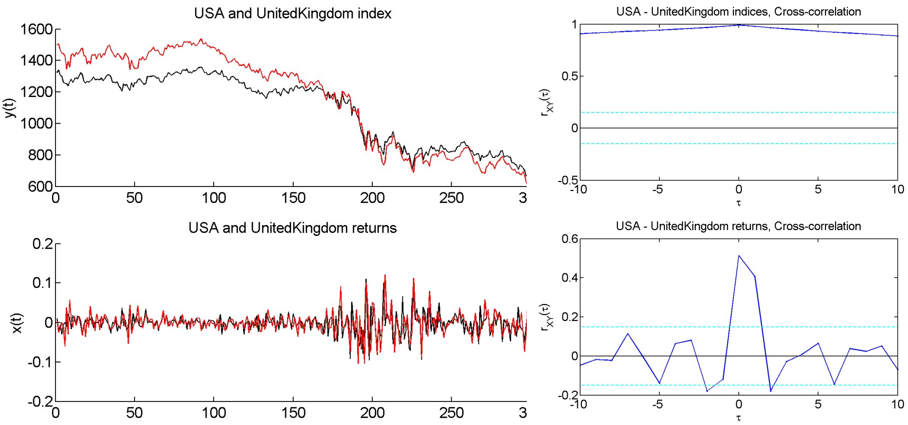

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Large cross-correlations (X: USA, Y: UK):

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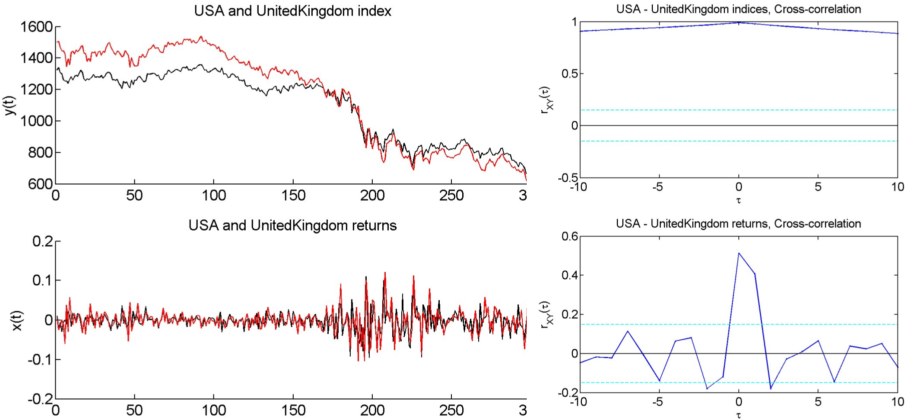

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$r_{X,Y}(0) = \mathrm{Corr}(X_t, Y_t)$: USA and UK returns are instantly correlated.

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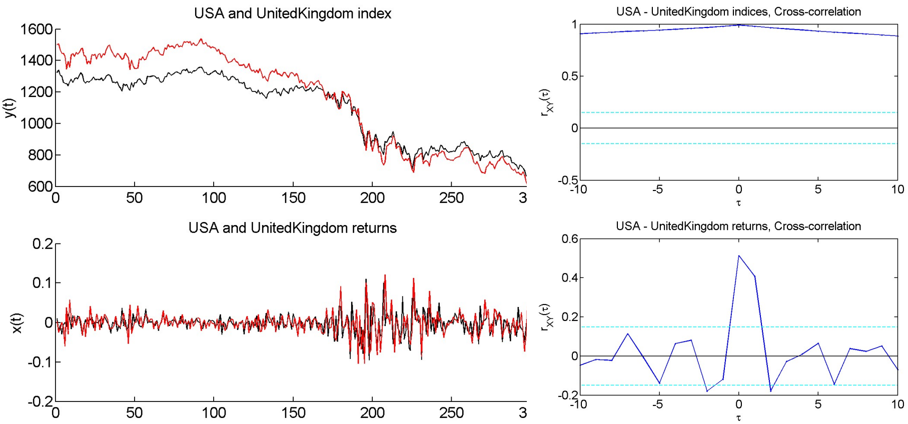

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Large cross-correlations (X: USA, Y: UK):

$r_{X,Y}(0) = \text{Corr}(X_t, Y_t)$: USA and UK returns are instantly correlated.

$r_{X,Y}(1) = \text{Corr}(X_t, Y_{t+1})$: USA return is correlated to UK return a day ahead $\Longrightarrow$ USA returns influence UK returns.

Measure of nonlinear cross-correlation

Cross-covariance: $\gamma_{XY}(\tau) = \text{Cov}[X_t, Y_{t+\tau}] = \text{E}[(X_t - \mu_X)(Y_{t+\tau} - \mu_Y)]$,

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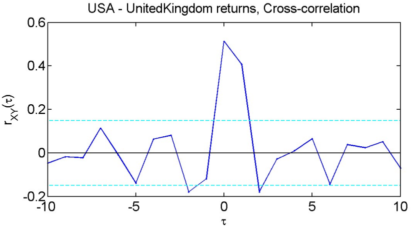
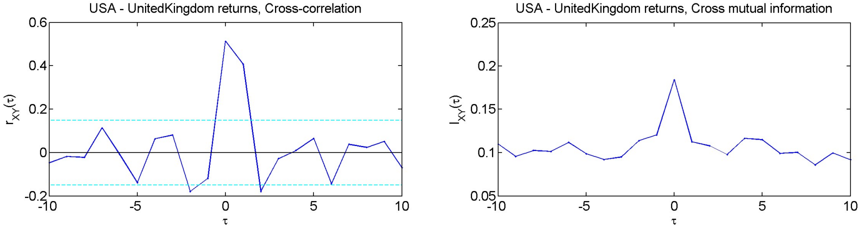
... or better, standardize both time series and use the same partition for each.

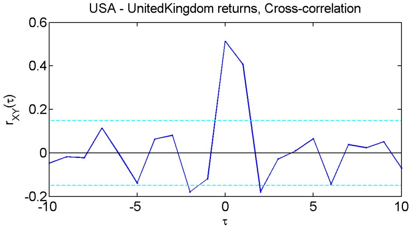
Example: two world stock indices

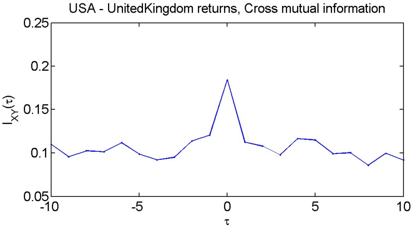


Example: two world stock indices



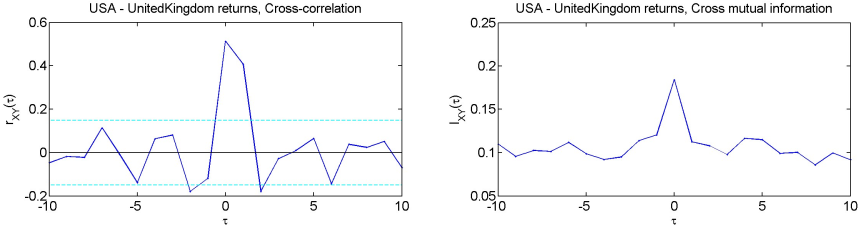
Example: two world stock indices





Limits of significance for $I_{XY}(\tau)$?

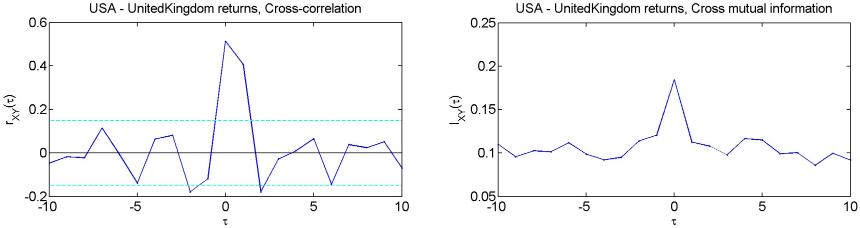
Example: two world stock indices



Limits of significance for $\mathrm{I}_{XY}(\tau)$?

$r_{XY}(0)$ and $\mathrm{I}_{XY}(0)$: USA and UK returns are instantly correlated (linearly and nonlinearly).

Example: two world stock indices



Limits of significance for $I_{XY}(\tau)$?

$r_{XY}(0)$ and $I_{XY}(0)$: USA and UK returns are instantly correlated (linearly and nonlinearly).

$r_{XY}(1)$ large but $I_{XY}(1)$ not large: do USA returns influence UK returns?

Exercise 6: Correlation of two financial indices

Find the correlation between two financial indices

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Find the correlation between two financial indices

1. Choose two of the eights markets in file `WorldMarkets.dat` (1. USA, 2. Australia, 3. UnitedKingdom, 4. Germany, 5. Greece, 6. Malaysia, 7. SouthAfrica, 8. Croatia)
2. Compute the cross-correlation between the two indices and between their returns.
3. Decide for the statistical significant cross-correlation between the two markets (use a parametric significance test).

Literature

[1] Chatfield C (2004) *The Analysis of Time Series, An Introduction*, Sixth Edition, Chapman & Hall.

[2] Brockwell PJ and Davis RA (2002) *Introduction to Time Series and Forecasting*, Second Edition, Springer.

[3] Kantz H and Schreiber T (2003) *Nonlinear Time Series Analysis*, Second Edition, Cambridge.