X ΜΕΘΟΔΟΣ PICARD

Αν (i) $|f(x,y)| < M$

(ii) $\left|\dfrac{\partial f}{\partial y}\right| < \kappa$

για $|x-x_0|<\alpha$, $|y-y_0|<\beta$ υπάρχει μία μόνο λύση στο πρόβλημα

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0)=y_0$$

για ένα διάστημα $|x-x_0|<\rho = \min(\alpha, \beta/M)$

ή
$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt.$$

$$y(x) = \lim_{n\to\infty} \varphi_n(x)$$

$$\varphi_0(x) = y_0$$
$$\varphi_n(x) = y_0 + \int_{x_0}^{x} f(t, \varphi_{n-1}(t))\, dt$$

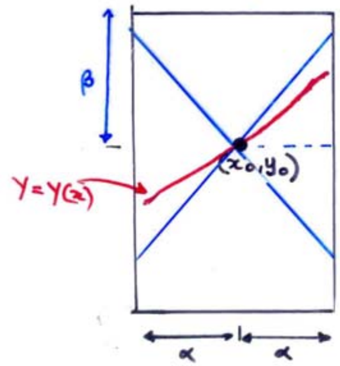
ΚΑΤΑΣΚΕΥΗ Picard Υποθέσεις

(i) $\boxed{|f(x,y)| < M \quad \text{για} \quad |x-x_0|<\alpha \quad \text{και} \quad |y-y_0|<\beta}$

Αν υπάρχει κάποια λύση τέτοια ώστε

$$y' = f(x,y) \qquad y(x_0)=y_0$$

τότε θα παρίσταται από μία καμπύλη:



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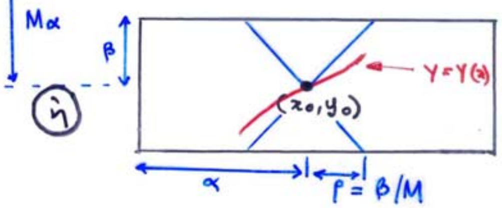


Μόνο αν $|x-x_0|<\rho$

όπου $\rho = \min\{\alpha, \beta/M\}$

(ii) $\boxed{\left|\dfrac{\partial f}{\partial y}\right| < \kappa} \Rightarrow$

$$f(x,y) - f(x,y') = \left.\frac{\partial f}{\partial y}\right|_{(x,\eta)} (y-y')$$ Θεώρημα Μέσης Τιμής

$$|f(x,y) - f(x,y')| < \kappa |y-y'|$$ Lipschitz condition

ΚΑΤΑΣΚΕΥΗ Picard

$$\varphi_0(x) = y_0$$

$$\varphi_1(x) = y_0 + \int_{x_0}^{x} f(t, \varphi_0(t))\,dt$$

$$\varphi_2(x) = y_0 + \int_{x_0}^{x} f(t, \varphi_1(t))\,dt$$

$$\cdots$$

$$\varphi_n(x) = y_0 + \int_{x_0}^{x} f(t, \varphi_{n-1}(t))\,dt$$

$$\left|\varphi_1(x) - \varphi_0(x)\right| = \left|\int_{x_0}^{x} f(t, y_0)\,dt\right| < M(x - x_0)$$

$$\left|\varphi_2(x) - \varphi_1(x)\right| = \left|\int_{x_0}^{x} \left(f(t, \varphi_1(t)) - f_2(t, \varphi_0(t))\right)dt\right| \leqslant$$

$$\leqslant k \int_{x_0}^{x} \left|\varphi_1(t) - \varphi_0(t)\right| dt \leqslant kM \frac{(x - x_0)^2}{2!}$$

$$\left|\varphi_3(x) - \varphi_2(x)\right| = \left|\int_{x_0}^{x} \left(f(t, \varphi_2(t)) - f(t, \varphi_1(t))\right)dt\right| \leqslant$$

$$\leqslant k \int_{x_0}^{x} \left|\varphi_2(t) - \varphi_1(t)\right| dt \leqslant M k^2 \frac{(x - x_0)^3}{3!}$$

$$\boxed{\left|\varphi_n(x) - \varphi_{n-1}(x)\right| \leqslant M k^{n-1} \frac{(x - x_0)^n}{n!}}$$

$$\varphi_n(x) = \varphi_0(x) + \sum_{\ell=1}^{n} \left(\varphi_\ell(x) - \varphi_{\ell-1}(x)\right)$$

$$y = \lim_{n \to \infty} \varphi_n(x) = \varphi_0(x) + \sum_{\ell=1}^{\infty} \left(\varphi_\ell(x) - \varphi_{\ell-1}(x)\right)$$

Συγκλίνει ομοιόμορφα

Weierstrass

$$\sum_{\ell=1}^{\infty} \left|\varphi_\ell(x) - \varphi_{\ell-1}(x)\right| \leqslant \frac{M}{k} \sum_{\ell=1}^{\infty} \frac{k^\ell |x - x_0|^\ell}{\ell!}$$

$$\leqslant \frac{M}{k} \left(\exp\left(k|x - x_0|\right) - 1\right)$$

$$y - \varphi_n(x) = \sum_{\ell=n+1}^{\infty} \left(\varphi_\ell(x) - \varphi_{\ell-1}(x)\right)$$

$$\left|y - \varphi_n(x)\right| \leqslant \sum_{\ell=n+1}^{\infty} \frac{M}{k} \frac{k^\ell \rho^\ell}{\ell!} = \left(\sum_{\ell=n+1}^{\infty} \frac{t^\ell}{\ell!}\right) \frac{M}{k}, \quad t = k\rho$$

$$\frac{t^{n+1}}{(n+1)!} + \frac{t^{n+2}}{(n+2)!} + \cdots = \frac{t^n}{n!} \left(\frac{t}{n+1} + \frac{t^2}{(n+1)(n+2)} + \cdots\right) \leqslant$$

$$\leqslant \frac{t^n}{n!} \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) = \frac{t^n}{n!} \left(e^t - 1\right)$$

$$\left|y - \varphi_n(x)\right| \leqslant \frac{M}{k} \frac{(k\rho)^n}{n!} \left(e^{k\rho} - 1\right)$$

Example: y' = 2x(y + 1), y(0) = 0

$\varphi_0(x) = 0$

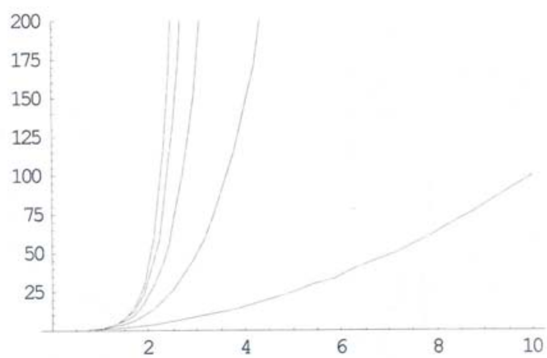
$\varphi_1(x) = x^2$

$\varphi_2(x) = x^2 + \frac{x^4}{2}$

$\varphi_3(x) = x^2 + \frac{x^4}{2} + \frac{x^6}{6}$

$\varphi_4(x) = x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24}$

$\varphi_5(x) = x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \frac{x^{10}}{120}$



$$y = e^{x^2} - 1 = \lim_{n\to\infty} \varphi_n(x)$$