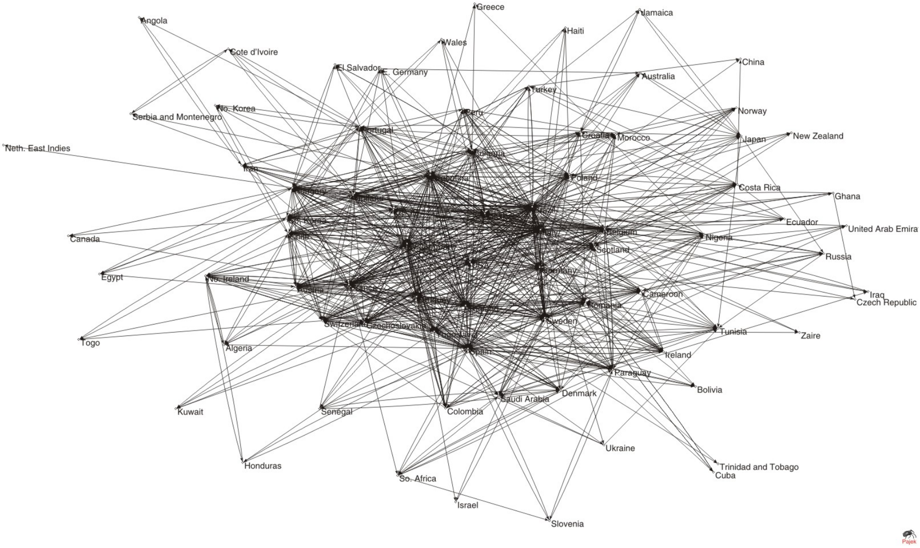
Statistical Analysis of Networks - Networks, correlation and time series

Dimitris Kugiumtzis

November 7, 2018

Contents

Introduction - Example: Games of world cup 1930 - 2006



Data from: `http://gd2006.org/contest/details.php#worldcup` see [1]

Introduction - Example: Flight connections

Data from: `https://au.pinterest.com/pin/488077678338752549`

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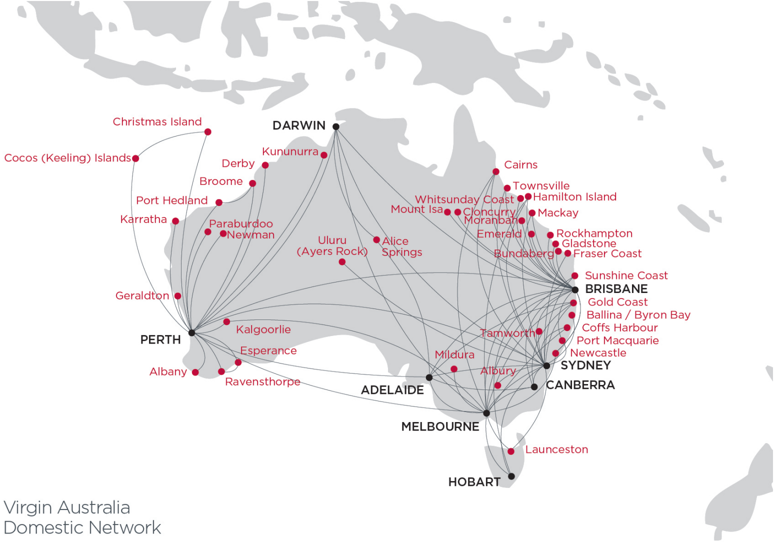
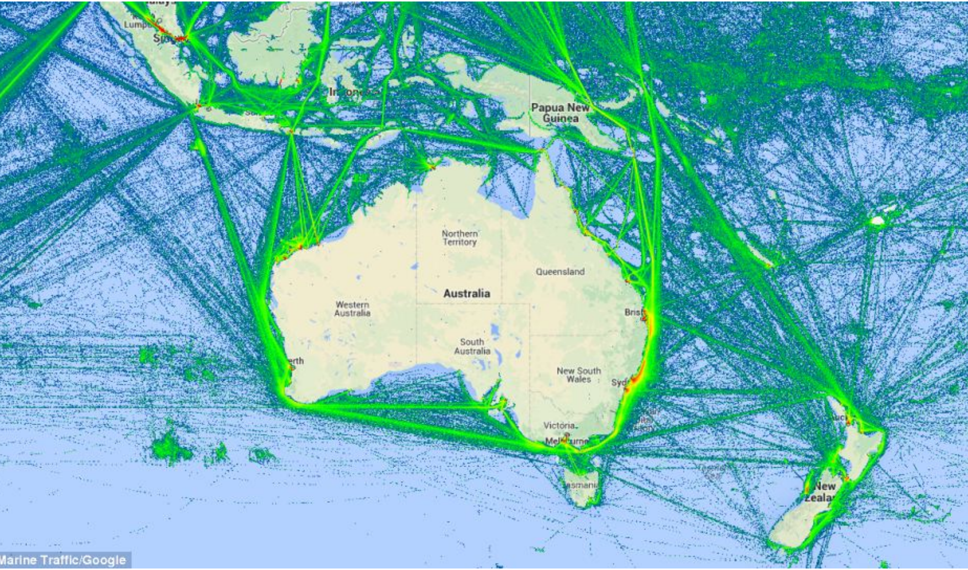
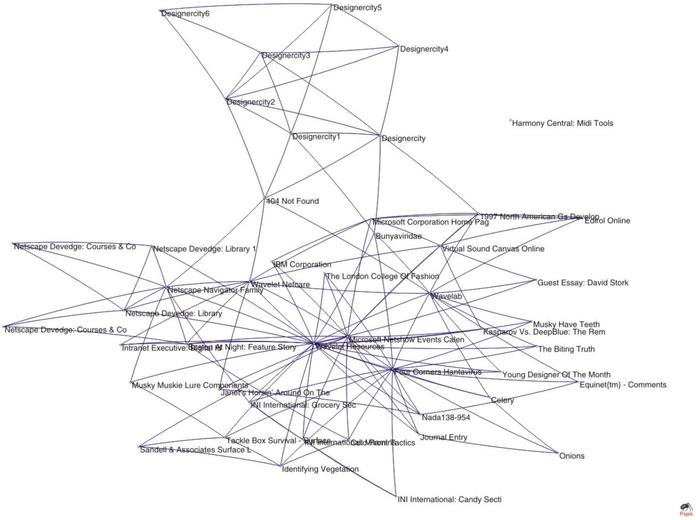


Introduction - Example: Ship connections



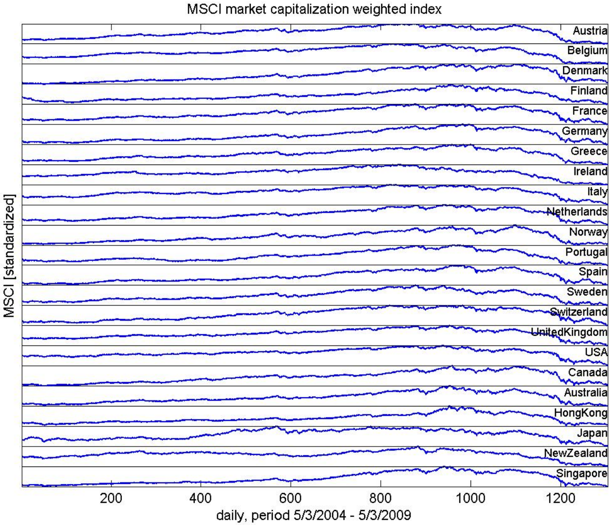
Introduction - Example: similar web-pages



Data from: `http://vlado.fmf.uni-lj.si/pub/networks/data/GD/gd97/B97.net`

see [1]

Introduction - Example: Finance



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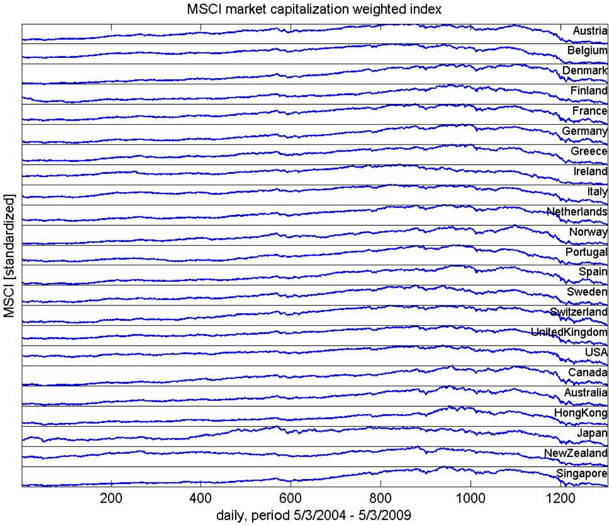
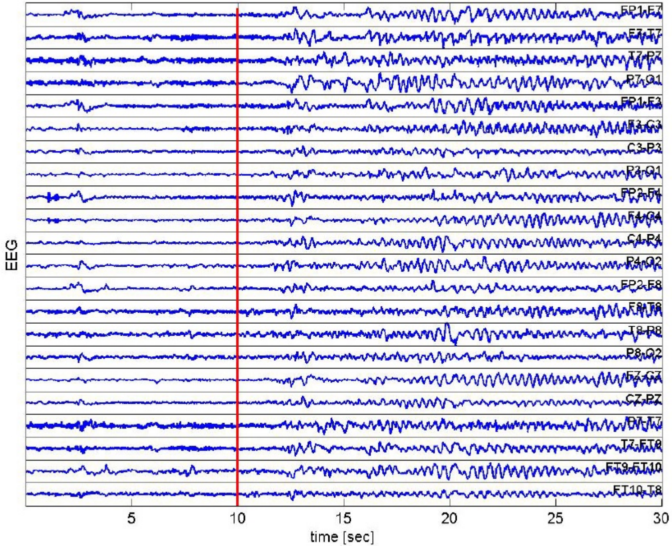


Network ?

Introduction - Example: Brain Data



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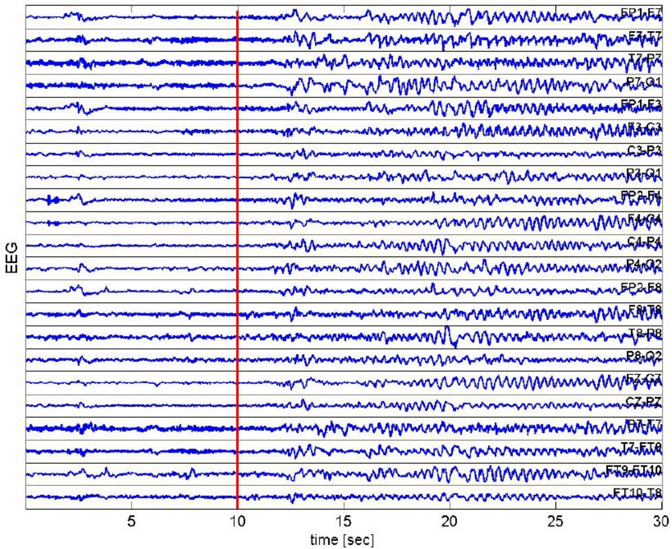

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Introduction - Example: brain network

PHYSICAL REVIEW E **79**, 061916 (2009)

Network inference with confidence from multivariate time series

Mark A. Kramer,[1,*] Uri T. Eden,[1] Sydney S. Cash,[2] and Eric D. Kolaczyk[1]
[1]*Department of Mathematics and Statistics, Boston University, Boston, Massachusetts 02215, USA*
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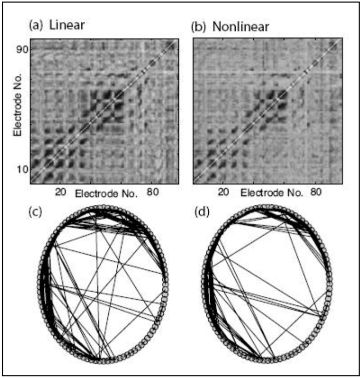

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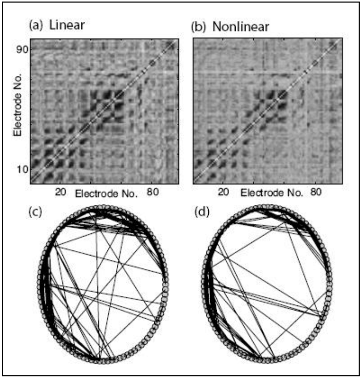

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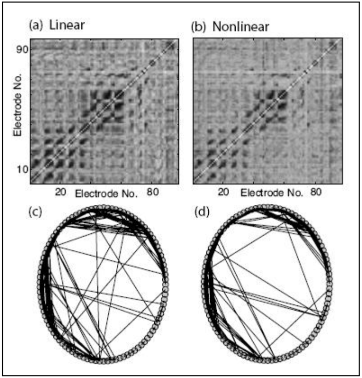

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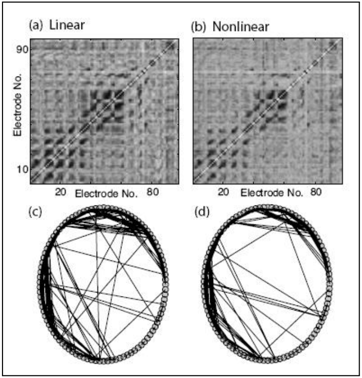

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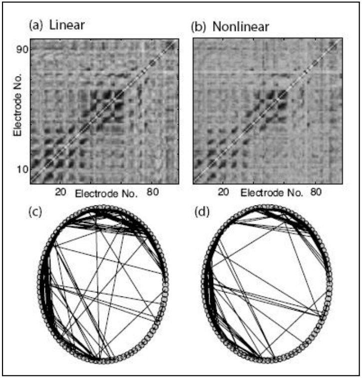

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- Use appropriate measure of correlation / association / causality.
- Assess the significance of the measure.

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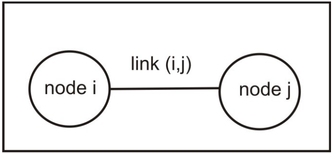

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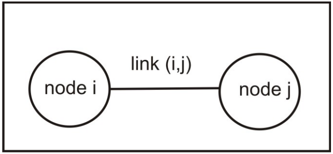

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Node: national team, web-page, ...

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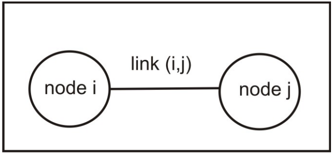

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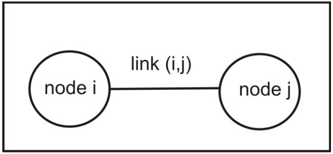

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Each node is an entity and the link denotes a connection between two entities.

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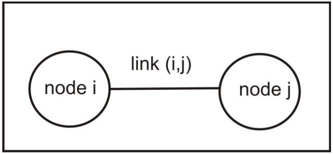

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Here, we will study a different (specific) type of nodes and links:

Each node type is a variable.

The link denotes some form of association or correlation between the variables.

Association Network (link: association rule) see [2], Sec 7.3

Link: association level between attributes of the two nodes.

Association is not necessarily determined by a statistical measure.

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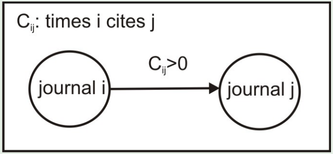

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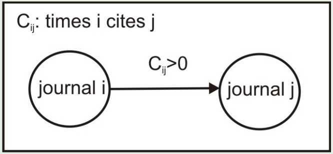

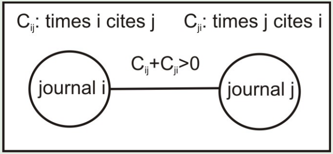

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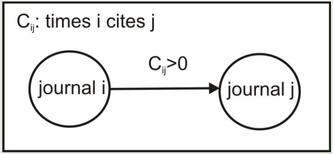
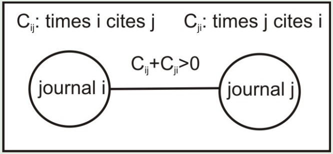
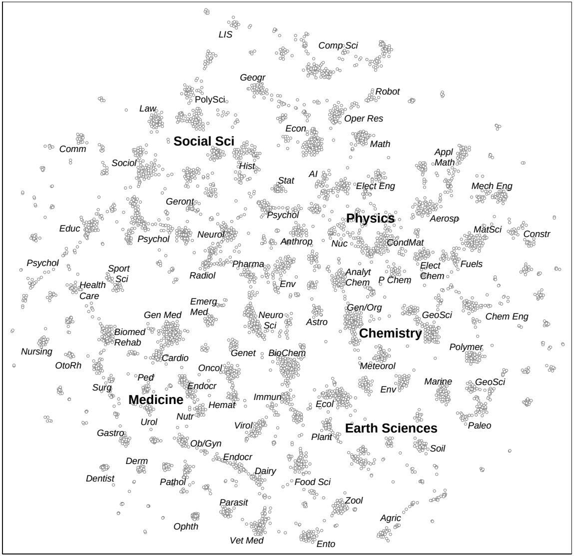




Another association rule is the Jaccard measure:

$$\mathrm{JAC}_{ij} = \mathrm{JAC}_{ji} = \frac{C_{ij}+C_{ji}}{\sum_{k\neq j} C_{ik} + \sum_{k\neq i} C_{jk}}$$

A "backbone" map of Science and Social Science: 7121 journals from year 2000



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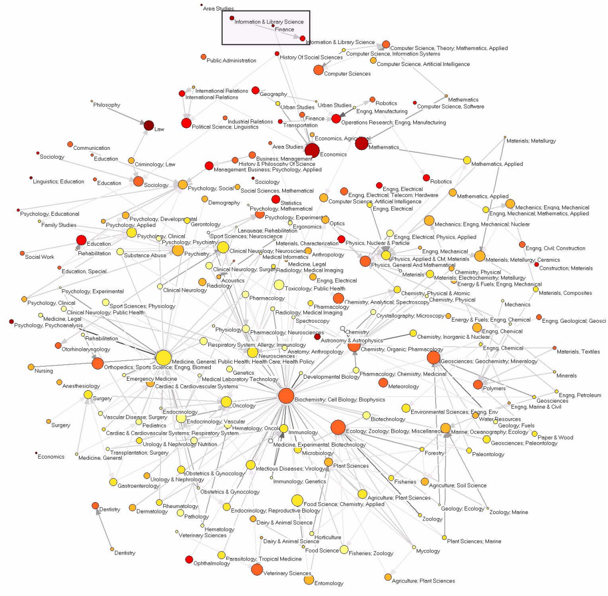
Source: `http://grants.nih.gov/grants/KM/OERRM/OER_KM_events/Borner.pdf`

The 212 nodes represent clusters of journals for different disciplines



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Each node i is presented with a vector $\mathbf{x}_i$ of n observed attributes

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$\mathrm{sim}(i,j)$ is not directly observable but can be inferred by the information in $\mathbf{x}_i$ and $\mathbf{x}_j$.

Exercise 1: Profile of Department web-sites

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You may use a software (e.g. `pajek`) to draw the network.

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Link: regulatory relationship, $\text{sim}(i, j) := \text{Corr}(X_i, X_j) = r_{X_i, X_j} = r_{ij}$

Example: Gene Regulation from Microarray Data

Data from: `http://m3d.bu.edu/cgi-bin/web/array/index.pl` see [2], Sec 7.3.1

Three genes: lrp, aroG, tyrR, and 41 experiments.

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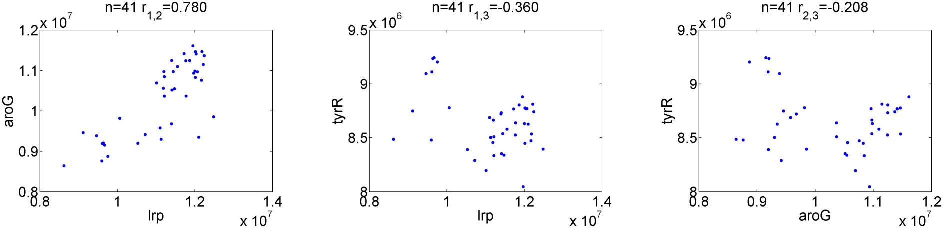

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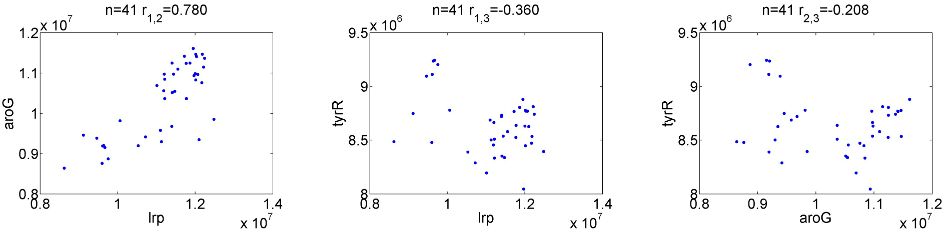

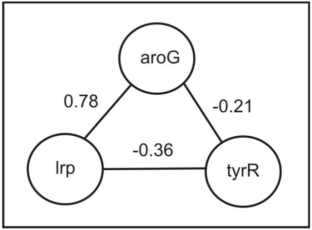

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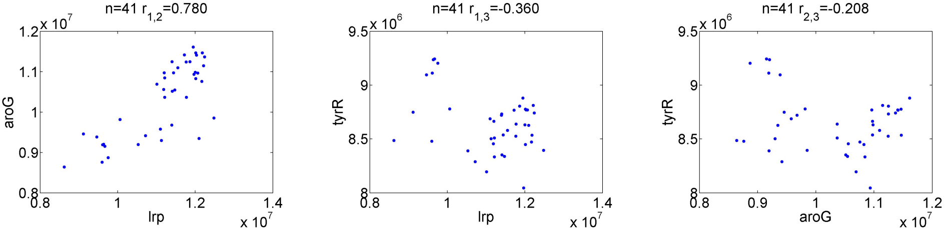

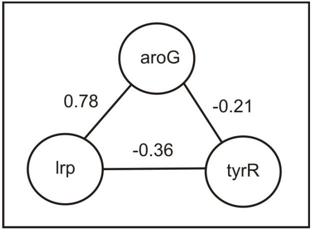

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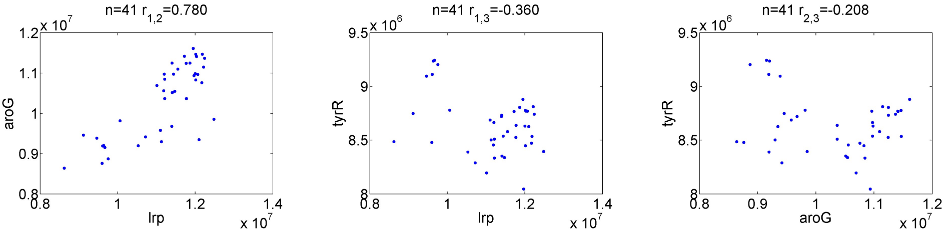

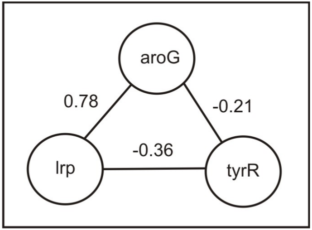

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Significance test for correlation coefficient?

Significance of correlation (parametric test)

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Test all pairs at the significance level α? Multiple testing?

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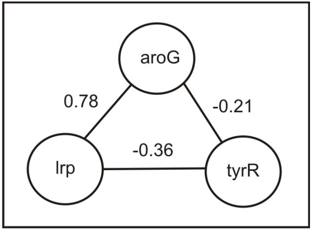

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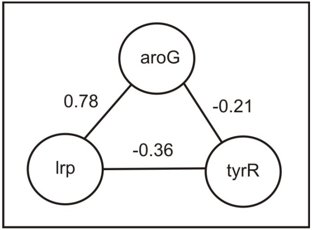

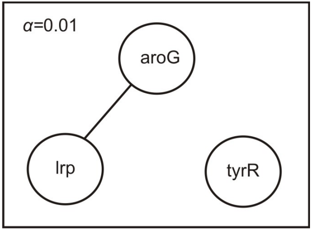

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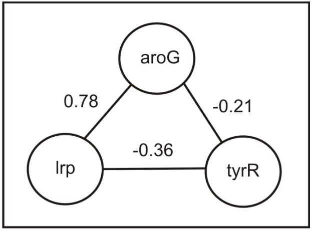

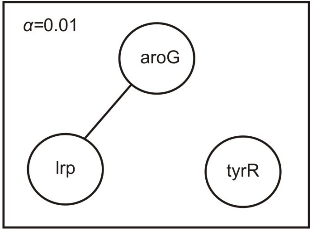

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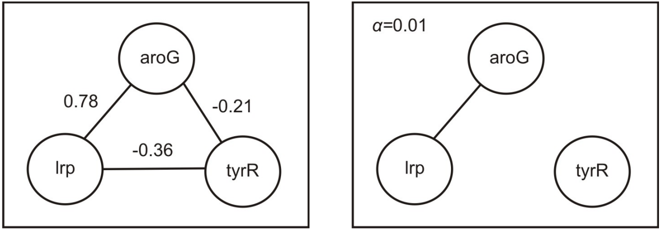

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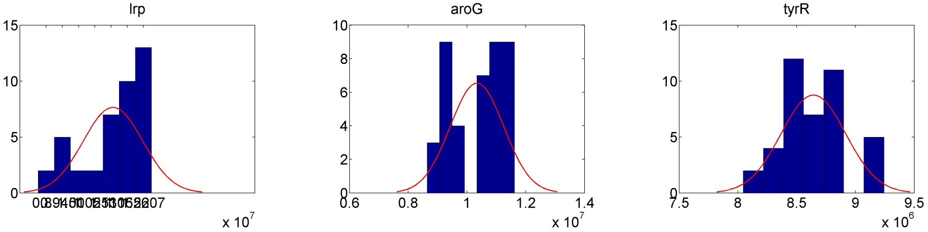

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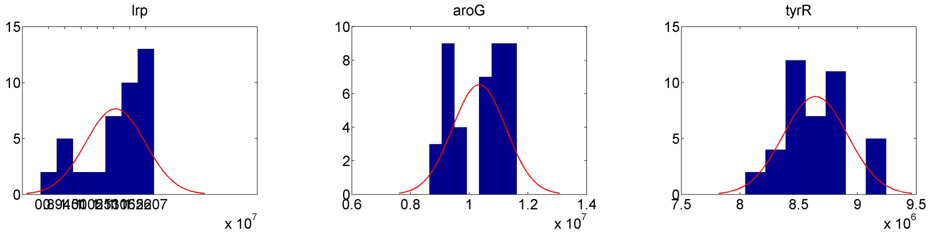

The results of the parametric testing are called into question!

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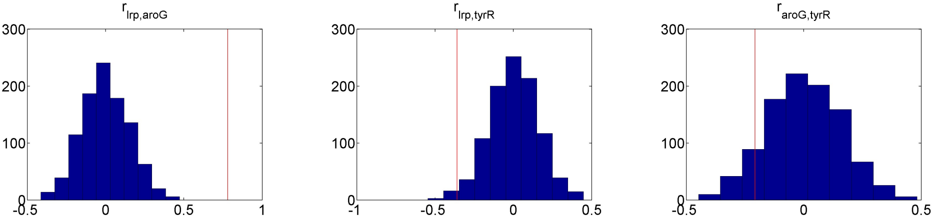

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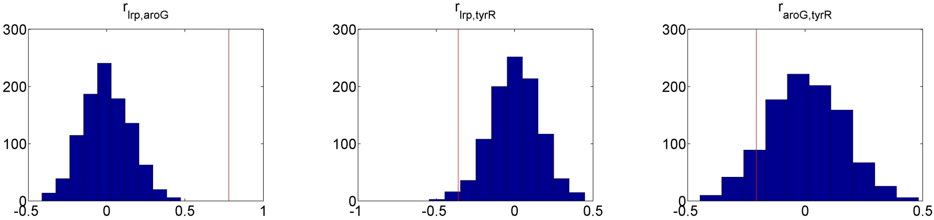

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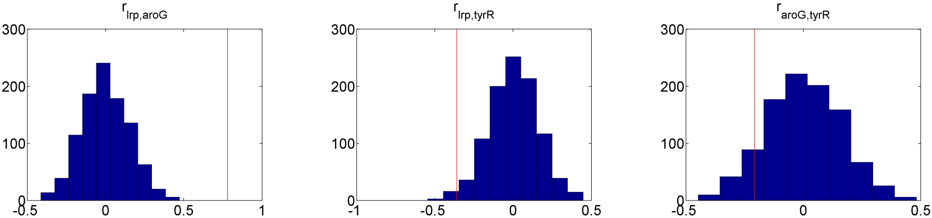

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Are the link(s) found statistically significant also "non-trivial"?

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matlab:

- for the Pearson correlation coefficient you may use the function `corrcoef`
- for random shuffling you may use the function `randperm`

Partial Correlation Network see [2], Sec 7.3.1

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To maintain links of only direct dependence, the appropriate similarity measure is the partial correlation

$$\rho_{ij|K} = \frac{\sigma_{ij|K}}{\sigma_{ii|K}\sigma_{jj|K}}$$

Partial Correlation Network see [2], Sec 7.3.1

If X_i and X_j are found to have a large r_{ij}:

1. There is direct dependence of X_i on X_j, or of X_j on X_i, or both.
2. Both X_i and X_j are dependent on an other variable (node) X_k or on m other variables (nodes) $\mathbf{X}_K = \{X_{k_1}, \ldots, X_{k_m}\}$, where $K = \{k_1, \ldots, k_m\}$.

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$\rho_{ij|K} = 0$ if X_i and X_j are independent, conditional to $\mathbf{X}_K$.

Partial variance / covariance

$\sigma_{ij|K}$, $\sigma_{ii|K}$ and $\sigma_{jj|K}$ are components of the 2×2 partial covariance matrix

$$\Sigma_{11|2} = \begin{bmatrix} s^2_{ii|K} & s_{ij|K} \\ s_{ij|K} & s^2_{jj|K} \end{bmatrix}$$

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The matrices Σ_{11}, Σ_{12}, Σ_{22} and Σ_{21} are components of the partitioned covariance matrix

$$\mathrm{Cov}(\mathbf{W}) = \left[\begin{array}{cc} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{array} \right]$$

of all involved variables partitioned as $\mathbf{W} = [\mathbf{W}_1, \mathbf{W}_2]'$, and $\mathbf{W}_1 = [X_i, X_j]'$, $\mathbf{W}_2 = \mathbf{X}_K$.

How to select the variables (nodes), to which the correlation between X_i and X_j is to be conditioned on?

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3. $r_{ij|K} = r_{\mathbf{e}_i,\mathbf{e}_j}$, the correlation coefficient of $\mathbf{e}_i$ and $\mathbf{e}_j$.

Example: Gene Regulation (continuing)

Partial correlation for the three genes: lrp, aroG, tyrR, and 41 experiments.

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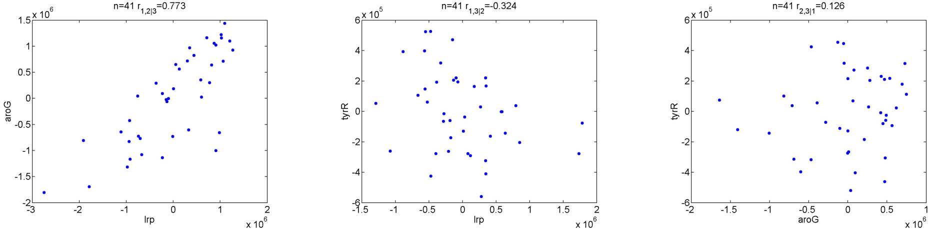

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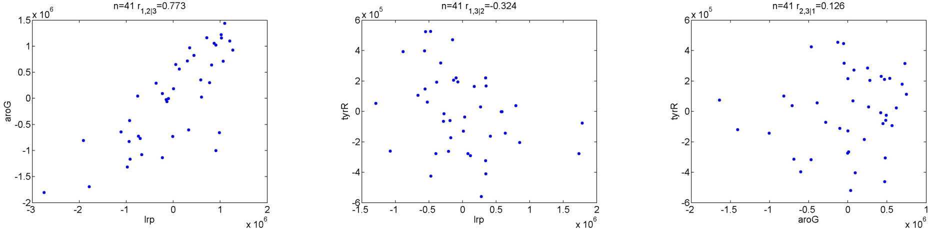

gene pair	r_{ij}	$r_{ij\|k}$	z-statistic (p-value)	rank (p-value)
lrp-aroG	0.78	0.77	6.25 (0.0000)	1001 (0.0013)
lrp-tyrR	-0.36	-0.32	-2.04 (0.0411)	23 (0.0453)
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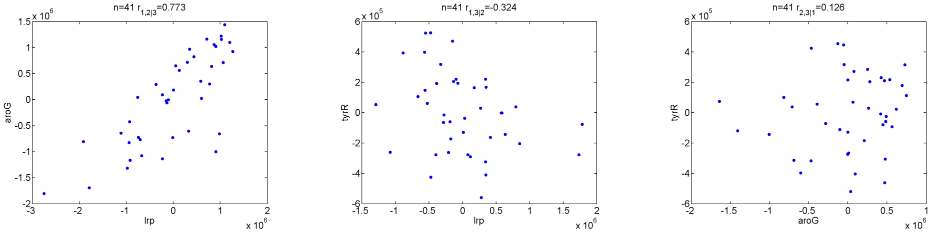

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Only the partial correlation of aroG-tyrR is substantially different from the correlation coefficient.

Exercise 3: Micro-array Data

Do the same as in Exercise 2 but using the partial correlation as similarity matrix.

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matlab: for the partial correlation you may use the function `parcorr` (in the Econometrics toolbox)

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Others ???

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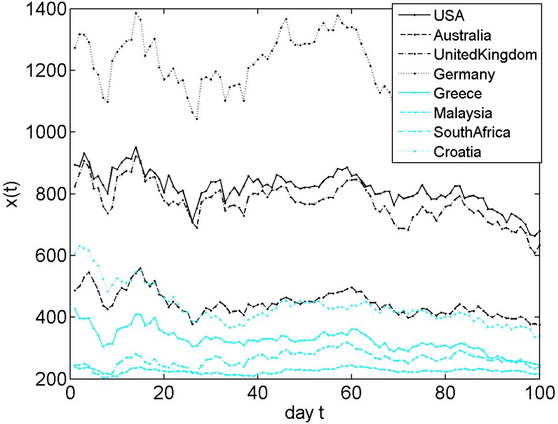

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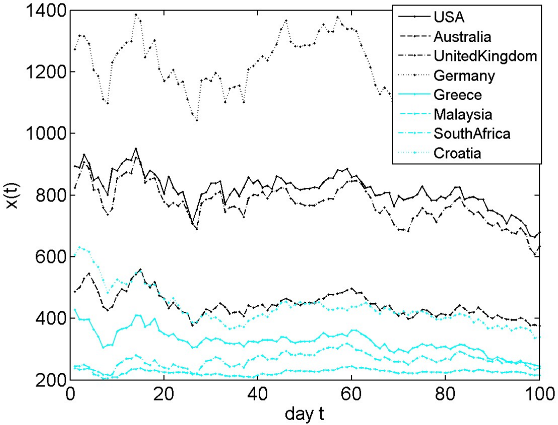

Similar indices, links among world stock markets?

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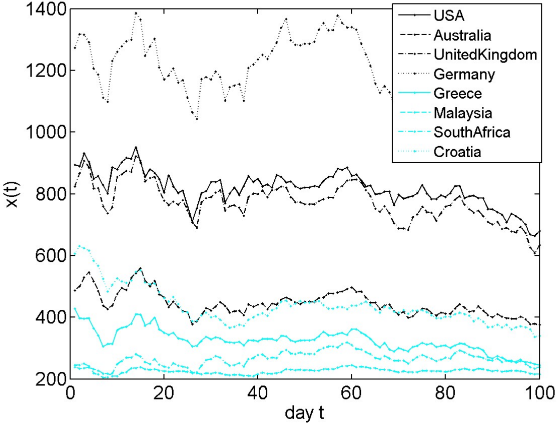

Similar indices, links among world stock markets?

Can we use the same similarity measure as for time-independent observations?

Example: World financial markets, correlation coefficient

Upper triangular: sample correlation coefficient r_{ij}.
Lower triangular: p-value for significance test for ρ_{ij} (z-statistic)

	USA	AUS	UK	GER	GRE	MAL	SAF	CRO
USA		0.86	0.92	0.88	0.89	0.33	0.27	0.75
AUS	0		0.91	0.82	0.90	0.56	0.27	0.83
UK	0	0		0.88	0.92	0.40	0.31	0.74
GER	0	0	0		0.84	0.44	0.53	0.61
GRE	0	0	0	0		0.40	0.16	0.82
MAL	0.0008	0	0	0	0		0.54	0.38
SAF	0.0057	0.0065	0.0017	0	0.1154	0		-0.15
CRO	0	0	0	0	0	0.0001	0.1408	

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GRE	0	0	0	0		0.40	0.16	0.82
MAL	0.0008	0	0	0	0		0.54	0.38
SAF	0.0057	0.0065	0.0017	0	0.1154	0		-0.15
CRO	0	0	0	0	0	0.0001	0.1408	

Almost all indices are strongly correlated.

Example: World financial markets, correlation network

Adjacency matrix, threshold at $\alpha = 0.01$ (multiple testing?)

	USA	AUS	UK	GER	GRE	MAL	SAF	CRO
USA	0	1	1	1	1	1	1	1
AUS	1	0	1	1	1	1	1	1
UK	1	1	0	1	1	1	1	1
GER	1	1	1	0	1	1	1	1
GRE	1	1	1	1	0	1	0	1
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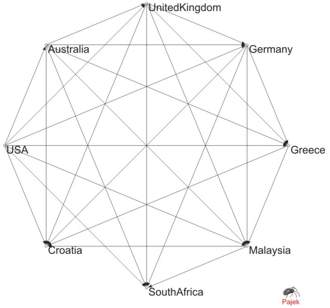

Example: World financial markets, partial correlation

Upper triangular: partial correlation $r_{ij|K}$, conditioned on all $|K| = 6$ rest variables.
Lower triangular: p-value for significance test for $\rho_{ij|K}$ (z-statistic)

	USA	AUS	UK	GER	GRE	MAL	SAF	CRO
USA		0.01	0.37	0.27	0.07	-0.27	0.11	0.27
AUS	0.9378		0.42	-0.02	0.15	0.30	0.10	0.38
UK	0.0002	0		0.08	0.36	-0.16	0.08	-0.11
GER	0.0081	0.8469	0.4693		0.38	-0.31	0.66	0.26
GRE	0.4946	0.1392	0.0003	0.0001		0.19	-0.36	0.01
MAL	0.0083	0.0033	0.1232	0.0026	0.0710		0.68	0.46
SAF	0.2908	0.3554	0.4321	0	0.0003	0		-0.70
CRO	0.0079	0.0002	0.3149	0.0099	0.9083	0	0	

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Correlation between any two indices decreased when conditioned on all others.

Example: Financial markets, partial correlation network

Adjacency matrix, threshold at $\alpha = 0.01$

	USA	AUS	UK	GER	GRE	MAL	SAF	CRO
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AUS	0	0	1	0	0	1	0	1
UK	1	1	0	0	1	0	0	0
GER	1	0	1	0	1	1	1	1
GRE	0	0	1	1	0	0	1	0
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MAL	1	1	1	1	0	0	1	1
SAF	0	0	1	1	1	1	0	1
CRO	1	1	1	1	0	1	1	0

Literature

[1] Data sets for `pajek` software,
http://vlado.fmf.uni-lj.si/pub/networks/data.

[2] Kolaczyk ED (2009) Statistical Analysis of Network Data, Springer.