THE USE OF ‘MENTAL’ BRACKETS WHEN CALCULATING ARITHMETIC EXPRESSIONS

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In this paper, the influence of the written format of an arithmetical expression on the way the students evaluate this expression, as well as a possible connection between this way of evaluation and an understanding of structure, are examined. Students from two countries evaluated a small number of rational expressions. The findings show that the rational form guided the students in their evaluation, temporarily leaving aside the rules for the order of operation. Instead, they used ‘mental’ brackets that mask a possible, or actual, structure sense.

INTRODUCTION

Brackets in arithmetic can be used by students in a manner that is either procedural or conceptual. The former is related to the rules for the order of operations in an arithmetic expression, indicating priority in the order of operations. For example in 12/(4+2) brackets are a signal to “do this first”. The latter considers brackets as structural elements, which determine the relation between the different parts of an expression. For example, in $\frac{6+4}{5+3}=(6+4)\div(5+3)$ brackets are used to preserve the structure of the rational expression and determine the relation between the two terms of the fraction. Many studies show that students exhibit both a poor procedural knowledge and a lack of an understanding of structure (Kieran, 1989; Linchevski & Livneh, 1999). For structural understanding, it is important for the student to be able to parse the expression correctly and identify the relation between the constituent parts as well as between the parts and the whole. In this paper, we examine the possible connection between the way students evaluate a rational expression and an understanding of structure given.

THEORETICAL BACKGROUND

Typically, brackets are introduced to the students alongside with the rules for the order of operations, suggesting what should be calculated first. However, they can also be necessary to preserve an expression’s mathematical structure (for example when a rational expression is rewritten horizontally) since they show how terms are grouped. Hence, students’ use of brackets can reveal their understanding of mathematical structure (Linchevski & Livneh, 1999). Hoch and Dreyfus (2004), working with students in the context of solving equations, found that students react differently to the presence or absence of brackets. More specifically, they found that the presence of

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brackets, by giving a clue of where to look and by focusing the students' attention, affected positively the students' structure sense. Brackets help students 'looking' before 'doing', which is a feature of using structure sense. Brackets focus the students' attention to recognize relationships between the parts of the expression as well as to consider a compound term as a single entity. They 'close' an expression by indicating its total, and therefore certain parts of the expression are considered as a whole, which is important for obtaining structure sense (Marchini & Papadopoulos, 2011). Despite the importance of understanding brackets in structure of arithmetic expressions, students seem to face difficulties in comprehending their role. Sometimes they ignore them, thus violating the priority of the involved operations. In their study, Blando, Kelly, Schneider, and Sleeman (1989), working with grade 7 students from a middle school, found that in the item $8-(2+4)$ some students calculated this as $6+4=10$ which means that they ignored the set of brackets and calculated first the subtraction $8-2$. Hewitt (2005) found that students, when reading written mathematical expressions with brackets, ignored the mathematical structure and the intended meaning of the expressions. Linchevski and Livneh (1999) claim that this lack of structure sense could result in that students focus on the numbers rather than on the structure or the operations. They explain that when the students have to deal with expressions of the form $a \pm b \times c$, it is necessary to make them detach the middle number (*b*) from the preceding addition/subtraction. They suggest that the use of brackets can resolve this issue, i.e., $\pm(b \times c)$. Finally, another way to make students focus on the structure of an expression is the use of 'useless' brackets to help students see algebraic structure (Hoch & Dreyfus, 2004) and to increase success rates in arithmetic expressions (Marchini & Papadopoulos, 2011). However, there are instances where useless brackets could cause impediment for the learning of the order of operations (Gunnarsson, Sönnerhed & Hernell, 2016).

In this study, our interest lies on the use of 'mental' brackets when rational expressions are written horizontally. 'Mental' brackets were introduced in the work of Linchevski and Livneh (1999). They noticed that some of their students, in their effort to solve the equation $926-167+167=?$, put 'mental' brackets around $167+167$. It seems that students imagined these brackets (not physically present) and view the equation as $926-(167+167)$. Additionally, 37% of their students put 'mental' brackets around the multiplicative terms in the expression $24 \div 3 \times 2$ in contradiction to the order of operations. Hence, Linchevski and Livneh (1999) successfully could understand students' behaviour by introducing the concept of 'mental' brackets. Therefore, we intend to apply this concept to study a different but adjacent topic. The influence of the written form of an expression upon how the same expression is evaluated, is a topic that so far has been less explored, but where research is needed. So, in this setting, our research questions are: How do the written form of an expression govern the way students evaluate arithmetic expressions? Is the interpretation of the written form related to the structural understanding of these expressions?

SETTING OF THE STUDY

The study took place in Sweden and Greece. The participants (11-12 years old) were 112 grade-6 students from Greece and 123 grade-5 students from Sweden. All the students had been taught the rules for the order of operations. A collection of groups of activities was designed aiming to reveal how the students understand the role and use of brackets while evaluating arithmetic expressions. In this paper, we examine the results from one group of activities. All the activities in this group invite students to initially re-write a rational (fractional) expression in horizontal form and then to evaluate this horizontal expression (Fig. 1). The aim of this group is to shed light on whether there is a connection between the format of the written expression and the way the students evaluate them.

You know that $\frac{8}{4}$ *can be written on a single line as 8/4.First re-write each fraction below on a single line, and then evaluate each arithmetical expression. Example:* $\frac{8}{4} = 8/4 = 2$

1) $\frac{12}{4} + 2$ = ________________ = ________

2) $\frac{12}{4+2}$ = ________________ = ________

3) $\frac{8+12}{3+2}$ = ________________ = ________

4) $\frac{20}{\frac{4}{2}}$ = ________________ = ________

5) $\frac{12+2\cdot 3}{3}$ = ________________ = ________

Figure 1: Rational expressions used in the study

A pilot study was conducted, and the findings were used to refine and decide the final form of the activities. The process was the same in both countries: sufficient time (no time limit) and same instructions. The students' worksheets constituted our data and the analysis took place in two levels: qualitative and quantitative. The qualitative part was based on content analysis (Mayring, 2014), aiming at organizing the students' answers in categories based on the solution strategies used for the expressions' evaluation. Each activity was examined separately, and the data were post-coded independently by the two authors. The coding results were compared, codes were clarified, and some data were recoded until agreement. The quantitative part is limited to the frequencies of the answers that belong in each solution strategy.

RESULTS

The re-writing of the rational expressions horizontally needs the use of brackets to preserve the structure of the rational expression and ensure that both the numerator and the denominator will be evaluated separately. For example, the 3rd activity must be written as $(8+12)\div(3+2)$ to be considered mathematically correct. The analysis resulted in nine categories of answers. Four of them correspond to correct answers and they are (i) *correct result using brackets* (C-brackets), (ii) *correct result without brackets* (C-No brackets), (iii) *correct result in the second step* since the students make the necessary calculations on the fraction's terms before the horizontal re-writing (C-2nd step), and (iv) *correct result based on the knowledge of operations of fractions* (C-fraction operations). The distribution of the students' answers into these four categories can be seen in Table 1. (Data represent absolute number of answers and the sums in each column do not add up to 123 (Swe) and 112 (Gre), because wrong or blank answers are not included.)

	Activity-1		Activity-2		Activity-3		Activity-4		Activity-5	
	Swe	Gre	Swe	Gre	Swe	Gre	Swe	Gre	Swe	Gre
C-brackets	0	3	2	5	3	7	3	10	1	6
C-No brack	62	52	33	82	32	76	28	54	31	56
C-2nd step	15	8	73	6	73	9	59	15	69	12
C-fr. oper.	0	9	0	0	0	1	0	1	0	0

Table 1: Arithmetical data of correct answers across the four categories

The wrong answers (not included in Table 1) have been divided into three categories, but it is out of the scope of this paper to present them in detail. In brief, the reasons for these wrong answers were miscalculations, left-to-right calculations, and lack of knowledge of the concept of fraction. There was also one category for the unanswered items and one for items that were not codable.

The 'C-brackets' strategy

This strategy refers to the correct use of the necessary brackets in the horizontal expression to preserve the structure of the rational expression. An example of students' answers based on this strategy can be seen in Fig. 2. It is interesting that the specific student used brackets for all the items except for the first one.

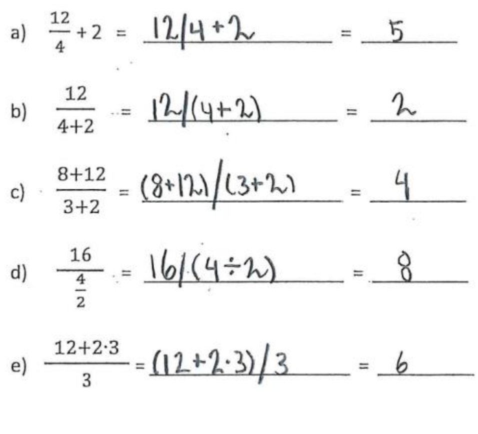

Figure 2: Answers that use brackets and preserve structure

The reason was that any pair of brackets around the division $12 \div 4$ would be 'useless' since the rules for the order of operation do not violate the structure of the given expression. Division must be done first, and this is in alignment with the structure of the rational expression. However, as it can be seen in Table 1, very few students found the correct answer using the necessary brackets (C-brackets row) in the horizontal form. An average (over all items) of 1.8 Swedish students and 6.2 Greek students used necessary brackets, thus preserving the structure of the given rational expression.

The 'C-No brackets' strategy

The students who applied this strategy found the correct arithmetic result without the use of brackets. Their evaluation of the horizontal expression is mathematically incorrect since it violates the order of operations. However, they manage to get the correct result (Fig. 3). A large number of students in both countries found the correct answers without the use of brackets. An average of 37 and 64 Swedish and Greek students, respectively, used this strategy.

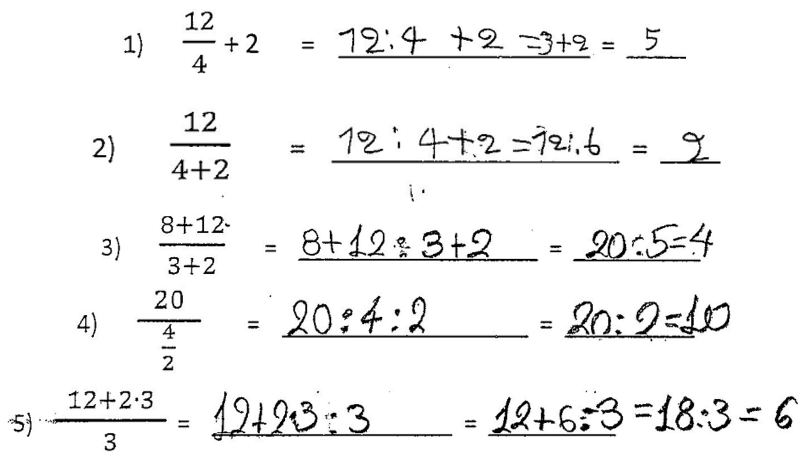

Figure 3: Correct result, but incorrect process

More specifically, the students wrote the horizontal expression without brackets. Therefore, it seems that structure is not preserved in the written form, but the result is correct. The first interesting thing to notice is that the first two items in their horizontal form are identical. However, the first one is evaluated as 5 while the second one as 2. As it is evident from the specific student's calculations, in the first item, the expression $12 \div 4 + 2$ was evaluated as $3 + 2$ whereas in the second as $12 \div 6$. The student seems to feel comfortable with these two expressions that look the same but give different results. Both final answers show an alignment with the structure of the initial rational expression of these items. In the third item, if one follows the rules for the order of operations, the result is $8 + 12 \div 3 + 2 = 8 + 4 + 2 = 14$. However, it seems that the student calculated separately the sums $8 + 12$ and $3 + 2$ (i.e., the terms of the fraction), before making the division $20 \div 5 = 4$. In a similar way, the fourth item included operations of the same priority. As it is written, the conventions are that the expression should be evaluated from left to right, resulting in $20 \div 4 \div 2 = 5 \div 2 = 2.5$, instead of 10. The last item is the most demanding, since its numerator included an arithmetic expression which needs the knowledge of the precedence rules for its evaluation. The result is again correct (6 and not 14, as it should be according to a formally correct evaluation of the horizontal expression, i.e., $12+2\times3\div3=12+6\div3=12+2=14$).

The 'C-2nd step' strategy

This strategy, as well as the next one, is not fully aligned with the task's instruction. A large number of the Swedish students (an average of 57.8 students over all items) initially made the necessary calculation either on numerator or denominator before writing the fraction horizontally (Fig. 4). The corresponding average for the Greek students was 10 students.

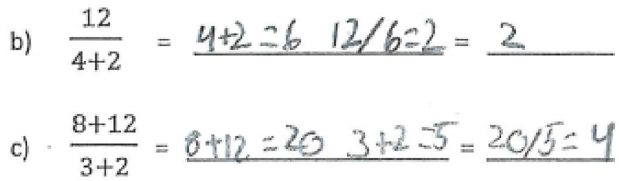

Figure 4: Correct answer with an intermediate calculation in the 2nd step

The fact is that by initially doing the calculation for either one of the terms or both, the horizontal form does not pose a dilemma to the students. For example, in Figure 4, second item, the student calculated initially the sum $4+2=6$ and therefore the horizontal form was simply asking for the division $12\div6$. Similarly, in the third item, the student made initially the calculation for each one of the terms and therefore the horizontal form is a simple division.

The 'C-fraction operations' strategy

This strategy was applied only by a very small number of Greek student who ignored the instructions of the tasks and worked out the tasks using their knowledge about the operations of fractions (Fig. 5). So, in the first example (Fig.5, left), the student made equivalent fractions with common denominator to perform the addition. In the second example (Fig. 5, right), the student follows a rule about the simplification of a whole number over a fraction (complex fractions) that is usually taught in Greek classrooms (and this rather explains why only Greek students applied this strategy).

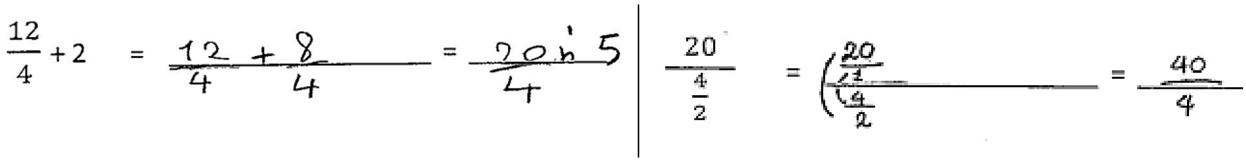

Figure 5: Correct answers based on operations of fractions

DISCUSSION

Our main interest lies in the 'C-No Brackets' category. The students' preference to this strategy is connected rather to the fact that the expressions were presented in rational form and this guided their evaluation. The students' conceptual understanding of fractions makes their evaluation of the expression straightforward as they seem to translate $\frac{a}{b}$ is translated into $a \div b$. So, in this case, the students write $8+12\div3+2$ (third item), but they evaluate the expression in a way that reflects an implicit presence of brackets that preserve the structure of the initial expression. We believe that in this case, the students use 'mental' brackets around the two sums as a kind of grouping mechanism. The same can be said for all the items of the group. The first two items were designed intentionally to contrast similar expressions in their horizontal form, $12\div4+2$. So, it can be said that in the first case the 'mental' brackets are used around the division $12\div4$ whereas in the second around the sum $4+2$.

This raises the question: Is the use of the 'C-No Brackets' strategy a mere consequence of the influence of the expressions' written format, thus indicating a lack of knowledge about the precedence rules? The knowledge of fractions is sufficient to correctly evaluate the first four items without necessarily knowing the precedence rules. Indeed, the rational form of the expression imposes the separate calculation of each term of the fraction, and given that these terms in the first four items include only one operation, it is easy to obtain the correct result. However, this knowledge is not sufficient for the last item. Its numerator includes an expression that demands an understanding of the rules for the order of operations. The fact that 31 and 56 Swedish and Greek students, respectively, used this strategy and found the correct result, is an indication that this cannot be attributed to the sole impact of the expression's written format. Therefore, it would useful to follow the students who did not use brackets in the first four items of the group, and correlate that with their answers in the fifth one. So, from the 31 Swedish students who solved correctly the fifth item, 28 did not use brackets in the first four items. The corresponding numbers for the Greek students were 51 students out of 56. But, if these students know the rules for the order of operations, how could their unorthodox behavior of the use of the 'C-No Brackets' strategy then be explained?

We argue that the rational form of the arithmetic expression imposes the way it is perceived and evaluated. The students respect the form and do not check the mathematical accuracy of their evaluation. Therefore, in the 'C-No brackets' strategy they are guided by the fractional form. So, by using mental brackets, they perceive the expressions as they should be perceived, but their writing does not preserve the structure of the initial expression. The students' understanding of the precedence rules are of minor importance in the first four tasks, because their format seem to trigger a correct evaluation accompanied by a rewriting that is not in agreement with the conventions. The students, however, who rely only on the fractional form of the expression, fail to succeed when the written format include more complex terms, such as the fifth item does, and that requires an understanding of the order of operations. Our data shows that there are students who used the 'C-No Brackets' strategy in the first four activities but were able to turn to the precedence rules when necessary (i.e., in the fifth expression). For these students, the written expressions were apparently violating the order of operations, but we believe that they mentally put brackets in the expressions when they evaluated them. Hence, the mental brackets made the students find the results of the calculations before figuring out how to write the horizontal expression. We conjecture, that in some way they see the writing as obsolete and therefore do not reflect on the structure of the expression in relation to the results of the calculation. However, even if they do so, they can turn to the precedence rules when the knowledge provoked by the format (fractions) is not sufficient for evaluating the whole expression.

CONCLUSIONS

The use of brackets is considered important for evaluating arithmetic expressions and exhibiting a structure sense (Linchevski & Livneh, 1999). We argue that this does not necessarily mean that the absence of these necessary brackets shows lack of structural understanding. Our data give evidence that when students write rational expressions in horizontal form, they do not use the brackets in the written form (to preserve the formal structure of the expression) but add the brackets mentally, in their own evaluation of the expression. Indeed, this can be interpreted as lack of structure sense. But, we argue that the way the students evaluate these horizontal expressions, even though they follow a seemingly unorthodox process that violates the order of operations, show that the structure is preserved through the use of 'mental' brackets.

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